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Abstract. We give a criterion for the vanishing of the Iwasawa $\lambda$ invariants of totally real number fields $K$ based on the class number of $K(\zeta_p)$ by evaluating the $p$-adic $L$ functions at $s = -1$.

1. Introduction

Let $K$ be a real abelian number field and let $p$ be an odd prime. Set $F = K(\zeta_p)$ where $\zeta_p$ is a primitive $p$-th rooth of unity and $H = \text{Gal}(F/K)$. Set, moreover, $G = \text{Gal}(F/\mathbb{Q})$ and $\omega = \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. So the diagram of our extensions is as follows:

Let $\tilde{\omega} : H \to \mathbb{Z}_p^\times$ and $\omega : \varpi \to \mathbb{Z}_p^\times$ be the Teichmüller characters of $K$ and $\mathbb{Q}$, respectively. We give (Theorem 2.3) a criterion under which a set of odd Iwasawa invariants associated to $F$ vanish: by means of a Spiegelungssatz, these odd invariants make their even mirrors vanish too. In the case $p = 3$ (Corollary 2.5) or $p = 5$ and $[K : \mathbb{Q}] = 2$ (Theorem 2.7) this allows us to verify a conjecture of Greenberg for the fields satisfying our criterion.

2. Main result

Proposition 2.1. The following equality holds

$$rk_p(K_2(\mathcal{O}_K)) = rk_p((\text{Cl}_F')_{\omega^{-1}}) + |S|$$

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where $K_2(\mathcal{O}_K)$ is the tame kernel of $K$-theory, $\text{Cl}_F'$ is the class group of the ring $\mathcal{O}_F[1/p]$ (and we take its $\bar{\omega}^{-1}$-component for the action of $H$) and $S$ is the set of $p$-adic primes of $K$ which split completely in $F$.

**Proof.** This result dates back to Tate: for an explicit reference see [Gra] Theorem 7.7.3.1. □

**Proposition 2.2.** Suppose that $\mathbb{Q}(\zeta_p)$ is linearly disjoint from $K$ over $\mathbb{Q}$. Then the following equalities holds

$$v_p(|K_2(\mathcal{O}_K)|) = v_p(\zeta_K(-1)) \quad \text{if } p \geq 5$$

$$v_3(|K_2(\mathcal{O}_K)|) = v_3(\zeta_K(-1)) + 1$$

where $v_p$ denotes the standard $p$-adic valuation and $\zeta_K$ is the Dedekind zeta function for $K$.

**Proof.** The Birch-Tate conjecture which has been proved by Mazur, Wiles and by Greither (since it is a consequence of the Main Conjecture in Iwasawa theory) tells that

$$|K_2(\mathcal{O}_K)| = \zeta_K(-1)$$

where

$$w_2 = \max\{n \in \mathbb{N} | \text{the exponent of Gal}(K(\zeta_n)/K) = 2\}$$

By our hypothesis, $\mathbb{Q}(\zeta_p)$ is linearly disjoint from $K$ over $\mathbb{Q}$. Hence $F/K$ is Galois with cyclic Galois group of order $p - 1$. If $p = 3$, then for the same argument $3|w_2$ but $9 \nmid w_2$ since $K(\zeta_9)/K$ has degree 6. Taking $p$-adic valuation we get the claim. □

**Theorem 2.3.** Let $p \geq 5$. Suppose that the following holds

- $K$ and $\mathbb{Q}(\zeta_p)$ are linearly disjoint over $\mathbb{Q}$;
- the set $S$ of Proposition 2.1 is empty;
- the Main Conjecture of Iwasawa theory holds for $F$.

Then, if $p$ does not divide the order of $\text{Cl}_F(\bar{\omega}^{-1})$, $\lambda_{\chi,\omega}^p(F) = 0$ for all characters $\chi$ of $\Delta$.

**Proof.** First of all, we should just prove the theorem for non-trivial characters of $\Delta$, since $\lambda_{\omega,2} = 0$ as it corresponds to the $\omega^2$-part of the cyclotomic extension of $\mathbb{Q}(\zeta_p)$, which is always trivial: indeed, $B_{1/2} = -1/2$, and then Herbrand’s theorem and Leopoldt’s Spiegelungssatz ([Was], theorems 6.7 and 10.9) give $\lambda_{\omega,2} = 0$.

By hypothesis, the set $S$ of Proposition (2.1) is empty. Therefore $rk_p(K_2(\mathcal{O}_K)) = 0$ and Proposition (2.2) (that we can apply because $K$ verifies its hypothesis) together with $p \geq 5$ tells us that $v_p(\zeta_K(-1)) = 0$. Since we can factor

$$\zeta_K(s) = \prod_{(\chi \in \Delta)} L(s, \chi) = \zeta(s) \prod_{\chi \neq 1} L(s, \chi)$$

we find that

$$v_p(\zeta_K(-1)) = \sum_{\chi \neq 1} v_p(L(-1, \chi)) = 0.$$  \hspace{1cm} (2.1)

The interpolation formula for the $p$-adic $L$-function (see [Was], chapter 5) tells us that

$$L_p(-1, \chi) = (1 - \chi\omega^{-2}(p)p)L(-1, \chi\omega^{-2});$$  \hspace{1cm} (2.2)
now we invoke the Main Conjecture as stated in ([Gre], page 452) to relate these \( L \) functions with the characteristic polynomials of some sub-modules of the Iwasawa module \( X_\infty(F) \). Observe that the hypothesis of linear disjointness tells us that \( G \cong \Delta \times \varpi \) so we can split

\[
X_\infty(F) \cong \bigoplus_{\chi \in \Delta} \bigoplus_{i=1}^{p-1} X_\infty(F)(\chi\omega^i)
\]

where \( G \) acts on \( X_\infty(F)(\chi\omega^i) \) as \( g \cdot x = (\chi\omega^i)(g)x \) for all \( g \in G \) and \( x \in X \). Then the Main Conjecture for \( F \) allows us to write \( L_p(-1, \chi\omega^i) = f(-p/(1+p), \chi^{-1}\omega^i) \) for all even \( 2 \leq i \leq p-1 \), where \( f(T, \chi^{-1}\omega^i) \in \mathbb{Z}_p[T] \) is the characteristic polynomial of \( X_\infty(F)(\chi^{-1}\omega^i) \); thus \( L_p(-1, \chi\omega^i) \) is \( \mathbb{Z}_p \)-integral. Applying this for \( i = 2 \) and plugging it in (2.2) we find \( v_p(L(-1, \chi)) \geq 0 \) for all \( \chi \), and thanks to (2.1) we indeed find \( v_p(L(-1, \chi)) = 0 \) for all \( \chi \in \hat{\Delta} \), so

\[
v_p(L_p(-1, \chi\omega^2)) = 0 \quad \forall \chi \in \hat{\Delta}.
\]

If we now apply again the Main Conjecture we find that this corresponds to

\[
v_p\left(f\left(\frac{1}{1+p} - 1, \chi^{-1}\omega^{-1}\right)\right) = v_p\left(f\left(\frac{-p}{1+p}, \chi^{-1}\omega^{-1}\right)\right) = 0 \quad \forall \chi \in \hat{\Delta}.
\]

Since \( f(T, \chi^{-1}\omega^{-1}) \in \mathbb{Z}_p[T] \), is distinguished (see [Was], chapter 7) this is possible if and only if \( \deg_T(f(T, \chi^{-1}\omega^{-1})) = 0 \); but this is precisely the Iwasawa invariant \( \lambda_{\chi^{-1}\omega^{-1}} \), so we have

\[
\lambda_{\chi^{-1}\omega^{-1}} = 0 \quad \forall \chi \in \hat{\Delta}.
\]

Since the inequality \( \lambda_{\chi^{-1}\omega^{-1}} \geq \lambda_{\chi\omega^2} \) is classical and well-known (see, for instance, [BN] section 4), we achieve the proof.

**Remark 2.4.** We should ask that the Main Conjecture holds for \( K \) to apply it in the form of [Gre]. For this, it is enough that there exists a field \( E \) that is unramified at \( p \) and such that \( F = E(\zeta_p) \), as it is often the case in the applications. Moreover, we remark that the hypotheses of the theorem are trivially fulfilled if \( p \) is unramified in \( K/\mathbb{Q} \).

**Corollary 2.5.** Assume \( p = 3 \). If \( 3 \) does not divide the order of \( \text{Cl}_F(\omega^{-1}) \) and it is unramified in \( K \), then \( \lambda(K) = \lambda(F) = 0 \).

**Proof.** First of all, the Theorem applies for \( p = 3 \) also, since we still have (2.1) thanks to \( \zeta(-1) = -1/12 \); moreover, \( K \) is clearly disjoint from \( \mathbb{Q}(\sqrt{-3}) = \mathbb{Q}(\zeta_3) \), as it is unramified, and \( F/K \) is ramified, so \( S = \emptyset \). But in this case we have \( \omega^2 = 1 \), so the statement of the Theorem is that all Iwasawa invariants \( \lambda_{\chi} \) vanish for \( \chi \in \hat{\Delta} \) and their sum is precisely \( \lambda(K) \). Concerning \( \lambda(F) \), in the proof of the Theorem we first prove that all \( \lambda_{\chi\omega} \) vanish, and deduce from it the vanishing of their “mirror” parts.

**Remark 2.6.** In the case \( K = \mathbb{Q}(\sqrt{d}) \) is real quadratic, this is a classical result of Scholtz (although it is expressed in term of Iwasawa invariants), see [Was] Theorem 10.10.

**Theorem 2.7.** Let \( K \) be a real quadratic field and suppose that \( 5 \nmid |\text{Cl}_F| \). Then \( \lambda(K) = 0 \).
Proof. Write $K = \mathbb{Q} (\sqrt{d})$ and let $\chi$ be its non-trivial character: the result being well-known if $d = 5$ we assume throughout that $d \neq 5$. Then we should consider two cases, namely $5 \mid d$ and $5 \nmid d$. We have the following diagram of fields (we don’t draw the whole of it):

\[
\begin{array}{c}
F = K(\zeta_5) = K^*(\zeta_5) \\
\downarrow \\
Q(\zeta_5) \\
\downarrow \\
Q(\sqrt{5}) \\
\downarrow \\
K = \mathbb{Q}(\sqrt{d}) \\
\downarrow \\
K^* = \mathbb{Q}(\sqrt{5d})
\end{array}
\]

Suppose first of all that $5 \mid d$ or that $5$ is inert in $K/\mathbb{Q}$. Since $5 \nmid [F : K]$, our hypothesis implies that $5 \nmid \text{Cl}_K$ (see [Was] Lemma 16.15). But then we would trivially have $\lambda(K) = 0$ as an easy application of Nakayama’s Lemma (see [Was] Proposition 13.22). We can thus suppose that $5$ splits in $K/\mathbb{Q}$. We then apply Theorem 2.3 to $K^*$ instead of $K$: since $\mathbb{Q}(\sqrt{5}) \subseteq Q(\zeta_5)$, our field is linearly disjoint over $Q$ from $Q(\zeta_5)$ and $S = \emptyset$ thanks to degree computations. Moreover the Main Conjecture holds for $F$ since $F = K(\zeta_5)$ and $K$ is totally real and unramified at $5$. We find that $\lambda_{\omega^2\chi} = 0$ where $\chi^* = \chi \omega^2$ so $\lambda_\chi = 0$. Since the Iwasawa invariant associated to the trivial character is $\lambda(\mathbb{Q}) = 0$ we have $\lambda(K) = \lambda(\mathbb{Q}) + \lambda_\chi = 0$. □

References


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