Bell polynomials in combinatorial Hopf algebras

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Word Bell Polynomials

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Abstract

Partial multivariate Bell polynomials have been defined by E.T. Bell in 1934. These polynomials have numerous applications in Combinatorics, Analysis, Algebra, Probabilities *etc.* Many of the formulæ on Bell polynomials involve combinatorial objects (set partitions, set partitions into lists, permutations *etc*). So it seems natural to investigate analogous formulæ in some combinatorial Hopf algebras with bases indexed by these objects. In this paper we investigate the connexions between Bell polynomials and several combinatorial Hopf algebras: the Hopf algebra of symmetric functions, the Faà di Bruno algebra, the Hopf algebra of word symmetric functions etc. We show that Bell polynomials can be defined in all these algebras and we give analogues of classical results. To this aim, we construct and study a family of combinatorial Hopf algebras whose bases are indexed by colored set partitions.

1 Introduction

Partial multivariate Bell polynomials (Bell polynomials for short) have been defined by E.T. Bell in [1] in 1934. But their name is due to Riordan [29] which studied the Faà di Bruno formula [11, 12] allowing one to write the nth derivative of a composition $f \circ g$ in terms of the derivatives of f and g [28]. The applications of Bell polynomials in Combinatorics, Analysis, Algebra, Probabilities *etc.* are so numerous that it should be very long to detail them in the paper. Let us give only a few seminal examples.

- The main applications to Probabilities follow from the fact that the nth moment of a probability distribution is a complete Bell polynomial of the cumulants.
- Partial Bell polynomials are linked to the Lagrange inversion. This follows from the Faà di Bruno formula.
- Many combinatorial formulæ on Bell polynomials involve classical combinatorial numbers like Stirling numbers, Lah numbers *etc.*

The Faà di Bruno formula and many combinatorial identities can be found in [7]. The PhD thesis of M. Mihoubi [24] contains a rather complete survey of the applications of these polynomials together with numerous formulæ.

Some of the simplest formulæ are related to the enumeration of combinatorial objects (set partitions, set partitions into lists, permutations *etc.*). So it seems natural to investigate analogous formulæ in some combinatorial Hopf algebras with bases indexed by these objects. We recall that combinatorial Hopf algebras are graded bigebras with bases indexed by combinatorial objects such that the product and the coproduct have some compatibilities.

The paper is organized as follows. In Section 2, we investigate the combinatorial properties of the colored set partitions. Section 3 is devoted to the study of the Hopf algebras of

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colored set partitions. After having introduced this family of algebras, we give some special cases which can be found in the literature. The main application explains the connections with Sym, the algebra of symmetric functions. This explains that we can recover some identities on Bell polynomials when the variables are specialized to combinatorial numbers from analogous identities in some combinatorial Hopf algebras. We show that the algebra **WSym** of word symmetric functions has an important role for this construction. In Section 4, we give a few analogues of complete and partial Bell polynomials in **WSym**, ΠQSym = **WSym*** and $\mathbb{C}\langle\mathbb{A}\rangle$ and investigate their main properties. Finally, in Section 5 we investigate the connection with other noncommutative analogues of Bell polynomials defined by Munthe-Kass [33].

2 Definition, background and basic properties of colored set partitions

2.1 Colored set partitions

Let $a = (a_m)_{m\geq 1}$ be a sequence of nonnegative integers. A *colored set partition* associated to the sequence a is a set of couples

$$\Pi = \{[\pi_1, i_1], [\pi_2, i_2], \dots, [\pi_k, i_k]\}$$

such that $\pi = \{\pi_1, \dots, \pi_k\}$ is a partition of $\{1, \dots, n\}$ for some $n \in \mathbb{N}$ and $1 \leq i_\ell \leq a_{\#\pi_\ell}$ for each $1 \leq \ell \leq k$. The integer n is the *size* of Π. We will write $|\Pi| = n$, $\Pi \vDash n$ and $\Pi \Rrightarrow \pi$. We will denote by $\mathcal{CP}_n(a)$ the set of colored partitions of size n associated to the sequence a. Notice that these sets are finite. We will set also $\mathcal{CP}(a) = \bigcup_n \mathcal{CP}_n(a)$. We endow $\mathcal{CP}$ with the additional statistic $\#\Pi$ and set $\mathcal{CP}_{n,k}(a) = \{\Pi \in \mathcal{CP}_n(a) : \#\Pi = k\}$.

Example 1. Consider the sequence whose first terms are $a = (1, 2, 3, \dots)$. The colored partitions of size 3 associated to a are

$$\begin{aligned}\mathcal{CP}_3(a) = \{&\{[\{1,2,3\},1]\}, \{[\{1,2,3\},2]\}, \{[\{1,2,3\},3]\}, \{[\{1,2\},1],[\{3\},1]\},\\ &\{[\{1,2\},2],[\{3\},1]\}, \{[\{1,3\},1],[\{2\},1]\}, \{[\{1,3\},2],[\{2\},1]\},\\ &\{[\{2,3\},1],[\{1\},1]\}, \{[\{2,3\},2],[\{1\},1]\}, \{[\{1\},1],[\{2\},1],[\{3\},1]\}\}.\end{aligned}$$

The colored partitions of size 3 and cardinality 2 are

$$\begin{aligned}\mathcal{CP}_{3,2}(a) \quad = \quad &\{\{[\{1,2\},1],[\{3\},1]\}, \{[\{1,2\},2],[\{3\},1]\}, \{[\{1,3\},1],[\{2\},1]\},\\ &\{[\{1,3\},2],[\{2\},1]\}, \{[\{2,3\},1],[\{1\},1]\}, \{[\{2,3\},2],[\{1\},1]\}\}.\end{aligned}$$

It is well known that the number of colored set partitions of size n for a given sequence $a = (a_n)_n$ is equal to the evaluation of the complete Bell polynomial $A_n(a_1, \dots, a_m, \dots)$ and that the number of colored set partitions of size n and cardinality k is given by the evaluation of the partial Bell polynomial $B_n(a_1, a_2, \dots, a_m, \dots)$. That is

$$\#\mathcal{CP}_n(a) = A_n(a_1, a_2, \dots) \text{ and } \#\mathcal{CP}_{n,k}(a) = B_{n,k}(a_1, a_2, \dots).$$

Now, let $\Pi = \{[\pi_1, i_1], \dots [\pi_k, i_k]\}$ be a set such that the π_j are finite sets of nonnegative integers such that no integer belongs to more than one π_j, and $1 \leq i_j \leq a_{\#(\pi_j)}$ for each j. Then, the *standardized* $\mathrm{std}(\Pi)$ of Π is well defined as the unique colored set partition obtained by replacing the ith smallest integer in the π_j by i.

Example 2. For instance:

$$\begin{aligned}&\mathrm{std}(\{[\{1,4,7\},1],[\{3,8\},1],[\{5\},3],[\{10\},1]\}) =\\ &\{[\{1,3,5\},1],[\{2,6\},1],[\{4\},3],[\{7\},1]\}\end{aligned}$$

We define two binary operations $\uplus : \mathcal{CP}_{n,k}(a) \otimes \mathcal{CP}_{n',k'}(a) \longrightarrow \mathcal{CP}_{n+n',k+k'}(a)$,

$$\Pi \uplus \Pi' = \Pi \cup \Pi'[n],$$

where $\Pi'[n]$ means that we add n to each integer occurring in the sets of Π' and $\Cup$: $\mathcal{CP}_{n,k} \otimes \mathcal{CP}_{n',k'} \longrightarrow \mathcal{P}(\mathcal{CP}_{n+n',k+k'})$ by

$$\Pi \Cup \Pi' = \{\hat{\Pi} \cup \hat{\Pi}' \in \mathcal{CP}_{n+n',k+k'}(a) : \mathrm{std}(\hat{\Pi}) = \Pi \text{ and } \mathrm{std}(\hat{\Pi}') = \Pi'\}.$$

Example 3. We have

$$\{[\{1,3\},5],[\{2\},3]\} \uplus \{[\{1\},2],[\{2,3\},4]\} = \{[\{1,3\},5],[\{2\},3],[\{4\},2],[\{5,6\},4]\},$$

and

$$\begin{aligned}&\{[\{1\},5],[\{2\},3]\} \Cup \{[\{1,2\},2]\} = \{\{[\{1\},5],[\{2\},3],[\{3,4\},2]\},\\ &\{[\{1\},5],[\{3\},3],[\{2,4\},2]\},\{[\{1\},5],[\{4\},3],[\{2,3\},2]\},\\ &\{[\{2\},5],[\{3\},3],[\{1,4\},2]\},\{[\{2\},5],[\{4\},3],[\{1,3\},2]\},\{[\{3\},5],[\{4\},3],[\{1,2\},2]\}\}.\end{aligned}$$

The operator $\Cup$ provides an algorithm which computes all the colored partitions:

$$\mathcal{CP}_{n,k}(a) = \bigcup_{i_1+\cdots+i_k=n} \bigcup_{j_1=1}^{a_{i_1}} \cdots \bigcup_{j_k=1}^{a_{i_k}} \{[\{1,\ldots,i_1\},j_1]\} \Cup \cdots \Cup \{[\{1,\ldots,i_k\},j_k]\}. \tag{1}$$

Nevertheless each colored partition is generated more than once using this process. For a triple (Π,Π',Π'') we will denote by $\alpha^{\Pi}_{\Pi',\Pi''}$ the number of pairs of disjoint subsets $(\hat{\Pi}', \hat{\Pi}'')$ of Π such that $\hat{\Pi}' \cup \hat{\Pi}'' = \Pi$, $\mathrm{std}(\hat{\Pi}') = \Pi'$ and $\mathrm{std}(\hat{\Pi}'') = \Pi''$.

Remark 4. Notice that for $a = \mathbf{1} = (1,1,\ldots)$ (*i.e.* the ordinary set partitions), there is an alternative simple way to construct efficiently the set $\mathcal{CP}_n(\mathbf{1})$. It suffices to use the induction

$$\begin{aligned}\mathcal{CP}_0(\mathbf{1}) = &\quad \{\emptyset\},\\ \mathcal{CP}_{n+1}(\mathbf{1}) = &\quad \{\pi \cup \{\{n+1\}\} : \pi \in \mathcal{CP}_n(\mathbf{1})\} \cup \{(\pi \setminus \{e\})\\ &\quad \cup \{e \cup \{n+1\}\} : \pi \in \mathcal{CP}_n(\mathbf{1}),\ e \in \pi\}.\end{aligned} \tag{2}$$

Applying this recurrence, the set partitions of $\mathcal{CP}_{n+1}(\mathbf{1})$ are each obtained exactly once from the set partitions of $\mathcal{CP}_n(\mathbf{1})$.

2.2 Generating series

The generating series of the colored set partitions $\mathcal{CP}(a)$ is obtained from the cycle generating function for the species of colored set partitions. The construction is rather classical, see e.g. [3]. Recall first that a species of structures is a rule F which produces for each finite set U, a finite set $F[U]$ and for each bijection $\phi : U \longrightarrow V$, a function $F[\phi] : F[U] \longrightarrow F[V]$ satisfying the following properties:

- for all pairs of bijections $\phi : U \longrightarrow V$ and $\psi : V \longrightarrow W$, $F[\psi \circ \phi] = F[\psi] \circ F[\phi]$
- if Id_U denotes the identity map on U, then $F[Id_U] = Id_{F[U]}$.

An element $s \in F[U]$ is called an F-structure on U. The cycle generating function of a species F is the formal power series in infinitely independent many variables $p_1, p_2, \ldots$ (called power sums) defined by the formula

$$Z_F(p_1,p_2,\cdots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} |F([n])^{\sigma}| p^{\mathrm{cycle\_type}(\sigma)}, \tag{3}$$

where $F([n])^{\sigma}$ denotes the set of F-structures on $[n] := \{1,\ldots,n\}$ which are fixed by the permutation σ, and $p^{\lambda} = p_{\lambda_1} \cdots p_{\lambda_k}$ if λ is the vector $[\lambda_1,\ldots,\lambda_k]$. For instance, the trivial species TRIV has only one TRIV-structure on every n. Hence, its cycle generating function is nothing else but the Cauchy function

$$\sigma_1 := \exp\left\{\sum_{n\geq 1} \frac{p_n}{n} t^n\right\} = \sum_{n\geq 0} h_n. \tag{4}$$

Here h_n denotes the complete function $h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p^{\lambda}$ where $\lambda \vdash n$ means that the sum is over the partitions λ of n and $z_\lambda = \prod_i i^{m_i(\lambda(i))} m_i(\lambda(i))!$ if $m_i(\lambda(i))$ is the multiplicity of the part i in λ.

We consider also the species $\mathrm{NCS}(a)$ of non-empty colored sets having a_n $\mathrm{NCS}(a)$-structures on $[n]$ which are invariant by permutations. Its cycle generating function is

$$Z_{\mathrm{NCS}(a)} = \sum_{n\geq 1} a_n h_n. \tag{5}$$

As a species, $\mathcal{CP}(a)$ is a composite $\mathrm{TRIV} \circ \mathrm{NCS}(a)$. Hence, its cycle generating function is obtained by computing the plethysm

$$Z_{\mathrm{NCS}(a)}(p_1, p_2, \dots) = \sigma_1[Z_{\mathrm{NCS}(a)}] = \exp\left\{\sum_{n>0} \frac{1}{n} \sum_{k>0} a_k p_n[h_k]\right\}. \tag{6}$$

The exponential generating function of $\mathcal{CP}(a)$ is obtained by setting $p_1 = t$ and $p_i = 0$ for $i > 1$ in (6):

$$\sum_{n\geq 0} A_n(a_1, a_2, \dots)\frac{t^n}{n!} = \exp\left\{\sum_{i>0} \frac{a_i}{i!} t^i\right\}. \tag{7}$$

We deduce easlily that $A_n(a_1, a_2, \dots)$ are multivariate polynomials in the variables a_i's. These polynomials are known under the name of complete Bell polynomials [1]. The double generating function of $\mathrm{card}(\mathcal{CP}_{n,k}(a))$ is easily deduced from (7) by

$$\sum_{n\geq 0}\sum_{k\geq 0} B_{n,k}(a_1, a_2, \dots)\frac{x^k t^n}{n!} = \exp\left\{x \sum_{i>0} \frac{a_i}{i!} t^i\right\}. \tag{8}$$

Hence,

$$\sum_{n\geq k} B_{n,k}(a_1, a_2, \dots)\frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{i>0} \frac{a_i}{i!} t^i\right)^n. \tag{9}$$

So, one has

$$A_n(a_1, a_2, \dots) = \sum_{k=1}^{n} B_{n,k}(a_1, a_2, \dots), \forall n \geqslant 1 \text{ and } A_0(a_1, a_2, \dots) = 1. \tag{10}$$

The multivariate polynomials $B_{n,k}(a_1, a_2, \dots)$ are known under the name partial Bell polynomials [1]. Let $S_{n,k}$ denote the Stirling number of the second kind which counts the number of ways to partition a set of n objects into k nonempty subsets. The following identity holds

$$B_{n,k}(1, 1, \dots) = S_{n,k}. \tag{11}$$

Note also that $A_n(x, x, \dots) = \sum_{k=0}^{n} S_{n,k} x^k$ is the classical univariate Bell polynomial denoted by $\phi_n(x)$ in [1]. Several other identities involve combinatorial numbers. For instance, one has

$$B_{n,k}(1!, 2!, 3!, \dots) = \binom{n-1}{k-1}\frac{n!}{k!}, \text{ Unsigned Lah numbers A105278 in [30]}, \tag{12}$$

$$B_{n,k}(1, 2, 3, \dots) = \binom{n}{k} k^{n-k}, \text{ Idempotent numbers A059297 in [30]}, \tag{13}$$

$$B_{n,k}(0!, 1!, 2!, \dots) = |s_{n,k}|, \text{ Stirling numbers of the first kind A048994 in [30]}. \tag{14}$$

We can also find many other examples in [1, 7, 23, 34, 25].

Remark 5. Without loss of generality, when needed, we will suppose $a_1 = 1$ in the remainder of the paper. Indeed, if $a_1 \neq 0$, then the generating function gives

$$B_{n,k}(a_1, \dots, a_p, \dots) = a_1^k B_{n,k}\left(1, \frac{a_2}{a_1}, \cdots, \frac{a_p}{a_1}\right) \tag{15}$$

and when $a_1 = 0$,

$$B_{n,k}(0, a_2, \dots, a_p, \dots) = \begin{cases} 0 \text{ if } n < k \\ \frac{n!}{(n-k)!} B_{n,k}(a_2, \dots, a_p, \dots) \quad \text{if } n \geq k. \end{cases} \tag{16}$$

Notice that the ordinary series of the isomorphism types of $\mathcal{CP}(a)$ is obtained by setting $p_i = t^i$ in (6). Remarking that under this specialization we have $p_k[h_n] = t^{nk}$, we obtain, unsurprisingly, the ordinary generating series of colored (integer) partitions

$$\prod_{i>0} \frac{1}{(1-t^i)^{a_i}}. \tag{17}$$

2.3 Bell polynomials and symmetric functions

The algebra of symmetric functions [22, 20] is isomorphic to its polynomial realization $Sym(\mathbb{X})$ on an infinite set $\mathbb{X} = \{x_1, x_2, \dots\}$ of commuting variables, so the algebra $Sym(\mathbb{X})$ is defined as the set of polynomials invariant under permutation of the variables. As an algebra, $Sym(\mathbb{X})$ is freely generated by the power sum symmetric functions $p_n(\mathbb{X})$, defined by $p_n(\mathbb{X}) = \sum_{i\geqslant 1} x_i^n$, or the complete symmetric functions h_n, where h_n is the sum of all the monomials of total degree n in the variables $x_1, x_2, \dots$. The generating function for the h_n, called *Cauchy function*, is

$$\sigma_t(\mathbb{X}) = \sum_{n\geqslant 0} h_n(\mathbb{X})t^n = \prod_{i\geqslant 1}(1 - x_i t)^{-1}. \tag{18}$$

The relationship between the two families $(p_n)_{n\in\mathbb{N}}$ and $(h_n)_{n\in\mathbb{N}}$ is described in terms of generating series by the *Newton formula*:

$$\sigma_t(\mathbb{X}) = \exp\{\sum_{n\geqslant 1} p_n(\mathbb{X})\frac{t^n}{n}\}. \tag{19}$$

Notice that Sym is the free commutative algebra generated by $p_1, p_2 \dots$ *i.e.* $Sym = \mathbb{C}[p_1, p_2, \dots]$ and $Sym(\mathbb{X}) = \mathbb{C}[p_1(\mathbb{X}), p_2(\mathbb{X}), \dots]$ when $\mathbb{X}$ is an infinite alphabet without relations on the variables. As a consequence of the Newton Formula (19), it is also the free commutative algebra generated by $h_1, h_2, \dots$. The freeness of the algebra provides a mechanism of specialization. For any sequence of commutative scalars $u = (u_n)_{n\in\mathbb{N}}$, there is a morphism of algebra ϕ_u sending each p_n to u_n (resp. sending h_n to a certain v_n which can be deduced from u). These morphisms are manipulated *as if* there exists an underlying alphabet (so called *virtual alphabet*) $\mathbb{X}_u$ such that $p_n(\mathbb{X}_u) = u_n$ (resp. $h_n(\mathbb{X}_u) = v_n$). The interest of such a vision is that one defines operations on sequences and symmetric functions by manipulating alphabets.

The bases of Sym are indexed by the *partitions* $\lambda \vdash n$ of all the integers n. A partition λ of n is a finite noncreasing sequence of positive integers $(\lambda_1 \geq \lambda_2 \geq \dots)$ such that $\sum_i \lambda_i = n$.

By specializing either the power sums p_i or the complete functions h_i to the numbers $\frac{a_i}{i!}$, the partial and complete Bell polynomials are identified with well known bases.

The algebra Sym is usually endowed with three coproducts:

- the coproduct Δ such that the power sums are Lie-like ($\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n$);
- the coproduct Δ' such that the power sums are group-like ($\Delta'(p_n) = p_n \otimes p_n$);
- the coproduct of Faà di Bruno (see e.g. [9, 18]).

Most of the formulas on Bell polynomials can be stated and proved using specializations and these three coproducts. Since this is not really the purpose of our article, we have deferred to Appendix A a list of examples which are rewrites of existing proofs in terms of symmetric functions. One of the aims of our paper is to rise some of these identities to other combinatorial Hopf algebras.

3 Hopf algebras of colored set partitions

3.1 The Hopf algebras CWSym(a) and CΠQSym(a)

Let **CWSym**(a) (**CWSym** for short when there is no ambiguity) be the algebra defined by its basis $(\Phi_\Pi)_{\Pi\in\mathcal{CP}(a)}$ indexed by colored set partitions associated to the sequence $a = (a_m)_{m\geq 1}$ and the product

$$\Phi_\Pi \Phi_{\Pi'} = \Phi_{\Pi\uplus\Pi'}. \tag{20}$$

Example 6. One has

$$\Phi_{\{[\{1,3,5\},3],[\{2,4\},1]\}}\Phi_{\{[\{1,2,5\},4],[\{3\},1],[\{4\},2]\}} = \Phi_{\{[\{1,3,5\},3],[\{2,4\},1],[\{6,7,10\},4],[\{8\},1],[\{9\},2]\}}.$$

Let $\mathbf{CWSym}_n$ be the subspace generated by the elements Φ_Π such that $\Pi \vDash n$.

For each n we consider an infinite alphabet $\mathbb{A}_n$ of noncommuting variables and we suppose $\mathbb{A}_n \cap \mathbb{A}_m = \emptyset$ when $n \neq m$. For each colored set partition $\Pi = \{[\pi_1, i_1], [\pi_2, i_2], \dots, [\pi_k, i_k]\}$, we construct a polynomial $\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots) \in \mathbb{C}\langle\bigcup_n \mathbb{A}_n\rangle$

$$\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots) := \sum_{\mathtt{w}=\mathtt{a}_1\dots\mathtt{a}_n} \mathtt{w}, \tag{21}$$

where the sum is over the words $\mathtt{w} = \mathtt{a}_1 \dots \mathtt{a}_n$ satisfying

- For each $1 \leq \ell \leq k$, $\mathtt{a}_j \in \mathbb{A}_{i_\ell}$ if and only if $j \in \pi_\ell$.
- If $j_1, j_2 \in \pi_\ell$ then $\mathtt{a}_{j_1} = \mathtt{a}_{j_2}$.

Example 7.

$$\Phi_{\{[\{1,3\},3],[\{2\},1],[\{4\},3]\}}(\mathbb{A}_1, \mathbb{A}_2, \dots) = \sum_{\substack{\mathtt{a}_1, \mathtt{a}_2 \in \mathbb{A}_3 \\ \mathtt{b} \in \mathbb{A}_1}} \mathtt{a}_1 \mathtt{b} \mathtt{a}_1 \mathtt{a}_2.$$

Proposition 8. *The family*

$$\Phi(a) := (\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots))_{\Pi \in \mathcal{CP}(a)}$$

spans a subalgebra of $\mathbb{C}\langle \bigcup_n \mathbb{A}_n \rangle$ *which is isomorphic to* $\mathbf{CWSym}(a)$.

Proof. First, remark that $\mathrm{span}(\Phi(a))$ is stable under concatenation. Indeed,

$$\Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots)\Phi_{\Pi'}(\mathbb{A}_1, \mathbb{A}_2, \dots) = \Phi_{\Pi \uplus \Pi'}(\mathbb{A}_1, \mathbb{A}_2, \dots).$$

Furthermore, this shows that $\mathrm{span}(\Phi(a))$ is homomorphic to $\mathbf{CWSym}(a)$ and that an explicit (onto) morphism is given by $\Phi_\Pi \longrightarrow \Phi_\Pi(\mathbb{A}_1, \mathbb{A}_2, \dots)$. Observing that the family $\Phi(a)$ is linearly independent, the fact that the algebra $\mathbf{CWSym}(a)$ is graded in finite dimension implies the result. $\square$

We turn **CWSym** into a Hopf algebra by considering the coproduct

$$\Delta(\Phi_\Pi) = \sum_{\substack{\hat{\Pi}_1 \cup \hat{\Pi}_2 = \Pi \\ \hat{\Pi}_1 \cap \hat{\Pi}_2 = \emptyset}} \Phi_{\mathrm{std}(\hat{\Pi}_1)} \otimes \Phi_{\mathrm{std}(\hat{\Pi}_2)} = \sum_{\Pi_1, \Pi_2} \alpha^{\Pi}_{\Pi_1, \Pi_2} \Phi_{\Pi_1} \otimes \Phi_{\Pi_2}. \tag{22}$$

Indeed, **CWSym** splits as a direct sum of finite dimension spaces

$$\mathbf{CWSym} = \bigoplus_n \mathbf{CWSym}_n.$$

This defines a natural graduation on **CWSym**. Hence, since it is a connected algebra, it suffices to verify that it is a bigebra. More precisely:

$$\begin{aligned} \Delta(\Phi_\Pi \Phi_{\Pi'}) &= \Delta(\Phi_{\Pi \uplus \Pi'}) \\ &= \sum_{\substack{\hat{\Pi}_1 \cup \hat{\Pi}_2 = \Pi, \hat{\Pi}'_1 \cup \hat{\Pi}'_2 = \Pi'[n] \\ \hat{\Pi}_1 \cap \hat{\Pi}_2 = \emptyset, \hat{\Pi}'_1 \cap \hat{\Pi}'_2 = \emptyset}} \Phi_{\mathrm{std}(\hat{\Pi}_1) \uplus \mathrm{std}(\hat{\Pi}'_1)} \otimes \Phi_{\mathrm{std}(\hat{\Pi}_2) \uplus \mathrm{std}(\hat{\Pi}'_2)} \\ &= \Delta(\Phi_\Pi)\Delta(\Phi_{\Pi'}). \end{aligned}$$

Notice that Δ is cocommutative.

Example 9. For instance,

$$\begin{aligned} \Delta\left(\Phi_{\{[\{1,3\},5],[\{2\},3]\}}\right) &= \Phi_{\{[\{1,3\},5],[\{2\},3]\}} \otimes 1 + \Phi_{\{[\{1,2\},5]\}} \otimes \Phi_{\{[\{1\},3]\}} + \\ &\quad \Phi_{\{[\{1\},3]\}} \otimes \Phi_{\{[\{1,2\},5]\}} + 1 \otimes \Phi_{\{[\{1,3\},5],[\{2\},3]\}}. \end{aligned}$$

The graded dual $\mathrm{C\Pi QSym}(a)$ (which will be called $\mathrm{C\Pi QSym}$ for short when there is no ambiguity) of **CWSym** is the Hopf algebra generated as a space by the dual basis $(\Psi_\Pi)_{\Pi \in \mathcal{CP}(a)}$ of $(\Phi_\Pi)_{\Pi \in \mathcal{CP}(a)}$. Its product and its coproduct are given by

$$\Psi_{\Pi'}\Psi_{\Pi''} = \sum_{\Pi \in \Pi' \uplus \Pi''} \alpha^{\Pi}_{\Pi', \Pi''} \Psi_\Pi \text{ and } \Delta(\Psi_\Pi) = \sum_{\Pi' \uplus \Pi'' = \Pi} \Psi_{\Pi'} \otimes \Psi_{\Pi''}.$$

Example 10. For instance, one has

$$\begin{aligned} \Psi_{\{[\{1,2\},3]\}}\Psi_{\{[\{1\},4],[\{2\},1]\}} &= \Psi_{\{[\{1,2\},3],[\{3\},4],[\{4\},1]\}} + \Psi_{\{[\{1,3\},3],[\{2\},4],[\{4\},1]\}} \\ &\quad + \Psi_{\{[\{1,4\},3],[\{2\},4],[\{3\},1]\}} + \Psi_{\{[\{2,3\},3],[\{1\},4],[\{4\},1]\}} \\ &\quad + \Psi_{\{[\{2,4\},3],[\{1\},4],[\{3\},1]\}} + \Psi_{\{[\{3,4\},3],[\{1\},4],[\{2\},1]\}} \end{aligned}$$

and

$$\begin{aligned} \Delta(\Psi_{\{[\{1,3\},3],[\{2\},4],[\{4\},1]\}}) &= 1 \otimes \Psi_{\{[\{1,3\},3],[\{2\},4],[\{4\},1]\}} + \Psi_{\{[\{1,3\},3],[\{2\},4]\}} \otimes \Psi_{\{[\{1\},1]\}} \\ &\quad + \Psi_{\{[\{1,3\},3],[\{2\},4],[\{4\},1]\}} \otimes 1. \end{aligned}$$

3.2 Special cases

In this section, we investigate a few interesting special cases of the construction.

3.2.1 Word symmetric functions

The most prominent example follows from the specialization $a_n = 1$ for each n. In this case, the Hopf algebra **CWSym** is isomorphic to **WSym**, the Hopf algebra of word symmetric functions. Let us briefly recall its construction. The algebra of word symmetric functions is a way to construct a noncommutative analogue of the algebra Sym. Its bases are indexed by set partitions. After the seminal paper [32], this algebra was investigated in [2, 16] as well as an abstract algebra as in its realization with noncommutative variables. Its name comes from its realization as a subalgebra of $\mathbb{C}\langle\mathbb{A}\rangle$ where $\mathbb{A} = \{\mathtt{a}_1, \dots, \mathtt{a}_n \cdots\}$ is an infinite alphabet.

Consider the family of functions $\Phi := \{\Phi_\pi\}_\pi$ whose elements are indexed by set partitions of $\{1, \dots, n\}$. The algebra **WSym** is formally generated by Φ using the shifted concatenation product: $\Phi_\pi \Phi_{\pi'} = \Phi_{\pi\pi'[n]}$ where π and π' are set partitions of $\{1, \dots, n\}$ and $\{1, \dots, m\}$, respectively, and $\pi'[n]$ means that we add n to each integer occurring in π'. The polynomial realization $\mathbf{WSym}(\mathbb{A}) \subset \mathbb{C}\langle\mathbb{A}\rangle$ is defined by $\Phi_\pi(\mathbb{A}) = \sum_{\mathtt{w}} \mathtt{w}$ where the sum is over the words $\mathtt{w} = \mathtt{a}_1 \cdots \mathtt{a}_n$ where $i, j \in \pi_\ell$ implies $\mathtt{a}_i = \mathtt{a}_j$, if $\pi = \{\pi_1, \dots, \pi_k\}$ is a set partition of $\{1, \dots, n\}$.

Example 11. For instance, one has $\Phi_{\{\{1,4\}\{2,5,6\}\{3,7\}\}}(\mathbb{A}) = \sum_{\mathtt{a},\mathtt{b},\mathtt{c} \in \mathbb{A}} \mathtt{abcabbc}$.

Although the construction of $\mathbf{WSym}(\mathbb{A})$, the polynomial realization of **WSym**, seems to be close to $Sym(\mathbb{X})$, the structures of the two algebras are quite different since the Hopf algebra **WSym** is not autodual. Surprisingly, the graded dual $\Pi\mathrm{QSym} := \mathbf{WSym}^*$ of **WSym** admits a realization in the same subspace $(\mathbf{WSym}(\mathbb{A}))$ of $\mathbb{C}\langle\mathbb{A}\rangle$ but for the shuffle product.

With no surprise, we notice the following fact:

Proposition 12.

- *The algebras* $\mathbf{CWSym}(1, 1, \dots)$, **WSym** *and* $\mathbf{WSym}(\mathbb{A})$ *are isomorphic.*
- *The algebras* $\mathrm{C}\Pi\mathrm{QSym}(1, 1, \dots)$, $\Pi\mathrm{QSym}$ *and* $(\mathbf{WSym}(\mathbb{A}), \sqcup\!\sqcup)$ *are isomorphic.*

In the rest of the paper, when there is no ambiguity, we will identify the algebras **WSym** and $\mathbf{WSym}(\mathbb{A})$.

The word analogue of the basis $(c_\lambda)_\lambda$ of Sym is the dual basis $(\Psi_\pi)_\pi$ of $(\Phi_\pi)_\pi$.

Other bases are known, for example, the word monomial functions defined by $\Phi_\pi = \sum_{\pi \leq \pi'} M_{\pi'}$, where $\pi \leq \pi'$ indicates that π is finer than π', *i.e.*, that each block of π' is a union of blocks of π.

Example 13. For instance,

$$\begin{aligned}\Phi_{\{\{1,4\}\{2,5,6\}\{3,7\}\}} = {} & M_{\{\{1,4\}\{2,5,6\}\{3,7\}\}} + M_{\{\{1,2,4,5,6\}\{3,7\}\}} + M_{\{\{1,3,4,7\}\{2,5,6\}\}} \\ & + M_{\{\{1,4\}\{2,3,5,6,7\}\}} + M_{\{\{1,2,3,4,5,6,7\}\}}.\end{aligned}$$

From the definition of the M_π, we deduce that the polynomial representation of the word monomial functions is given by $M_\pi(\mathbb{A}) = \sum_{\mathtt{w}} \mathtt{w}$ where the sum is over the words $\mathtt{w} = \mathtt{a}_1 \cdots \mathtt{a}_n$ where $i, j \in \pi_\ell$ if and only if $\mathtt{a}_i = \mathtt{a}_j$, where $\pi = \{\pi_1, \dots, \pi_k\}$ is a set partition of $\{1, \dots, n\}$.

Example 14. $M_{\{\{1,4\}\{2,5,6\}\{3,7\}\}}(\mathbb{A}) = \sum_{\substack{\mathtt{a},\mathtt{b},\mathtt{c} \in \mathbb{A} \\ \mathtt{a} \neq \mathtt{b}, \mathtt{a} \neq \mathtt{c}, \mathtt{b} \neq \mathtt{c}}} \mathtt{abcabbc}$.

The analogue of complete symmetric functions is the basis $(S_\pi)_\pi$ of $\Pi\mathrm{QSym}$ which is the dual of the basis $(M_\pi)_\pi$ of **WSym**.

The algebra $\Pi\mathrm{QSym}$ is also realized in the space $\mathbf{WSym}(\mathbb{A})$: it is the subalgebra of $(\mathbb{C}\langle\mathbb{A}\rangle, \sqcup\!\sqcup)$ generated by $\Psi_\pi(\mathbb{A}) = \pi!\Phi_\pi(\mathbb{A})$ where $\pi! = \#\pi_1! \cdots \#\pi_k!$ for $\pi = \{\pi_1, \dots, \pi_k\}$. Indeed, the linear map $\Psi_\pi \longrightarrow \Psi_\pi(\mathbb{A})$ is a bijection sending $\Psi_{\pi_1}\Psi_{\pi_2}$ to

$$\begin{aligned}\sum_{\substack{\pi = \pi_1' \cup \pi_2',\ \pi_1' \cap \pi_2' = \emptyset \\ \pi_1 = \mathrm{std}(\pi_1'),\ \pi_2 = \mathrm{std}(\pi_2')}} \Psi_\pi(\mathbb{A}) &= \pi_1!\pi_2! \sum_{\substack{\pi = \pi_1' \cup \pi_2',\ \pi_1' \cap \pi_2' = \emptyset \\ \pi_1 = \mathrm{std}(\pi_1'),\ \pi_2 = \mathrm{std}(\pi_2')}} \Phi_\pi(\mathbb{A}) \\ &= \pi_1!\pi_2! \Phi_{\pi_1}(\mathbb{A}) \sqcup\!\sqcup \Phi_{\pi_2}(\mathbb{A}) = \Psi_{\pi_1}(\mathbb{A}) \sqcup\!\sqcup \Psi_{\pi_2}(\mathbb{A}).\end{aligned}$$

With these notations the image of S_π is $S_\pi(\mathbb{A}) = \sum_{\pi' \leq \pi} \Psi_{\pi'}(\mathbb{A})$. For our realization, the duality bracket $\langle\ |\ \rangle$ implements the scalar product $\langle\ |\ \rangle$ on the space $\mathbf{WSym}(\mathbb{A})$ for which $\langle S_{\pi_1}(\mathbb{A}) | M_{\pi_2}(\mathbb{A})\rangle = \langle \Phi_{\pi_1}(\mathbb{A}) | \Psi_{\pi_2}(\mathbb{A})\rangle = \delta_{\pi_1,\pi_2}$.

The subalgebra of $(\mathbf{WSym}(\mathbb{A}), \sqcup\!\sqcup)$ generated by the complete functions $S_{\{\{1,\dots,n\}\}}(\mathbb{A})$ is isomorphic to Sym. Therefore, we define $\sigma_t^W(\mathbb{A})$ and $\phi_t^W(\mathbb{A})$ by

$$\sigma_t^W(\mathbb{A}) = \sum_{n\geq 0} S_{\{\{1,\dots,n\}\}}(\mathbb{A})t^n$$

and

$$\phi_t^W(\mathbb{A}) = \sum_{n\geq 1} \Psi^{\{\{1,\dots,n\}\}}(\mathbb{A})t^{n-1}.$$

These series are linked by the equality

$$\sigma_t^W(\mathbb{A}) = \exp_{\sqcup\!\sqcup}\left(\phi_t^W(\mathbb{A})\right), \tag{23}$$

where $\exp_{\sqcup\!\sqcup}$ is the exponential in $(\mathbf{WSym}(\mathbb{A}), \sqcup\!\sqcup)$. Furthermore, the coproduct of $\mathbf{WSym}$ consists in identifying the algebra $\mathbf{WSym} \otimes \mathbf{WSym}$ with $\mathbf{WSym}(\mathbb{A}+\mathbb{B})$, where $\mathbb{A}$ and $\mathbb{B}$ are two alphabets such that the letters of $\mathbb{A}$ commute with those of $\mathbb{B}$. Hence, one has $\sigma_t^W(\mathbb{A}+\mathbb{B}) = \sigma_t^W(\mathbb{A}) \sqcup\!\sqcup \sigma_t^W(\mathbb{B})$. In particular, we define the multiplication of an alphabet $\mathbb{A}$ by a constant $k \in \mathbb{N}$ by

$$\sigma_t^W(k\mathbb{A}) = \sum_{n\geq 0} S_{\{\{1,\dots,n\}\}}(k\mathbb{A})t^n = \sigma_t^W(\mathbb{A})^k.$$

Nevertheless, the notion of specialization is subtler to define than in Sym. Indeed, the knowledge of the complete functions $S_{\{\{1,\dots,n\}\}}(\mathbb{A})$ does not allow us to recover all the polynomials using uniquely the algebraic operations. In [5], we made an attempt to define virtual alphabets by reconstituting the whole algebra using the action of an operad. Although the general mechanism remains to be defined, the case where each complete function $S_{\{\{1,\dots,n\}\}}(\mathbb{A})$ is specialized to a sum of words of length n can be understood via this construction. More precisely, we consider the family of multilinear k-ary operators $\sqcup\!\sqcup_\Pi$ indexed by set compositions (a set composition is a sequence $[\pi_1,\dots,\pi_k]$ of subsets of $\{1,\dots,n\}$ such that $\{\pi_1,\dots,\pi_k\}$ is a set partition of $\{1,\dots,n\}$) acting on words by $\sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(a_1^1\cdots a_{n_1}^1,\dots,a_1^k\cdots a_{n_k}^k) = b_1\cdots b_n$ with $b_{i_\ell^p} = a_\ell^p$ if $\pi_p = \{i_1^p < \cdots < i_{n_p}^p\}$ and $\sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(a_1^1\cdots a_{n_1}^1,\dots,a_1^k\cdots a_{n_k}^k) = 0$ if $\#\pi_p \neq n_p$ for some $1 \leq p \leq k$.

Let $P = (P_n)_{n\geq 1}$ be a family of a homogeneous word polynomials such that $\deg(P_n) = n$ for each n. We set $S_{\{\{1,\dots,n\}\}}\left[\mathbb{A}^{(P)}\right] = P_n$ and

$$S_{\{\pi_1,\dots,\pi_k\}}\left[\mathbb{A}^{(P)}\right] = \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(S_{\{\{1,\dots,\#\pi_1\}\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{\{1,\dots,\#\pi_k\}\}}\left[\mathbb{A}^{(P)}\right]).$$

The space $\mathbf{WSym}\left[\mathbb{A}^{(P)}\right]$ generated by the polynomials $S_{\{\pi_1,\dots,\pi_k\}}\left[\mathbb{A}^{(P)}\right]$ and endowed with the two products $\cdot$ and $\sqcup\!\sqcup$ is homomorphic to the double algebra $(\mathbf{WSym}(\mathbb{A}), \cdot, \sqcup\!\sqcup)$. Indeed, let $\pi = \{\pi_1,\dots,\pi_k\} \vDash n$ and $\pi' = \{\pi'_1,\dots,\pi'_{k'}\} \vDash n'$ be two set partitions, one has

$$\begin{aligned}
S_\pi\left[\mathbb{A}^{(P)}\right] \cdot S_{\pi'}\left[\mathbb{A}^{(P)}\right] &= \sqcup\!\sqcup_{[\{1,\dots,n\},\{n+1,\dots,n+n'\}]}\left(S_\pi\left[\mathbb{A}^{(P)}\right], S_{\pi'}\left[\mathbb{A}^{(P)}\right]\right) \\
&= \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k,\pi'_1[n],\dots,\pi'_{k'}[n]]}\Big(S_{\{1,\dots,\#\pi_1\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{1,\dots,\#\pi_k\}}\left[\mathbb{A}^{(P)}\right], \\
&\qquad S_{\{1,\dots,\#\pi'_1\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{1,\dots,\#\pi'_{k'}\}}\left[\mathbb{A}^{(P)}\right]\Big) \\
&= S_{\pi\uplus\pi'}\left[\mathbb{A}^{(P)}\right]
\end{aligned}$$

and

$$\begin{aligned}
S_\pi\left[\mathbb{A}^{(P)}\right] \sqcup\!\sqcup S_{\pi'}\left[\mathbb{A}^{(P)}\right] &= \sum_{I\cup J=\{1,\dots,n+n'\},\ I\cap J=\emptyset} \sqcup\!\sqcup_{[I,J]}\left(S_\pi\left[\mathbb{A}^{(P)}\right], S_{\pi'}\left[\mathbb{A}^{(P)}\right]\right) \\
&= \sum \sqcup\!\sqcup_{[\pi''_1,\dots,\pi''_{k+k'}]}\Big(S_{\{1,\dots,\#\pi_1\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{1,\dots,\#\pi_k\}}\left[\mathbb{A}^{(P)}\right], \\
&\qquad S_{\{1,\dots,\#\pi'_1\}}\left[\mathbb{A}^{(P)}\right],\dots,S_{\{1,\dots,\#\pi'_{k'}\}}\left(\mathbb{A}^{(P)}\right]\Big),
\end{aligned}$$

where the second sum is over the partitions $\{\pi''_1,\dots,\pi''_{k+k'}\} \in \pi \uplus \pi'$ satisfying, for each $k+1 \leq i \leq k+k'$, $\mathrm{std}(\{\pi''_1,\dots,\pi''_k\}) = \pi$, $\mathrm{std}(\{\pi''_{k+1},\dots,\pi''_{k+k'}\}) = \pi'$, $\#\pi''_i = \pi_i$. Hence,

$$S_\pi\left[\mathbb{A}^{(P)}\right] \sqcup\!\sqcup S_{\pi'}\left[\mathbb{A}^{(P)}\right] = \sum_{\pi''\in\pi\uplus\pi'} S_{\pi''}\left[\mathbb{A}^{(P)}\right].$$

In other words, we consider the elements of $\mathbf{WSym}\left[\mathbb{A}^{(P)}\right]$ as word polynomials in the virtual alphabet $\mathbb{A}^{(P)}$ specializing the elements of $\mathbf{WSym}(\mathbb{A})$.

3.2.2 Biword symmetric functions

The bi-indexed word algebra **BWSym** was defined in [5]. We recall its definition here: the bases of **BWSym** are indexed by set partitions into lists, which can be constructed from a set partition by ordering each block. We will denote by $\mathcal{PL}_n$ the set of the set partitions of $\{1,\dots,n\}$ into lists.

Example 15. $\{[1,2,3],[4,5]\}$ and $\{[3,1,2],[5,4]\}$ are two distinct set partitions into lists of the set $\{1,2,3,4,5\}$.

The number of set partitions into lists of an n-element set (or set partitions into lists of size n) is given by Sloane's sequence A000262 [30]. The first values are

$$1,1,3,13,73,501,4051,\dots$$

If $\hat{\Pi}$ is a set partition into lists of $\{1,\dots,n\}$, we will write $\Pi \Vdash n$. Set $\hat{\Pi}\uplus\hat{\Pi}' = \hat{\Pi}\cup\{[l_1+n,\dots,l_k+n] : [l_1,\dots,l_k]\in\hat{\Pi}'\} \Vdash n+n'$. Let $\hat{\Pi}'\subset\hat{\Pi} \Vdash n$, since the integers appearing in $\hat{\Pi}'$ are all distinct, the standardized $\mathrm{std}(\hat{\Pi}')$ of $\hat{\Pi}'$ is well defined as the unique set partition into lists obtained by replacing the ith smallest integer in $\hat{\Pi}$ by i. For example, $\mathrm{std}(\{[5,2],[3,10],[6,8]\}) = \{[3,1],[2,6],[4,5]\}$.

The Hopf algebra **BWSym** is formally defined by its basis $(\Phi_{\hat{\Pi}})$ where the $\hat{\Pi}$ are set partitions into lists, its product $\Phi_{\hat{\Pi}}\Phi_{\hat{\Pi}'} = \Phi_{\hat{\Pi}\uplus\hat{\Pi}'}$ and its coproduct

$$\Delta(\Phi_{\hat{\Pi}}) = \sum^{\wedge} \Phi_{\mathrm{std}(\hat{\Pi}')}\otimes\Phi_{\mathrm{std}(\hat{\Pi}'')}, \tag{24}$$

where the $\sum^{\wedge}$ means that the sum is over the $(\hat{\Pi}',\hat{\Pi}'')$ such that $\hat{\Pi}'\cup\hat{\Pi}''=\hat{\Pi}$ and $\hat{\Pi}'\cap\hat{\Pi}''=\emptyset$.

The product of the graded dual BΠQSym of **BWSym** is completely described in the dual basis $(\Psi_{\hat{\Pi}})_{\hat{\Pi}}$ of $(\Phi_{\hat{\Pi}})_{\hat{\Pi}}$ by

$$\Psi_{\hat{\Pi}_1}\Psi_{\hat{\Pi}_2} = \sum_{\substack{\hat{\Pi}=\hat{\Pi}_1'\cup\hat{\Pi}_2',\ \hat{\Pi}_1'\cap\hat{\Pi}_2' \\ \mathrm{std}(\hat{\Pi}_1')=\hat{\Pi}_1,\ \mathrm{std}(\hat{\Pi}_1')=\hat{\Pi}_1}} \Psi_{\hat{\Pi}}.$$

Now consider a bijection ι_n from $\{1,\dots,n\}$ to the symmetric group $\mathfrak{S}_n$. The linear map $\kappa : \mathcal{CP}(1!,2!,3!,\dots)\longrightarrow\mathcal{CL}$ sending

$$\{[\{i_1^1,\dots,i_{n_1}^1\},m_1],\dots,[\{i_1^k,\dots,i_{n_k}^k\},m_1]\}\in\mathcal{CP}_n(1!,2!,3!,\dots),$$

with $i_1^j\le\dots\le i_{n_j}^j$, to

$$\{[i^1_{\iota_{n_1}(1)},\dots,i^1_{\iota_{n_1}(n_1)}],\dots,[i^\ell_{\iota_{n_\ell}(1)},\dots,i^\ell_{\iota_{n_\ell}(n_\ell)}]\}$$

is a bijection. Hence, a fast checking shows that the linear map sending Ψ^{Π} to $\Psi^{\kappa(\Pi)}$ is an isomorphism. So, we have

Proposition 16.

- *The Hopf algebras* **CWSym**$(1!,2!,3!,\dots)$ *and* **BWSym** *are isomorphic.*
- *The Hopf algebras* CΠQSym$(1!,2!,3!,\dots)$ *and* BΠQSym *are isomorphic.*

3.2.3 Word symmetric functions of level 2

We consider the algebra $\mathbf{WSym}_{(2)}$ which is spanned by the Φ_Π where Π is a set partition of level 2, that is, a partition of a partition π of $\{1,\dots,n\}$ for some n. The product of this algebra is given by $\Phi_\Pi\Phi_{\Pi'} = \Phi_{\Pi\cup\Pi'[n]}$ where $\Pi'[n]=\{e[n] : e\in\Pi'\}$. The dimensions of this algebra are given by the exponential generating function

$$\sum_i b_i^{(2)}\frac{t^i}{i!} = \exp(\exp(\exp(t)-1)-1).$$

The first values are

$$1,3,12,60,358,2471,19302,167894,1606137,\dots$$

see sequence A000258 of [30].

The coproduct is defined by $\Delta(\Phi_\Pi) = \sum_{\substack{\Pi'\cup\Pi''=\Pi\\ \Pi'\cap\Pi''=\emptyset}} \Phi_{\mathrm{std}(\Pi')}\otimes\Phi_{\mathrm{std}(\Pi'')}$ where, if Π is a partition of a partition of $\{i_1,\dots,i_k\}$, $\mathrm{std}(\Pi)$ denotes the standardized of Π, that is the partition of partition of $\{1,\dots,k\}$ obtained by substituting each occurrence of i_j by j in Π. The coproduct being co-commutative, the dual algebra $\Pi\mathrm{QSym}_{(2)} := \mathbf{WSym}^*_{(2)}$ is commutative. The algebra $\Pi\mathrm{QSym}_{(2)}$ is spanned by a basis $(\Psi_\Pi)_\Pi$ satisfying $\Psi_\Pi\Psi_{\Pi'} = \sum_{\Pi''} C^{\Pi''}_{\Pi,\Pi'}\Psi_{\Pi''}$ where $C^{\Pi''}_{\Pi,\Pi'}$ is the number of ways to write $\Pi'' = A\cup B$ with $A\cap B=\emptyset$, $\mathrm{std}(A)=\Pi$ and $\mathrm{std}(B)=\Pi'$.

Let b_n be the nth Bell number $A_n(1,1,\dots)$. Considering a bijection from $\{1,\dots,b_n\}$ to the set of the set partitions of $\{1,\dots,n\}$ for each n, we obtain, in the same way as in the previous subsection, the following result

Proposition 17.

- *The Hopf algebras* $\mathbf{CWSym}(b_1,b_2,b_3,\dots)$ *and* $\mathbf{WSym}_{(2)}$ *are isomorphic.*
- *The Hopf algebras* $\mathrm{C}\Pi\mathrm{QSym}(b_1,b_2,b_3,\dots)$ *and* $\Pi\mathrm{QSym}_{(2)}$ *are isomorphic.*

3.2.4 Cycle word symmetric functions

We consider the Grossman-Larson Hopf algebra of heap-ordered trees $\mathfrak{S}\mathbf{Sym}$ [15]. The combinatorics of this algebra has been extensively investigated in [16]. This Hopf algebra is spanned by the Φ_σ where σ is a permutation. We identify each permutation with the set of its cycles (for example, the permutation 321 is $\{(13),(2)\}$). The product in this algebra is given by $\Phi_\sigma\Phi_\tau = \Phi_{\sigma\cup\tau[n]}$, where n is the size of the permutation σ and $\tau[n] = \{(i_1+n, i_2+n,\dots,i_k+n) \mid (i_1,\dots,i_k)\in\tau\}$. The coproduct is given by

$$\Delta(\Phi_\sigma) = \sum \Phi_{\mathrm{std}(\sigma|_I)}\otimes\Phi_{\mathrm{std}(\sigma|_J)}, \tag{25}$$

where the sum is over the partitions of $\{1,\dots,n\}$ into 2 sets I and J such that the action of σ lets the sets I and J globally invariant, $\sigma|_I$ denotes the restriction of the permutation σ to the set I and $\mathrm{std}(\sigma|_I)$ is the permutation obtained from $\sigma|_I$ by replacing the ith smallest label by i in $\sigma|_I$.

Example 18.

$$\Delta(\Phi_{3241}) = \Phi_{3241}\otimes 1 + \Phi_1\otimes\Phi_{231} + \Phi_{231}\otimes\Phi_1 + 1\otimes\Phi_{3241}.$$

The basis (Φ_σ) and its dual basis (Ψ_σ) are respectively denoted by (S^σ) and (M_σ) in [16]. The Hopf algebra $\mathfrak{S}\mathbf{Sym}$ is not commutative but it is cocommutative, so it is not autodual and not isomorphic to the Hopf algebra of free quasi-symmetric functions.
Let ι_n be a bijection from the set of the cycles of $\mathfrak{S}_n$ to $\{1,\dots,(n-1)!\}$. We define the bijection $\kappa : \mathfrak{S}_n \leftrightarrow \mathcal{CP}(0!,1!,2!,\dots)$ by

$$\kappa(\sigma) = \{[\mathrm{support}(c_1), \iota_{\#\mathrm{support}(c_1)}(\mathrm{std}(c_1))],\dots,[\mathrm{support}(c_k), \iota_{\#\mathrm{support}(c_k)}(\mathrm{std}(c_k))]\},$$

if $\sigma = c_1\dots c_k$ is the decomposition of σ into cycles and $\mathrm{support}(c)$ denotes the support of the cycle c, *i.e.* the set of the elements which are permuted by the cycle.

Example 19. For instance, set

$$\iota_1(1) = 1,\ \iota_3(231) = 2,\ \text{and}\ \iota_3(312) = 1.$$

One has

$$\kappa(32415867) = \{[\{2\},1],[\{1,3,4\},2],[\{5\},1],[\{6,7,8\},1]\}.$$

The linear map $K : \mathfrak{S}\mathbf{Sym} \longrightarrow \mathbf{CWSym}(0!,1!,2!,\dots)$ sending Φ_σ to $\Phi_{\kappa(\sigma)}$ is an isomorphism of algebra. Indeed, it is straightforward to see that it is a bijection and furthermore $\kappa(\sigma\cup\tau[n]) = \kappa(\sigma)\uplus\kappa(\tau)$. Moreover, if $\sigma\in\mathfrak{S}_n$ is a permutation and $\{I,J\}$ is a partition of $\{1,\dots,n\}$ into two subsets such that the action of σ lets I and J globally invariant, we check that $\kappa(\sigma) = \Pi_1\cup\Pi_2$ with $\Pi_1\cap\Pi_2=\emptyset$, $\mathrm{std}(\Pi_1) = \kappa(\mathrm{std}(\sigma|_I))$ and $\mathrm{std}(\Pi_2) = \kappa(\mathrm{std}(\sigma|_J))$. Conversely, if $\kappa(\sigma) = \Pi_1\cup\Pi_2$ with $\Pi_1\cap\Pi_2=\emptyset$ then there exists a partition $\{I,J\}$ of $\{1,\dots,n\}$ into two subsets such that the action of σ lets I and J globally invariant and $\mathrm{std}(\Pi_1) = \kappa(\mathrm{std}(\sigma|_I))$ and $\mathrm{std}(\Pi_2) = \kappa(\mathrm{std}(\sigma|_J))$.

In other words,

$$\Delta(\Phi_{\kappa(\sigma)}) = \sum \Phi_{\kappa(\mathrm{std}(\sigma|_I))}\otimes\Phi_{\kappa(\mathrm{std}(\sigma|_J)},$$

where the sum is over the partitions of $\{1,\dots,n\}$ into 2 sets I and J such that the action of σ lets the sets I and J globally invariant. Hence K is a morphism of cogebras and, as with the previous examples, one has

Proposition 20.

- *The Hopf algebras* $\mathbf{CWSym}(0!, 1!, 2!, \dots)$ *and* $\mathfrak{S}\mathbf{Sym}$ *are isomorphic.*
- *The Hopf algebras* $\mathrm{C\Pi QSym}(0!, 1!, 2!, \dots)$ *and* $\mathfrak{S}\mathbf{Sym}^*$ *are isomorphic.*

3.2.5 Miscellanous subalgebras of the Hopf algebra of endofunctions

We denote by End the combinatorial class of endofunctions (an endofunction of size $n \in \mathbb{N}$ is a function from $\{1, \dots, n\}$ to itself). Given a function f from a finite subset A of $\mathbb{N}$ to itself, we denote by $\mathrm{std}(f)$ the endofunction $\phi \circ f \circ \phi^{-1}$, where ϕ is the unique increasing bijection from A to $\{1, 2, \dots, \mathrm{card}(A)\}$. Given a function g from a finite subset B of $\mathbb{N}$ (disjoint from A) to itself, we denote by $f \cup g$ the function from $A \cup B$ to itself whose f and g are respectively the restrictions to A and B. Finally, given two endofunctions f and g, respectively of size n and m, we denote by $f \bullet g$ the endofunction $f \cup \tilde{g}$, where $\tilde{g}$ is the unique function from $\{n+1, n+2, \dots, n+m\}$ to itself such that $\mathrm{std}(\tilde{g}) = g$.
Now, let EQSym be the Hopf algebra of endofunctions [16]. This Hopf algebra is defined by its basis (Ψ_f) indexed by endofunctions, the product

$$\Psi_f \Psi_g = \sum_{\mathrm{std}(\tilde{f})=f, \mathrm{std}(\tilde{g})=g, \tilde{f}\cup\tilde{g}\in\mathrm{End}} \Psi_{\tilde{f}\cup\tilde{g}} \tag{26}$$

and the coproduct

$$\Delta(\Psi_h) = \sum_{f\bullet g=h} \Psi_f \otimes \Psi_g. \tag{27}$$

This algebra is commutative but not cocommutative. We denote by $\mathbf{ESym} := \mathrm{EQSym}^*$ its graded dual, and by (Φ_f) the basis of $\mathbf{ESym}$ dual to (Ψ_f). The bases (Φ_σ) and (Ψ_σ) are respectively denoted by (S^σ) and (M_σ) in [16]. The product and the coproduct in $\mathbf{ESym}$ are respectively given by

$$\Phi_f \Phi_g = \Phi_{f\bullet g} \tag{28}$$

and

$$\Delta(\Phi_h) = \sum_{f\cup g=h} \Phi_{\mathrm{std}(f)} \otimes \Phi_{\mathrm{std}(g)}. \tag{29}$$

Remark : The Ψ_f, where f is a bijective endofunction, span a Hopf subalgebra of EQSym obviously isomorphic to $\mathfrak{S}\mathrm{QSym} := \mathfrak{S}\mathbf{Sym}^*$, that is isomorphic to $\mathrm{C\Pi QSym}(0!, 1!, 2!, \dots)$ from (3.2.4).

As suggested by [16], we investigate a few other Hopf subalgebras of EQSym.

- The Hopf algebra of idempotent endofunctions is isomorphic to the Hopf algebra $\mathrm{C\Pi QSym}(1, 2, 3, \dots)$. The explicit isomorphism sends Ψ_f to $\Psi_{\phi(f)}$, where for any idempotent endofunction f of size n,

$$\phi(f) = \left\{ \left[f^{-1}(i), \mathrm{card}(\{j \in f^{-1}(i) \mid j \le i\}) \right] \middle| 1 \le i \le n, f^{-1}(i) \neq \emptyset \right\}. \tag{30}$$

- The Hopf algebra of involutive endofunctions is isomorphic to

$$\mathrm{C\Pi QSym}(1, 1, 0, \dots, 0, \dots) \hookrightarrow \Pi\mathrm{QSym}.$$

  Namely, it is a Hopf subalgebra of $\mathfrak{S}\mathrm{QSym}$, and the natural isomorphism from $\mathfrak{S}\mathrm{QSym}$ to $\mathrm{C\Pi QSym}(0!, 1!, 2!, \dots)$ sends it to the sub algebra $\mathrm{C\Pi QSym}(1, 1, 0, \dots, 0, \dots)$.
- In the same way the endofunctions such that $f^3 = \mathrm{Id}$ generate a Hopf subalgebra of $\mathfrak{S}\mathrm{QSym} \hookrightarrow \mathrm{EQSym}$ isomorphic to the Hopf algebra $\mathrm{C\Pi QSym}(1, 0, 2, 0, \dots, 0, \dots)$.
- More generally, the endofunctions such that $f^p = \mathrm{Id}$ generate a Hopf subalgebra of $\mathfrak{S}\mathrm{QSym} \hookrightarrow \mathrm{EQSym}$ isomorphic to $\mathrm{C\Pi QSym}(\tau(p))$ where $\tau(p)_i = (i-1)!$ if $i \mid p$ and $\tau(p)_i = 0$ otherwise.

3.3 About specializations

The aim of this section is to show how the specialization $c_n \longrightarrow \frac{a_n}{n!}$ factorizes through $\Pi\mathrm{QSym}$ and $\mathrm{C\Pi QSym}$.

Notice first that the algebra Sym is isomorphic to the subalgebra of $\Pi\mathrm{QSym}$ generated by the family $(\Psi_{\{\{1,\dots,n\}\}})_{n\in\mathbb{N}}$; the explicit isomorphism α sends c_n to $\Psi_{\{\{1,\dots,n\}\}}$. The image of h_n is $S_{\{\{1,\dots,n\}\}}$ and the image of $c_\lambda = \frac{1}{\lambda!} c_{\lambda_1} \cdots c_{\lambda_k}$ is $\sum_{\pi\vDash\lambda} \Psi_\pi$ where $\pi \vDash \lambda$

means that $\pi = \{\pi_1, \dots, \pi_k\}$ is a set partition such that $\#\pi_1 = \lambda_1, \ \dots, \#\pi_k = \lambda_k$ and $\lambda^! = \frac{\lambda_1 \cdots \lambda_k}{z_\lambda} = \prod_i \mathfrak{m}_i(\lambda)!$ where $\mathfrak{m}_i(\lambda)$ denotes the multiplicity of i in λ. Indeed, c_λ is mapped to $\frac{1}{\lambda^!}\Psi_{\{\{1,\dots,\lambda_1\}\}} \cdots \Psi_{\{\{1,\dots,\lambda_k\}\}}$ and $\Psi_{\{\{1,\dots,\lambda_1\}\}} \cdots \Psi_{\{\{1,\dots,\lambda_k\}\}} = \lambda^! \sum_{\pi \vDash \lambda} \Psi_\pi$.

Now the linear map $\beta_a : \Pi\mathrm{QSym} \longrightarrow \mathrm{C}\Pi\mathrm{QSym}(a)$ sending each Ψ_π to the element $\sum_{\Pi \Rrightarrow \pi} \Psi_\Pi$ is a morphism of algebra and the subalgebra $\widetilde{\Pi\mathrm{QSym}} := \beta_a(\Pi\mathrm{QSym})$ is isomorphic to $\Pi\mathrm{QSym}$ if and only if $a \in (\mathbb{N} \setminus \{0\})^{\mathbb{N}}$.

Let $\gamma_a : \mathrm{C}\Pi\mathrm{QSym}(a) \longrightarrow \mathbb{C}$ be the linear map sending Ψ_Π to $\frac{1}{|\Pi|!}$. We have

$$\gamma_a(\Psi_{\Pi_1}\Psi_{\Pi_2}) = \sum_{\substack{\Pi = \Pi_1' \cup \Pi_2', \Pi_1' \cap \Pi_2' = \emptyset \\ \mathrm{std}(\Pi_1') = \Pi_1, \mathrm{std}(\Pi_2') = \Pi_2}} \gamma_a(\Psi_\Pi).$$

From any subset A of $\{1, 2, \dots, |\Pi_1| + |\Pi_2|\}$ of cardinality n, one has $\mathrm{std}(\Pi_1') = \Pi_1$, $\mathrm{std}(\Pi_2') = \Pi_2$ and $\Pi_1' \cap \Pi_2' = \emptyset$, where Π_1' is obtained from Π_1 by replacing each label i by the ith smallest element of A, and Π_2' is obtained from Π_2 by replacing each label i by the ith smallest element of $\{1, \dots, |\Pi_1| + |\Pi_2|\} \setminus A$. Since there are $\binom{|\Pi_1| + |\Pi_2|}{|\Pi_1|}$ ways to construct A, one has

$$\gamma_a(\Psi_{\Pi_1}\Psi_{\Pi_2}) = \frac{1}{(|\Pi_1| + |\Pi_2|)!}\binom{|\Pi_1| + |\Pi_2|}{|\Pi_1|} = \frac{1}{|\Pi_1|!|\Pi_2|!} = \gamma_a(\Psi_{\Pi_1})\gamma_a(\Psi_{\Pi_2}).$$

In other words, γ_a is a morphism of algebra. Furthermore, the restriction $\hat{\gamma}_a$ of γ_a to $\widetilde{\Pi\mathrm{QSym}}$ is a morphism of algebra that sends $\beta_a(\Psi_{\{\{1,\dots,n\}\}})$ to $\frac{a_n}{n!}$. It follows that if $f \in Sym$, then one has

$$f(\mathbb{X}^{(a)}) = \tilde{\gamma}_a(\beta_a(\alpha(f))). \tag{31}$$

The following theorem summarizes the section:

Theorem 21. *The diagram*

$$\begin{tikzcd}
\mathrm{C}\Pi\mathrm{QSym}(a) \arrow[dr, "\gamma_a"'] & \widetilde{\Pi\mathrm{QSym}} \arrow[l, hook'] \arrow[d, "\hat{\gamma}_a"] & \Pi\mathrm{QSym} \arrow[l, two heads, "\beta_a"'] \\
& \mathbb{C} = Sym[\mathbb{X}^{(a)}] & Sym \arrow[l, two heads] \arrow[u, hook, "\alpha"']
\end{tikzcd}$$

is commutative.

4 Word Bell polynomials

4.1 Bell polynomials in $\Pi\mathrm{QSym}$

Since Sym is isomorphic to the subalgebra of $\Pi\mathrm{QSym}$ generated by the elements $\Psi_{\{\{1,\dots,n\}\}}$, we can compute $A_n(\Psi_{\{\{1\}\}}, \Psi_{\{\{1,2\}\}}, \dots, \Psi_{\{\{1,\dots,m\}\}}, \dots)$. From (23), we have

$$A_n(1!\Psi_{\{\{1\}\}}, 2!\Psi_{\{\{1,2\}\}}, \dots, m!\Psi_{\{\{1,\dots,m\}\}}, \dots) = n!S_{\{\{1,\dots,n\}\}} = n!\sum_{\pi \vDash n} \Psi_\pi. \tag{32}$$

Notice that, from the previous section, the image of the Bell polynomial

$$A_n(\Psi_{\{\{1\}\}}, \Psi_{\{\{1,2\}\}}, \dots, \Psi_{\{\{1,\dots,m\}\}}, \dots)$$

by the morphism γ sending $\Psi_{\{\{1,\dots,n\}\}}$ to $\frac{1}{n!}$ is

$$\gamma(A_n(1!\Psi_{\{\{1\}\}}, 2!\Psi_{\{\{1,2\}\}}, \dots, m!\Psi_{\{\{1,\dots,m\}\}}, \dots)) = b_n = A_n(1, 1, \dots).$$

In the same way, we have

$$B_{n,k}(1!\Psi_{\{\{1\}\}}, 2!\Psi_{\{\{1,2\}\}}, \dots, m!\Psi_{\{\{1,\dots,m\}\}}, \dots) = n! \sum_{\substack{\pi \vDash n \\ \#\pi = k}} \Psi_\pi. \tag{33}$$

If $(F_n)_n$ is a homogeneous family of elements of $\Pi\mathrm{QSym}$, such that $|F_n| = n$, we define

$$\mathcal{A}_n(F_1, F_2, \dots) = \frac{1}{n!}A_n(1!F_1, 2!F_2, \dots, m!F_m, \dots) \tag{34}$$

and

$$\mathcal{B}_{n,k}(F_1, F_2, \dots) = \frac{1}{n!} B_{n,k}(1!F_1, 2!F_2, \dots, m!F_m, \dots). \tag{35}$$

Considering the map $\beta_a \circ \alpha$ as a specialization of Sym, the following identities hold in CΠQSym(a):

$$\mathcal{A}_n \left(\sum_{1 \le i \le a_1} \Psi_{\{[\{1\},i]\}}, \sum_{1 \le i \le a_2} \Psi_{\{[\{1,2\},i]\}}, \dots, \sum_{1 \le i \le a_m} \Psi_{\{[\{1,\dots,m\},i]\}}, \dots \right) = \sum_{\Pi \vDash n} \Psi_\Pi$$

and

$$\mathcal{B}_{n,k} \left(\sum_{1 \le i \le a_1} \Psi_{\{[\{1\},i]\}}, \sum_{1 \le i \le a_2} \Psi_{\{[\{1,2\},i]\}}, \dots, \sum_{1 \le i \le a_m} \Psi_{\{[\{1,\dots,m\},i]\}}, \dots \right) = \sum_{\substack{\Pi \vDash n \\ \#\Pi = k}} \Psi_\Pi.$$

Example 22. In BΠQSym $\sim$ CΠQSym$(1!, 2!, \dots)$, we have

$$\mathcal{B}_{n,k} \left(\Psi_{\{[1]\}}, \Psi_{\{[1,2]\}} + \Psi_{\{[2,1]\}}, \dots, \sum_{\sigma \in \mathfrak{S}_m} \Psi_{\{[\sigma]\}}, \dots \right) = \sum_{\substack{\hat{\Pi} \vDash n \\ \#\hat{\Pi} = k}} \Psi_{\hat{\Pi}},$$

where the sum on the right is over the set partitions of $\{1, \dots, n\}$ into k lists. Considering the morphism sending $\Psi_{\{[\sigma_1,\dots,\sigma_n]\}}$ to $\frac{1}{n!}$, Theorem 21 allows us to recover $B_{n,k}(1!, 2!, 3!, \dots) = L_{n,k}$, the number of set partitions of $\{1, \dots, n\}$ into k lists.

Example 23. In ΠQSym$_{(2)}$ $\sim$ CΠQSym$(b_1, b_2, \dots)$, we have

$$\mathcal{B}_{n,k} \left(\Psi_{\{\{\{1\}\}\}}, \Psi_{\{\{\{1,2\}\}\}} + \Psi_{\{\{\{1\},\{2\}\}\}}, \dots, \sum_{\pi \vDash m} \Psi_{\{\pi\}}, \dots \right) = \sum_{\substack{\Pi \text{ partition of } \pi \vDash n \\ \#\hat{\Pi} = k}} \Psi_\Pi,$$

where the sum on the right is over the set partitions of $\{1, \dots, n\}$ of level 2 into k blocks. Considering the morphism sending $\Psi_{\{\pi\}}$ to $\frac{1}{n!}$ for $\pi \vDash n$, Theorem 21 allows us to recover $B_{n,k}(b_1, b_2, b_3, \dots) = S^{(2)}_{n,k}$, the number of set partitions into k sets of a partition of $\{1, \dots, n\}$.

Example 24. In $\mathfrak{S}$**Sym*** $\sim$ CΠQSym$(0!, 1!, 2! \dots)$, we have

$$\mathcal{B}_{n,k} \left(\Psi_{[1]}, \Psi_{[2,1]}, \Psi_{[2,3,1]} + \Psi_{[3,1,2]}, \dots, \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma \text{ is a cycle}}} \Psi_{\{\pi\}}, \dots \right) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \sigma \text{ has } k \text{ cycles}}} \Psi_\sigma,$$

where the sum on the right is over the permutations of size n having k cycles. Considering the morphism sending Ψ_σ to $\frac{1}{n!}$ for $\sigma \in \mathfrak{S}_n$, Theorem 21 allows us to recover $B_{n,k}(0!, 1!, 2!, \dots) = s_{n,k}$, the number of permutations of $\mathfrak{S}_n$ having exactly k cycles.

Example 25. In the Hopf algebra of idempotent endofunctions, we have

$$\mathcal{B}_{n,k} \left(\Psi_{f_{1,1}}, \Psi_{f_{2,1}} + \Psi_{f_{2,2}}, \Psi_{f_{3,1}} + \Psi_{f_{3,2}} + \Psi_{f_{3,3}}, \dots, \textstyle\sum_{i=1}^n \Psi_{f_{n,i}}, \dots \right) = \sum_{|f|=n, \mathrm{card}(f(\{1,\dots,n\}))=k} \Psi_f,$$

where for $i \ge j \ge 1$, $f_{i,j}$ is the constant endofunction of size i and of image $\{j\}$. Here, the sum on the right is over idempotent endofuctions f of size n such that the cardinality of the image of f is k. Considering the morphism sending Ψ_f to $\frac{1}{n!}$ for $|f| = n$, Theorem 21 allows us to recover that $B_{n,k}(1, 2, 3, \dots)$ is the number of these idempotent endofunctions, that is the idempotent number $\binom{n}{k} k^{n-k}$ [17, 31].

4.2 Bell polynomials in WSym

Recursive descriptions of Bell polynomials are given in [10]. In this section we rewrite this result and other ones related to these polynomials in the Hopf algebra of word symmetric functions **WSym**. We define the operator ∂ acting linearly on the left on **WSym** by

$$1\partial = 0 \text{ and } \Phi_{\{\pi_1,\dots,\pi_k\}}\partial = \sum_{i=1}^{k} \Phi_{(\{\pi_1,\dots,\pi_k\} \setminus \pi_i) \cup \{\pi_i \cup \{n+1\}\}}.$$

In fact, the operator ∂ acts on Φ_π almost as the multiplication of $M_{\{\{1\}\}}$ on M_π. More precisely :

Proposition 26. *We have:*

$$\partial = \phi^{-1} \circ \mu \circ \phi - \mu,$$

where ϕ is the linear operator satisfying $M_\pi \phi = \Phi_\pi$ and μ is the multiplication by $\Phi_{\{\{1\}\}}$.

Example 27. For instance, one has

$$\begin{aligned}
\Phi_{\{\{1,3\},\{2,4\}\}}\partial &= \Phi_{\{\{1,3\},\{2,4\}\}}(\phi^{-1}\mu\phi - \mu) \\
&= M_{\{\{1,3\},\{2,4\}\}}\mu\phi - \Phi_{\{\{1,3\},\{2,4\},\{5\}\}} \\
&= (M_{\{\{1,3,5\},\{2,4\}\}} + M_{\{\{1,3\},\{2,4,5\}\}} + M_{\{\{1,3\},\{2,4\},\{5\}\}})\phi \\
&\quad -\Phi_{\{\{1,3\},\{2,4\},\{5\}\}} \\
&= \Phi_{\{\{1,3,5\},\{2,4\}\}} + \Phi_{\{\{1,3\},\{2,4,5\}\}} + \Phi_{\{\{1,3\},\{2,4\},\{5\}\}} \\
&\quad -\Phi_{\{\{1,3\},\{2,4\},\{5\}\}} \\
&= \Phi_{\{\{1,3,5\},\{2,4\}\}} + \Phi_{\{\{1,3\},\{2,4,5\}\}}.
\end{aligned}$$

Following Remark 4, we define recursively the elements $\mathfrak{A}_n$ of **WSym** as

$$\mathfrak{A}_0 = 1,\ \mathfrak{A}_{n+1} = \mathfrak{A}_n(\Phi_{\{\{1\}\}} + \partial). \tag{36}$$

So we have

$$\mathfrak{A}_n = 1(\Phi_{\{1\}} + \partial)^n. \tag{37}$$

Easily, one shows that $\mathfrak{A}_n$ provides an analogue of complete Bell polynomials in **WSym**.

Proposition 28.

$$\mathfrak{A}_n = \sum_{\pi \vDash n} \Phi_\pi.$$

Noticing that the multiplication by $\Phi_{\{\{1\}\}}$ adds one part to each partition, we give the following analogue for partial Bell polynomials.

Proposition 29. *If we set*

$$\mathfrak{B}_{n,k} = [t^k]1(t\Phi_{\{\{1\}\}} + \partial)^n, \tag{38}$$

then we have $\mathfrak{B}_{n,k} = \sum_{\substack{\pi \vDash n \\ \#\pi = k}} \Phi_\pi$.

Example 30. We have

$$\begin{aligned}
&1(t\Phi_{\{\{1\}\}} + \partial)^4 = t^4\Phi_{\{\{1\},\{2\},\{3\},\{4\}\}} + t^3(\Phi_{\{\{1,2\},\{3\},\{4\}\}} + \Phi_{\{\{1,3\},\{2\},\{4\}\}} \\
&+\Phi_{\{\{1\},\{2,3\},\{4\}\}} + \Phi_{\{\{1,4\},\{2\},\{3\}\}} + \Phi_{\{\{1\},\{2,4\},\{3\}\}} + \Phi_{\{\{1\},\{2\},\{3,4\}\}}) + \\
&t^2(\Phi_{\{\{1,3,4\},\{2\}\}} + \Phi_{\{\{1,2,3\},\{4\}\}} + \Phi_{\{\{1,2,4\},\{3\}\}} + \Phi_{\{\{1,2\},\{3,4\}\}} + \Phi_{\{\{1,3\},\{2,4\}\}} \\
&+\Phi_{\{\{1,4\},\{2,3\}\}} + \Phi_{\{\{1\},\{2,3,4\}\}}) + t\Phi_{\{\{1\},\{3\},\{4\},\{2\}\}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathfrak{B}_{4,2} &= \Phi_{\{\{1,3,4\},\{2\}\}} + \Phi_{\{\{1,2,3\},\{4\}\}} + \Phi_{\{\{1,2,4\},\{3\}\}} + \Phi_{\{\{1,2\},\{3,4\}\}} \\
&\quad +\Phi_{\{\{1,3\},\{2,4\}\}} + \Phi_{\{\{1,4\},\{2,3\}\}} + \Phi_{\{\{1\},\{2,3,4\}\}}.
\end{aligned}$$

4.3 Bell polynomials in $\mathbb{C}\langle\mathbb{A}\rangle$

Both **WSym** and ΠQSym admit word polynomial realizations in a subspace **WSym**($\mathbb{A}$) of the free associative algebra $\mathbb{C}\langle\mathbb{A}\rangle$ over an infinite alphabet $\mathbb{A}$. When endowed with the concatenation product, **WSym**($\mathbb{A}$) is isomorphic to **WSym** and when endowed with the shuffle product, it is isomorphic to ΠQSym. Alternatively to the definitions of partial Bell numbers in ΠQSym (35) and in **WSym**, we set, for any sequence of polynomials $(F_i)_{i\in\mathbb{N}}$ in $\mathbb{C}\langle\mathbb{A}\rangle$,

$$\sum_{n\geq 0} \mathtt{B}_{n,k}(F_1, \dots, F_m, \dots)t^n = \frac{1}{k!}\left(\sum_i F_i t^i\right)^{\sqcup\!\sqcup k} \tag{39}$$

and

$$\mathtt{A}_n(F_1, \dots, F_m, \dots) = \sum_{k\geq 1} \mathtt{B}_{n,k}(F_1, \dots, F_m, \dots). \tag{40}$$

This definition generalizes (35) and (38) in the following sense:

Proposition 31. *We have*

$$\mathtt{B}_{n,k}(\Psi_{\{\{1\}\}}(\mathbb{A}), \dots, \Psi_{\{\{1,\dots,m\}\}}(\mathbb{A}), \dots) = \mathcal{B}_{n,k}(\Psi_{\{\{1\}\}}(\mathbb{A}), \dots, \Psi_{\{\{1,\dots,m\}\}}(\mathbb{A}), \dots)$$

and

$$\mathtt{B}_{n,k}(\Phi_{\{\{1\}\}}(\mathbb{A}), \dots, \Phi_{\{\{1,\dots,m\}\}}(\mathbb{A}), \dots) = \mathfrak{B}_{n,k}(\mathbb{A}).$$

Proof. The two identities follow from

$$\Psi_{\pi_1}(\mathbb{A}) \sqcup\!\sqcup \Psi_{\pi_2}(\mathbb{A}) = \sum_{\substack{\pi=\pi_1'\cup\pi_2',\ \pi_1'\cap\pi_2'=\emptyset \\ \mathrm{std}(\pi_1')=\pi_1,\ \mathrm{std}(\pi_2')=\pi_2}} \Psi_{\pi}(\mathbb{A}).$$

□

Equality (39) allows us to show more general properties. For instance, let $\mathbb{A}'$ and $\mathbb{A}''$ be two disjoint subalphabets of $\mathbb{A}$ and set

$$S_n^{\mathbb{A}'}(\mathbb{A}'') = S_{\{\{1\}\}}(\mathbb{A}') \sqcup\!\sqcup S_{\{\{1,\dots,n-1\}\}}(\mathbb{A}'').$$

Remarking that

$$\sum_n \mathtt{B}_{n,k}(S_1^{\mathbb{A}'}(\mathbb{A}''),\dots,S_m^{\mathbb{A}'}(\mathbb{A}''),\dots)t^n =$$
$$t^k S_{\{\{1\},\dots,\{k\}\}}(\mathbb{A}') \sqcup\!\sqcup (\sum_{n\geq 0} S_{\{\{1,\dots,n\}\}}(\mathbb{A}'')t^n)^{\sqcup\!\sqcup k} = t^k S_{\{\{1\},\dots,\{k\}\}}(\mathbb{A}') \sqcup\!\sqcup \sigma_t^W(k\mathbb{A}''),$$

we obtain a word analogue of the formula allowing one to write a Bell polynomial as a symmetric function (see eq. (58) in Appendix A):

Proposition 32.

$$\mathtt{B}_{n,k}(S_1^{\mathbb{A}'}(\mathbb{A}''),\dots,S_m^{\mathbb{A}'}(\mathbb{A}''),\dots) = S_{\{\{1\},\dots,\{k\}\}}(\mathbb{A}') \sqcup\!\sqcup S_{\{\{1,\dots,n-k\}\}}(k\mathbb{A}'').$$

For simplicity, let us write $\mathtt{B}_{n,k}^{\mathbb{A}'}(\mathbb{A}'') := \mathtt{B}_{n,k}(S_1^{\mathbb{A}'}(\mathbb{A}''),\dots,S_m^{\mathbb{A}'}(\mathbb{A}''),\dots)$.
Let $k = k_1 + k_2$. From

$$S_{\{\{1\},\dots,\{k_1\}\}}(\mathbb{A}') \sqcup\!\sqcup S_{\{\{1\},\dots,\{k_2\}\}}(\mathbb{A}') = \binom{k}{k_1} S_{\{\{1\},\dots,\{k\}\}}(\mathbb{A}')$$

and

$$S_{\{\{1,\dots,n-k\}\}}(k\mathbb{A}'') = \sum_{i+j=n-k} S_{\{\{1,\dots,i\}\}}(k_1\mathbb{A}'') \sqcup\!\sqcup S_{\{\{1,\dots,i\}\}}(k_2\mathbb{A}''),$$

we deduce an analogue of the binomiality of the partial Bell polynomials (see eq. (59) in Appendix A):

Corollary 33. *Let $k = k_1 + k_2$ be three nonnegative integers. We have*

$$\binom{k}{k_1} \mathtt{B}_{n,k}^{\mathbb{A}'}(\mathbb{A}'') = \sum_{i=0}^{n} \mathtt{B}_{i,k_1}^{\mathbb{A}'}(\mathbb{A}'') \sqcup\!\sqcup \mathtt{B}_{n-i,k_2}^{\mathbb{A}'}(\mathbb{A}''). \tag{41}$$

Example 34. Consider a family of functions $(f_k)_k$ such that $f_k : \mathbb{N} \longrightarrow \mathbb{C}\langle\mathbb{A}\rangle$ satisfying

$$f_0 = 1 \text{ and } f_n(\alpha+\beta) = \sum_{n=i+j} f_i(\alpha) \sqcup\!\sqcup f_j(\beta). \tag{42}$$

From (39), we obtain

$$\mathtt{B}_{n,k}(f_0(a),\dots,f_{m-1}(a),\dots)t^n = \frac{1}{k!} \sum_{i_1+\dots+i_k=n-k} f_{i_1}(a) \sqcup\!\sqcup \dots \sqcup\!\sqcup f_{i_k}(a).$$

Hence, iterating (42), we deduce

$$\mathtt{B}_{n,k}(f_0(a),\dots,f_{m-1}(a),\dots) = \frac{1}{k!} f_{n-k}(ka).$$

Set $f_n(k) = k!\mathtt{B}_{n,k}^{\mathbb{A}'}(\mathbb{A}'')$ and $f_0(k) = 1$. By (41), the family $(f_n)_{n\in\mathbb{N}}$ satisfies (42). Hence we obtain an analogue of composition formula (see eq. (63) in Appendix A):

$$k_1!\mathtt{B}_{n,k_1}(1,\dots,k_2!\mathtt{B}_{m-1,k_2}^{\mathbb{A}'}(\mathbb{A}''),\dots) = (k_1k_2)!\mathtt{B}_{n-k_1,k_1k_2}^{\mathbb{A}'}(\mathbb{A}'').$$

Suppose now $\mathbb{A}'' = \mathbb{A}_1'' + \mathbb{A}_2''$.
By $S_{\{\{1,\dots,n\}\}}(\mathbb{A}'') = \sum_{i=0}^{n} S_{\{\{1,\dots,i\}\}}(\mathbb{A}_1'') \sqcup\!\sqcup S_{\{\{1,\dots,n-i\}\}}(\mathbb{A}_2'')$, Proposition (32) allows us to write a word analogue of the convolution formula for Bell polynomials (see formula (60) in Appendix A):

Corollary 35.

$$S_{\{\{1\},\dots,\{k\}\}}(\mathbb{A}')\sqcup\!\sqcup \mathtt{B}^{\mathbb{A}'}_{n,k}(\mathbb{A}'') = \sum_{i=0}^{n} \mathtt{B}^{\mathbb{A}'}_{i,k}(\mathbb{A}''_1)\sqcup\!\sqcup \mathtt{B}^{\mathbb{A}'}_{n-i,k}(\mathbb{A}''_2). \tag{43}$$

Let k_1 and k_2 be two positive integers. We have

$$\sum_n \mathtt{B}_{n,k_1}(\mathtt{B}^{\mathbb{A}'}_{k_2,k_2}(\mathbb{A}''),\dots,\mathtt{B}^{\mathbb{A}'}_{k_2+m-1,k_2}(\mathbb{A}''),\dots)t^n =$$
$$\frac{1}{k_1!}\left(\sum_{m\geq 1}\mathtt{B}^{\mathbb{A}'}_{k_2+m-1,k_2}(\mathbb{A}'')t^m\right)^{\sqcup\!\sqcup k_1} = t^{k_1}S_{\{\{1\},\dots,\{k_2k_1\}\}}(\mathbb{A}')\sqcup\!\sqcup \sigma^W_t(k_1k_2\mathbb{A}'').$$

Hence,

Proposition 36.

$$\mathtt{B}_{n,k_1}(\mathtt{B}^{\mathbb{A}'}_{k_2,k_2}(\mathbb{A}''),\dots,\mathtt{B}^{\mathbb{A}'}_{k_2+m-1,k_2}(\mathbb{A}''),\dots) = \mathtt{B}^{\mathbb{A}'}_{n-k_1+k_1k_2,k_1k_2}(\mathbb{A}''). \tag{44}$$

4.4 Specialization again

In [5] we have shown that one can construct a double algebra which is homomorphic to $(\mathbf{WSym}(\mathbb{A}),.,\sqcup\!\sqcup)$. This is a general construction which is an attempt to define properly the concept of virtual alphabet for **WSym**. In our context the construction is simpler, let us briefly recall it.

Let $F=(F_\pi(\mathbb{A}))_\pi$ be a basis of $\mathbf{WSym}(\mathbb{A})$. We will say that F is *shuffle*-compatible if

$$F_{\{\pi_1,\dots,\pi_k\}}(\mathbb{A}) = \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}\left(F_{\{\{1,\dots,\#\pi_1\}\}}(\mathbb{A}),\dots,F_{\{\{1,\dots,\#\pi_k\}\}}(\mathbb{A})\right).$$

Hence, one has

$$F_{\pi_1}(\mathbb{A})\sqcup\!\sqcup F_{\pi_2}(\mathbb{A}) = \sum_{\substack{\pi=\pi'_1\cup\pi'_2,\pi'_1\cap\pi'_2=\emptyset\\ \mathrm{std}(\pi'_1)=\pi_1,\mathrm{std}(\pi'_2)=\pi_2}} F_\pi \text{ and } F_{\pi_1}(\mathbb{A}).F_{\pi_2}(\mathbb{A}) = F_{\pi_1\uplus\pi_2}(\mathbb{A}).$$

Example 37. The bases $(S_\pi(\mathbb{A}))_\pi$, $(\Phi_\pi(\mathbb{A}))_\pi$ and $(\Psi_\pi(\mathbb{A}))_\pi$ are shuffle-compatible but not the basis $(M_\pi(\mathbb{A}))_\pi$.

Straightforwardly, one has:

Claim 38. *Let $(F_\pi(\mathbb{A}))_\pi$ be a* shuffle*-compatible basis of $\mathbf{WSym}(\mathbb{A})$. Let $\mathbb{B}$ be another alphabet and let $P=(P_k)_{k>0}$ be a family of noncommutative polynomials of $\mathbb{C}\langle\mathbb{B}\rangle$ such that $\deg P_k = k$. Then, the space spanned by the polynomials*

$$F_{\{\pi_1,\dots,\pi_k\}}[\mathbb{A}^{(P)}_F] := \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(P_{\#\pi_1},\dots,P_{\#\pi_k})$$

is stable under concatenation and shuffle product in $\mathbb{C}\langle\mathbb{B}\rangle$. So it is a double algebra which is homomorphic to $(\mathbf{WSym}(\mathbb{A}),.,\sqcup\!\sqcup)$. We will call $\mathbf{WSym}[\mathbb{A}^{(P)}_F]$ this double algebra and $f[\mathbb{A}^{(P)}_F]$ will denote the image of an element $f\in\mathbf{WSym}(\mathbb{A})$ by the morphism $\mathbf{WSym}(\mathbb{A})\longrightarrow\mathbf{WSym}[\mathbb{A}^{(P)}_F]$ sending $F_{\{\pi_1,\dots,\pi_k\}}$ to $F_{\{\pi_1,\dots,\pi_k\}}[\mathbb{A}^{(P)}_F]$.

With these notations, one has

$$\mathtt{B}_{n,k}(P_1,\dots,P_m,\dots) = \mathfrak{B}_{n,k}[\mathbb{A}^{(P)}_\Phi].$$

Example 39. We define a specialization by setting

$$\Phi_{\{\{1,\dots,n\}\}}[\mathfrak{S}] = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \sigma_1=1}} \mathtt{b}_{\sigma[1]}\dots\mathtt{b}_{\sigma[n]},$$

where the letters $\mathtt{b}_i$ belong to an alphabet $\mathbb{B}$. Let $\sigma\in\mathfrak{S}_n$ be a permutation and $\sigma = c_1\circ\dots\circ c_k$ its decomposition into cycles. Each cycle $c^{(i)}$ is denoted by a sequence of integers $(n^{(i)}_1,\dots,n^{(i)}_{\ell_i})$ such that $n^{(i)}_1=\min\{n^{(i)}_1,\dots,n^{(i)}_{\ell_i}\}$. Let $\widetilde{c^{(i)}}\in\mathfrak{S}_{\ell_i}$ be the permutation which is the standardized of the sequence $n^{(i)}_1\dots n^{(i)}_{\ell_i}$. The cycle support of σ is the partition

$$\mathrm{support}(\sigma) = \{\{n^{(1)}_1,\dots,n^{(1)}_{\ell(i)}\},\dots,\{n^{(k)}_1,\dots,n^{(k)}_{\ell(k)}\}\}.$$

We define $w[c^{(i)}] = \mathtt{b}_{\widetilde{c^{(i)}}[1]} \cdots \mathtt{b}_{\widetilde{c^{(i)}}[\ell^{(i)}]}$ and $w[\sigma] = \sqcup\!\sqcup_{[\pi_1,\dots,\pi_k]}(w[c^{(1)}],\dots,w[c^{(k)}])$ where $\pi_i = \{n_1^{(i)},\dots,n_{\ell^{(i)}}^{(i)}\}$ for each $1 \leq i \leq k$.

For instance, if $\sigma = 312654 = (132)(46)(5)$ we have $w[(132)] = \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2$, $w[(46)] = \mathtt{b}_1\mathtt{b}_2$, $w[(5)] = \mathtt{b}_1$ and $w[\sigma] = \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2\mathtt{b}_1\mathtt{b}_1\mathtt{b}_2$.

So, we have

$$\mathtt{B}_{n,k}(\Phi_{\{\{1\}\}}[\mathfrak{S}],\dots,\Phi_{\{\{1,\dots,m\}\}}[\mathfrak{S}],\dots) = \mathfrak{B}_{n,k}[\mathfrak{S}] = \sum_{\substack{\pi\vDash n\\ \#\pi=k}} \Phi_\pi[\mathfrak{S}] = \sum_{\substack{\sigma\in\mathfrak{S}_n\\ \#\mathrm{support}(\sigma)=k}} w[\sigma].$$

For instance

$$\begin{aligned}
&\mathtt{B}_{4,2}(\mathtt{b}_1, \mathtt{b}_1\mathtt{b}_2, \mathtt{b}_1\mathtt{b}_2\mathtt{b}_3 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2, \mathtt{b}_1\mathtt{b}_2\mathtt{b}_3\mathtt{b}_4 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2\mathtt{b}_4 + \mathtt{b}_1\mathtt{b}_2\mathtt{b}_4\mathtt{b}_3 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_4\mathtt{b}_2 + \mathtt{b}_1\mathtt{b}_4\mathtt{b}_2\mathtt{b}_3\\
&\qquad + \mathtt{b}_1\mathtt{b}_4\mathtt{b}_3\mathtt{b}_2,\dots)\\
&= \Phi_{\{\{1\},\{2,3,4\}\}}[\mathfrak{S}] + \Phi_{\{\{2\},\{1,3,4\}\}}[\mathfrak{S}] + \Phi_{\{\{3\},\{1,2,4\}\}}[\mathfrak{S}] + \Phi_{\{\{4\},\{1,2,3\}\}}[\mathfrak{S}] + \Phi_{\{\{1,2\},\{3,4\}\}}[\mathfrak{S}]\\
&\qquad + \Phi_{\{\{1,3\},\{2,4\}\}}[\mathfrak{S}] + \Phi_{\{\{1,4\},\{2,3\}\}}[\mathfrak{S}]\\
&= 2\mathtt{b}_1\mathtt{b}_1\mathtt{b}_2\mathtt{b}_3 + 2\mathtt{b}_1\mathtt{b}_1\mathtt{b}_3\mathtt{b}_2 + \mathtt{b}_1\mathtt{b}_2\mathtt{b}_1\mathtt{b}_3 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2\mathtt{b}_1 + \mathtt{b}_1\mathtt{b}_2\mathtt{b}_3\mathtt{b}_1 + \mathtt{b}_1\mathtt{b}_3\mathtt{b}_2\mathtt{b}_1 + \mathtt{b}_1\mathtt{b}_2\mathtt{b}_1\mathtt{b}_2 + 2\mathtt{b}_1\mathtt{b}_1\mathtt{b}_2\mathtt{b}_2.
\end{aligned}$$

Notice that the sum of the coefficients of the words occurring in the expansion of $\mathfrak{B}_{n,k}[\mathfrak{S}]$ is equal to the Stirling number $s_{n,k}$. Hence, this specialization gives another word analogue of formula (14).

5 Munthe-Kaas polynomials

5.1 Munthe-Kaas polynomials from $WSym$

In order to generalize the Runge-Kutta method to integration on manifolds, Munthe-Kaas [33] introduced a noncommutative version of Bell polynomials. We recall here the construction in a slightly different variant adapted to our notation, the operators acting on the left. Consider an alphabet $\mathbb{D} = \{d_1, d_2, \dots\}$. The algebra $\mathbb{C}\langle\mathbb{D}\rangle$ is equipped with the derivation defined by $d_i\partial = d_{i+1}$. The noncommutative Munthe-Kaas Bell polynomials are defined by setting $t = 1$ in $\mathtt{MB}_n(t) = 1.(td_1 + \partial)^n$. The partial noncommutative Bell polynomial $\mathtt{MB}_{n,k}$ is the coefficient of t^k in $\mathtt{MB}_n(t)$.

Example 40.
- $\mathtt{MB}_1(t) = d_1 t$,
- $\mathtt{MB}_2(t) = d_1^2t^2 + d_2t$,
- $\mathtt{MB}_3(t) = d_1^3t^3 + (2d_2d_1 + d_1d_2)t^2 + d_3t$,
- $\mathtt{MB}_4(t) = d_1^4t^4 + (3d_2d_1^2 + 2d_1d_2d_1 + d_1^2d_2)t^3 + (3d_3d_1 + 3d_2^2 + d_1d_3)t^2 + d_4t$.

We consider the map χ which sends each set partition π to the integer composition $[\mathrm{card}(\pi_1),\dots,\mathrm{card}(\pi_k)]$ if $\pi = \{\pi_1,\dots,\pi_k\}$ where $\min(\pi_i) < \min(\pi_{i+1})$ for any $0 < i < k$. The linear map Ξ sending Φ_π to $d_{\chi(\pi)[1]}\cdots d_{\chi(\pi)[k]}$ is a morphism of algebra. Hence, we deduce

Proposition 41.

$$\Xi\left(\mathfrak{B}_{n,k}\right) = \mathtt{MB}_{n,k}. \tag{45}$$

Example 42.

$$\Xi\left(\mathfrak{B}_{3,2}\right) = \Xi\left(\Phi_{\{\{1\},\{2,3\}\}} + \Phi_{\{\{1,3\},\{2\}\}} + \Phi_{\{\{1,2\},\{3\}\}}\right) = d_1d_2 + 2d_2d_1 = \mathtt{MB}_{3,2}.$$

We recover a result due to Ebrahimi-Fard *et al.* [10].

Theorem 43. *If $j_1+\cdots+j_k = n$, the coefficient of $d_{j_1}\cdots d_{j_k}$ in $\mathtt{MB}_{n,k}$ is equal to the number of partitions of $\{1,2,\dots,n\}$ into parts $\pi_1,\dots,\pi_k$ such that $\mathrm{card}(\pi_\ell) = j_\ell$ for each $1 \leq \ell \leq k$ and $\min(\pi_1) < \cdots < \min(\pi_k)$.*

5.2 Dendriform structure and quasideterminant formula

The algebra ΠQSym is equipped with a Zinbiel structure. The notion of Zinbiel algebra is due to Loday [21]. This is an algebra equipped with two nonassociative products $\prec$ and $\succ$ satisfying

- $(u \prec v) \prec w = u \prec (v \prec w) + u \prec (v \succ w)$,
- $(u \succ v) \prec w = u \succ (v \prec w)$,

- $u \succ (v \succ w) = (u \prec v) \succ w + (u \succ v) \succ w$,
- $u \prec v = v \succ u$.

The Zinbiel structure on ΠQSym is defined for any $\pi \vDash n$ and $\pi' \vDash m$ by $\Phi_\pi \prec \Phi_{\pi'} = \sum' \Phi^{\pi[I]\cup\pi'[J]}$ (resp. $\Phi_\pi \succ \Phi_{\pi'} = \sum'' \Phi^{\pi[I]\cup\pi'[J]}$) where $\sum'$ (resp. $\sum''$) means the sum is over the partitions $I \cup J = \{1, \dots, n+m\}$ ($I \cap J = \emptyset$) such that $\mathrm{card}(I) = n$, $\mathrm{card}(J) = m$, and $1 \in A$ (resp. $1 \in B$), and $\Phi[I]$ is obtained from π by substituting each ℓ by i_ℓ if $I = \{i_1, \dots, i_n\}$ and $i_1 < \cdots < i_n$. See also [26, 27, 13] for other combinatorial Hopf algebras with a dendriform structure.

We notice that one has

$$\sum_n \mathcal{B}_{n,k}(\Phi_{\{\{1\}\}}, \Phi_{\{\{1,2\}\}}, \dots)t^n = \left(\sum_i \Phi\{\{1, \dots, i\}\}t^i\right)^{\overrightarrow{\prec} k} \tag{46}$$

with $u^{\overrightarrow{\prec} k} = u^{\overrightarrow{\prec} k-1} \prec u$ and $u^{\overrightarrow{\prec} 0} = 1$.

Definition 44. Let $A_n = (a_{ij})_{1\le i\le j\le n}$ be an upper triangular matrix whose entries are in a Zinbiel algebra. We define the polynomial

$$\mathrm{P}(A_n; t) = t\sum_{k=1}^{n} \mathrm{P}(A_{k-1}) \prec a_{k,n} \text{ and } \mathrm{P}(A_0) = 1. \tag{47}$$

Example 45.

$$\begin{aligned}
\mathrm{P}(A_4, t) &= t\mathrm{P}(A_3) \prec a_{44} + t\mathrm{P}(A_2) \prec a_{34} + t\mathrm{P}(A_1) \prec a_{24} + t\mathrm{P}(A_0) \prec a_{14} \\
&= t^4((a_{11} \prec a_{22}) \prec a_{33}) \prec a_{44} + t^3(a_{11} \prec a_{23}) \prec a_{44} \\
&\quad + t^3(a_{12} \prec a_{33}) \prec a_{44} + t^3(a_{11} \prec a_{22}) \prec a_{34} + t^2 a_{12} \prec a_{34} + a t_{11}^2 \prec a_{24} \\
&\quad + t a_{14}.
\end{aligned}$$

By induction, we find

$$\mathrm{P}(A_n; t) = t a_{1n} + \sum_{\substack{1\le j_1<j_2 \\ <\cdots<j_k}} t^k(\cdots(a_{1j_1} \prec (a_{j_1+1,j_2}) \prec (a_{j_2+1,j_3}) \prec \cdots \prec a_{j_{k-1}+1,j_k}) \prec a_{j_k+1,n}). \tag{48}$$

Setting $M_n := (\Phi_{\{\{1,\dots,j-i+1\}\}})_{1\le i\le j\le n}$, we find

Proposition 46.

$$\mathcal{B}_{n,k}(\Phi_{\{\{1\}\}}, \Phi_{\{\{1,2\}\}}, \dots) = [t^k]\mathrm{P}(M_n; t). \tag{49}$$

Example 47. We have $\mathrm{P}(A_3; t) = t^3(a_{11} \prec a_{22}) \prec a_{33} + t^2(a_{11} \prec a_{23} + a_{12} \prec a_{33}) + t a_{13}$. Hence

$$\begin{aligned}
\mathrm{P}(M_3; t) &= t^3(\Phi_{\{\{1\}\}} \prec \Phi_{\{\{1\}\}}) \prec \Phi_{\{\{1\}\}} + t^2(\Phi_{\{\{1\}\}} \prec \Phi_{\{\{1,2\}\}} + \Phi_{\{\{1,2\}\}}) \prec \Phi_{\{\{1\}\}})) \\
&\quad + t\Phi_{\{\{1,2,3\}\}} \\
&= t^3\Phi_{\{\{1\},\{2\},\{3\}\}} + t^2(\Phi_{\{\{1\},\{2,3\}} + \Phi_{\{\{1,2\},\{3\}\}} + \Phi_{\{\{1,3\},\{2\}\}}) + t\Phi_{\{\{1,2,3\}\}} \\
&= t^3\mathcal{B}_{3,3} + t^2\mathcal{B}_{3,2} + t\mathcal{B}_{3,1}.
\end{aligned}$$

Formula (48) has reminiscence of a well known result on quasideterminants.

Proposition 48 (Gelfand *et al.* [14])**.**

$$\begin{vmatrix}
a_{11} & a_{12} & a_{13} & \cdots & \boxed{a_{1n}} \\
-1 & a_{22} & a_{23} & \cdots & a_{2n} \\
0 & -1 & a_{33} & \cdots & a_{3n} \\
 & \cdots & & & \\
0 & \cdots & 0 & -1 & a_{nn}
\end{vmatrix} = a_{1n} + \sum_{1\le j_1<\cdots<j_k<n} a_{1j_1}a_{j_1+1,2}\cdots a_{j_k+1,n} \tag{50}$$

Furthermore, formula (49) is an analogue of the result of Ebrahimi *et al.*.

Theorem 49 (Ebrahimi *et al.* [10])**.**

$$\mathrm{MB}_n(1) = \begin{vmatrix}
\binom{n-1}{0}d_1 & \binom{n-1}{1}d_2 & \binom{n-1}{2}d_3 & \cdots & \boxed{\binom{n-1}{n-1}d_n} \\
-1 & \binom{n-2}{0}d_1 & \binom{n-2}{1}d_2 & \cdots & \binom{n-2}{n-2}d_{n-1} \\
0 & -1 & \binom{n-3}{0}d_1 & \cdots & \binom{n-3}{n-3}d_{n-3} \\
 & \cdots & & & \\
0 & \cdots & 0 & -1 & \binom{0}{0}d_1
\end{vmatrix}.$$

The connection between all these results remains to be investigated.

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References

[1] Eric Temple Bell. Exponential polynomials. *Annals of Mathematics*, pages 258–277, 1934.

[2] Nantel Bergeron, Christophe Reutenauer, Mercedes Rosas, and Mike Zabrocki. Invariants and coinvariants of the symmetric groups in noncommuting variables. *Canad. J. Math.*, 60(2):266–296, 2008.

[3] François Bergeron and Gilbert Labelle and Pierre Leroux, *Introduction to the Theory of Species of Structures*, UQÀM Univerité du Québec à Montréal, 2013.

[4] Sadek Bouroubi and Moncef Abbas. New identities for Bell's polynomials. New approaches. In *Rostock. Math. Kolloq*, volume 61, pages 49–55, 2006.

[5] Jean-Paul Bultel, Ali Chouria, Jean-Gabriel Luque, and Olivier Mallet. Word symmetric functions and the Redfield-Pólya theorem. *FPSAC'2013, DMTCS Proceedings*, (01):563–574, 2013.

[6] R. M. Corless, G. H. Gonnet, D. E. Hare, D. J. Jeffrey, and D. E. Knuth. On the Lambert W function. *Advances in Computational Mathematics*, 5:329–359, 1996.

[7] Louis Comtet. *Advanced Combinatorics: The art of finite and infinite expansions.* Springer, 1974.

[8] Djurdje Cvijović. New identities for the partial Bell polynomials. *Applied Mathematics Letters*, 24(9):1544–1547, 2011.

[9] Peter Doubilet. A Hopf algebra arising from the lattice of partitions of a set. *Journal of Algebra*, 28(1):127–132, 1974.

[10] Kurusch Ebrahimi-Fard, Alexander Lundervold, and Dominique Manchon. Noncommutative Bell polynomials, quasideterminants and incidence Hopf algebras. *International Journal of Algebra and Computation*, 24(05):671–705, 2014.

[11] Francesco Faà di Bruno. Sullo sviluppo delle funzioni. *Annali di Scienze Matematiche e Fisiche*, 6:479–480, 1855.

[12] Francesco Faà di Bruno. Note sur une nouvelle formule de calcul différentiel. *Quarterly J. Pure Appl.Math.*, 1:359–360, 1857.

[13] L. Foissy, *Bidendriform bialgebras, trees, and free quasi-symmetric functions*, J.Pure Appl. Algebra 209 no. 2: 439-459, 2007.

[14] Israel Gelfand, Sergei Gelfand, Vladimir Retakh, and R Wilson, *Quasideterminants*, Advances in Mathematics, 193(1):56–141, 2005.

[15] Robert Grossman and Richard G Larson. Hopf-algebraic structure of families of trees. *Journal of Algebra*, 126(1):184–210, 1989.

[16] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon. Commutative combinatorial Hopf algebras. *Journal of Algebraic Combinatorics*, 28(1):65–95, 2008.

[17] Bernhard Harris and Lowell Schoenfeld. The number of idempotent elements in symmetric semigroups. *Journal of Combinatorial Theory*, 3(2):122–135, 1967.

[18] SA Joni and Gian-Carlo Rota. Coalgebras and bialgebras in combinatorics. *Stud. Appl. Math*, 61(2):93–139, 1979.

[19] S. Khelifa and Y. Cherruault. New results for the Adomian method. *Kybernetes.*, 29(n3):332–355, 2000.

[20] A. Lascoux. *Symmetric Functions and Combinatorial Operators on Polynomials.* CBMS Regional Conference Series in Mathematics, American Mathematical Society, 2003.

[21] J.-L. Loday, *Cup-product for Leibniz cohomology and dual Leibniz algebras* , Math. Scand., vol. 77, no 2: 189-196, 1995.

[22] Ian Grant Macdonald. *Symmetric functions and Hall polynomials.* Oxford University Press, 1998.

[23] Miloud Mihoubi. Bell polynomials and binomial type sequences. *Discrete Mathematics*, 308(12):2450–2459, 2008.

[24] Miloud Mihoubi. Polynômes multivariés de Bell et polynômes de type binomial. *Thèse de Doctorat, L'Université des Sciences et de la Technologie Houari Boumediene*, 2008.

[25] Miloud Mihoubi. Partial Bell polynomials and inverse relations. *J. Integer Seq*, 13(4), 2010.

[26] J.-C. Novelli, J.-Y. Thibon , *Construction de trigèbres dendriformes* , C. R. Acad. Sci, Paris, Sér. I, 342 , (2006), 365–369.

[27] J.-C. Novelli, J.-Y. Thibon , *Hopf algebras and dendriform structures arising from parking functions*, Fundamenta Mathematicæ193: 189-241, 2007

[28] John Riordan. Derivatives of composite functions. *Bulletin of American Mathematical Society*, 52:664–667, 1946.

[29] John Riordan. *An Introduction to Combinatorial Analysis.* John Wiley & Sons, New York, 1958 ; Princeton University Press, Princeton, NJ, 1980.

[30] N. J. A. Sloane. The on-line encyclopedia of integer sequences, 2003.

[31] Melvin Tainiter. Generating functions on idempotent semigroups with application to combinatorial analysis. *Journal of Combinatorial Theory*, 5(3):273–288, 1968.

[32] Margarete C Wolf. Symmetric functions of non-commutative elements. *Duke Mathematical Journal*, 2(4):626–637, 1936.

[33] H. Munthe-Kaas, *Lie-Butcher theory for Runge-Kutta methods*, BIT Numerical Mathematics, 35(4):572–587, 1995.

[34] Weiping Wang and Tianming Wang. General identities on Bell polynomials. *Computers and Mathematics with Applications*, 58(1):104–118, 2009.

A Bell polynomials and coproducts in Sym

In fact, most of the identities on Bell polynomials can be obtained by manipulating generating functions and are closely related to some other identities occurring in literature. Typically, the relation between the complete Bell polynomials $A_n(a_1, a_2, \dots)$ and the variables $a_1, a_2, \dots$ is very closely related to the Newton Formula which links the generating functions of complete symmetric functions h_n (Cauchy series) to those of the power sums p_n. The symmetric functions form a commutative algebra Sym freely generated by the complete functions h_n or the power sum functions p_n. So, specializing the variable a_n to some numbers is equivalent to specializing the power sum functions p_n. More soundly, the algebra Sym can be endowed with coproducts conferring to it a structure of Hopf algebra. For instance, the coproduct for which the power sums are primitive turns Sym into a self-dual Hopf algebra. The coproduct can be translated in terms of generating functions by a product of two Cauchy series. This kind of manipulations appears also in the context of Bell polynomials, for instance when computing the complete Bell polynomials of the sum of two sequences of variables a_1+b_1, $a_2+b_2, \dots$. Another coproduct turns Sym into a non-cocommutative Hopf algebra called the Faà di Bruno algebra which is related to the Lagrange inversion. Finally, the coproduct such that the power sums are group-like can be related also to a few other formulae on Bell polynomials. The aim of this section is to investigate these connexions and in particular to restate some known results in terms of symmetric functions and virtual alphabets. We also give a few new results that are difficult to prove without the help of symmetric functions.

A.1 Bell polynomials as symmetric functions

First, let us recall some operations on alphabets. Given two alphabets $\mathbb{X}$ and $\mathbb{Y}$, we also define (see *e.g.* [20]) the alphabet $\mathbb{X}+\mathbb{Y}$ by:

$$p_n(\mathbb{X}+\mathbb{Y}) = p_n(\mathbb{X}) + p_n(\mathbb{Y}) \tag{51}$$

and the alphabet $\alpha\mathbb{X}$ (resp $\mathbb{X}\mathbb{Y}$), for $\alpha \in \mathbb{C}$ by:

$$p_n(\alpha\mathbb{X}) = \alpha p_n(\mathbb{X})(\text{ resp } p_n(\mathbb{X}\mathbb{Y}) = p_n(\mathbb{X})p_n(\mathbb{Y})). \tag{52}$$

In terms of Cauchy functions, these transforms imply

$$\sigma_t(\mathbb{X}+\mathbb{Y}) = \sigma_t(\mathbb{X})\sigma_t(\mathbb{Y}) \tag{53}$$

and

$$\sigma_t(\mathbb{X}\mathbb{Y}) = \sum_\lambda \frac{1}{z_\lambda} p^\lambda(\mathbb{X}) p^\lambda(\mathbb{Y}) t^{|\lambda|}. \tag{54}$$

In fact $\sigma_t(\mathbb{X}\mathbb{Y})$ encodes the kernel of the scalar product defined by $\langle p^\lambda, c_\mu \rangle = \delta_{\lambda,\mu}$ with $c_\lambda = \frac{p^\lambda}{z_\lambda}$. Notice that $c_n = \frac{p_n}{n}$ and

$$Sym = \mathbb{C}[c_1, c_2, \dots]. \tag{55}$$

From (7) and (19), we obtain

Proposition 50. $h_n = \frac{1}{n!} A_n(1!c_1, 2!c_2, \dots)$.

Conversely, Equality (55) implies that the morphism ϕ_a sending each c_i to $\frac{a_i}{i!}$ is well defined for any sequence of numbers $a = (a_i)_{i \in \mathbb{N}\setminus\{0\}}$ and $\phi_a(h_n) = \frac{1}{n!} A_n(a_1, a_2, \dots)$. Let us define also $h_n^{(k)}(\mathbb{X}) = [\alpha^k] h_n(\alpha \mathbb{X})$. From (19) and (52) we have

$$h_n^{(k)} = \sum_{\lambda = [\lambda_1, \dots, \lambda_k] \vdash n} c_\lambda = [t^n] \frac{1}{k!} \left(\sum_{i \geq 1} c_i t^i \right)^k$$

and so, everything works as if we use a special (virtual) alphabet $\mathbb{X}^{(a)}$ satisfying $c_n(\mathbb{X}^{(a)}) = n! a_n$. More precisely:

Proposition 51.

$$\phi_a(h_n^{(k)}) = h_n^{(k)}(\mathbb{X}^{(a)}) = \frac{1}{n!} B_{n,k}(a_1, \dots, a_k, \dots). \tag{56}$$

Example 52. Let $\mathbf{1}$ be the virtual alphabet defined by $c_n(\mathbf{1}) = \frac{1}{n}$ for each $n \in \mathbb{N}$. In this case the Newton Formula gives $h_n(\mathbf{1}) = 1$. Hence $A_n(0!, 1!, 2!, \dots, (m-1)!, \dots) = n!$ and $B_{n,k}(0!, 1!, 2!, \dots, (m-1)!, \dots) = n![\alpha^k][t^n] \left(\frac{1}{1-t}\right)^\alpha = s_{n,k}$, the Stirling number of the first kind.

Example 53. A more complicated example is treated in [4, 19] where $a_i = i^{i-1}$. In this case, the specialization gives $\sigma_t(\alpha \mathbb{X}^{(a)}) = \exp\{-\alpha W(-t)\}$ where $W(t) = \sum_{n=1}^\infty (-n)^{n-1} \frac{t^n}{n!}$ is the Lambert W function satisfying $W(t) \exp\{W(t)\} = t$ (see *e.g.* [6]). Hence, $\sigma_t(\alpha \mathbb{X}^{(a)}) = \left(\frac{W(-t)}{-t}\right)^\alpha$. But the expansion of the series $\left(\frac{W(t)}{t}\right)^\alpha$ is known to be:

$$\left(\frac{W(t)}{t}\right)^\alpha = 1 + \sum_{n=1}^\infty \frac{1}{n!} \alpha(\alpha+n)^{n-1}(-t)^n. \tag{57}$$

Hence, we obtain $B_{n,k}(1, 2, 3^2, \dots, m^{m-1}, \dots) = \binom{n-1}{k-1} n^{n-k}$. Note that the expansion of $W(t)$ and (57) are usually obtained by the use of the Lagrange inversion.

Example 54. With these notations we have $B_{n,k}(a_1 + b_1, \dots) = \frac{1}{n!} h_n^{(k)}(\mathbb{X}^{(a)} + \mathbb{X}^{(b)})$, and classical properties of Bell polynomials can be deduced from symmetric functions through this formalism. For instance, the equalities $c_n(\mathbb{X}^{(a)} + \mathbb{X}^{(b)}) = c_n(\mathbb{X}^{(a)}) + c_n(\mathbb{X}^{(b)})$ and $h_n(\mathbb{X}^{(a)} + \mathbb{X}^{(b)}) = \sum_{i+j=n} h_i(\mathbb{X}^{(a)}) h_j(\mathbb{X}^{(b)})$ give

$$A_n(a_1 + b_1, \dots) = \sum_{i+j=n} \binom{n}{i} A_i(a_1, a_2, \dots) A_j(b_1, b_2, \dots)$$

and

$$B_{n,k}(a_1 + b_1, \dots) = \sum_{r+s=k} \sum_{i+j=n} \binom{n}{i} B_{i,r}(a_1, a_2, \dots) B_{j,s}(b_1, b_2, \dots).$$

Example 55. Another example is given by

$$A_n(1a_1b_1, 2a_2b_2, \dots, ma_mb_m, \dots) = n! \sum_{\lambda \vdash n} \det \left| \frac{A_{\lambda_i - i + j}(a_1, a_2, \dots)}{(\lambda_i - i + j)!} \right| \times \det \left| \frac{A_{\lambda_i - i + j}(b_1, b_2, \dots)}{(\lambda_i - i + j)!} \right|,$$

using the convention $A_{-n} = 0$ for $n > 0$. This formula is an emanation of the Jacobi-Trudi formula and is derived from the Cauchy kernel (54), remarking that $c_n(\mathbb{X}^{(a)}\mathbb{X}^{(b)}) = nc_n(\mathbb{X}^{(a)})c_n(\mathbb{X}^{(b)})$ and

$$h_n(\mathbb{X}^{(a)}\mathbb{X}^{(b)}) = \sum_{\lambda\vdash n} s_\lambda(\mathbb{X}^{(a)})s_\lambda(\mathbb{X}^{(b)}) = \sum_{\lambda\vdash n} \det\left|h^{(a)}_{\lambda_i-i+j}(\mathbb{X}^a)\right| \det\left|h^{(b)}_{\lambda_i-i+j}(\mathbb{X}^b)\right|,$$

where $s_\lambda = \det\left|h^{(a)}_{\lambda_i-i+j}\right|$ is a Schur function (see *e.g.*[22]).

A.2 Other interpretations

First we focus on the identity (9) and we interpret it as the Cauchy function $\sigma_t(k\hat{\mathbb{X}}^{(a)})$ where $\hat{\mathbb{X}}^{(a)}$ is the virtual alphabet such that $h_{i-1}(\hat{\mathbb{X}}^{(a)}) = \frac{a_i}{i!}$. This means that we consider the morphism $\hat{\phi}_a : Sym \longrightarrow \mathbb{C}$ sending h_i to $\frac{a_{i+1}}{(i+1)!}$. We suppose that $a_1 = 1$ otherwise we use (15) and (16). With these notations we have

Proposition 56.

$$B_{n,k}(a_1, a_2, \dots) = \frac{n!}{k!}h_{n-k}(k\hat{\mathbb{X}}^{(a)}). \tag{58}$$

Example 57. If $a_i = i$ we have $h_i(\hat{\mathbb{X}}^{(a)}) = \frac{1}{i!}$ and so $\sigma_t(k\hat{\mathbb{X}}^{(a)}) = \exp(kt)$. Hence, we recover the classical result

$$B_{n,k}(1, 2, \dots, m, \dots) = \binom{n}{k}k^{n-k}.$$

From $h_n(\mathbb{X} + \mathbb{Y}) = \sum_{i+j=n} h_i(\mathbb{X})h_j(\mathbb{Y})$ we deduce two classical identities:

$$\binom{k_1+k_2}{k_1}B_{n,k_1+k_2}(a_1, a_2, \dots) = \sum_{i=0}^{n}\binom{n}{i}B_{i,k_1}(a_1, a_2, \dots)B_{n-i,k_2}(a_1, a_2, \dots) \tag{59}$$

and

$$\begin{aligned}&\binom{n}{k}B_{n-k,k}\left(a_1b_1, \dots, \frac{1}{m+1}\sum_{i=1}^{m}\binom{m+1}{i}a_ib_{m+1-i}, \dots\right) = \\ &\sum_{i=k}^{n-k}\binom{n}{i}B_{i,k}(a_1, a_2, \dots)B_{i,k}(b_1, b_2, \dots).\end{aligned} \tag{60}$$

Indeed, formula (59) is obtained by setting $\mathbb{X} = k_1\hat{\mathbb{X}}^{(a)}$ and $\mathbb{X} = k_2\hat{\mathbb{X}}^{(a)}$. Formula (60) is called the convolution formula for Bell polynomials (see *e.g.* [24]) and is obtained by setting $\mathbb{X} = \hat{\mathbb{X}}^{(a)}$ and $\mathbb{Y} = \hat{\mathbb{X}}^{(b)}$ in the left hand side and $\mathbb{X} = k\hat{\mathbb{X}}^{(a)}$ and $\mathbb{Y} = k\hat{\mathbb{X}}^{(b)}$ in the right hand side.

Example 58. The partial Bell polynomials are known to be involved in interesting identities on binomial functions. Let us first recall that a binomial sequence is a family of functions $(f_n)_{n\in\mathbb{N}}$ satisfying $f_0(x) = 1$ and

$$f_n(a + b) = \sum_{k=0}^{n}\binom{n}{k}f_k(a)f_{n-k}(b), \tag{61}$$

for all $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$. Setting $h_n(\mathbb{A}) := \frac{f_n(a)}{n!}$ and $h_n(\mathbb{B}) := \frac{f_n(b)}{n!}$, with these notations $f_n(ka) = n!h_n(k\mathbb{A})$. Hence,

$$B_{n,k}(1, \dots, if_{i-1}(a), \dots) = \frac{n!}{k!}h_{n-k}(k\mathbb{A}) = \binom{n}{k}f_{n-k}(ka). \tag{62}$$

Notice that from (59), $f_n(k) = \begin{cases}\binom{n}{k}^{-1}B_{n,k}(a_1, a_2, \dots) & \text{if } n > 0\\ 1 & \text{if } n = 0\end{cases}$, is binomial and we obtain

$$\begin{aligned}&\binom{n}{k_1k_2}^{-1}B_{n,k_1}(1, \dots, i\binom{i-1}{k}^{-1}B_{i-1,k_2}(a_1, a_2, \dots), \dots) = \\ &\binom{n-k_1}{k_1k_2}^{-1}B_{n-k_1,k_1k_2}(a_1, a_2, \dots).\end{aligned} \tag{63}$$

Several related identities are compiled in [24].

Example 59. Taking the coefficient of t^{n-k-1} in the left hand side and the right hand side of the equality $\frac{d}{dt}\sigma_t((k+1)\mathbb{X}) = (k+1)\left(\frac{d}{dt}\sigma_t(\mathbb{X})\right)\sigma_t(k\mathbb{X})$, we obtain

$$(n-k)h_{n-k}((k+1)\mathbb{X}) = (k+1)\sum_{i=1}^{n-k} ih_i(\mathbb{X})h_{n-i-k}(\mathbb{X})$$

and we recover the identity (see *e.g.* [8]):

$$B_{n,k}(a_1,a_2,\dots) = \frac{1}{n-k}\sum_{i=1}^{n-k}\binom{n}{i}\left[(k+1)-\frac{n+1}{i+1}\right](i+1)a_iB_{n-i,k}(a_1,a_2,\dots). \tag{64}$$

Example 60. Let $(a_n)_{n>0}$ and $(b_n)_{n>0}$ be two sequences of numbers such that $a_1 = b_1 = 1$ and $d_n = n!\sum_{\lambda\vdash n-1}\det\left|\frac{a_{\lambda_i-i+j+1}}{(\lambda_i+j+1)!}\right|\det\left|\frac{b_{\lambda_i-i+j+1}}{(\lambda_i+j+1)!}\right|$ with the convention $a_{-n} = b_{-n} = 0$ if $n \geq 0$. The Cauchy kernel and the orthogonality of Schur functions give

$$B_{n,k}(d_1,d_2,\dots) = \tfrac{n!}{k!}\sum_{\lambda\vdash n-k}(k_1!k_2!)^{\ell(\lambda)}\det\left|\frac{B_{\lambda_i-i+j+k_1,k_1}(a_1,a_2,\dots)}{(\lambda_i-i+j+k_1)!}\right| \det\left|\frac{B_{\lambda_i-i+j+k_1,k_1}(b_1,b_2,\dots)}{(\lambda_i-i+j+k_1)!}\right|,$$

for any $k_1k_2 = k$. Indeed, it suffices to use the fact that

$$h_n(k\mathbb{X}^{(a)}\mathbb{X}^{(b)}) = \sum_{\lambda\vdash n}s_\lambda(k_1\mathbb{X}^{(a)})s_\lambda(k_2\mathbb{X}^{(b)}).$$

The sum $\mathbb{X}+\mathbb{Y}$ and the product $\mathbb{X}\mathbb{Y}$ of alphabets are two examples of coproducts endowing Sym with a structure of Hopf algebra. The sum of alphabets encodes the coproduct Δ for which the power sums are of type Lie (*i.e.* $\Delta(p_n) = p_n\otimes 1+1\otimes p_n \sim p_n(\mathbb{X}+\mathbb{Y}) = p_n(\mathbb{X})+p_n(\mathbb{Y})$ by identifying $f\otimes g$ with $f(\mathbb{X})g(\mathbb{Y})$) whilst the product of alphabets encodes the coproduct Δ' for which the power sums are group-like (*i.e.* $\Delta'(p_n) = p_n\otimes p_n \sim p_n(\mathbb{X}\mathbb{Y}) = p_n(\mathbb{X})p_n(\mathbb{Y})$).

The algebra of symmetric functions can be endowed with another coproduct that confers a structure of Hopf algebra: this is the Faà di Bruno algebra [9, 18]. This algebra is rather important since it is related to the Lagrange-Bürmann formula. The Bell polynomials also appear in this context. As a consequence, one can define a new operation on alphabets corresponding to the composition of Cauchy generating functions. Let $\mathbb{X}$ and $\mathbb{Y}$ be two alphabets and set $f(t) = t\sigma_t(\mathbb{X})$ and $g(t) = t\sigma_t(\mathbb{Y})$. The composition $\mathbb{X}\circ\mathbb{Y}$ is defined by $\sigma_t(\mathbb{X}\circ\mathbb{Y}) = \frac{1}{t}f\circ g(t)$. The relationship with Bell polynomials can be established by observing that we have

$$\frac{1}{t}f\circ g = \sum_{n\geq 0}\left(\sum_{k=1}^{n+1}\frac{k!}{(n+1)!}h_{k-1}(\mathbb{X})B_{n+1,k}(1,2h_1(\mathbb{Y}),3!h_2(\mathbb{Y}),\dots)\right)t^n.$$

Equivalently, $h_n(\mathbb{X}\circ\mathbb{Y}) = \sum_{k=0}^{n}\frac{(k+1)!}{(n+1)!}h_k(\mathbb{X})B_{n+1;k+1}(1,2!h_1(\mathbb{Y}),3!h_2(\mathbb{Y}),\dots)$.

The antipode of the Faà di Bruno algebra is also described in terms of alphabets as the operation which associates to each alphabet $\mathbb{X}$ the alphabet $\mathbb{X}^{\langle -1\rangle}$ satisfying $\sigma_t(\mathbb{X}\circ\mathbb{X}^{\langle -1\rangle}) = 1$. More explicitly, one has

$$h_n(\mathbb{X}^{\langle -1\rangle}) = \frac{n!}{(2n+1)!(n+1)}B_{2n+1,n}(1,-2!e_1(\mathbb{X}),3!e_2(\mathbb{X}),\dots), \tag{65}$$

where $e_n(\mathbb{X})$ is the elementary symmetric function defined by $\sum_n e_n(\mathbb{X})t^n = \frac{1}{\sigma_{-t}(\mathbb{X})}$.

Example 61. Let $\omega(t) = t\sigma_t(\mathbb{X})$. The Lagrange inversion consists in finding an alphabet $\mathbb{X}'$ such that $\phi(t) = \sigma_t(\mathbb{X}')$. According to (65), it suffices to set $\mathbb{X}' = -\mathbb{X}^{\langle -1\rangle}$.
Let $F(t) = \sigma_t(\mathbb{Y})$. When stated in terms of alphabets, the Lagrange-Bürmann formula reads

$$F(\omega(t)) = 1+\sum_{n\geq 1}\frac{d^{n-1}}{du^{n-1}}[\sigma'_u(\mathbb{Y})\sigma_u(-n\mathbb{X}^{\langle -1\rangle})]|_{u=0}\frac{t^n}{n!}.$$

In other words one has, $h_{n-k}(-n\mathbb{X}^{\langle -1\rangle}) = \frac{(k-1)!}{(n-1)!}B_{n,k}(1,2!h_1(\mathbb{X}),3!h_3(\mathbb{X}),\dots)$. So we recover a result due to Sadek Bouroubi and Moncef Abbas [4]:

$$B_{n,k}(1,h_1(2\mathbb{X}),\dots,(m-1)!h_{m-1}(m\mathbb{X}),\dots) = \frac{(n-1)!}{(k-1)!}h_{n-k}(n\mathbb{X}).$$