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Word Bell Polynomials

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Abstract

Partial multivariate Bell polynomials have been defined by E.T. Bell in 1934. These polynomials have numerous applications in Combinatorics, Analysis, Algebra, Probabilities etc. Many of the formulae on Bell polynomials involve combinatorial objects (set partitions, set partitions into lists, permutations etc). So it seems natural to investigate analogous formulæ in some combinatorial Hopf algebras with bases indexed by these objects. In this paper we investigate the connections between Bell polynomials and several combinatorial Hopf algebras: the Hopf algebra of symmetric functions, the Faà di Bruno algebra, the Hopf algebra of word symmetric functions etc. We show that Bell polynomials can be defined in all these algebras and we give analogues of classical results. To this aim, we construct and study a family of combinatorial Hopf algebras whose bases are indexed by colored set partitions.

1 Introduction

Partial multivariate Bell polynomials (Bell polynomials for short) have been defined by E.T. Bell in [1] in 1934. But their name is due to Riordan [29] which studied the Faà di Bruno formula [11, 12] allowing one to write the nth derivative of a composition \( f \circ g \) in terms of the derivatives of \( f \) and \( g \) [28]. The applications of Bell polynomials in Combinatorics, Analysis, Algebra, Probabilities etc. are so numerous that it should be very long to detail them in the paper. Let us give only a few seminal examples.

• The main applications to Probabilities follow from the fact that the nth moment of a probability distribution is a complete Bell polynomial of the cumulants.

• Partial Bell polynomials are linked to the Lagrange inversion. This follows from the Faà di Bruno formula.

• Many combinatorial formulæ on Bell polynomials involve classical combinatorial numbers like Stirling numbers, Lah numbers etc.

The Faà di Bruno formula and many combinatorial identities can be found in [7]. The PhD thesis of M. Mihoubi [24] contains a rather complete survey of the applications of these polynomials together with numerous formulæ.

Some of the simplest formulæ are related to the enumeration of combinatorial objects (set partitions, set partitions into lists, permutations etc.). So it seems natural to investigate analogous formulæ in some combinatorial Hopf algebras with bases indexed by these objects. We recall that combinatorial Hopf algebras are graded bigebras with bases indexed by combinatorial objects such that the product and the coproduct have some compatibilities.

The paper is organized as follows. In Section 2, we investigate the combinatorial properties of the colored set partitions. Section 3 is devoted to the study of the Hopf algebras of
colored set partitions. After having introduced this family of algebras, we give some special cases which can be found in the literature. The main application explains the connections with $Sym$, the algebra of symmetric functions. This explains that we can recover some identities on Bell polynomials when the variables are specialized to combinatorial numbers from analogous identities in some combinatorial Hopf algebras. We show that the algebra $WSym$ of word symmetric functions has an important role for this construction. In Section 4, we give a few analogues of complete and partial Bell polynomials in $WSym$, $IQSym = WSym^*$ and $C(A)$ and investigate their main properties. Finally, in Section 5 we investigate the connection with other noncommutative analogues of Bell polynomials defined by Munthe-Kass [33].

2 Definition, background and basic properties of colored set partitions

2.1 Colored set partitions

Let $a = (a_n)_{n \geq 1}$ be a sequence of nonnegative integers. A colored set partition associated to the sequence $a$ is a set of couples

$$\Pi = \{[\pi_1, i_1], [\pi_2, i_2], \ldots, [\pi_k, i_k]\}$$

such that $\pi = \{\pi_1, \ldots, \pi_k\}$ is a partition of $\{1, \ldots, n\}$ for some $n \in \mathbb{N}$ and $1 \leq i_\ell \leq a_{\#\pi_\ell}$ for each $1 \leq \ell \leq k$. The integer $n$ is the size of $\Pi$. We will write $|\Pi| = n$, $\Pi \vdash n$ and $\Pi \models \pi$. We will denote by $\Pi_n(a)$ the set of colored partitions of size $n$ associated to the sequence $a$. Notice that these sets are finite. We will set also $\Pi(a) = \bigcup_n \Pi_n(a)$. We endow $\Pi$ with the additional statistic $\#\Pi$ and set $\Pi_{n,k}(a) = \{\Pi \in \Pi_n(a) : \#\Pi = k\}$. 

**Example 1.** Consider the sequence whose first terms are $a = (1, 2, 3, \ldots)$. The colored partitions of size 3 associated to $a$ are

$$\Pi_3(2) = \{\{\{1, 2, 3\}, 1\}, \{\{1, 2, 3\}, 2\}, \{\{1, 2, 3\}, 3\}, \{\{1, 2\}, 1\}, \{\{1, 2\}, 3\}, \{\{1, 3\}, 1\}, \{\{1, 3\}, 2\}, \{\{2\}, 1\}, \{\{2\}, 2\}, \{\{1\}, 1\}, \{\{1\}, 3\}, \{\{3\}, 1\}\}.$$ 

The colored partitions of size 3 and cardinality 2 are

$$\Pi_{3,2}(a) = \{\{\{1, 2\}, 1\}, \{\{1, 3\}, 1\}, \{\{1, 2\}, 2\}, \{\{2\}, 1\}, \{\{1, 3\}, 2\}, \{\{1\}, 1\}, \{\{2\}, 2\}, \{\{1\}, 3\}\}.$$ 

It is well known that the number of colored set partitions of size $n$ for a given sequence $a = (a_n)_n$ is equal to the evaluation of the complete Bell polynomial $A_n(a_1, \ldots, a_m, \ldots)$ and that the number of colored set partitions of size $n$ and cardinality $k$ is given by the evaluation of the partial Bell polynomial $B_n(a_1, a_2, \ldots, a_m, \ldots)$. That is

$$\#\Pi_n(a) = A_n(a_1, a_2, \ldots) \text{ and } \#\Pi_{n,k}(a) = B_{n,k}(a_1, a_2, \ldots).$$

Now, let $\Pi = \{[\pi_1, i_1], \ldots, [\pi_k, i_k]\}$ be a set such that the $\pi_j$ are finite sets of nonnegative integers such that no integer belongs to more than one $\pi_j$, and $1 \leq i_j \leq a_{\#\pi_j}$ for each $j$. Then, the *standardized* $\text{std}(\Pi)$ of $\Pi$ is well defined as the unique colored set partition obtained by replacing the $i$th smallest integer in the $\pi_j$ by $i$.

**Example 2.** For instance:

$$\text{std}([\{1, 4, 7\}, \{3, 8\}, \{5\}, \{10\}, 1]) = \{\{1, 3, 5\}, 1\}, \{2, 6, 1\}, \{4, 3\}, \{7\}, 1\}.$$ 

We define two binary operations $\otimes : \Pi_{n,k}(a) \otimes \Pi_{n', k'}(a) \rightarrow \Pi_{n+n', k+k'}(a)$, 

$$\Pi \otimes \Pi' = \Pi \cup \Pi'[n],$$

where $\Pi'[n]$ means that we add $n$ to each integer occurring in the sets of $\Pi'$ and $\otimes : \Pi_{n,k} \otimes \Pi_{n', k'} \rightarrow \mathcal{P}(\Pi_{n+n', k+k'})$ by

$$\Pi \otimes \Pi' = \{\hat{\Pi} \cup \hat{\Pi}' \in \Pi_{n+n', k+k'}(a) : \text{std}(\hat{\Pi}) = \Pi \text{ and } \text{std}(\hat{\Pi}') = \Pi'\}. $$
Example 3. We have
\[
\{[1, 3], 5], [[2], 3]\} \cup \{[1, 2], [2, 3], 4]\} = \{[1, 3], 5], [[2], 3], [4], 2\}, [5, 6], 4\},
\]
and
\[
\{[1, 3], 5], [[2], 3]\} \cup \{[1, 2], 2\} = \{[1, 3], 5], [[2], 3], [4], 2\},
\{[1, 3], 5], [[2], 3], [4], 2\}, \{[1, 3], 5], [[2], 3], [4], 2\},
\{[1, 3], 5], [[2], 3], [4], 2\}, \{[1, 3], 5], [[2], 3], [4], 2\}, \{[3], 5], [4], 3\}, [1, 2], 2\}.\]

The operator \(\Psi\) provides an algorithm which computes all the colored partitions:
\[
\mathcal{CP}_n(a) = \bigcup_{i_1 + \cdots + i_k = n} \bigcup_{j_1 = 1}^{a_{i_1}} \cdots \bigcup_{j_k = 1}^{a_{i_k}} \{[1, \ldots, i_k], j_1\} \cup \ldots \cup \{[1, i_k], j_k\}. \tag{1}
\]

Nevertheless each colored partition is generated more than once using this process. For a triple \((\Pi', \Pi'', \Pi'')\) we will denote by \(a^{\Pi'}_{\Pi''}\), the number of pairs of disjoint subsets \((\Pi', \Pi'')\) of \(\Pi\) such that \(\Pi' \cup \Pi'' = \Pi\), \(\text{std}(\Pi') = \Pi'\) and \(\text{std}(\Pi'') = \Pi''\).

Remark 4. Notice that for \(a = 1, 1, \ldots\) (i.e. the ordinary set partitions), there is an alternative simple way to construct efficiently the set \(\mathcal{CP}_n(1)\). It suffices to use the induction
\[
\mathcal{CP}_0(1) = \{\emptyset\},
\mathcal{CP}_{n+1}(1) = \{\pi \cup \{n+1\} : \pi \in \mathcal{CP}_n(1)\} \cup \{(\pi \setminus \{e\}) \cup \{e \cup \{n+1\}\} : \pi \in \mathcal{CP}_n(1), e \in \pi\}. \tag{2}
\]

Applying this recurrence, the set partitions of \(\mathcal{CP}_{n+1}(1)\) are each obtained exactly once from the set partitions of \(\mathcal{CP}_n(1)\).

2.2 Generating series

The generating series of the colored set partitions \(\mathcal{CP}(a)\) is obtained from the cycle generating function for the species of colored set partitions. The construction is rather classical, see e.g. [3]. Recall first that a species of structures is a rule \(F\) which produces for each finite set \(U\), a finite set \(F[U]\) and for each bijection \(\phi : U \to V\), a function \(F[\phi] : F[U] \to F[V]\) satisfying the following properties:

- for all pairs of bijections \(\phi : U \to V\) and \(\psi : V \to W\), \(F[\psi \circ \phi] = F[\psi] \circ F[\phi]\)
- if \(Id_U\) denotes the identity map on \(U\), then \(F[Id_U] = Id_{F[U]}\).

An element \(s \in F[U]\) is called an \(F\)-structure on \(U\). The cycle generating function of a species \(F\) is the formal power series in infinitely independent many variables \(p_1, p_2, \ldots\) (called power sums) defined by the formula
\[
Z_F(p_1, p_2, \ldots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\sigma \in S_n} |F([n])|^\sigma p_{\text{cycle-type}(\sigma)}, \tag{3}
\]
where \(F([n])^\sigma\) denotes the set of \(F\)-structures on \([n] := \{1, \ldots, n\}\) which are fixed by the permutation \(\sigma\), and \(p^\lambda = p_{\lambda_1} \cdots p_{\lambda_k}\), if \(\lambda\) is the vector \([\lambda_1, \ldots, \lambda_k]\). For instance, the trivial species \(\text{TRIV}\) has only one \(\text{TRIV}\)-structure on every \(n\). Hence, its cycle generating function is nothing else but the Cauchy function
\[
\sigma_1 := \exp \left\{ \sum_{n \geq 1} \frac{p_n}{n} t^n \right\} = \sum_{n \geq 0} h_n. \tag{4}
\]

Here \(h_n\) denotes the complete function \(h_n = \sum_{\lambda \vdash n} \frac{1}{z_\lambda} p^\lambda\) where \(\lambda \vdash n\) means that the sum is over the partitions \(\lambda\) of \(n\) and \(z_\lambda = \prod_i i^{m_i(\lambda)} m_i(\lambda)\) if \(m_i(\lambda)\) is the multiplicity of the part \(i\) in \(\lambda\).

We consider also the species \(\text{NCS}(a)\) of non-empty colored sets having \(a\) \(\text{NCS}(a)\)-structures on \([n]\) which are invariant by permutations. Its cycle generating function is
\[
Z_{\text{NCS}(a)} = \sum_{n \geq 1} a_n h_n. \tag{5}
\]
As a species, \( \mathcal{CP}(a) \) is a composite \( \text{TRIV} \circ \text{NCS}(a) \). Hence, its cycle generating function is obtained by computing the plethysm

\[
Z_{\text{NCS}(a)}(p_1, p_2, \ldots) = \sigma_1[Z_{\text{NCS}(a)}] = \exp \left\{ \sum_{n>0} \frac{1}{n} \sum_{k>0} a_k p_n h_k \right\}.
\]

The exponential generating function of \( \mathcal{CP}(a) \) is obtained by setting \( p_1 = t \) and \( p_i = 0 \) for \( i > 1 \) in (6):

\[
\sum_{n \geq 0} A_n(a_1, a_2, \ldots) \frac{t^n}{n!} = \exp \left\{ \sum_{i > 0} \frac{a_i}{i!} t^i \right\}.
\]

We deduce easily that \( A_n(a_1, a_2, \ldots) \) are multivariate polynomials in the variables \( a_i \)'s. These polynomials are known under the name of complete Bell polynomials [1]. The double generating function of \( \text{card}(\mathcal{CP}_{n,k}(a)) \) is easily deduced from (7) by

\[
\sum_{n \geq k} \sum_{k \geq 0} B_{n,k}(a_1, a_2, \ldots) \frac{x^k t^n}{n!} = \exp \left\{ x \sum_{i > 0} \frac{a_i}{i!} t^i \right\}.
\]

Hence,

\[
\sum_{n \geq k} B_{n,k}(a_1, a_2, \ldots) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{i > 0} \frac{a_i}{i!} t^i \right)^n.
\]

So, one has

\[
A_n(a_1, a_2, \ldots) = \sum_{k=n}^n B_{n,k}(a_1, a_2, \ldots), \forall n \geq 1 \text{ and } A_0(a_1, a_2, \ldots) = 1. \quad (10)
\]

The multivariate polynomials \( B_{n,k}(a_1, a_2, \ldots) \) are known under the name partial Bell polynomials [1]. Let \( S_{n,k} \) denote the Stirling number of the second kind which counts the number of ways to partition a set of \( n \) objects into \( k \) nonempty subsets. The following identity holds

\[
B_{n,k}(1, 1, \ldots) = S_{n,k}. \quad (11)
\]

Note also that \( A_n(x, x, \ldots) = \sum_{k=0}^n S_{n,k} x^k \) is the classical univariate Bell polynomial denoted by \( \phi_n(x) \) in [1]. Several other identities involve combinatorial numbers. For instance, one has

\[
B_{n,k}(1!, 2!, 3!, \ldots) = \binom{n-1}{k-1} \frac{n!}{k!}, \text{ Unsigned Lah numbers } A105278 \text{ in [30]},
\]

\[
B_{n,k}(1, 2, 3, \ldots) = \binom{n}{k} k^{n-k}, \text{ Idempotent numbers } A059297 \text{ in [30]},
\]

\[
B_{n,k}(0!, 1!, 2!, \ldots) = |s_{n,k}|, \text{ Stirling numbers of the first kind } A048994 \text{ in [30]}. \quad (14)
\]

We can also find many other examples in [1, 7, 23, 34, 25].

Remark 5. Without loss of generality, when needed, we will suppose \( a_1 = 1 \) in the remainder of the paper. Indeed, if \( a_1 \neq 0 \), then the generating function gives

\[
B_{n,k}(a_1, \ldots, a_p, \ldots) = a_1^k B_{n,k} \left( 1, \frac{a_2}{a_1}, \ldots, \frac{a_p}{a_1} \right) \quad (15)
\]

and when \( a_1 = 0 \),

\[
B_{n,k}(0, a_2, \ldots, a_p, \ldots) = \begin{cases} 0 & \text{if } n < k \\ \frac{n!}{(n-k)!} B_{n,k}(a_2, \ldots, a_p, \ldots) & \text{if } n \geq k. \end{cases} \quad (16)
\]

Notice that the ordinary series of the isomorphism types of \( \mathcal{CP}(a) \) is obtained by setting \( p_i = t^i \) in (6). Remarking that under this specialization we have \( p_i | h_n \equiv t^{nk} \), we obtain, unsurprisingly, the ordinary generating series of colored (integer) partitions

\[
\prod_{i > 0} \frac{1}{(1-t^{a_i})}.
\]
2.3 Bell polynomials and symmetric functions

The algebra of symmetric functions [22, 20] is isomorphic to its polynomial realization $Sym(\mathbb{X})$ on an infinite set $\mathbb{X} = \{x_1, x_2, \ldots \}$ of commuting variables, so the algebra $Sym(\mathbb{X})$ is defined as the set of polynomials invariant under permutation of the variables. As an algebra, $Sym(\mathbb{X})$ is freely generated by the power sum symmetric functions $p_n(\mathbb{X})$, defined by $p_n(\mathbb{X}) = \sum_{i \geq 1} x_i^n$, or the complete symmetric functions $h_n$, where $h_n$ is the sum of all the monomials of total degree $n$ in the variables $x_1, x_2, \ldots$. The generating function for the $h_n$, called Cauchy function, is

$$\sigma_t(\mathbb{X}) = \sum_{n \geq 0} h_n(\mathbb{X}) t^n = \prod_{i \geq 1} (1 - x_i t)^{-1}. \quad (18)$$

The relationship between the two families $(p_n)_{n \in \mathbb{N}}$ and $(h_n)_{n \in \mathbb{N}}$ is described in terms of generating series by the Newton formula:

$$\sigma_t(\mathbb{X}) = \exp \{ \sum_{n \geq 1} p_n(\mathbb{X}) \frac{t^n}{n} \}. \quad (19)$$

Notice that $Sym$ is the free commutative algebra generated by $p_1, p_2, \ldots$ i.e. $Sym = \mathbb{C}[p_1, p_2, \ldots]$ and $Sym(\mathbb{X}) = \mathbb{C}[p_1(\mathbb{X}), p_2(\mathbb{X}), \ldots]$ when $\mathbb{X}$ is an infinite alphabet without relations on the variables. As a consequence of the Newton Formula (19), it is also the free commutative algebra generated by $h_1, h_2, \ldots$. The freeness of the algebra provides a mechanism of specialization. For any sequence of commutative scalars $u = (u_n)_{n \in \mathbb{N}}$, there is a morphism of algebra $\phi_u$ sending each $p_n$ to $u_n$ (resp. sending $h_n$ to a certain $v_n$ which can be deduced from $u$). These morphisms are manipulated as if there exists an underlying alphabet (so called virtual alphabet) $\mathbb{X}_u$ such that $p_n(\mathbb{X}_u) = u_n$ (resp. $h_n(\mathbb{X}_u) = v_n$). The interest of such a vision is that one defines operations on sequences and symmetric functions by manipulating alphabets.

The bases of $Sym$ are indexed by the partitions $\lambda \vdash n$ of all the integers $n$. A partition $\lambda$ of $n$ is a finite nonincreasing sequence of positive integers $(\lambda_1 \geq \lambda_2 \geq \ldots)$ such that $\sum_1 \lambda_i = n$.

By specializing either the power sums $p_i$ or the complete functions $h_i$ to the numbers $\frac{a_i}{i!}$, the partial and complete Bell polynomials are identified with well known bases.

The algebra $Sym$ is usually endowed with three coproducts:

- the coproduct $\Delta$ such that the power sums are Lie-like ($\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n$);
- the coproduct $\Delta'$ such that the power sums are group-like ($\Delta'(p_n) = p_n \otimes p_n$);
- the coproduct of Faà di Bruno (see e.g. [9, 18]).

Most of the formulas on Bell polynomials can be stated and proved using specializations and these three coproducts. Since this is not really the purpose of our article, we have deferred to Appendix A a list of examples which are rewrites of existing proofs in terms of symmetric functions. One of the aims of our paper is to rise some of these identities to other combinatorial Hopf algebras.

3 Hopf algebras of colored set partitions

3.1 The Hopf algebras CWSym$(a)$ and CIQSym$(a)$

Let CWSym$(a)$, (CWSym for short when there is no ambiguity), be the algebra defined by its basis $(\Phi_\Pi)_{\Pi \subseteq P(a)}$ indexed by colored set partitions associated to the sequence $a = (a_m)_{m \geq 1}$ and the product

$$\Phi_\Pi \Phi_{\Pi'} = \Phi_{\Pi \cap \Pi'}. \quad (20)$$

Example 6. One has

$$\Phi_{\{(1,3,5,3),(2,4,1)\}} \Phi_{\{(1,2,5,4),(3,1),(4,2)\}} \Phi_{\{(1,3,5,3),(2,4,1),(6,7,10,4),(8,1),(9,2)\}}.$$

Let CWSym$_n$ be the subspace generated by the elements $\Phi_\Pi$ such that $\Pi \vdash n$.

For each $n$ we consider an infinite alphabet $\mathcal{A}_n$ of noncommuting variables and we suppose $\mathcal{A}_n \cap \mathcal{A}_m = \emptyset$ when $n \neq m$. For each colored set partition $\Pi = \{[\pi_1, i_1], [\pi_2, i_2], \ldots, [\pi_k, i_k]\}$, we construct a polynomial $\Phi_\Pi(\mathcal{A}_1, \mathcal{A}_2, \ldots) \in \mathbb{C}(\bigcup_n \mathcal{A}_n)$

$$\Phi_\Pi(\mathcal{A}_1, \mathcal{A}_2, \ldots) := \sum_{\mathcal{V} = a_1 \ldots a_n} \mathcal{W}, \quad (21)$$

where the sum is over the words $\mathcal{W} = a_1 \ldots a_n$ satisfying
• For each $1 \leq \ell \leq k$, $a_j \in \mathbb{A}_\ell$, if and only if $j \in \pi_\ell$.
• If $j_1, j_2 \in \pi_\ell$ then $a_{j_1} = a_{j_2}$.

**Example 7.**

$$
\phi_{\{(1,3),\{(2,1),\{(4,3)\}\}}(\lambda_1, \lambda_2, \ldots) = \sum_{a_1a_2 \in \lambda_3 \quad \beta \in \lambda_1} a_1b_1a_2.
$$

**Proposition 8.** The family

$$
\Phi(a) := (\Phi(\lambda_1, \lambda_2, \ldots))_{\Pi \in CP(a)}
$$

spans a subalgebra of $\mathbb{C}(\bigcup_n \mathbb{A}_n)$ which is isomorphic to $\text{CWSym}(a)$.

**Proof.** First, remark that $\text{span}(\Phi(\lambda))$ is stable under concatenation. Indeed,

$$
\text{span}(\Phi(\lambda_1, \lambda_2, \ldots)) = \text{span}(\Phi(\lambda_1, \lambda_2, \ldots)).
$$

Furthermore, this shows that $\text{span}(\Phi(\lambda))$ is homomorphic to $\text{CWSym}(a)$ and that an explicit (onto) morphism is given by $\Phi(\lambda) \to \Phi(\lambda_1, \lambda_2, \ldots)$. Observing that the family $\Phi(\lambda)$ is linearly independent, the fact that the algebra $\text{CWSym}(a)$ is graded in finite dimension implies the result. 

We turn $\text{CWSym}$ into a Hopf algebra by considering the coproduct

$$
\Delta(\Phi(\Pi)) = \sum_{\Pi_1 \cup \Pi_2 = \Pi \quad \Pi_1 \cap \Pi_2 = \emptyset} \Phi_{\text{std}(\Pi_1)} \otimes \Phi_{\text{std}(\Pi_2)} = \sum_{\Pi_1, \Pi_2} \alpha_{\Pi_1, \Pi_2} \Phi_{\Pi_1} \otimes \Phi_{\Pi_2}.
$$

Indeed, $\text{CWSym}$ splits as a direct sum of finite dimension spaces

$$
\text{CWSym} = \bigoplus_n \text{CWSym}_n.
$$

This defines a natural graduation on $\text{CWSym}$. Hence, since it is a connected algebra, it suffices to verify that it is a bigebra. More precisely:

$$
\Delta(\Phi(\Pi) \Phi(\Pi')) = \Delta(\Phi(\Pi \Pi')) = \sum_{\Pi_1 \cup \Pi_2 = \Pi \quad \Pi_1 \cap \Pi_2 = \emptyset} \Phi_{\text{std}(\Pi_1)} \otimes \text{std}(\Pi_2) = \Delta(\Phi(\Pi)) \Delta(\Phi(\Pi')).
$$

Notice that $\Delta$ is comultiplicative.

**Example 9.** For instance,

$$
\Delta(\Phi_{\{(1,3),\{(2,3)\}}}) = \Phi_{\{(1,3),\{(2,3)\}} \otimes 1 + \Phi_{\{(1,2),\{(3)\}\}} \otimes \Phi_{\{(1,3)\}} + \Phi_{\{(1,3)\}} \otimes \Phi_{\{(1,2),\{(3)\}\}} + 1 \otimes \Phi_{\{(1,3),\{(2,3)\}}.
$$

The graded dual $\text{CIQSym}^\ast(a)$ (which will be called CIQSym for short when there is no ambiguity) of $\text{CWSym}$ is the Hopf algebra generated as a space by the dual basis $(\Phi(\Pi))_{\Pi \in CP(a)}$ of $(\Phi(\Pi))_{\Pi \in CP(a)}$. Its product and its coproduct are given by

$$
\Psi_{\Pi} \Psi_{\Pi'} = \sum_{\Pi \otimes \Pi' = \Pi'} \alpha_{\Pi_1, \Pi_2} \Psi_{\Pi_1} \Psi_{\Pi_2} \text{ and } \Delta(\Phi(\Pi)) = \sum_{\Pi \otimes \Pi' = \Pi} \Psi_{\Pi} \otimes \Psi_{\Pi'}.
$$

**Example 10.** For instance, one has

$$
\Psi_{\{(1,2),\{(3)\}} \Psi_{\{(1),\{(2,4)\}} = \Psi_{\{(1,2),\{(3),\{(4)\}\}} + \Psi_{\{(1,3),\{(2,4)\}} + \Psi_{\{(1,4),\{(2,3)\}} + \Psi_{\{(1),\{(4)\}\}} + \Psi_{\{(2,4),\{(1,3)\}} + \Psi_{\{(4),\{(1,2)\}}
$$

and

$$
\Delta(\Psi_{\{(1,3),\{(2,4)\}}}) = 1 \otimes \Psi_{\{(1,3),\{(2,4)\}} + \Psi_{\{(1,4),\{(2,3)\}} \otimes \Psi_{\{(1,1)\}} + \Psi_{\{(1,3),\{(2,4)\}} \otimes 1.
$$
3.2 Special cases

In this section, we investigate a few interesting special cases of the construction.

3.2.1 Word symmetric functions

The most prominent example follows from the specialization \( a_n = 1 \) for each \( n \). In this case, the Hopf algebra \( \text{CWSym} \) is isomorphic to \( \text{WSym} \), the Hopf algebra of word symmetric functions. Let us briefly recall its construction. The algebra of word symmetric functions is a way to construct a noncommutative analogue of the algebra \( \text{Sym} \). Its bases are indexed by set partitions. After the seminal paper \([32]\), this algebra was investigated in \([2, 16]\) as well as an abstract algebra as in its realization with noncommutative variables. Its name comes from its realization as a subalgebra of \( \mathbb{C}(\Lambda) \) where \( \Lambda = \{ a_1, \ldots, a_n \cdots \} \) is an infinite alphabet.

Consider the family of functions \( \Phi := \{ \Phi_{\pi} \}_\pi \) whose elements are indexed by set partitions of \( \{1, \ldots, n\} \). The algebra \( \text{WSym} \) is formally generated by \( \Phi \) using the shifted concatenation product: \( \Phi_{\pi} \Phi_{\pi'} = \Phi_{\pi \pi'|n} \) where \( \pi \) and \( \pi' \) are set partitions of \( \{1, \ldots, n\} \) and \( \{1, \ldots, m\} \), respectively, and \( \pi'|n \) means that we add \( n \) to each integer occurring in \( \pi' \). The polynomial realization \( \text{WSym}(\Lambda) \subset \mathbb{C}(\Lambda) \) is defined by \( \Phi_{\pi}(\Lambda) = \sum \omega \) where the sum is over the words \( \omega = a_1 \cdots a_n \) where \( i, j \in \pi_e \) implies \( a_i = a_j \) if \( \pi = \{ \pi_1, \ldots, \pi_k \} \) is a set partition of \( \{1, \ldots, n\} \).

**Example 11.** For instance, one has \( \Phi_{\{(1,4)\}(2,5,6)\}(3,7)}(\Lambda) = \sum_{a, b, c \in A} a b c a b c. \)

Although the construction of \( \text{WSym}(\Lambda) \), the polynomial realization of \( \text{WSym} \), seems to be close to \( \text{Sym}(X) \), the structures of the two algebras are quite different since the Hopf algebra \( \text{WSym} \) is not automodular. Surprisingly, the graded dual \( \text{IIQSym} := \text{WSym}^* \) of \( \text{WSym} \) admits a realization in the same subspace \( (\text{WSym}(\Lambda)) \) of \( \mathbb{C}(\Lambda) \) but for the shuffle product.

With no surprise, we notice the following fact:

**Proposition 12.**

- The algebras \( \text{CWSym}(1, \ldots, n) \), \( \text{WSym} \) and \( \text{WSym}(\Lambda) \) are isomorphic.
- The algebras \( \text{CIIQSym}(1, \ldots, n) \), \( \text{IIQSym} \) and \( (\text{WSym}(\Lambda), [\cdot, \cdot]) \) are isomorphic.

In the rest of the paper, when there is no ambiguity, we will identify the algebras \( \text{WSym} \) and \( \text{IIQSym}(\Lambda) \).

The word analogue of complete symmetric functions is the basis \( (c_\pi)_\pi \) of \( \text{Sym} \) is the dual basis \( (\Psi_\pi)_{\pi} \) of \( (\Phi_{\pi})_{\pi} \).

Other bases are known, for example, the word monomial functions defined by \( \Phi_{\pi} = \sum_{\pi \leq \pi'} M_{\pi} \), where \( \pi \leq \pi' \) indicates that \( \pi \) is finer than \( \pi' \) i.e., that each block of \( \pi' \) is a union of blocks of \( \pi \).

**Example 13.** For instance,

\[
\Phi_{\{(1,4)\}(2,5,6)\}(3,7)} = M_{\{(1,4)\}(2,5,6)\}(3,7)} + M_{\{(1,2,4,5,6)\}(3,7)} = M_{\{(1,3,4,7)\}(2,5,6)} + M_{\{(1,4)\}(2,3,5,6,7)} + M_{\{(1,2,3,4,5,6,7)\}}.
\]

From the definition of \( M_{\pi} \), we deduce that the polynomial representation of the word monomial functions is given by \( M_{\pi}(\Lambda) = \sum \omega \) where the sum is over the words \( \omega = a_1 \cdots a_n \) where \( i, j \in \pi_e \) if and only if \( a_i = a_j \) where \( \pi = \{ \pi_1, \ldots, \pi_k \} \) is a set partition of \( \{1, \ldots, n\} \).

**Example 14.** \( M_{\{(1,4)\}(2,5,6)\}(3,7)}(\Lambda) = \sum_{a, b, c \in A} a b c a b c. \)

The analogue of complete symmetric functions is the basis \( (s_\pi)_\pi \) of \( \text{IIQSym} \) which is the dual of the basis \( (M_{\pi})_\pi \) of \( \text{WSym} \).

The algebra \( \text{IIQSym} \) is also realized in the space \( \text{WSym}(\Lambda) \): it is the subalgebra of \( (\mathbb{C}(\Lambda), [\cdot, \cdot]) \) generated by \( \Psi_{\pi}(\Lambda) = \pi! \Phi_{\pi}(\Lambda) \) where \( \pi! = \# \pi_1 \cdots \# \pi_k \) for \( \pi = \{ \pi_1, \ldots, \pi_k \} \). Indeed, the linear map \( \Psi_{\pi} : B (\Lambda) \to \Psi_{\pi}(\Lambda) \) is a bijection sending \( \Psi_{\pi_1} \Psi_{\pi_2} \) to

\[
\sum_{\substack{\pi = \pi_1 \cup \pi_2, \pi_1 \cap \pi_2 = \emptyset \\ \\
\pi_1 = \text{std}(\pi_1), \pi_2 = \text{std}(\pi_2)}} \Psi_{\pi}(\Lambda) = \pi_1! \pi_2! \sum_{\substack{\pi = \pi_1 \cup \pi_2, \pi_1 \cap \pi_2 = \emptyset \\ \\
\pi_1 = \text{std}(\pi_1), \pi_2 = \text{std}(\pi_2)}} \Phi_{\pi}(\Lambda)
\]

\[
= \pi_1! \pi_2! \Phi_{\pi_1}(\Lambda) [\Psi_{\pi_2}(\Lambda) = \Psi_{\pi_1}(\Lambda) [\Psi_{\pi_2}(\Lambda)] = \delta_{\pi_1, \pi_2}.
\]

With these notations the image of \( S_{\pi} \) is \( S_{\pi}(\Lambda) = \sum_{\pi - \pi} \Psi_{\pi}(\Lambda) \). For our realization, the duality bracket \( \langle \cdot | \cdot \rangle \) implements the scalar product \( \langle \cdot | \cdot \rangle \) on the space \( \text{WSym}(\Lambda) \) for which

\[
\langle S_{\pi_1}(\Lambda)| M_{\pi_2}(\Lambda) \rangle = \langle \Phi_{\pi_1}(\Lambda)| \Psi_{\pi_2}(\Lambda) \rangle = \delta_{\pi_1, \pi_2}.
\]
The subalgebra of $(\text{WSym}(\mathbb{A}), \|\|$ generated by the complete functions $S_{\{1,\ldots,n\}}(\mathbb{A})$ is isomorphic to $\text{Sym}$. Therefore, we define $\sigma^W_t(\mathbb{A})$ and $\phi^W_t(\mathbb{A})$ by
\[ \sigma^W_t(\mathbb{A}) = \sum_{n \geq 0} S_{\{1,\ldots,n\}}(\mathbb{A}) t^n \]
and
\[ \phi^W_t(\mathbb{A}) = \sum_{n \geq 1} \psi^{\{1,\ldots,n\}}(\mathbb{A}) t^{n-1}. \]
These series are linked by the equality
\[ \sigma^W_t(\mathbb{A}) = \exp\|\phi^W_t(\mathbb{A})\|, \quad (23) \]
where $\exp\|$ is the exponential in $(\text{WSym}(\mathbb{A}), \|\|)$. Furthermore, the coproduct of $\text{WSym}$ consists in identifying the algebra $\text{WSym} \otimes \text{WSym}$ with $\text{WSym}(\mathbb{A} + \mathbb{B})$, where $\mathbb{A}$ and $\mathbb{B}$ are two alphabets such that the letters of $\mathbb{A}$ commute with those of $\mathbb{B}$. Hence, one has
\[ \sigma^W_t(\mathbb{A} + \mathbb{B}) = \sigma^W_t(\mathbb{A}) \| \sigma^W_t(\mathbb{B}) \] In particular, we define the multiplication of an alphabet $\mathbb{A}$ by a constant $k \in \mathbb{N}$ by
\[ \sigma^W_t(k\mathbb{A}) = \sum_{n \geq 0} S_{\{1,\ldots,n\}}(k\mathbb{A}) t^n = \sigma^W_t(\mathbb{A})^k. \]

Nevertheless, the notion of specialization is subtler to define than in $\text{Sym}$. Indeed, the knowledge of the complete functions $S_{\{1,\ldots,n\}}(\mathbb{A})$ does not allow us to recover all the polynomials using uniquely the algebraic operations. In [5], we made an attempt to define virtual alphabets by reconstituting the whole algebra using the action of an operad. Although the general mechanism remains to be defined, the case where each complete function $S_{\{1,\ldots,n\}}(\mathbb{A})$ is specialized to a sum of words of length $n$ can be understood via this construction. More precisely, we consider the family of multilinear $k$-ary operators $\mathbb{W}_\Pi$ indexed by set compositions (a set composition is a sequence $[\pi_1, \ldots, \pi_k]$ of subsets of $\{1,\ldots,n\}$ such that $\{\pi_1, \ldots, \pi_k\} = \text{set partition of}\{1,\ldots,n\}$ acting on words by $\mathbb{W}_{[\pi_1,\ldots,\pi_k]}(a_1^{\pi_1} \cdot a_2^{\pi_2} \cdots a_k^{\pi_k}) = b_1 \cdots b_n$ with $b_\pi = a_\pi$ if $\pi = [\emptyset \leq \cdots \leq \emptyset_{n_p}]$ and $\mathbb{W}_{[\pi_1,\ldots,\pi_k]}(a_1^{\pi_1} \cdot a_2^{\pi_2} \cdots a_k^{\pi_k}) = 0$ if $\#\pi_p \neq \#_n_p$ for some $1 \leq p \leq k$.

Let $P = (P_n)_{n \geq 1}$ be a family of a homogeneous word polynomials such that $\deg(P_n) = n$ for each $n$. We set $S_{\{1,\ldots,n\}}(\mathbb{A}^{(P)}) = P_n$ and
\[ S_{\{\pi_1,\ldots,\pi_k\}}(\mathbb{A}^{(P)}) = \mathbb{W}_{[\pi_1,\ldots,\pi_k]}(S_{\{1,\ldots,\#\pi_1\}}(\mathbb{A}^{(P)}), \ldots, S_{\{1,\ldots,\#\pi_k\}}(\mathbb{A}^{(P)})). \]
The space $\text{WSym}(\mathbb{A}^{(P)})$ generated by the polynomials $S_{\{\pi_1,\ldots,\pi_k\}}(\mathbb{A}^{(P)})$ and endowed with the two products $\cdot$ and $\|$ is isomorphic to the double algebra $(\text{WSym}(\mathbb{A}), \cdot, \|)$. Indeed, let $\pi = \{\pi_1, \ldots, \pi_k\} \vdash n$ and $\pi' = \{\pi_1', \ldots, \pi_k'\} \vdash n'$ be two set partitions, one has
\[ S_{\pi}(\mathbb{A}^{(P)}), S_{\pi'}(\mathbb{A}^{(P)}) = \mathbb{W}_{[\pi_1,\ldots,\pi_k]}(S_{\{1,\ldots,\#\pi_1\}}(\mathbb{A}^{(P)}), \ldots, S_{\{1,\ldots,\#\pi_k\}}(\mathbb{A}^{(P)})) \]
and
\[ S_{\pi}(\mathbb{A}^{(P)}) \| S_{\pi'}(\mathbb{A}^{(P)}) = \sum_{I \cup J = \{1,\ldots,n+n'\}, \cap I \cap J = \emptyset} \mathbb{W}_{[I,J]}(S_{\pi}(\mathbb{A}^{(P)}), S_{\pi'}(\mathbb{A}^{(P)})) \]
where the second sum is over the partitions $\{\pi_1'', \ldots, \pi_{k+k}''\} \in \pi \uplus \pi'$ satisfying, for each $k+1 \leq i \leq k+k'$, $\text{std}([\pi_1'', \ldots, \pi_{k+k}'']) = \pi$, $\text{std}([\pi_{k+1}'', \ldots, \pi_{k+k}'']) = \pi'$, $\#\pi_i'' = \pi_i$. Hence,
\[ S_{\pi}(\mathbb{A}^{(P)}), S_{\pi'}(\mathbb{A}^{(P)}) = \sum_{\pi'' \in \pi \uplus \pi'} S_{\pi''}(\mathbb{A}^{(P)}). \]
In other words, we consider the elements of $\text{WSym}(\mathbb{A}^{(P)})$ as word polynomials in the virtual alphabet $\mathbb{A}^{(P)}$ specializing the elements of $\text{WSym}(\mathbb{A})$. 

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3.2.2 Biword symmetric functions

The bi-indexed word algebra \( \text{BWSym} \) was defined in [5]. We recall its definition here: the bases of \( \text{BWSym} \) are indexed by set partitions into lists, which can be constructed from a set partition by ordering each block. We will denote by \( \mathcal{P}_\ell \) the set of the set partitions of \( \{1, \ldots, n\} \) into lists.

**Example 15.** \( \{[1, 2, 3], [4, 5]\} \) and \( \{[3, 1, 2], [5, 4]\} \) are two distinct set partitions into lists of the set \( \{1, 2, 3, 4, 5\} \).

The number of set partitions into lists of an \( n \)-element set (or set partitions into lists of size \( n \)) is given by Sloane’s sequence \( \text{A000262} \) [30]. The first values are

\[
1, 1, 3, 13, 73, 501, 4051, \ldots
\]

If \( \hat{\Pi} \) is a set partition into lists of \( \{1, \ldots, n\} \), we will write \( \Pi \vdash n \). Set \( \hat{\Pi} \cup \hat{\Pi}' = \hat{\Pi} \cup \{[l_1 + n, \ldots, l_k + n] : [l_1, \ldots, l_k] \in \Pi'\} \vdash n + n' \). Let \( \Pi' \subset \hat{\Pi} \vdash n \), since the integers appearing in \( \hat{\Pi}' \) are all distinct, the standardized \( \text{std}(\hat{\Pi}') \) of \( \hat{\Pi}' \) is well defined as the unique set partition into lists obtained by replacing the \( i \)th smallest integer in \( \hat{\Pi} \) by \( i \). For example, \( \text{std}([5, 2], [3, 10], [6, 8]) = ([3, 1], [2, 6], [4, 5]) \).

The Hopf algebra \( \text{BWSym} \) is formally defined by its basis \( (\Phi_{\hat{\Pi}})_{\hat{\Pi}} \) where the \( \hat{\Pi} \) are set partitions into lists, its product \( \Phi_{\hat{\Pi}} \Phi_{\hat{\Pi}'} = \Phi_{\hat{\Pi} \cup \hat{\Pi}'} \) and its coproduct

\[
\Delta(\Phi_{\hat{\Pi}}) = \sum_{\substack{n=n_1+\cdots+n_k, n'_1+\cdots+n'_k, \, \text{std}(\hat{\Pi}_1)\vdash n_1, \, \text{std}(\hat{\Pi}_2)\vdash n_2}} \Psi_{\hat{\Pi}_1} \Psi_{\hat{\Pi}_2},
\]

where the \( \sum_n \) means that the sum is over the \( (\hat{\Pi}', \hat{\Pi}'') \) such that \( \hat{\Pi}' \cup \hat{\Pi}'' = \hat{\Pi} \) and \( \hat{\Pi}' \cap \hat{\Pi}'' = \emptyset \).

The product of the graded dual BIQSym of \( \text{BWSym} \) is completely described in the dual basis \( (\Psi_{\hat{\Pi}})_{\hat{\Pi}} \) of \( (\Phi_{\hat{\Pi}})_{\hat{\Pi}} \) by

\[
\Psi_{\hat{\Pi}_1} \Psi_{\hat{\Pi}_2} = \sum_{\substack{n=n_1+\cdots+n_k, n'_1+\cdots+n'_k, \, \text{std}(\hat{\Pi}_1)\vdash n_1, \, \text{std}(\hat{\Pi}_2)\vdash n_2}} \Psi_{\hat{\Pi}'_1} \Psi_{\hat{\Pi}'_2},
\]

where \( \hat{\Pi}'_1 \) and \( \hat{\Pi}'_2 \) are two distinct set partitions into lists of \( \{1, \ldots, n\} \) such that \( \hat{\Pi}'_1 \cup \hat{\Pi}'_2 = \hat{\Pi} \) and \( \hat{\Pi}'_1 \cap \hat{\Pi}'_2 = \emptyset \).

Now consider a bijection \( \iota_n \) from \( \{1, \ldots, n\} \) to the symmetric group \( \mathfrak{S}_n \). The linear map \( \kappa : \mathcal{CP}(1!, 2!, 3!, \ldots) \rightarrow \mathcal{CL} \) sending

\[
\left\{ [[i^1_1, i^1_{n_1}], m_1], \ldots, [[i^k_1, i^k_{n_k}], m_k] \right\} \in \mathcal{CP}_n(1!, 2!, 3!, \ldots),
\]

with \( i_{1}^1 \leq \cdots \leq i_{n_1}^1 \), to

\[
\left\{ [[i^1_{m_1(1)}, \ldots, i^1_{m_1(n_1)}], \ldots, [i^k_{m_k(1)}, \ldots, i^k_{m_k(n_k)}]] \right\}
\]

is a bijection. Hence, a fast checking shows that the linear map sending \( \Psi_{\hat{\Pi}} \) to \( \Psi_{\kappa(\hat{\Pi})} \) is an isomorphism. So we have

**Proposition 16.**

- The Hopf algebras \( \text{CWSym}(1!, 2!, 3!, \ldots) \) and \( \text{BWSym} \) are isomorphic.

- The Hopf algebras \( \text{CIQSym}(1!, 2!, 3!, \ldots) \) and \( \text{BIQSym} \) are isomorphic.

3.2.3 Word symmetric functions of level 2

We consider the algebra \( \text{WSym}_{(2)} \) which is spanned by the \( \Phi_{\hat{\Pi}} \) where \( \hat{\Pi} \) is a set partition of level 2, that is, a partition of a partition \( \pi \) of \( \{1, \ldots, n\} \) for some \( n \). The product of this algebra is given by \( \Phi_{\hat{\Pi}} \Phi_{\hat{\Pi}'} = \Phi_{\hat{\Pi} \cup \hat{\Pi}'} \) where \( \hat{\Pi}'[n] = \{e[n] : e \in \hat{\Pi}'\} \). The dimensions of this algebra are given by the exponential generating function

\[
\sum_i \frac{b_i^{(2)} t^i}{i!} = \exp(\exp(\exp(t) - 1) - 1).
\]

The first values are

\[
1, 3, 12, 60, 358, 2471, 19302, 167894, 1606137, \ldots
\]

see sequence \( \text{A000258} \) of [30].
The coproduct is defined by $\Delta(\Phi_\Pi) = \sum_{\Pi' \cup \Pi'' = \Pi} \Phi_{\text{std}(\Pi')} \otimes \Phi_{\text{std}(\Pi'')}$, where, if $\Pi$ is a partition of a partition of $\{i_1, \ldots, i_k\}$, $\text{std}(\Pi)$ denotes the standardized of $\Pi$, that is the partition of partition of $\{1, \ldots, k\}$ obtained by substituting each occurrence of $i_j$ by $j$ in $\Pi$. The coproduct being co-commutative, the dual algebra $\Pi\Omega\Pi_{(2)} := W_{\Pi\Omega\Pi_{(2)}}$ is commutative. The algebra $\Pi\Omega\Pi_{(2)}$ is spanned by a basis $(\Psi_{\Pi})_{\Pi}$ satisfying $\Psi_{\Pi} \Psi_{\Pi'} = \sum_{\Pi''} C_{\Pi''}^{\Pi'\Pi} \Psi_{\Pi''}$, where $C_{\Pi''}^{\Pi'\Pi}$ is the number of ways to write $\Pi'' = \Pi \cup B$ with $\Pi \cap B = \emptyset$, $\text{std}(\Pi) = \Pi$ and $\text{std}(B) = \Pi'$.

Let $b_n$ be the $n$th Bell number $A_n(1, 1, \ldots)$. Considering a bijection from $\{1, \ldots, b_n\}$ to the set of the set partitions of $\{1, \ldots, n\}$ for each $n$, we obtain, in the same way as in the previous subsection, the following result.

**Proposition 17.**
- The Hopf algebras $C\Pi\Omega\Pi(b_1, b_2, b_3, \ldots)$ are isomorphic.
- The Hopf algebras $C\Pi\Omega\Pi(b_1, b_2, b_2, \ldots)$ and $\Pi\Omega\Pi_{(2)}$ are isomorphic.

### 3.2.4 Cycle word symmetric functions

We consider the Grossman-Larson Hopf algebra of heap-ordered trees $\mathbb{G}Sym$ [15]. The combinatorics of this algebra has been extensively investigated in [16]. This Hopf algebra is spanned by the $\Phi$, where $\sigma$ is a permutation. We identify each permutation with the set of its cycles (for example, the permutation $3241$ is $\{(13), (2)\}$). The product in this algebra is given by $\Phi_\sigma \Phi_\tau = \Phi_{\sigma \cup \tau[\text{size}(\sigma)]}$, where $\tau[\text{size}(\sigma)]$ denotes the restriction of the permutation $\sigma$ to the set $I$ and $\text{std}(\sigma[\text{size}(\sigma)])$ is the permutation obtained from $\sigma[\text{size}(\sigma)]$ by replacing the $i$th smallest label by $i$ in $\sigma[\text{size}(\sigma)]$.

**Example 18.**

$$\Delta(\Phi_{3241}) = \Phi_{3241} \otimes 1 + \Phi_1 \otimes \Phi_{231} + \Phi_{231} \otimes \Phi_1 + 1 \otimes \Phi_{3241}.$$ 

The basis $(\Phi_\sigma)$ and its dual basis $(\Psi_\sigma)$ are respectively denoted by $(S^\sigma)$ and $(M_\sigma)$ in [16]. The Hopf algebra $\mathbb{G}Sym$ is not commutative but it is co-commutative, so it is not automorphic and not isomorphic to the Hopf algebra of free quasi-symmetric functions. Let $\kappa : \mathbb{G}Sym \leftrightarrow CP(0, 1, 2, \ldots)$ by

$$\kappa(\sigma) = \{[\text{support}(c_1), \text{size}(\text{support}(c_1))], \ldots, [\text{support}(c_k), \text{size}(\text{support}(c_k))])\},$$

if $\sigma = c_1 \ldots c_k$ is the decomposition of $\sigma$ into cycles and $\text{support}(c)$ denotes the support of the cycle $c$, i.e. the set of the elements which are permuted by the cycle.

**Example 19.** For instance, set

$$\kappa(1) = 1, \quad \kappa(321) = 2, \quad \kappa(312) = 1.$$ 

One has $\kappa(321456789) = \{[[2], 1], [[1, 3, 4], 2], [[5], 1], [[6, 7, 8], 1]\}$.

The linear map $K : \mathbb{G}Sym \rightarrow C\Pi\Omega\Pi(0, 1, 2, \ldots)$ sending $\Phi_\sigma$ to $\Phi_{\kappa(\sigma)}$ is an isomorphism of algebra. Indeed, it is straightforward to see that it is a bijection and furthermore $\kappa(\sigma \cup \tau[\text{size}(\tau)]) = \kappa(\sigma) \cup \kappa(\tau)$. Moreover, if $\sigma \in \mathbb{G}Sym$ is a permutation and $\{I, J\}$ is a partition of $\{1, \ldots, n\}$ into two subsets such that the action of $\sigma$ does not $I$ and $J$ global invariant, we check that $\kappa(\sigma) = I_1 \cup I_2$ with $\Pi_1 \cap I_2 = \emptyset$, $\text{std}(\Pi_1) = \kappa(\text{std}(\sigma[\text{size}(\sigma)])$ and $\text{std}(\Pi_2) = \kappa(\text{std}(\sigma[j]))$. Conversely, if $\kappa(\sigma) = I_1 \cup I_2$ with $\Pi_1 \cap I_2 = \emptyset$ then there exists a partition $\{I, J\}$ of $\{1, \ldots, n\}$ into two subsets such that the action of $\sigma$ does not $I$ and $J$ global invariant and $\text{std}(\Pi_1) = \kappa(\text{std}(\sigma[\text{size}(\sigma)])$ and $\text{std}(\Pi_2) = \kappa(\text{std}(\sigma[j])$.

In other words,

$$\Delta(\Phi_{\kappa(\sigma)}) = \sum \Phi_{\kappa(\text{std}(\sigma[\text{size}(\sigma)]) \otimes \Phi_{\kappa(\text{std}(\sigma[j]))},$$

where the sum is over the partitions of $\{1, \ldots, n\}$ into 2 sets $I$ and $J$ such that the action of $\sigma$ does not $I$ and $J$ global invariant. Hence $K$ is a morphism of cogebras and, as with the previous examples, one has
Proposition 20.
- The Hopf algebras $\text{CWSym}(0!, 1!, 2!, \ldots)$ and $\mathcal{S}\text{Sym}$ are isomorphic.
- The Hopf algebras $\text{CIQSym}(0!, 1!, 2!, \ldots)$ and $\mathcal{S}\text{Sym}^*$ are isomorphic.

3.2.5 Miscellaneous subalgebras of the Hopf algebra of endofunctions

We denote by $\text{End}$ the combinatorial class of endofunctions (an endofunction of size $n \in \mathbb{N}$ is a function from $\{1, \ldots, n\}$ to itself). Given a function $f$ from a finite subset $A$ of $\mathbb{N}$ to itself, we denote by $\text{std}(f)$ the endofunction $\phi \circ f \circ \phi^{-1}$, where $\phi$ is the unique increasing bijection from $A$ to $\{1, 2, \ldots, \text{card}(A)\}$. Given a function $g$ from a finite subset $B$ of $\mathbb{N}$ (disjoint from $A$) to itself, we denote by $f \cup g$ the function from $A \cup B$ to itself whose $f$ and $g$ are respectively the restrictions to $A$ and $B$. Finally, given two endofunctions $f$ and $g$, respectively of size $n$ and $m$, we denote by $f \bullet g$ the endofunction $f \cup \tilde{g}$, where $\tilde{g}$ is the unique function from $\{n+1, n+2, \ldots, n+m\}$ to itself such that $\text{std}(\tilde{g}) = g$.

Now, let $\text{EQSym}$ be the Hopf algebra of endofunctions [16]. This Hopf algebra is defined by its basis $(\Psi_f)$ indexed by endofunctions, the product

$$\Psi_f \Psi_g = \sum_{\text{std}(\tilde{f}) = \text{std}(\tilde{g}) = g, f \cup \tilde{g} \in \text{End}} \Psi_{f \cup \tilde{g}}$$

(26)

and the coproduct

$$\Delta(\Psi_h) = \sum_{f \bullet g = h} \Psi_f \otimes \Psi_g.$$

(27)

This algebra is commutative but not cocommutative. We denote by $\text{ESym} := \text{EQSym}^*$ its graded dual, and by $(\Phi_f)$ the basis of $\text{ESym}$ dual to $(\Psi_f)$. The bases $(\Phi_f)$ and $(\Psi_f)$ are respectively denoted by $(S^n)$ and $(M_n)$ in [16]. The product and the coproduct in $\text{ESym}$ are respectively given by

$$\Phi_f \Phi_g = \Phi_{f \bullet g}$$

(28)

and

$$\Delta(\Phi_h) = \sum_{f \bullet g = h} \Phi_{\text{std}(f)} \otimes \Phi_{\text{std}(g)}.$$

(29)

Remark: The $\Psi_f$, where $f$ is a bijective endofunction, span a Hopf subalgebra of $\text{EQSym}$ obviously isomorphic to $\mathcal{S}\text{QSym} := \mathcal{S}\text{Sym}^*$, that is isomorphic to $\text{CIQSym}(0!, 1!, 2!, \ldots)$ from (3.2.4).

As suggested by [16], we investigate a few other Hopf subalgebras of $\text{EQSym}$.
- The Hopf algebra of idempotent endofunctions is isomorphic to the Hopf algebra $\text{CIQSym}(1, 2, 3, \ldots)$. The explicit isomorphism sends $\Psi_f$ to $\Psi_{\phi(f)}$, where for any idempotent endofunction $f$ of size $n$,

$$\phi(f) = \left\{ f^{-1}(i), \text{card}([j \in f^{-1}(i) \mid j \leq i]) \right\} 1 \leq i \leq n, f^{-1}(i) \neq \emptyset.\right.$$  

(30)

- The Hopf algebra of involutive endofunctions is isomorphic to

$\text{CIQSym}(1, 1, 0, \ldots, 0, \ldots) \hookrightarrow \text{IIQSym}.$

Namely, it is a Hopf subalgebra of $\mathcal{S}\text{QSym}$, and the natural isomorphism from $\mathcal{S}\text{QSym}$ to $\text{CIQSym}(1, 1, 0, \ldots, 0, \ldots)$ sends it to the sub algebra $\text{CIQSym}(1, 1, 0, \ldots, 0, \ldots)$.

- In the same way the endofunctions such that $f^3 = \text{Id}$ generate a Hopf subalgebra of $\mathcal{S}\text{QSym}$ isomorphic to the Hopf algebra $\text{CIQSym}(1, 2, 0, \ldots, 0, \ldots)$.

- More generally, the endofunctions such that $f^p = \text{Id}$ generate a Hopf subalgebra of $\mathcal{S}\text{QSym}$ isomorphic to $\text{CIQSym}(\tau(p))$ where $\tau(p)_i = (i-1)!$ if $i \mid p$ and $\tau(p)_i = 0$ otherwise.

3.3 About specializations

The aim of this section is to show how the specialization $c_n \longrightarrow \frac{a_n}{n!}$ factorizes through $\text{IIQSym}$ and $\text{CIQSym}$.

Notice first that the algebra $\text{Sym}$ is isomorphic to the subalgebra of $\text{IIQSym}$ generated by the family $(\Psi_{\{1, \ldots, n\}})_{n \in \mathbb{N}}$; the explicit isomorphism $\alpha$ sends $c_n$ to $\Psi_{\{1, \ldots, n\}}$. The image of $h_n$ is $S_{\{1, \ldots, n\}}$ and the image of $c_\lambda = \frac{1}{\lambda!} c_{\lambda_1} \cdots c_{\lambda_k}$ is $\sum_{\pi \vdash \lambda} \Psi_{\pi}$ where $\pi = \lambda$. 

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means that $\pi = \{\pi_1, \ldots, \pi_k\}$ is a set partition such that $\#\pi_1 = \lambda_1$, $\ldots$, $\#\pi_k = \lambda_k$ and

$$\lambda' = \frac{\lambda_1 \cdot \lambda_2 \cdots \lambda_k}{\pi_1 ! \pi_2 ! \cdots \pi_k !} = \prod_i m_i(\lambda) !$$

where $m_i(\lambda)$ denotes the multiplicity of $i$ in $\lambda$. Indeed, $c_{\lambda}$ is mapped to $\mathbb{X}_{\Gamma}(\lambda' \Psi_{\{1, \ldots, n\}}) \cdot \Psi_{\{1, \ldots, \lambda_1\}} \cdots \Psi_{\{1, \ldots, \lambda_k\}} = \lambda' \sum_{\pi \sim \lambda} \Psi_{\pi}$.

Now the linear map $\beta_a : \Pi\Psi_{\text{Sym}} \rightarrow \text{CIQsym}(\alpha)$ sending each $\Psi_{\pi}$ to the element

$$\sum_{\Pi \in \pi} \Psi_{\Pi}$$

is a morphism of algebra and the subalgebra $\Pi\Psi_{\text{Sym}} := \beta_{\pi}(\Pi\Psi_{\text{Sym}})$ is isomorphic to $\Pi\Psi_{\text{Sym}}$ if and only if $a \in (N \setminus \{0\})^N$.

Let $\gamma_a : \text{CIQsym}(\alpha) \rightarrow \mathbb{C}$ be the linear map sending $\Psi_{\Pi}$ to $\frac{1}{|\Pi|}$. We have

$$\gamma_a(\Psi_{\Pi}) = \sum_{\Pi_2 \in \pi} \gamma_a(\Psi_{\Pi_2}).$$

From any subset $A$ of $\{1, 2, \ldots, |\Pi_1| + |\Pi_2|\}$ of cardinality $n$, one has $\text{std}(\Pi'_1) = \Pi_1$, $\text{std}(\Pi'_2) = \Pi_2$ and $\Pi'_1 \cap \Pi'_2 = \emptyset$, where $\Pi'_i$ is obtained from $\Pi_i$ by replacing each label $i$ by the $i$th smallest element of $\{1, \ldots, |\Pi_1| + |\Pi_2|\} \setminus A$. Since there are $|\Pi_1| + |\Pi_2|$ ways to construct $A$, one has

$$\gamma_a(\Psi_{\Pi_1})\gamma_a(\Psi_{\Pi_2}) = \frac{1}{(|\Pi_1| + |\Pi_2|)!} \left(\binom{|\Pi_1| + |\Pi_2|}{|\Pi_1|}\right) = \frac{1}{|\Pi_2|!} = \gamma_a(\Psi_{\Pi_1})\gamma_a(\Psi_{\Pi_2}).$$

In other words, $\gamma_a$ is a morphism of algebra. Furthermore, the restriction $\tilde{\gamma}_a$ of $\gamma_a$ to $\Pi\Psi_{\text{Sym}}$ is a morphism of algebra that sends $\beta_{\pi}(\Psi_{\{1, \ldots, n\}})$ to $\frac{\pi_n}{n!}$. It follows that if $f \in \text{Sym}$, then one has

$$f(\chi^{(a)}) = \tilde{\gamma}_a(\beta_a(f)).$$

The following theorem summarizes the section:

**Theorem 21. The diagram**

$$\begin{array}{ccc}
\text{CIQsym}(\alpha) & \xleftarrow{\beta_a} & \Pi\Psi_{\text{Sym}} \\
\gamma_a \downarrow & & \gamma_a \downarrow \\
\mathbb{C} = \text{Sym}[\chi^{(a)}] & \hookrightarrow & \text{Sym}
\end{array}$$

is commutative.

## 4 Word Bell polynomials

### 4.1 Bell polynomials in $\Pi\Psi_{\text{Sym}}$

Since $\text{Sym}$ is isomorphic to the subalgebra of $\Pi\Psi_{\text{Sym}}$ generated by the elements $\Psi_{\{1, \ldots, n\}}$, we can compute $A_n(\Psi_{\{1\}}, \Psi_{\{1, 2\}}, \ldots, \Psi_{\{1, \ldots, n\}}, \ldots)$. From (23), we have

$$A_n(1!\Psi_{\{1\}}, 2!\Psi_{\{1, 2\}}, \ldots, m!\Psi_{\{1, \ldots, m\}}, \ldots) = n!S_{\{1, \ldots, n\}} = n! \sum_{\pi \sim \lambda} \Psi_{\pi}.$$ (32)

Notice that, from the previous section, the image of the Bell polynomial

$$A_n(\Psi_{\{1\}}, \Psi_{\{1, 2\}}, \ldots, \Psi_{\{1, \ldots, m\}}, \ldots)$$

by the morphism $\gamma$ sending $\Psi_{\{1, \ldots, n\}}$ to $\frac{1}{n!}$ is

$$\gamma(A_n(1!\Psi_{\{1\}}, 2!\Psi_{\{1, 2\}}, \ldots, m!\Psi_{\{1, \ldots, m\}}, \ldots)) = b_n = A_n(1, 1, \ldots).$$

In the same way, we have

$$B_{n,k}(1!\Psi_{\{1\}}, 2!\Psi_{\{1, 2\}}, \ldots, m!\Psi_{\{1, \ldots, m\}}, \ldots) = n! \sum_{\pi \sim \lambda} \Psi_{\pi}.$$ (33)

If $(F_n)_n$ is a homogeneous family of elements of $\Pi\Psi_{\text{Sym}}$, such that $|F_n| = n$, we define

$$A_n(F_1, F_2, \ldots) = \frac{1}{n!}A_n(1!F_1, 2!F_2, \ldots, m!F_m, \ldots)$$ (34)
and

\[ B_{n,k}(F_1, F_2, \ldots) = \frac{1}{n!} B_{n,k}(1! F_1, 2! F_2, \ldots, m! F_m, \ldots). \] (35)

Considering the map \( \beta_n \circ \alpha \) as a specialization of \( \text{Sym} \), the following identities hold in \( \text{CIQSym}(a) \):

\[
A_n \left( \sum_{1 \leq i \leq a_1} \Psi_{\{i\}, i}, \sum_{1 \leq i \leq a_2} \Psi_{\{i\}, i}, \ldots, \sum_{1 \leq i \leq a_m} \Psi_{\{i\}, i}, \ldots \right) = \sum_{\Pi \vdash n} \Psi_{\Pi}
\]

and

\[
B_{n,k} \left( \sum_{1 \leq i \leq a_1} \Psi_{\{i\}, i}, \sum_{1 \leq i \leq a_2} \Psi_{\{i\}, i}, \ldots, \sum_{1 \leq i \leq a_m} \Psi_{\{i\}, i}, \ldots \right) = \sum_{\Pi \vdash n, |\Pi| = k} \Psi_{\Pi}.
\]

**Example 22.** In \( \text{BIQSym} \sim \text{CIQSym}(1!, 2!, \ldots) \), we have

\[
B_{n,k} \left( \psi_{\{1\}}, \psi_{\{1,2\}} + \psi_{\{2,1\}}, \ldots, \sum_{\pi \vdash m} \psi_{\pi}, \ldots \right) = \sum_{\Pi \vdash n, |\Pi| = k} \psi_{\Pi},
\]

where the sum on the right is over the set partitions of \( \{1, \ldots, n\} \) into \( k \) lists. Considering the morphism sending \( \psi_{\{\sigma_1, \ldots, \sigma_n\}} \) to \( L_{n,k} \), Theorem 21 allows us to recover \( B_{n,k}(1!, 2!, 3!, \ldots) = L_{n,k} \), the number of set partitions of \( \{1, \ldots, n\} \) into \( k \) lists.

**Example 23.** In \( \text{IQSym}(2) \sim \text{CIQSym}(b_1, b_2, \ldots) \), we have

\[
B_{n,k} \left( \psi_{\{1\}}, \psi_{\{1,2\}} + \psi_{\{2,1\}}, \sum_{\pi \vdash m} \psi_{\pi}, \ldots \right) = \sum_{\Pi \vdash n, |\Pi| = k} \psi_{\Pi},
\]

where the sum on the right is over the set partitions of \( \{1, \ldots, n\} \) of level 2 into \( k \) blocks. Considering the morphism sending \( \psi_{\pi} \) to \( \frac{1}{\pi} \) for \( \pi \vdash n \), Theorem 21 allows us to recover \( B_{n,k}(b_1, b_2, b_3, \ldots) = S_{n,k}^{(2)} \), the number of set partitions into \( k \) sets of a partition of \( \{1, \ldots, n\} \).

**Example 24.** In \( \mathcal{S} \text{Sym}^* \sim \text{CIQSym}(0!, 1!, 2!, \ldots) \), we have

\[
B_{n,k} \left( \psi_{[1]}, \psi_{[2,1]}, \psi_{[2,3,1]} + \psi_{[3,1,2]}, \ldots, \sum_{\sigma \in \mathcal{S}_n} \psi_{\sigma}, \ldots \right) = \sum_{\sigma \in \mathcal{S}_n, \sigma \text{ has } k \text{ cycles}} \psi_{\sigma},
\]

where the sum on the right is over the permutations of size \( n \) having \( k \) cycles. Considering the morphism sending \( \psi_{\sigma} \) to \( \frac{1}{\sigma} \) for \( \sigma \in \mathcal{S}_n \), Theorem 21 allows us to recover \( B_{n,k}(0!, 1!, 2!, \ldots) = s_{n,k} \), the number of permutations of \( \mathcal{S}_n \) having exactly \( k \) cycles.

**Example 25.** In the Hopf algebra of idempotent endofunctions, we have

\[
B_{n,k} \left( \psi_{f_{1,1}}, \psi_{f_{2,1}} + \psi_{f_{2,2}} + \psi_{f_{3,1}}, \ldots, \sum_{i=1}^{n} \psi_{f_{i,n}}, \ldots \right) = \sum_{|f| = n, \text{card}(f(\{1,\ldots,n\})) = k} \psi_{f},
\]

where for \( i \geq j \geq 1 \), \( f_{i,j} \) is the constant endofunction of size \( i \) and of image \( \{j\} \). Here, the sum on the right is over idempotent endofunctions \( f \) of size \( n \) such that the cardinality of the image of \( f \) is \( k \). Considering the morphism sending \( \psi_{f} \) to \( \frac{1}{f} \) for \( |f| = n \), Theorem 21 allows us to recover that \( B_{n,k}(1, 2, 3, \ldots) \) is the number of these idempotent endofunctions, that is the idempotent number \( \left( \begin{array}{c} n \\ k \end{array} \right) k^{n-k} \) [17, 31].

### 4.2 Bell polynomials in WSym

Recursive descriptions of Bell polynomials are given in [10]. In this section we rewrite this result and other ones related to these polynomials in the Hopf algebra of word symmetric functions WSym. We define the operator \( \partial \) acting linearly on the left on WSym by

\[ 1\partial = 0 \text{ and } \Phi_{\{\pi_1, \ldots, \pi_n\}} \partial = \sum_{i=1}^{k} \Phi_{\{\pi_1, \ldots, \pi_k \} \setminus \{\pi_i\}; \{\pi_1, \ldots, (n+1)\}}. \]

In fact, the operator \( \partial \) acts on \( \Phi_{\pi} \) almost as the multiplication of \( M_{\{1\}} \) on \( M_{\pi} \). More precisely:
Proposition 26. We have:
\[ \partial = \phi^{-1} \circ \mu \circ \phi - \mu, \]
where \( \phi \) is the linear operator satisfying \( M_x \phi = \Phi_x \) and \( \mu \) is the multiplication by \( \Phi_{\{1\}} \).

Example 27. For instance, one has
\[
\Phi_{\{1,3\}, \{2,4\}} \partial = \Phi_{\{1,3\}, \{2,4\}} (\phi^{-1} \mu \phi - \mu) = M_{\{1,3\}, \{2,4\}} \mu \phi - \Phi_{\{1,3\}, \{2,4\}} = (M_{\{1,3,5\}, \{2,4\}} + M_{\{1,3\}, \{2,4,5\}} + M_{\{1,3\}, \{2,4,5\}}) \phi - \Phi_{\{1,3\}, \{2,4,5\}} = \Phi_{\{1,3,5\}, \{2,4\}} + \Phi_{\{1,3\}, \{2,4,5\}} + \Phi_{\{1,3,5\}, \{2,4,5\}}.
\]

Following Remark 4, we define recursively the elements \( \mathfrak{A}_n \) of \( \text{WSym} \) as
\[
\mathfrak{A}_0 = 1, \quad \mathfrak{A}_{n+1} = \mathfrak{A}_n (\Phi_{\{1\}} + \partial).
\]

So we have
\[
\mathfrak{A}_n = 1 (\Phi_{\{1\}} + \partial)^n.
\]

Easily, one shows that \( \mathfrak{A}_n \) provides an analogue of complete Bell polynomials in \( \text{WSym} \).

Proposition 28.
\[ \mathfrak{A}_n = \sum_{\pi \in \mathfrak{S}_n} \Phi_\pi. \]

Proposition 29. If we set
\[ \mathfrak{B}_{n,k} = [t^k] 1(t \Phi_{\{1\}} + \partial)^n, \]
then we have \( \mathfrak{B}_{n,k} = \sum_{\pi \in \mathfrak{S}_n} \Phi_\pi \).

Example 30. We have
\[
1(t \Phi_{\{1\}} + \partial)^4 = t^4 \Phi_{\{1\}, \{2\}, \{3\}, \{4\}} + t^3 (\Phi_{\{1,2\}, \{3\}, \{4\}} + \Phi_{\{1,3\}, \{2\}, \{4\}} + \Phi_{\{1,4\}, \{2\}, \{3\}}) + t^2 (\Phi_{\{1,2,3\}, \{4\}} + \Phi_{\{1,2,4\}, \{3\}} + \Phi_{\{1,3,4\}, \{2\}}) + t (\Phi_{\{1,2,3,4\}}) + \Phi_{\{1\}, \{2\}, \{3\}, \{4\}},
\]

Hence,
\[ \mathfrak{B}_{4,2} = \Phi_{\{1,3,4\}, \{2\}} + \Phi_{\{1,2,3\}, \{4\}} + \Phi_{\{1,2,4\}, \{3\}} + \Phi_{\{1,3,4\}, \{2\}} + \Phi_{\{1,2,3,4\}} + \Phi_{\{1\}, \{2\}, \{3\}, \{4\}}. \]

4.3 Bell polynomials in \( C(\mathbb{A}) \)

Both \( \text{WSym} \) and IIQSym admit word polynomial realizations in a subspace \( \text{WSym}(\mathbb{A}) \) of the free associative algebra \( C(\mathbb{A}) \) over an infinite alphabet \( \mathbb{A} \). When endowed with the concatenation product, \( \text{WSym}(\mathbb{A}) \) is isomorphic to \( \text{WSym} \) and when endowed with the shuffle product, it is isomorphic to IIQSym. Alternatively to the definitions of partial Bell numbers in IIQSym (35) and in \( \text{WSym} \), we set, for any sequence of polynomials \((F_i)_{i \in \mathbb{N}} \) in \( C(\mathbb{A}) \),
\[
\sum_{n \geq 0} \mathfrak{B}_{n,k}(F_1, \ldots, F_m, \ldots) t^n = \frac{1}{k!} \left( \sum_i F_i t^i \right)^{\mu_k} \quad (39)
\]
and
\[
\mathfrak{A}_n (F_1, \ldots, F_m, \ldots) = \sum_{k \geq 1} \mathfrak{B}_{n,k}(F_1, \ldots, F_m, \ldots). \quad (40)
\]

This definition generalizes (35) and (38) in the following sense:

Proposition 31. We have
\[ \mathfrak{B}_{n,k}(\Psi_{\{1\}}(\mathbb{A}), \ldots, \Psi_{\{1, \ldots, m\}}(\mathbb{A}), \ldots) = \mathfrak{B}_{n,k}(\Psi_{\{1\}}(\mathbb{A}), \ldots, \Psi_{\{1, \ldots, m\}}(\mathbb{A}), \ldots) = \mathfrak{B}_{n,k}(\mathbb{A}), \]

and
\[ \mathfrak{B}_{n,k}(\Phi_{\{1\}}(\mathbb{A}), \ldots, \Phi_{\{1, \ldots, m\}}(\mathbb{A}), \ldots) = \mathfrak{B}_{n,k}(\mathbb{A}), \]
Proof. The two identities follow from

\[ \Psi_{\pi_1}(A) \cup \Psi_{\pi_2}(A) = \sum_{\pi_1 \cap \pi_2 = \emptyset, \text{sym}} \Psi_{\pi}(A). \]

Equality (39) allows us to show more general properties. For instance, let \( \mathcal{A}' \) and \( \mathcal{A}'' \) be two disjoint subalphabets of \( \mathcal{A} \) and set

\[ S_n^{(A)}(\mathcal{A}'') = S_{(1)}(\mathcal{A}') \cup S_{(1,\ldots,n-1)}(\mathcal{A}''). \]

Remarking that

\[ \sum_{n \geq 0} S_n^{(A)}(\mathcal{A}'') t^n = \sum_{n \geq 0} S_{(1)}(\mathcal{A}') t^n \cup k^{(\mathcal{A}'')} S_{(1,\ldots,n)}(\mathcal{A}'') t^n \]

we obtain a word analogue of the formula allowing one to write a Bell polynomial as a symmetric function (see eq. (58) in Appendix A):

**Proposition 32.**

\[ B_{n,k}(S_n^{(A)}(\mathcal{A}''), \ldots, S_n^{(A)}(\mathcal{A}''), \ldots) = S_{(1)}(\mathcal{A}') \cup S_{(1,\ldots,n-k)}(\mathcal{A}'') \]

For simplicity, let us write \( B_{n,k}^{(A)}(\mathcal{A}'') := B_{n,k}(S_n^{(A)}(\mathcal{A}''), \ldots, S_n^{(A)}(\mathcal{A}''), \ldots) \).

Let \( k = k_1 + k_2 \). From

\[ S_{(1)}(\mathcal{A}') \cup S_{(1,\ldots,k_1)}(\mathcal{A}') = \left( \binom{k}{k_1} \right) \]

and

\[ S_{(1,\ldots,n-k)}(\mathcal{A}'') = \sum_{i+j=n-k} S_{(1,\ldots,i)}(\mathcal{A}'') \cup S_{(1,\ldots,i)}(\mathcal{A}''), \]

we deduce an analogue of the binomiality of the partial Bell polynomials (see eq. (59) in Appendix A):

**Corollary 33.** Let \( k = k_1 + k_2 \) be three nonnegative integers. We have

\[ \left( \binom{k}{k_1} \right) B_{n,k}^{(A)}(\mathcal{A}'') = \sum_{i=0}^{n} B_{i,k_1}^{(A)}(\mathcal{A}'') \cup B_{n-i,k_2}^{(A)}(\mathcal{A}''). \]  \hspace{1cm} (41)

**Example 34.** Consider a family of functions \( \{f_k\}_k \) such that \( f_k : \mathbb{N} \rightarrow \mathbb{C}^{(\mathcal{A})} \) satisfying

\[ f_0 = 1 \text{ and } f_n(\alpha + \beta) = \sum_{n=i+j} f_i(\alpha) \cup f_j(\beta). \]  \hspace{1cm} (42)

From (39), we obtain

\[ B_{n,k}(f_0(a), \ldots, f_{m-1}(a), \ldots) t^n = \frac{1}{k!} \sum_{i_1 + \cdots + i_k = n-k} f_{i_1}(a) \cup \cdots \cup f_{i_k}(a). \]

Hence, iterating (42), we deduce

\[ B_{n,k}(f_0(a), \ldots, f_{m-1}(a), \ldots) = \frac{1}{k!} f_{n-k}(ka). \]

Set \( f_n(k) = k! B_{n,k}^{(A)}(\mathcal{A}'') \) and \( f_0(k) = 1 \). By (41), the family \( \{f_n\}_{n \in \mathbb{N}} \) satisfies (42). Hence we obtain an analogue of composition formula (see eq. (63) in Appendix A):

\[ k_1 ! B_{n,k_1}(1, \ldots, k_2 B_{n-1,k_2}^{(A)}(\mathcal{A}''), \ldots) = (k_1 k_2)! B_{n-k_1,k_2}^{(A)}(\mathcal{A}''). \]

Suppose now \( \mathcal{A}' = \mathcal{A}'' = \mathcal{A}' \).

By \( S_{(1,\ldots,n)}(\mathcal{A}) = \sum_{i=0}^{n} S_{(1,\ldots,i)}(\mathcal{A}) \cup S_{(1,\ldots,n-i)}(\mathcal{A}) \), Proposition 32 allows us to write a word analogue of the convolution formula for Bell polynomials (see formula (60) in Appendix A):
Example 39. We define a specialization by setting

\[ \Phi_{\{1, \ldots, n\}}[\mathfrak{B}] = \sum_{\sigma \in \mathfrak{S}_n} b_{\sigma[1]} \cdots b_{\sigma[n]}, \]

where the letters \(b_i\) belong to an alphabet \(\mathfrak{B}\). Let \(\sigma \in \mathfrak{S}_n\) be a permutation and \(\sigma = c_{i_1} \circ \cdots \circ c_{i_k}\) its decomposition into cycles. Each cycle \(c^{(i)}\) is denoted by a sequence of integers \((n_1^{(i)}, \ldots, n_{\ell_i}^{(i)})\) such that \(n_1^{(i)} = \min\{n_1^{(i)}, \ldots, n_{\ell_i}^{(i)}\}\). Let \(c^{(i)}\) be the permutation which is the standardization of the sequence \(n_1^{(i)}, \ldots, n_{\ell_i}^{(i)}\). The cycle support of \(\sigma\) is the partition

\[ \text{support}(\sigma) = \{\{n_1^{(i)}, \ldots, n_{\ell_i}^{(i)}\}, \ldots, \{n_1^{(k)}, \ldots, n_{\ell_i}^{(k)}\}\}. \]

Corollary 35.

\[ S_{\{1, \ldots, k\}}(A') \sqcup B^\ell_{n,k}(A'') = \sum_{i=0}^{n} B^\ell_{n,k}(A'_i) \sqcup B^\ell_{n-i,k}(A''_i). \quad (43) \]

Let \(k_1\) and \(k_2\) be two positive integers. We have

\[ \sum_{n} B_{n,k_1}(B^\ell_{k_2+k_2}(A''_n), \ldots, B^\ell_{k_2+k_2-1,k_2}(A''_n), \ldots) t^n = \]

\[ \frac{1}{k_1^1} \left( \sum_{m \geq 1} B^\ell_{k_2+k_2-1,k_2}(A''_m) t^m \right) \text{wh}_{k_1} = t^{k_1} S_{\{1, \ldots, (k_2k_1)\}}(A') \sqcup A^\ell_{\pi}(k_1k_2A''). \]

Hence,

**Proposition 36.**

\[ B_{n,k_1}(B^\ell_{k_2+k_2}(A''), \ldots, B^\ell_{k_2+k_2-1,k_2}(A''), \ldots) = B^\ell_{n-k_1+k_1k_2k_2}(A''). \quad (44) \]

### 4.4 Specialization again

In [5] we have shown that one can construct a double algebra which is homomorphic to \((\text{WSym}(A), \sqcup)\). This is a general construction which is an attempt to define properly the concept of virtual alphabet for \text{WSym}. In our context the construction is simpler, let us briefly recall it.

Let \(F = (F_\pi)_{\pi}\) be a basis of \text{WSym}(A). We will say that \(F\) is shuffle-compatible if

\[ F_{\{\pi_1, \ldots, \pi_k\}}(A) = \sqcup_{\pi_1, \ldots, \pi_k} F_{\{1, \ldots, \#\pi_1\}}(A), \ldots, F_{\{1, \ldots, \#\pi_k\}}(A) \cdot (A) . \]

Hence, one has

\[ F_{\pi_1}(A) \sqcup F_{\pi_2}(A) = \sum_{\pi_1, \pi_2, \#\pi_1, \#\pi_2 = \emptyset} F_{\pi_1}(A).F_{\pi_2}(A) = F_{\pi_1 \cup \pi_2}(A). \]

**Example 37.** The bases \((S_\pi(A))_\pi\), \((\Phi_\pi(A))_\pi\) and \((\Psi_\pi(A))_\pi\) are shuffle-compatible but not the basis \((M_\pi(A))_\pi\).

Straightforwardly, one has:

**Claim 38.** Let \((F_\pi(A))_\pi\) be a shuffle-compatible basis of \text{WSym}(A). Let \(B\) be another alphabet and let \(P = (P_k)_{k \geq 0}\) be a family of noncommutative polynomials of \(\mathbb{C}([B])\) such that \(\deg P_k = k\). Then, the space spanned by the polynomials

\[ F_{\{\pi_1, \ldots, \pi_k\}}[A^{(P)}_{\pi}] := \sqcup_{\pi_1, \ldots, \pi_k} F_{\{1, \ldots, \#\pi_1\}, \ldots, F_{\{1, \ldots, \#\pi_k\}}(A) \cdot (A) \]

is stable under concatenation and shuffle product in \(\mathbb{C}([B])\). So it is a double algebra which is homomorphic to \((\text{WSym}(A), \sqcup)\). We will call \text{WSym}(A^{(P)}_{\pi}) this double algebra and \(f[A^{(P)}_{\pi}]\) will denote the image of an element \(f \in \text{WSym}(A)\) by the morphism \text{WSym}(A) \rightarrow \text{WSym}(A^{(P)}_{\pi})\) sending \(F_{\{\pi_1, \ldots, \pi_k\}}\) to \(F_{\{\pi_1, \ldots, \pi_k\}}[A^{(P)}_{\pi}]\).

With these notations, one has

\[ B_{n,k}(P_1, \ldots, P_m, \ldots) = \mathfrak{B}_{n,k}[A^{(P)}_{\pi}] . \]

**Example 39.**
We define \( w[e^{(i)}] = b_{\alpha^{(i)}[0]} \cdots b_{\alpha^{(i)}[\varrho]} \) and \( w[\sigma] = \bigcup_{\pi_1, \ldots, \pi_k}(w[e^{(1)}], \ldots, w[e^{(k)}]) \) where \( \pi_i = \{a_1^{(i)}, \ldots, a_n^{(i)}\} \) for each \( 1 \leq i \leq k \).

For instance, if \( \sigma = 312654 = (132)(46) \) we have \( w[(132)] = b_1 b_3 b_2, \ w[(46)] = b_1 b_2, \ w[(5)] = b_1 \) and \( w[\sigma] = b_1 b_3 b_2 b_1 b_2 b_1 b_2 b_2 \).

So, we have
\[
B_{n,k}(\Phi_{\{1\}}[\mathcal{S}], \ldots, \Phi_{\{1,\ldots,m\}}[\mathcal{S}], \ldots) = \mathfrak{B}_{n,k}[\mathcal{S}] = \sum_{\pi \in \mathcal{S}} \Phi_\pi[\mathcal{S}] = \sum_{\sigma \in \mathcal{S}} w[\sigma].
\]

For instance
\[
B_{4,2}(b_1, b_1 b_2 b_3 + b_1 b_3 b_4 + b_1 b_2 b_2 b_1 + b_1 b_3 b_2 b_4 + b_1 b_2 b_1 b_3 + b_1 b_2 b_1 b_2 b_3 + b_1 b_2 b_2 b_3 + b_1 b_2 b_3 + b_1 b_2 b_2 b_1 + b_1 b_2 b_3 + b_1 b_2 b_1 + b_1 b_2 b_2 b_1 + b_1 b_2 b_1 + b_1 b_2 b_2 b_1 + b_1 b_2 b_2 b_1 + b_1 b_2 b_2 b_1 + 2 b_1 b_1 b_2 b_2 + b_1 b_1 b_2 b_2 b_1 b_1 b_2 b_1) = \Phi_{\{1\}}[\mathcal{S}] + \Phi_{\{2\}}[\mathcal{S}] + \Phi_{\{3\}}[\mathcal{S}] + \Phi_{\{4\}}[\mathcal{S}] + \Phi_{\{1,2\}}[\mathcal{S}] + \Phi_{\{1,3\}}[\mathcal{S}] + \Phi_{\{1,4\}}[\mathcal{S}] + \Phi_{\{2,3\}}[\mathcal{S}].
\]

Notice that the sum of the coefficients of the words occurring in the expansion of \( \mathfrak{B}_{n,k}[\mathcal{S}] \) is equal to the Stirling number \( s_{n,k} \). Hence, this specialization gives another word analogue of formula (14).

5 Munthe-Kaas polynomials

5.1 Munthe-Kaas polynomials from \( W Sym \)

In order to generalize the Runge-Kutta method to integration on manifolds, Munthe-Kaas [33] introduced a noncommutative version of Bell polynomials. We recall here the construction in a slightly different variant adapted to our notation, the operators acting on the left. Consider an alphabet \( \mathbb{D} = \{d_1, d_2, \ldots\} \). The algebra \( \mathbb{C}(\mathbb{D}) \) is equipped with the derivation defined by \( d_1 \partial = d_2 \partial = \ldots \). The noncommutative Munthe-Kaas Bell polynomials are defined by setting \( t = 1 \) in \( \mathbb{M}_n(t) = 1.(t d_1 + \partial)^n \). The partial noncommutative Bell polynomial \( \mathbb{M}_{n,k} \) is the coefficient of \( t^k \) in \( \mathbb{M}_n(t) \).

**Example 40.**
- \( \mathbb{M}_1(t) = d_1 t \),
- \( \mathbb{M}_2(t) = d_1^2 t^2 + d_2 t \),
- \( \mathbb{M}_3(t) = d_1^3 t^3 + \left(2 d_3 d_1 + d_1 d_2\right) t^2 + d_3 t \),
- \( \mathbb{M}_4(t) = d_1^4 t^4 + \left(3 d_3 d_1^2 + 2 d_1 d_2 d_1 + d_2^2 d_1\right) t^3 + \left(3 d_3 d_1 + 3 d_2^2 + d_1 d_3\right) t^2 + d_4 t \).

We consider the map \( \Xi \) which sends each set partition \( \pi \) to the integer composition \( [\text{card}(\pi_1), \ldots, \text{card}(\pi_k)] \) if \( \pi = \{\pi_1, \ldots, \pi_k\} \) where \( \text{min}(\pi_i) < \text{min}(\pi_{i+1}) \) for any \( 0 < i < k \). The linear map \( \Xi \Xi \) sends \( \Phi_{\pi} \) to \( \Phi_\pi^{\{1\}} \cdots \Phi_\pi^{\{k\}} \) is a morphism of algebra. Hence, we deduce

**Proposition 41.**
\[
\Xi(\mathfrak{B}_{n,k}) = \mathbb{M}_{n,k}.
\]

**Example 42.**
\[
\Xi(\mathfrak{B}_{3,2}) = \Xi(\Phi_{\{1\},\{2,3\}} + \Phi_{\{1,3\},\{2\}} + \Phi_{\{1,2\},\{3\}}) = d_1 d_2 + 2 d_2 d_1 = \mathbb{M}_{4,2}.
\]

We recover a result due to Ebrahimi-Fard et al. [10].

**Theorem 43.** If \( j_1 + \cdots + j_k = n \), the coefficient of \( d_{j_1} \cdots d_{j_k} \) in \( \mathbb{M}_{n,k} \) is equal to the number of partitions of \( \{1, 2, \ldots, n\} \) into parts \( \pi_1, \ldots, \pi_k \) such that \( \text{card}(\pi_\ell) = j_\ell \) for each \( 1 \leq \ell \leq k \) and \( \text{min}(\pi_1) < \cdots < \text{min}(\pi_k) \).

5.2 Dendriform structure and quasideterminant formula

The algebra \( \Pi Sym \) is equipped with a Zinbiel structure. The notion of Zinbiel algebra is due to Loday [21]. This is an algebra equipped with two nonassociative products \( \prec \) and \( \succ \) satisfying
- \( (u \prec v) \prec w = u \prec (v \prec w) + u \lessdot (v \succ w) \),
- \( (u \succ v) \prec w = u \succ (v \prec w) \).


\[
\Phi_n = \begin{cases}
1 & \text{if } \ell = 0, \\
n & \text{if } \ell = 1, \\
3 & \text{if } \ell = 2, \\
\vdots & \\
(2I) & \text{if } \ell = I, \\
\vdots & \\
(2n-2) & \text{if } \ell = n-2, \\
(2n-1) & \text{if } \ell = n-1, \\
(2n) & \text{if } \ell = n, \\
\end{cases}
\]

We have 

\[
P(A_n; t) = \sum_{k=1}^{n} P(A_{k-1}) \prec a_{k,n} \text{ and } P(A_0) = 1.
\]

**Example 45.**

\[
P(A_4, t) = \sum_{k=1}^{n} P(A_{k-1}) \prec a_{k,n} \text{ and } P(A_0) = 1.
\]

**Proposition 46.**

\[
\sum_{k=1}^{n} B_{n,k}(\Phi_{\{1\}}, \Phi_{\{1,2\}}, \ldots) t^n = \left( \sum_{i} \Phi_{\{1,\ldots,i\}} n \right) t^n \Rightarrow k
\]

with \( u \lessdot k = u \lessdot k - 1 \lessdot u \) and \( u \lessdot 0 = 1 \).

**Definition 44.** Let \( A_n = (a_{ij})_{1 \leq i, j \leq n} \) be an upper triangular matrix whose entries are in a Zinbiel algebra. We define the polynomial

\[
P(A_n; t) = \sum_{k=1}^{n} P(A_{k-1}) \prec a_{k,n} \text{ and } P(A_0) = 1.
\]

**Example 47.** We have 

\[
P(A_3; t) = \sum_{k=1}^{n} P(A_{k-1}) \prec a_{k,n} \text{ and } P(A_0) = 1.
\]

**Proposition 48** (Gelfand et al. [14]).

\[
\begin{align*}
& a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
& -1 & a_{22} & a_{23} & \cdots & a_{2n} \\
& 0 & -1 & a_{33} & \cdots & a_{3n} \\
& \vdots & \vdots & \vdots & \ddots & \vdots \\
& 0 & \cdots & 0 & -1 & a_{nn}
\end{align*}
\]

Further, formula (49) is an analogue of the result of Ebrahimi et al.

**Theorem 49** (Ebrahimi et al. [10]).

\[
MB_n(1) = \begin{bmatrix}
\binom{n-1}{0} d_1 & \binom{n-1}{1} d_2 & \binom{n-1}{2} d_3 & \cdots & \binom{n-1}{n-1} d_n \\
-1 & \binom{n-2}{0} d_1 & \binom{n-2}{1} d_2 & \cdots & \binom{n-2}{n-2} d_{n-1} \\
0 & -1 & \binom{n-3}{0} d_1 & \cdots & \binom{n-3}{n-3} d_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -1 & \binom{0}{n} d_1
\end{bmatrix}
\]

The connection between all these results remains to be investigated.
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References


A Bell polynomials and coproducts in \( \text{Sym} \)

In fact, most of the identities on Bell polynomials can be obtained by manipulating generating functions and are closely related to some other identities occurring in literature. Typically, the relation between the complete Bell polynomials \( A_n(a_1, a_2, \ldots) \) and the variables \( a_1, a_2, \ldots \) is very closely related to the Newton Formula which links the generating functions of complete symmetric functions \( h_n \) (Cauchy series) to those of the power sums \( p_n \). The symmetric functions form a commutative algebra \( \text{Sym} \) freely generated by the complete functions \( h_n \) or the power sum functions \( p_n \). So, specializing the variable \( a_n \) to some numbers is equivalent to specializing the power sum functions \( p_n \). More soundly, the algebra \( \text{Sym} \) can be endowed with coproducts conferring to it a structure of Hopf algebra. For instance, the coproduct for which the power sums are primitive turns \( \text{Sym} \) into a self-dual Hopf algebra. The coproduct can be translated in terms of generating functions by a product of two Cauchy series. This kind of manipulations appears also in the context of Bell polynomials, for instance when computing the complete Bell polynomials of the sum of two sequences of variables \( a_1 + b_1, a_2 + b_2, \ldots \). Another coproduct turns \( \text{Sym} \) into a non-co-commutative Hopf algebra called the Faà di Bruno algebra which is related to the Lagrange inversion. Finally, the coproduct such that the power sums are group-like can be related also to a few other formulae on Bell polynomials. The aim of this section is to investigate these connections and in particular to restate some known results in terms of symmetric functions and virtual alphabets. We also give a few new results that are difficult to prove without the help of symmetric functions.

A.1 Bell polynomials as symmetric functions

First, let us recall some operations on alphabets. Given two alphabets \( X \) and \( Y \), we also define (see e.g. [20]) the alphabet \( X + Y \) by:

\[
p_n(X + Y) = p_n(X) + p_n(Y)
\]  

(51)

and the alphabet \( \alpha X \) (resp \( X \gamma \)), for \( \alpha \in \mathbb{C} \) by:

\[
p_n(\alpha X) = \alpha p_n(X) \quad (\text{resp} \quad p_n(X \gamma) = p_n(X)p_n(\gamma)).
\]  

(52)

In terms of Cauchy functions, these transforms imply

\[
\sigma_t(X + Y) = \sigma_t(X)\sigma_t(Y)
\]  

(53)
\( \sigma_t(\mathcal{X}) = \sum_{\lambda} \frac{1}{z_{\lambda}} p^\lambda(\mathcal{X}) p^\lambda(\mathcal{Y}) t^{||\lambda||}. \) 

(54)

In fact \( \sigma_t(\mathcal{X}) \) encodes the kernel of the scalar product defined by \( \langle p^\lambda, c_\mu \rangle = \delta_{\lambda, \mu} \) with \( c_\lambda = \frac{e^\lambda}{\lambda!} \). Notice that \( c_n = \frac{n^n}{n!} \) and

\[ S_{gm} = \mathbb{C}[x_1, c_2, \ldots]. \]

(55)

From (7) and (19), we obtain

**Proposition 50.** \( h_n = \frac{1}{n!} A_n(1, 1, 2, e_2, \ldots). \)

Conversely, Equality (55) implies that the morphism \( \phi_n \) sending each \( e_1 \) to \( \frac{n^n}{n!} \) is well defined for any sequence of numbers \( a = (a_i)_{i \in \mathbb{N}} \) and \( \phi_n(h_n) = \frac{1}{n!} A_n(a_1, a_2, \ldots). \) Let us define also \( h_n^{(k)}(\mathcal{X}) = [\alpha^k] h_n(\alpha \mathcal{X}) \). From (19) and (52) we have

\[ h_n^{(k)} = \sum_{\lambda=|\lambda_1, \ldots, \lambda_k|} c_\lambda = [t^n] \frac{1}{k!} \left( \sum_{i \geq 1} c_i t^i \right)^k \]

and so, everything works as if we use a special (virtual) alphabet \( \mathcal{X}(\alpha) \) satisfying \( c_n(\mathcal{X}(\alpha)) = n! a_n \). More precisely:

**Proposition 51.**

\[ \phi_n(h_n^{(k)}) = h_n^{(k)}(\mathcal{X}(\alpha)) = \frac{1}{n!} B_{n,k}(a_1, \ldots, a_k, \ldots). \]

(56)

**Example 52.** Let \( 1 \) be the virtual alphabet defined by \( c_n(1) = \frac{1}{n!} \) for each \( n \in \mathbb{N} \). In this case the Newton Formula gives \( h_n(1) = 1 \). Hence \( A_n(0!, 1!, 2!, \ldots, (m - 1)!, \ldots) = n! \) and \( B_{n,k}(0!, 1!, 2!, \ldots, (m - 1)!, \ldots) = n! [\alpha^k] [t^n] \left( \frac{1}{1 - t} \right)^\alpha = s_{n,k} \), the Stirling number of the first kind.

**Example 53.** A more complicated example is treated in [4, 19] where \( a_i = i^{i-1} \). In this case, the specialization gives \( \sigma_t(\alpha \mathcal{X}(\alpha)) = \exp\{-\alpha W\{-t\} \} \) where \( W(t) = \sum_{n=1}^{\infty} (-n)^{n-1} \frac{t^n}{n!} \)

is the Lambert W function satisfying \( W(t) \exp\{W(t)\} = t \) (see e. g. [6]). Hence, \( \sigma_t(\alpha \mathcal{X}(\alpha)) = \left[ \frac{W(-t)}{-t} \right]^\alpha \). But the expansion of the series \( \left[ \frac{W(t)}{t} \right]^\alpha \) is known to be:

\[ \left( \frac{W(t)}{t} \right)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} (\alpha + n)^{n-1} (t)^n. \]

(57)

Hence, we obtain \( B_{n,k}(1, 2, 3^2, \ldots, m^{m-1}, \ldots) = \frac{(n-1)!}{k!} n^{n-k} \). Note that the expansion of \( W(t) \) and (57) are usually obtained by the use of the Lagrange inversion.

**Example 54.** With these notations we have \( B_{n,k}(a_1 + b_1, \ldots) = \frac{1}{n!} h_n^{(k)}(\mathcal{X}(a) + \mathcal{X}(b)) \), and classical properties of Bell polynomials can be deduced from symmetric functions through this formalism. For instance, the equalities \( c_n(\mathcal{X}(a) + \mathcal{X}(b)) = c_n(\mathcal{X}(a)) + c_n(\mathcal{X}(b)) \) and \( h_n(\mathcal{X}(a) + \mathcal{X}(b)) = \sum_{i+j=n} h_i(\mathcal{X}(a)) h_j(\mathcal{X}(b)) \) give

\[ A_n(a_1 + b_1, \ldots) = \sum_{i+j=n} \binom{n}{i} A_i(a_1, a_2, \ldots) A_j(b_1, b_2, \ldots) \]

and

\[ B_{n,k}(a_1 + b_1, \ldots) = \sum_{i+j+k=n} \binom{n}{i} B_i(a_1, a_2, \ldots) B_{j,k}(b_1, b_2, \ldots). \]

**Example 55.** Another example is given by

\[ A_n(1a_1 b_1, 2a_2 b_2, \ldots, m a_m b_m, \ldots) = n! \sum_{\lambda \vdash n} \det \begin{bmatrix} A_{\lambda_i - i+j}(a_1, a_2, \ldots) \\ (\lambda_i - i + j)! \end{bmatrix} \times \det \begin{bmatrix} A_{\lambda_i - i+j}(b_1, b_2, \ldots) \\ (\lambda_i - i + j)! \end{bmatrix}, \]

where \( \lambda \vdash n \) lists the integer partitions of \( n \).
using the convention $A_{-n} = 0$ for $n > 0$. This formula is an emanation of the Jacobi-Trudi formula and is derived from the Cauchy kernel (54), remarking that $c_n(X^{(a)}X^{(b)}) = nc_n(X^{(a)})c_n(X^{(b)})$ and

$$h_n(X^{(a)}X^{(b)}) = \sum_{\lambda \vdash n} s_{\lambda}(X^{(a)})s_{\lambda}(X^{(b)}) = \sum_{\lambda \vdash n} \det \left| h_{\lambda_{i}}^{(a)}(X^{(a)}) \right| \det \left| h_{\lambda_{i}-1}^{(b)}(X^{(b)}) \right|,$$

where $s_{\lambda} = \det \left| h_{\lambda_{i}-1}^{(a)} \right|$ is a Schur function (see e.g. [22]).

### A.2 Other interpretations

First we focus on the identity (9) and we interpret it as the Cauchy function $\sigma_t(k\hat{X}^{(a)})$ where $\hat{X}^{(a)}$ is the virtual alphabet such that $h_{i-1}(\hat{X}^{(a)}) = \frac{a_i}{t}$. This means that we consider the morphism $\phi_a : \text{Sym} \rightarrow \mathbb{C}$ sending $h_i$ to $\frac{a_i}{(i+1)!}$. We suppose that $a_1 = 1$ otherwise we use (15) and (16). With these notations we have

**Proposition 56.**

$$B_{n,k}(a_1, a_2, \ldots) = \frac{n!}{k!} h_{n-k}(k\hat{X}^{(a)}). \quad (58)$$

**Example 57.** If $a_i = i$ we have $h_i(\hat{X}^{(a)}) = \frac{1}{n!}$ and so $\sigma_t(k\hat{X}^{(a)}) = \exp(kt)$. Hence, we recover the classical result

$$B_{n,k}(1, 2, \ldots, m, \ldots) = \binom{n}{k} k^{n-k}. \quad (59)$$

From $h_n(X + Y) = \sum_{i+j=n} h_i(X)h_j(Y)$ we deduce two classical identities:

$$\binom{k_1 + k_2}{k_1} B_{n,k_1+k_2}(a_1, a_2, \ldots) = \sum_{i=0}^{n} \binom{n}{i} B_{i,k_1}(a_1, a_2, \ldots)B_{n-i,k_2}(a_1, a_2, \ldots) \quad (59)$$

and

$$\binom{n}{k} B_{n-k,k}(a_1b_1, \ldots, \frac{1}{m+1} \sum_{i=1}^{m} \binom{m+1}{i} a_ib_{m+1-i}, \ldots) = \sum_{i=k}^{n-k} \binom{n}{i} B_{i,k}(a_1, a_2, \ldots)B_{i,k}(b_1, b_2, \ldots). \quad (60)$$

Indeed, formula (59) is obtained by setting $X = k_1\hat{X}^{(a)}$ and $Y = k_2\hat{X}^{(a)}$. Formula (60) is called the convolution formula for Bell polynomials (see e.g. [24]) and is obtained by setting $X = \hat{X}^{(a)}$ and $Y = \hat{X}^{(b)}$ in the left hand side and $X = k\hat{X}^{(a)}$ and $Y = k\hat{X}^{(b)}$ in the right hand side.

**Example 58.** The partial Bell polynomials are known to be involved in interesting identities on binomial functions. Let us first recall that a binomial sequence is a family of functions $(f_n)_{n \in \mathbb{N}}$ satisfying $f_0(x) = 1$ and

$$f_n(a + b) = \sum_{k=0}^{n} \binom{n}{k} f_k(a)f_{n-k}(b), \quad (61)$$

for all $a, b \in \mathbb{C}$ and $n \in \mathbb{N}$. Setting $h_n(A) := \frac{f_n(a)}{n!}$ and $h_n(B) := \frac{f_n(b)}{m!}$, with these notations $f_n(ka) = n!h_n(ka)$. Hence,

$$B_{n,k}(1, \ldots, i_{i-1}(a), \ldots) = \frac{n!}{k!} h_{n-k}(ka) = \binom{n}{k} f_{n-k}(ka). \quad (62)$$

Notice that from (59), $f_n(k) = \begin{cases} \binom{n}{k}^{-1} B_{n,k}(a_1, a_2, \ldots) & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$, is binomial and we obtain

$$\binom{n}{k}^{-1} B_{n,k_1}(1, \ldots, i_{i-1}^{-1} B_{i-1,k_2}(a_1, a_2, \ldots), \ldots) = \binom{n}{k}^{-1} B_{n-k_1,k_2}(a_1, a_2, \ldots). \quad (63)$$

Several related identities are compiled in [24].
Example 59. Taking the coefficient of $t^{n-k-1}$ in the left hand side and the right hand side of the equality $\frac{d}{dt} \sigma_t((k+1)X) = (k+1) \left( \frac{d}{dt} \sigma_t(X) \right) \sigma_t(kX)$, we obtain

$$(n-k)h_{n-k}((k+1)X) = (k+1) \sum_{i=1}^{n-k} ih_i(X)h_{n-i-k}(X)$$

and we recover the identity (see e.g. [8]):

$$B_n(a_1, a_2, \ldots) = \frac{1}{n!} \sum_{i=1}^{n-k} \binom{n}{i} \left( k+1 - \frac{n+i+1}{i+1} \right) (i+1) a_i B_{n-i,k}(a_1, a_2, \ldots). \quad (64)$$

Example 60. Let $(a_n)_{n \geq 1}$ and $(b_n)_{n \geq 1}$ be two sequences of numbers such that $a_1 = b_1 = 1$ and $d_n = n! \sum_{\lambda \vdash n} \det \left[ \frac{\lambda_i!}{(\lambda_i + j + k_i)!} \right] \det \left[ \frac{\lambda_i!}{(\lambda_i + j + k_i)!} \right]$ with the convention $a_{-n} = b_{-n} = 0$ if $n \geq 0$. The Cauchy kernel and the orthogonality of Schur functions give

$$B_n(a_1, a_2, \ldots) \det \left[ \frac{B_{\lambda_i+j+k_i}}{(\lambda_i + i + j + k_i)!} \right] \det \left[ \frac{B_{\lambda_i+j+k_i}}{(\lambda_i + i + j + k_i)!} \right],$$

for any $k_1, k_2 = k$. Indeed, it suffices to use the fact that

$$h_n(kX(\sigma(a)b)) = \sum_{\lambda \vdash n} \sigma_\lambda(k_1^X(\sigma(a))) \sigma_\lambda(k_2^X(b)).$$

The sum $X + Y$ and the product $XY$ of alphabets are two examples of coproducts endowing $\text{Sym}$ with a structure of Hopf algebra. The sum of alphabets encodes the coproduct $\Delta$ for which the power sums are of type Lie (i.e. $\Delta(p_n) = p_n \otimes 1 + 1 \otimes p_n \sim p_n(X + Y) = p_n(X) + p_n(Y)$ by identifying $f \otimes g$ with $f(X)g(Y)$) whilst the product of alphabets encodes the coproduct $\Delta'$ for which the power sums are group-like (i.e. $\Delta'(p_n) = p_n \otimes p_n \sim p_n(XY) = p_n(X)p_n(Y)$).

The algebra of symmetric functions can be endowed with another coproduct that confers a structure of Hopf algebra: this is the Faa di Bruno algebra [9, 18]. This algebra is rather important since it is related to the Lagrange-Bûrmann formula. The Bell polynomials also appear in this context. As a consequence, one can define a new operation on alphabets corresponding to the composition of Cauchy generating functions. Let $X$ and $Y$ be two alphabets and set $f(t) = \sigma_t(X)$ and $g(t) = \sigma_t(Y)$. The composition $X \circ Y$ is defined by $\sigma_t(X \circ Y) = \frac{1}{t} f \circ g(t)$. The relationship with Bell polynomials can be established by observing that we have

$$\frac{1}{t} f \circ g = \sum_{n \geq 0} \left( \sum_{k=1}^{n+1} \frac{k!}{(n+1)!} h_{k-1}(X) B_{n+1,k}(1, 2h_1(Y), 3h_2(Y), \ldots) \right) t^n.$$

Equivalently, $h_n(X \circ Y) = \sum_{k=0}^{n} \binom{(k+1)!}{(n+1)!} h_k(X) B_{n+1,k+1}(1, 2h_1(Y), 3h_2(Y), \ldots)$. The antipode of the Faa di Bruno algebra is also described in terms of alphabets as the operation which associates to each alphabet $X$ the alphabet $X^{-1}$ satisfying $\sigma_t(X^{-1}) = 1$. More explicitly, one has

$$h_n(X^{-1}) = \frac{n!}{(2n+1)!(n+1)} B_{2n+1,n}(1, -2e_1(X), 3e_2(X), \ldots), \quad (65)$$

where $e_n(X)$ is the elementary symmetric function defined by $\sum_{n \geq 0} e_n(X)t^n = \frac{1}{\sigma_t(X)}$.

Example 61. Let $\omega(t) = \sigma_t(X)$. The Lagrange inversion consists in finding an alphabet $X'$ such that $\phi(t) = \sigma_t(X')$. According to (65), it suffices to set $X' = -X^{-1}$. Let $F(t) = \sigma_t(Y)$. When stated in terms of alphabets, the Lagrange-Bûrmann formula reads

$$F(\omega(t)) = 1 + \sum_{n \geq 1} \frac{d^n(-1)}{dn^{-1}} \left[ \sigma_u(Y) \sigma_u(-nX^{-1}) \right]_{u=0} \frac{t^n}{n!}.$$

In other words one has, $h_{n-k}(-nX^{-1}) = \frac{(k-1)!}{(n-1)!} B_{n,k}(1, 2h_1(X), 3h_2(X), \ldots)$. So we recover a result due to Sadek Bouroubi and Moncef Abbas [4]:

$$B_{n,k}(1, h_1(2X), \ldots, (m-1)h_{m-1}(mX), \ldots) = \frac{(n-1)!}{(k-1)!} h_{n-k}(nX).$$