

Fractional coloring of triangle-free planar graphs

Zdeněk Dvořák, Jean-Sébastien Sereni, Jan Volec

▶ **To cite this version:**

Zdeněk Dvořák, Jean-Sébastien Sereni, Jan Volec. Fractional coloring of triangle-free planar graphs. The Electronic Journal of Combinatorics, 2015, 22 (4), pp.#P4.11. hal-00950493v2

HAL Id: hal-00950493
https://hal.science/hal-00950493v2

Submitted on 7 Jan 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Fractional coloring of triangle-free planar graphs*

Zdeněk Dvořák†
Computer Science Institute of Charles University
Prague, Czech Republic
rakdver@iuuk.mff.cuni.cz

Jean-Sébastien Sereni‡
Centre National de la Recherche Scientifique (LORIA)
Nancy, France
sereni@kam.mff.cuni.cz

Jan Volec§
Mathematics Institute and DIMAP
University of Warwick
Coventry, UK
honza@ucw.cz

Submitted: Feb 22, 2014; Accepted: ??? ?, 2015; Published: XX
Mathematics Subject Classifications: 05C15, 05C72

Abstract

We prove that every planar triangle-free graph on n vertices has fractional chromatic number at most $3 - \frac{3}{3n+1}$.

1 Introduction

The interest in the chromatic properties of triangle-free planar graphs originated with Grötzsch's theorem [6], stating that such graphs are 3-colorable. Since then, several simpler proofs have been given, e.g., by Thomassen [13, 14]. Algorithmic questions have also been addressed: while most proofs readily yield quadratic algorithms to 3-color such graphs, it takes considerably more effort to obtain asymptotically faster algorithms. Kowalik [10] proposed an algorithm running in time $O(n \log n)$, which relies on the design of an

*This research was supported by the Czech-French Laboratory LEA STRUCO.
†Supported by the Center of Excellence – Inst. for Theor. Comp. Sci., Prague, project P202/12/G061 of Czech Science Foundation.
‡This author's work was partially supported by the French *Agence Nationale de la Recherche* under reference ANR 10 JCJC 0204 01.
§This author's work was supported by a grant of the French Government.

advanced data structure. More recently, Dvořák, Kawarabayashi and Thomas [2] managed to obtain a linear-time algorithm, yielding at the same time a yet simpler proof of Grötzsch's theorem.

The fact that all triangle-free planar graphs admit a 3-coloring implies that all such graphs have an independent set containing at least one third of the vertices. Albertson, Bollobás and Tucker [1] had conjectured that there is always a larger independent set, which was confirmed by Steinberg and Tovey [12] even in a stronger sense: all triangle-free planar n-vertex graphs admit a 3-coloring where not all color classes have the same size, and thus at least one of them forms an independent set of size at least $\frac{n+1}{3}$. This bound turns out to be tight for infinitely many triangle-free graphs, as Jones [8] showed. As an aside, let us mention that the graphs built by Jones have maximum degree 4: this is no coincidence as Heckman and Thomas later established that all triangle-free planar n-vertex graphs with maximum degree at most 3 have an independent set of order at least $\frac{3n}{8}$, which again is a tight bound—actually attained by planar graphs of girth 5.

All these considerations naturally lead us to investigate the fractional chromatic number χ_f of triangle-free planar graphs. Indeed, this invariant is known to correspond to a weighted version of the independence ratio. In addition, since $\chi_f(G) \leqslant \chi(G)$ for every graph G, Grötzsch's theorem implies that $\chi_f(G) \leqslant 3$ whenever G is triangle-free and planar. On the other hand, Jones's construction shows the existence of triangle-free planar graphs with fractional chromatic number arbitrarily close to 3. Thus one wonders whether there exists a triangle-free planar graph with fractional chromatic number exactly 3. Let us note that this happens for the circular chromatic number χ_c, which is a different relaxation of the ordinary chromatic number such that $\chi_f(G) \leqslant \chi_c(G) \leqslant \chi(G)$ for every graph G.

The purpose of this work is to answer this question. We do so by establishing the following upper bound on the fractional chromatic number of triangle-free planar n-vertex graphs, which depends on n.

Theorem 1. *Every planar triangle-free graph on n vertices has fractional chromatic number at most $\frac{9n}{3n+1} = 3 - \frac{3}{3n+1}$.*

A consequence of Theorem 1 is that no (finite) triangle-free planar graph has fractional chromatic number equal to 3. How much is it possible to improve the bound of Theorem 1? The aforementioned construction of Jones [8] yields, for each $n \geqslant 2$ such that $n \equiv 2 \pmod 3$, a triangle-free planar graph G_n with $\alpha(G_n) = \frac{n+1}{3}$. Consequently, $\chi_f(G_n) \geqslant \frac{3n}{n+1} = 3 - \frac{3}{n+1}$. Therefore, the bound of form $3 - \frac{c}{n}$ for some c in Theorem 1 is qualitatively the best possible.

The bound can be improved for triangle-free planar graphs with maximum degree at most four, giving an exact result for such graphs.

Theorem 2. *Every planar triangle-free n-vertex graph of maximum degree at most four has fractional chromatic number at most $\frac{3n}{n+1}$.*

Furthermore, the graphs of Jones's construction contain a large number of separating 4-cycles (actually, all their faces have length five). We show that planar triangle-free graphs

of *maximum degree* 4 and *without separating* 4*-cycles* cannot have fractional chromatic number arbitrarily close to 3.

Theorem 3. *There exists $\delta > 0$ such that every planar triangle-free graph of maximum degree at most four and without separating* 4*-cycles has fractional chromatic number at most* $3 - \delta$*.*

Dvořák and Mnich [5] proved that there exists $\beta > 0$ such that all planar triangle-free n-vertex graphs without separating 4-cycles contain an independent set of size at least $n/(3-\beta)$. This gives evidence that the restriction on the maximum degree in Theorem 3 might not be necessary.

Conjecture 4. There exists $\delta > 0$ such that every planar triangle-free graph without separating 4-cycles has fractional chromatic number at most $3 - \delta$.

Faces of length four are usually easy to deal with in the proofs by collapsing; thus the following seemingly simpler variant of Conjecture 4 is likely to be equivalent to it.

Conjecture 5 (Dvořák and Mnich [5])**.** There exists $\delta > 0$ such that every planar graph of girth at least five has fractional chromatic number at most $3 - \delta$.

2 Notation and auxiliary results

Consider a graph G. For an integer $a \geqslant 1$, let $[a] = \{1, \ldots, a\}$. An *a-fractional coloring* of G is a function φ assigning to each vertex of G a subset of $[a]$, such that $\varphi(u) \cap \varphi(v) = \emptyset$ for all edges uv of G. Let $f\colon V(G) \to [a]$ be any function. If the a-fractional coloring φ satisfies $|\varphi(v)| \geqslant f(v)$ for every $v \in V(G)$, then φ is an (a, f)*-coloring* of G. If $|\varphi(v)| = f(v)$ for every $v \in V(G)$, then the (a, f)-coloring φ is *tight*. Note that if G has an (a, f)-coloring, then it also has a tight one. If f is the constant function assigning to each vertex of G the value $b \in [a]$, then an (a, f)-coloring is said to be an $(a : b)$*-coloring*. An a*-coloring* is an $(a : 1)$-coloring.

Let $f_1\colon V(G) \to [a_1]$ and $f_2\colon V(G) \to [a_2]$ be arbitrary functions, and let $f\colon V(G) \to [a_1 + a_2]$ be defined by $f(v) = f_1(v) + f_2(v)$ for all $v \in V(G)$. Suppose that φ_i is an (a_i, f_i)-coloring of G for $i \in \{1, 2\}$. Let φ be defined by setting $\varphi(v) = \varphi_1(v) \cup \{a_1 + c : c \in \varphi_2(v)\}$ for every $v \in V(G)$. Then φ is an $(a_1 + a_2, f)$-coloring of G, and we write $\varphi = \varphi_1 + \varphi_2$. For an integer $k \geqslant 1$, we define $k\varphi$ to be $\underbrace{\varphi + \cdots + \varphi}_{k \text{ times}}$.

The fractional chromatic number of a graph can be expressed in various equivalent ways, see [11] for details. In this paper, we use the following definition. The *fractional chromatic number* of G is

$$\chi_f(G) = \inf \left\{ \frac{a}{b} \,:\, G \text{ has an } (a : b)\text{-coloring} \right\}.$$

We need several results related to Grötzsch's theorem. The following lemma was proved for vertices of degree at most three by Steinberg and Tovey [12]. The proof for vertices of degree four follows from the results of Dvořák and Lidický [4], as observed by Dvořák, Král' and Thomas [3].

Lemma 6. *If G is a triangle-free planar graph and v is a vertex of G of degree at most four, then there exists a 3-coloring of G such that all neighbors of v have the same color.*

In fact, Dvořák, Král' and Thomas [3] proved the following stronger statement.

Lemma 7. *There exists an integer $D \geqslant 4$ with the following property. Let G be a triangle-free planar graph without separating 4-cycles and let X be a set of vertices of G of degree at most four. If the distance between every two vertices in X is at least D, then there exists a 3-coloring of G such that all neighbors of vertices of X have the same color.*

Let G be a triangle-free plane graph. A 5-face $f = v_1v_2v_3v_4v_5$ of G is *safe* if v_1, v_2, v_3 and v_4 have degree exactly three, their neighbors $x_1, \ldots, x_4$ (respectively) not incident with f are pairwise distinct and non-adjacent, and

- the distance between x_2 and v_5 in $G - \{v_1, v_2, v_3, v_4\}$ is at least four, and
- $G - \{v_1, v_2, v_3, v_4\}$ contains no path of length exactly three between x_3 and x_4.

Lemma 8 (Dvořák, Kawarabayashi and Thomas [2, Lemma 2.2])**.** *If G is a plane triangle-free graph of minimum degree at least three and all faces of G have length five, then G has a safe face.*

Finally, let us recall the folding lemma, which is frequently used in the coloring theory of planar graphs.

Lemma 9 (Klostermeyer and Zhang [9])**.** *Let G be a planar graph with odd-girth at least $g > 3$. If $C = v_0v_1 \ldots v_{r-1}$ is a facial circuit of G with $r \neq g$, then there is an integer $i \in \{0, \ldots, r-1\}$ such that the graph G' obtained from G by identifying v_{i-1} and v_{i+1} (where indices are taken modulo r) is also of odd-girth at least g.*

3 Proofs

First, let us show a lemma based on the idea of Hilton *et. al.* [7].

Lemma 10. *Let G be a planar triangle-free graph. For a vertex $v \in V(G)$, let $f_v \colon V(G) \to [3]$ be defined by $f_v(v) = 2$ and $f_v(w) = 1$ for $w \in V(G) \setminus \{v\}$. If v has degree at most 4, then G has a $(3, f_v)$-coloring.*

Proof. Lemma 6 implies that there exists a 3-coloring of G such that all neighbors of v have the same color, without loss of generality the color $\{1\}$. Hence, we can color v by the set $\{2, 3\}$. □

Theorem 2 now readily follows.

Proof of Theorem 2. Let $V(G) = \{v_1, \ldots, v_n\}$. For $i \in \{1, \ldots, n\}$, let $f_{v_i} \colon V(G) \to [3]$ be defined as in Lemma 10, and let φ_i be a $(3, f_{v_i})$-coloring of G. Then $\varphi_1 + \cdots + \varphi_n$ is a $(3n : n + 1)$-coloring of G. □

Similarly, Lemma 7 implies Theorem 3.

Proof of Theorem 3. Let D be the constant of Lemma 7, let $m = 4^D$ and let $\delta = \frac{3}{m+1}$. We show that every planar triangle-free graph G of maximum degree at most four and without separating 4-cycles has a $(3m : m+1)$-coloring, and thus $\chi_f(G) \leqslant \frac{3m}{m+1} = 3 - \delta$.

Let G' be the graph obtained from G by adding edges between all pairs of vertices at distance at most $D-1$. The maximum degree of G' is less than $4^D = m$, and thus G' has a coloring by at most m colors. Let $C_1, \ldots, C_m$ be the color classes of this coloring (some may be empty). For $i \in [m]$, let f_i be the function defined by $f_i(v) = 2$ for $v \in C_i$ and $f_i(v) = 1$ for $v \in V(G) \setminus C_i$. Note that the distance in G between any distinct vertices in C_i is at least D, and thus Lemma 7 ensures that G has a $(3, f_i)$-coloring φ_i. Then $\varphi_1 + \cdots + \varphi_m$ is a $(3m : m+1)$-coloring of G. □

The proof of Theorem 1 is somewhat more involved. Let G be a plane triangle-free graph. We say that G is a *counterexample* if there exists an integer $n \geqslant |V(G)|$ such that G does not have a $(9n : 3n+1)$-coloring. We say that G is a *minimal counterexample* if G is a counterexample and no plane triangle-free graph with fewer than $|V(G)|$ vertices is a counterexample. Observe that every minimal counterexample is connected.

Lemma 11. *If G is a minimal counterexample, then G is 2-connected. Consequently, the minimum degree of G is at least two.*

Proof. Let $n \geqslant |V(G)|$ be an integer such that G does not have a $(9n : 3n+1)$-coloring. Since $9n > 2(3n+1)$, it follows that G has at least three vertices. Hence, it suffices to prove that G is 2-connected, and the bound on the minimum degree will follow.

Suppose for a contradiction that G is not 2-connected, and let G_1 and G_2 be subgraphs of G such that $G = G_1 \cup G_2$, the graph G_1 intersects G_2 in exactly one vertex v, and $|V(G_1)|, |V(G_2)| < |V(G)|$. By the minimality of G, neither G_1 nor G_2 is a counterexample, and thus for $i \in \{1, 2\}$, there exists a $(9n : 3n+1)$-coloring φ_i of G_i. By permuting the colors, we can assume that $\varphi_1(v) = \varphi_2(v)$. Hence, $\varphi_1 \cup \varphi_2$ is a $(9n : 3n+1)$-coloring of G, which is a contradiction. □

Lemma 12. *If G is a minimal counterexample, then every face of G has length exactly 5.*

Proof. Let $n \geqslant |V(G)|$ be an integer such that G does not have a $(9n : 3n+1)$-coloring. Suppose for a contradiction that G has a face f of length other than 5. Since G is triangle-free, it has odd girth at least five, and by Lemma 9, there exists a path $v_1v_2v_3$ in the boundary of f such that the graph G' obtained by identifying v_1 with v_3 to a single vertex z has odd girth at least five as well. It follows that G' is triangle-free. Since G is a minimal counterexample, G' has a $(9n : 3n+1)$-coloring, and by giving both v_1 and v_3 the color of z, we obtain a $(9n : 3n+1)$-coloring of G. This is a contradiction. □

Lemma 13. *If G is a minimal counterexample, then G has minimum degree at least three.*

Proof. Let $n \geqslant |V(G)|$ be an integer such that G does not have a $(9n : 3n+1)$-coloring. By Lemma 11, the graph G has minimum degree at least two. Suppose for a contradiction

that $v \in V(G)$ has degree two. Let f_v be defined as in Lemma 10 and let φ_1 be a $(3, f_v)$-coloring of G.

Since G is a minimal counterexample and $|V(G-v)| \leqslant n-1$, there exists a tight $(9n-9 : 3n-2)$-coloring φ_2 of $G-v$. Let $f(x) = 3n-2$ for $x \in V(G-v)$ and $f(v) = 3n-5$. Since both neighbors of v are assigned sets of $3n-2$ colors, there are at least $(9n-9) - 2(3n-2) = 3n-5$ colors not appearing at any neighbor of v, and thus φ_2 can be extended to a $(9n : f)$-coloring of G.

However, $3\varphi_1 + \varphi_2$ is a $(9n : 3n+1)$-coloring of G, which is a contradiction. □

Lemma 14. *No minimal counterexample contains a safe* 5*-face.*

Proof. Let G be a minimal counterexample. Let $n \geqslant |V(G)|$ be an integer such that G does not have a $(9n : 3n+1)$-coloring. Suppose for a contradiction that f contains a safe 5-face $f = v_1v_2v_3v_4v_5$, and let $x_1, \ldots, x_4$ be the neighbors of $v_1, \ldots, v_4$ that are not incident with f, respectively. For $i \in \{1, \ldots, 4\}$, let f_{v_i} be defined as in Lemma 10 and let φ_i be a $(3, f_{v_i})$-coloring of G.

Let G' be the plane graph obtained from $G - \{v_1, v_2, v_3, v_4\}$ by identifying x_2 with v_5 into a new vertex u_1, and x_3 with x_4 into a new vertex u_2. Since f is safe, G' is triangle-free. Let $N = 9n-54$. Since G is a minimal counterexample and $|V(G')| \leqslant n-6$, we conclude that G' has a tight $(N : 3n-17)$-coloring φ_5. Let $f(x) = 3n-17$ for $x \in V(G - \{v_1, v_2, v_3, v_4\})$ and $f(v_i) = 3n-20$ for $i \in \{1, \ldots, 4\}$. We extend φ_5 to an (N, f)-coloring of G as follows.

Let $\varphi_5(x_2) = \varphi_5(v_5) = \varphi_5(u_1)$ and $\varphi_5(x_3) = \varphi_5(x_4) = \varphi_5(u_2)$. Note that $|\varphi_5(x_1) \cup \varphi_5(v_5)| \leqslant 2(3n-17)$, and thus we can choose $\varphi_5(v_1)$ as a subset of $[N] \setminus (\varphi_5(x_1) \cup \varphi_5(v_5))$ of size $3n-20$. Similarly, choose $\varphi_5(v_2)$ as a subset of $[N] \setminus (\varphi_5(x_2) \cup \varphi_5(v_1))$ of size $3n-20$. Let $M_3 = [N] \setminus (\varphi_5(v_2) \cup \varphi_5(x_3))$ and $M_4 = [N] \setminus (\varphi_5(v_5) \cup \varphi_5(x_4))$. Note that $|M_3| \geqslant 3n-20$ and $|M_4| \geqslant 3n-20$. Furthermore, since $\varphi_5(x_3) = \varphi_5(x_4)$ and $\varphi_5(v_2) \cap \varphi_5(v_5) = \varphi_5(v_2) \cap \varphi_5(x_2) = \emptyset$, we have $|M_3 \cup M_4| = N - |\varphi_5(x_3)| = N - (3n-17) > 2(3n-20)$. Let $\varphi_5(v_3) \subseteq M_3$ be a set of size $3n-20$ chosen so that $|\varphi_5(v_3) \cap M_4|$ is minimum. Observe that $|M_4 \setminus \varphi_5(v_3)| \geqslant 3n-20$, and thus we can choose a set $\varphi_5(v_4) \subseteq M_4 \setminus \varphi_5(v_3)$ of size $3n-20$. This gives an (N, f)-coloring of G.

Also, by Grötzsch's theorem, G has a $(3:1)$-coloring φ_6. However, $3(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) + \varphi_5 + 6\varphi_6$ is a $(9n : 3n+1)$-coloring of G, which is a contradiction. □

We can now establish Theorem 1.

Proof of Theorem 1. Suppose for a contradiction that there exists a planar triangle-free graph G on n vertices with fractional chromatic number greater than $3 - \frac{3}{3n+1}$. Then G has no $(9n : 3n+1)$-coloring, and thus G is a counterexample. Therefore, there exists a minimal counterexample G_0. Lemmas 13, 12 and 8 imply that G_0 has a safe 5-face. However, that contradicts Lemma 14. □

References

[1] M. Albertson, B. Bollobás, and S. Tucker. The independence ratio and the maximum degree of a graph. *Congr. Numer.*, 17:43–50, 1976.

[2] Z. Dvořák, K. Kawarabayashi, and R. Thomas. Three-coloring triangle-free planar graphs in linear time. *Trans. on Algorithms*, 7:article no. 41, 2011.

[3] Z. Dvořák, D. Král', and R. Thomas. Three-coloring triangle-free graphs on surfaces V. Coloring planar graphs with distant anomalies. arXiv:0911.0885v2, January 2015.

[4] Z. Dvořák and B. Lidický. 3-coloring triangle-free planar graphs with a precolored 8-cycle. *J. Graph Theory*, forthcoming.

[5] Z. Dvořák and M. Mnich. Large independent sets in triangle-free planar graphs. arXiv:1311.2749, November 2013.

[6] H. Grötzsch. Ein Dreifarbensatz für Dreikreisfreie Netze auf der Kugel. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe*, 8:109–120, 1958/1959.

[7] A. Hilton, R. Rado, and S. Scott. A (< 5)-colour theorem for planar graphs. *Bull. London Math. Soc.*, 5:302–306, 1973.

[8] K. F. Jones. Minimum independence graphs with maximum degree four. In *Graphs and applications (Boulder, Colo., 1982)*, Wiley-Intersci. Publ., pages 221–230. Wiley, 1985.

[9] W. Klostermeyer and C. Q. Zhang. $(2 + \epsilon)$-coloring of planar graphs with large odd-girth. *J. Graph Theory*, 33:109–119, 2000.

[10] Ł. Kowalik. Fast 3-coloring triangle-free planar graphs. In Susanne Albers and Tomasz Radzik, editors, *ESA*, volume 3221 of *Lecture Notes in Computer Science*, pages 436–447. Springer, 2004.

[11] E. R. Scheinerman and D. H. Ullman. *Fractional Graph Theory*. Dover Publications Inc., Mineola, NY, 2011.

[12] R. Steinberg and C. A. Tovey. Planar Ramsey numbers. *J. Combin. Theory, Ser. B*, 59(2):288–296, 1993.

[13] C. Thomassen. Grötzsch's 3-color theorem and its counterparts for the torus and the projective plane. *J. Combin. Theory, Ser. B*, 62:268–279, 1994.

[14] C. Thomassen. A short list color proof of Grötzsch's theorem. *J. Combin. Theory, Ser. B*, 88:189–192, 2003.