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A CRITERION FOR GREENBERG’S CONJECTURE

LUCA CAPUTO AND FILIPPO ALBERTO EDOARDO NUCCIO

Abstract. We give a criterion for the vanishing of the Iwasawa \( \lambda \) invariants of totally real number fields \( K \) based on the class number of \( K(\zeta_p) \) by evaluating the \( p \)-adic \( L \) functions at \( s = -1 \).

1. Introduction

Let \( K \) be a real abelian number field and let \( p \) be an odd prime. Set \( F = K(\zeta_p) \) where \( \zeta_p \) is a primitive \( p \)-th root of unity and \( H = \text{Gal}(F/K) \). Set, moreover, \( G = \text{Gal}(F/\mathbb{Q}) \) and \( \varpi = \text{Gal}((\mathbb{Q}(\zeta_p))/\mathbb{Q}) \). So the diagram of our extensions is as follows:

\[
\begin{array}{ccc}
F = K(\zeta_p) & \uparrow H & K \\
\downarrow G & & \\
\mathbb{Q}(\zeta_p) & \uparrow \varpi & \mathbb{Q} \\
\end{array}
\]

Let \( \tilde{\omega} : H \to \mathbb{Z}_p^\times \) and \( \omega : \varpi \to \mathbb{Z}_p^\times \) be the Teichmüller characters of \( K \) and \( \mathbb{Q} \), respectively. We give (Theorem 2.3) a criterion under which a set of odd Iwasawa invariants associated to \( F \) vanish: by means of a Spiegelungssatz, these odd invariants make their even mirrors vanish too. In the case \( p = 3 \) (Corollary 2.5) or \( p = 5 \) and \( [K : \mathbb{Q}] = 2 \) (Theorem 2.7) this allows us to verify a conjecture of Greenberg for the fields satisfying our criterion.

2. Main result

Proposition 2.1. The following equality holds

\[
\text{rk}_p(K_2(\mathcal{O}_K)) = \text{rk}_p((\text{Cl}_F')_{\omega^{-1}}) + |S|
\]
where $K_2(O_K)$ is the tame kernel of $K$-theory, $Cl'_p$ is the class group of the ring $O_F[1/p]$ (and we take its $\omega^{-1}$-component for the action of $H$) and $S$ is the set of $p$-adic primes of $K$ which split completely in $F$.

Proof. This result dates back to Tate: for an explicit reference see [Gra] Theorem 7.7.3.1. $\square$

Proposition 2.2. Suppose that $\mathbb{Q}(\zeta_p)$ is linearly disjoint from $K$ over $\mathbb{Q}$. Then the following equalities holds

$$v_p(|K_2(O_K)|) = v_p(\zeta_K(-1)) \text{ if } p \geq 5$$

$$v_3(|K_2(O_K)|) = v_3(\zeta_K(-1)) + 1$$

where $v_p$ denotes the standard $p$-adic valuation and $\zeta_K$ is the Dedekind zeta function for $K$.

Proof. The Birch-Tate conjecture which has been proved by Mazur, Wiles and by Greither (since it is a consequence of the Main Conjecture in Iwasawa theory) tells that

$$\frac{|K_2(O_K)|}{w_2} = \zeta_K(-1)$$

where

$$w_2 = \max\{n \in \mathbb{N} \mid \text{the exponent of } \text{Gal}(K(\zeta_n)/K) \text{ is } 2\}$$

By our hypothesis, $\mathbb{Q}(\zeta_p)$ is linearly disjoint from $K$ over $\mathbb{Q}$. Hence $F/K$ is Galois with cyclic Galois group of order $p-1$. If $p = 3$, then for the same argument $3 | w_2$ but $9 \nmid w_2$ since $K(\zeta_9)/K$ has degree 6. Taking $p$-adic valuation we get the claim. $\square$

Theorem 2.3. Let $p \geq 5$. Suppose that the following holds

- $K$ and $\mathbb{Q}(\zeta_p)$ are linearly disjoint over $\mathbb{Q}$;
- the set $S$ of Proposition 2.1 is empty;
- the Main Conjecture of Iwasawa theory holds for $F$.

Then, if $p$ does not divide the order of $Cl'_p(\omega^{-1})$, $\lambda_{\omega^2}(F) = 0$ for all characters $\chi$ of $\Delta$.

Proof. First of all, we should just prove the theorem for non-trivial characters of $\Delta$, since $\lambda_{\omega^2} = 0$ as it corresponds to the $\omega^2$-part of the cyclotomic extension of $\mathbb{Q}(\zeta_p)$, which is always trivial: indeed, $B_{1/2} = -1/2$, and then Herbrand’s theorem and Leopoldt’s Spiegelungssatz ([Was], theorems 6.7 and 10.9) give $\lambda_{\omega^2} = 0$.

By hypothesis, the set $S$ of Proposition (2.1) is empty. Therefore $rk_p(K_2(O_K)) = 0$ and Proposition (2.2) (that we can apply because $K$ verifies its hypothesis) together with $p \geq 5$ tells us that $v_p(\zeta_K(-1)) = 0$. Since we can factor

$$\zeta_K(s) = \prod_{\chi \in \Delta} L(s, \chi) = \zeta_Q(s) \prod_{\chi \neq 1} L(s, \chi)$$

we find that

$$v_p(\zeta_K(-1)) = \sum_{\chi \neq 1} v_p(L(-1, \chi)) = 0.$$  \hspace{1cm} (2.1)

The interpolation formula for the $p$-adic $L$-function (see [Was], chapter 5) tells us that

$$L_p(-1, \chi) = (1 - \chi \omega^{-2}(p)p)L(-1, \chi \omega^{-2});$$
now we invoke the Main Conjecture as stated in ([Gre], page 452) to relate these $L$ functions with the characteristic polynomials of some sub-modules of the Iwasawa module $X_{\infty}(F)$. Observe that the hypothesis of linear disjointness tells us that $G \cong \Delta \times \hat{\omega}$ so we can split

$$X_{\infty}(F) \cong \bigoplus_{\chi \in \Delta} \bigoplus_{i=1}^{p-1} X_{\infty}(F)(\chi \omega^i)$$

where $G$ acts on $X_{\infty}(F)(\chi \omega^i)$ as $g \cdot x = (\chi \omega^i)(g)x$ for all $g \in G$ and $x \in F$. Then the Main Conjecture for $F$ allows us to write $L_{\chi}(1, \Omega^i) = f(-p/(1+p), \chi^{-1}\omega^{i-1})$ for all even $2 \leq i \leq p-1$, where $f(T, \chi^{-1}\omega^{i-1}) \in \mathbb{Z}[T]$ is the characteristic polynomial of $X_{\infty}(F)(\chi^{-1}\omega^{1-1})$: thus $L_{\chi}(1, \Omega^i)$ is an integral. Applying this for $i = 2$ and plugging it into (2.2) we find $v_p(L(-1, \chi)) = 0$ for all $\chi$, and thanks to (2.1) we indeed find $v_p(L(-1, \chi)) = 0$ for all $\chi \in \Delta$, so

$$v_p(L_{\chi}(1, \Omega^{2})) = 0 \quad \forall \chi \in \Delta.$$

If we now apply again the Main Conjecture we find that this corresponds to

$$v_p\left( f\left(\frac{1}{1+p} - 1, \chi^{-1}\omega^{1-1}\right) \right) = v_p\left( f\left(\frac{-p}{1+p}, \chi^{-1}\omega^{1-1}\right) \right) = 0 \quad \forall \chi \in \Delta.$$

Since $f(T, \chi^{-1}\omega^{1-1}) \in \mathbb{Z}[T]$, is distinguished (see [Was], chapter 7) this is possible if and only if $\deg_T(f(T, \chi^{-1}\omega^{1-1})) = 0$; but this is precisely the Iwasawa invariant $\lambda_{\chi^{-1}\omega^{1-1}}$, so we have

$$\lambda_{\chi^{-1}\omega^{1-1}} = 0 \quad \forall \chi \in \Delta.$$

Since the inequality $\lambda_{\chi^{-1}\omega^{1-1}} \geq \lambda_{\Omega^{2}}$ is classical and well-known (see, for instance, [BN] section 4), we achieve the proof. \hfill $\Box$

**Remark 2.4.** We should ask that the Main Conjecture holds for $K$ to apply it in the form of [Gre]. For this, it is enough that there exists a field $E$ that is unramified at $p$ and such that $F = E(\zeta_p)$, as it is often the case in the applications. Moreover, we remark that the hypotheses of the theorem are trivially fulfilled if $p$ is unramified in $K/\mathbb{Q}$.

**Corollary 2.5.** Assume $p = 3$. If 3 does not divide the order of $Cl(F)(\omega^{-1})$ and it is unramified in $K$, then $\lambda(K) = \lambda(F) = 0$.

**Proof.** First of all, the Theorem applies for $p = 3$ also, since we still have (2.1) thanks to $\zeta(-1) = -1/12$; moreover, $K$ is clearly disjoint from $\mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3)$, as it is unramified, and $F/K$ is ramified, so $S = \emptyset$. But in this case we have $\omega^2 = 1$, so the statement of the Theorem is that all Iwasawa invariants $\lambda_{\chi}$ vanish for $\chi \in \Delta$ and their sum is precisely $\lambda(K)$. Concerning $\lambda(F)$, in the proof of the Theorem we first prove that all $\lambda_{\chi \omega}$ vanish, and deduce from it the vanishing of their “mirror” parts. \hfill $\Box$

**Remark 2.6.** In the case $K = \mathbb{Q}(\sqrt{-d})$ is real quadratic, this is a classical result of Scholtz (although it is expressed in term of Iwasawa invariants), see [Was] Theorem 10.10.

**Theorem 2.7.** Let $K$ be a real quadratic field and suppose that $5 \mid |Cl(F)|$. Then $\lambda(K) = 0$. 
Proof. Write $K = \mathbb{Q}(\sqrt{d})$ and let $\chi$ be its non-trivial character: the result being well-known if $d = 5$ we assume throughout that $d \neq 5$. Then we should consider two cases, namely $5 \mid d$ and $5 \nmid d$. We have the following diagram of fields (we don’t draw the whole of it):

Suppose first of all that $5 \mid d$ or that $5$ is inert in $K/\mathbb{Q}$. Since $5 \nmid [F : K]$, our hypothesis implies that $5 \nmid \text{Cl}_K$ (see [Was] Lemma 16.15). But then we would trivially have $\lambda(K) = 0$ as an easy application of Nakayama’s Lemma (see [Was] Proposition 13.22). We can thus suppose that $5$ splits in $K/\mathbb{Q}$. We then apply Theorem 2.3 to $K^*$ instead of $K$: since $\mathbb{Q}(\sqrt{5}) \subseteq \mathbb{Q}(\zeta_5)$, our field is linearly disjoint over $\mathbb{Q}$ from $\mathbb{Q}(\zeta_5)$ and $S = \emptyset$ thanks to degree computations. Moreover the Main Conjecture holds for $F$ since $F = K(\zeta_5)$ and $K$ is totally real and unramified at $5$. We find that $\lambda_{\omega^2} = 0$ where $\chi^*$ is the non-trivial character of $K^*$. But clearly $\chi^* = \chi \omega^2$ so $\lambda_{\chi} = 0$. Since the Iwasawa invariant associated to the trivial character is $\lambda(\mathbb{Q}) = 0$ we have $\lambda(K) = \lambda(\mathbb{Q}) + \lambda_{\chi} = 0$. \qed

References