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Frozen Boolean partial co-clones

Gustav Nordh∗ Bruno Zanuttini†

Abstract

We introduce and investigate the concept of frozen partial co-clones. Our main motivation for studying frozen partial co-clones is that they have important applications in complexity analysis of constraints. The frozen partial co-clones lie between the co-clones and partial co-clones in the sense that the partial co-clone lattice is a refinement of the frozen partial co-clone lattice, which in turn is a refinement of the co-clone lattice. We concentrate on the Boolean domain and determine large parts of the frozen partial co-clone lattice.

1 Introduction

A clone is a composition closed set of operations containing all projections. An operation f preserves a relation R (or is a polymorphism of R) if f applied coordinate-wise to any tuples from R gives a tuple in R. Given a set of relations Γ, Pol(Γ) is the set of operations preserving all relations in Γ. Pol(Γ) is always a clone and any clone can be defined as Pol(Γ) for some set of relations Γ. Inv(F) is the set of relations that are preserved by (invariant under) all the operations in F.

An n-ary relation R has a primitive positive (p.p.) definition (also called an implementation) in a set of relations Γ if R is the set of models of an existentially quantified conjunction of atomic formulas over Γ ∪ {=} (also called a Γ-formula), i.e., \( R(x_1, \ldots, x_n) \equiv \exists X \bigwedge_i R_i(x_{i1}, \ldots, x_{it}) \) where each \( R_i \in \Gamma \cup \{=\} \). Sets of relations closed under p.p. definability are called co-clones and the least co-clone containing Γ is denoted by \( \langle \Gamma \rangle \). There is a Galois connection between clones and sets of relations closed under p.p. definability (i.e., co-clones). In particular, given two sets of relations \( \Gamma_1 \) and \( \Gamma_2 \), then \( \langle \Gamma_1 \rangle \subseteq \langle \Gamma_2 \rangle \) if and only if \( \text{Pol}(\Gamma_2) \subseteq \text{Pol}(\Gamma_1) \), and for any set of relations \( \Gamma \) we have \( \langle \Gamma \rangle = \text{Inv}(\text{Pol}(\Gamma)) \). For more information on clones, co-clones, and the aforementioned Galois connection, we refer the reader to the books [14, 16].

The CSP(Γ) problem, where Γ is a set of relations (also called a constraint language), is the problem of deciding if a given set of variables subject to a set of constraints (given by atomic formulas over Γ) is satisfiable. The problem of classifying the computational complexity of CSP(Γ) with respect to Γ is an important open problem. The (so far) most successful approach to attack this problem is an algebraic approach which heavily relies on the following result.

Theorem 1 ([11, 12]) If \( \Gamma_1 \) is finite and \( \langle \Gamma_1 \rangle \subseteq \langle \Gamma_2 \rangle \), then CSP(Γ1) is polynomial-time reducible to CSP(Γ2).

Note that because of the Galois connection between clones and co-clones this result can be reformulated into: If \( \Gamma_1 \) is finite and \( \text{Pol}(\Gamma_2) \subseteq \text{Pol}(\Gamma_1) \), then CSP(Γ1) is polynomial-time reducible to CSP(Γ2). To illustrate the power of this result, note that Schaefer’s [18] seminal complexity classification (separating the cases in P from the NP-complete cases) for CSP(Γ) over the Boolean domain follows trivially from this result and Post’s classification of Boolean clones [15] (i.e., Post’s lattice). For more information on the connection between clones, co-clones, and the CSP(Γ) problem, we refer the reader to the survey articles [3, 4, 13].

This result is not so useful for more fine grained complexity analysis of CSP(Γ) (and similar) problems. The reason is that it does not preserve the size of the problem instances (in terms of the number of variables). The crux is that in order for the proof of Theorem 1 to work, the existential quantifiers in p.p. definitions are eliminated by introducing new variables. So the reduction in Theorem 1 maps a conjunction of m constraints on n variables to a conjunction of bn constraints on n + am variables, where a and b are constants which depend on the languages \( \Gamma_1 \) and \( \Gamma_2 \).

As an example of a problem for which this is too coarse a reduction, compare 3-SAT (i.e., CSP(\( \Gamma_{3SAT} \)) where \( \Gamma_{3SAT} \) consists of the relations corresponding to clauses on at most three variables) with 1-in-3-SAT (i.e., CSP(\( \Gamma_{1/3} \)) where \( \Gamma_{1/3} \) consists of the relation \{001, 010, 100\}). By Post’s lattice it is easy to verify that \( \langle \Gamma_{1/3} \rangle = \langle \Gamma_{3SAT} \rangle \) and hence, CSP(\( \Gamma_{3SAT} \)) is polynomial-time equivalent to

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CSP(Γ_{1/3}) according to Theorem 1. Despite this, 3-SAT (solvable in time $O(1.473^n)$) [1, 7]) seems to be a much harder problem than 1-in-3-SAT which can be solved in time $O(1.1003^n)$ [2] (where $n$ is the number of variables). Hence, it is clear that to get a better understanding of the complexity of CSP($\Gamma$) (and similar) problems, we need reductions/implementations/tools where the blow-up in instance size can be controlled.

With this in mind we consider partial co-clones instead. A partial clone is a composition closed set of partial functions containing all (total) projections. A partial operation $f$ preserves a relation $R$ (or is a partial polymorphism of $R$) if $f$ applied coordinate-wise to any tuples from $R$ gives a tuple in $R$ whenever $f$ is defined on all the arguments. $pPol(\Gamma)$ is the set of (partial) operations preserving all relations in $\Gamma$. Partial co-clones can be defined as the sets of relations that are closed under p.p. definitions not using existential quantification, and the least partial co-clone containing $\Gamma$ is denoted by $\langle \Gamma \rangle_p$. There is also a Galois connection between partial co-clones and partial polymorphisms, in particular we have the following result.

**Theorem 2 ([9, 8, 17])** Let $\Gamma_1$ and $\Gamma_2$ be sets of relations. Then $\langle \Gamma_1 \rangle_p \subseteq \langle \Gamma_2 \rangle_p$ if and only if $pPol(\Gamma_2) \subseteq pPol(\Gamma_1)$.

For formal definitions and more information on partial clones and partial co-clones we refer the reader to [14].

Since partial co-clones do not utilize existential quantification, it can be verified that if we replace co-clones by partial co-clones in Theorem 1 then the reductions are size-preserving (in terms of the number of variables). Hence, if $\langle \Gamma_1 \rangle_p \subseteq \langle \Gamma_2 \rangle_p$ and $\Gamma_1$ is finite then CSP($\Gamma_1$) is solvable in time $O(f(n))$ if CSP($\Gamma_2$) is solvable in time $O(f(n))$ (where $n$ is the number of variables). This result seems more useful than what it is since the structure of partial co-clones is very complicated and only small portions of the lattice are known even for the Boolean domain [10]. We remark in passing that there are other applications of partial clones in complexity analysis of CSP($\Gamma$) problems [19].

What we would like to have is a concept that combines the good features of partial co-clones (e.g., size-preserving reductions for CSP($\Gamma$)) and the good features of co-clones (e.g., simpler structure especially over the Boolean domain). In this paper we propose and investigate such a concept that we call frozen partial co-clones. Frozen partial co-clones can be defined as the sets of relations that are closed under p.p. definitions where only the variables that take the same value in every model of the p.p. definition/formula (so called frozen variables) can be existentially quantified. The least frozen partial co-clone containing $\Gamma$ is denoted $\langle \Gamma \rangle_{fr}$. Hence, the frozen partial co-clones lie between the co-clones and partial co-clones in the sense that the partial co-clone lattice is a refinement of the frozen partial co-clone lattice, which in turn is a refinement of the co-clone lattice. Moreover, if we replace co-clones by frozen partial co-clones in Theorem 1 then it can be verified that if $\langle \Gamma_1 \rangle_{fr} \subseteq \langle \Gamma_2 \rangle_{fr}$ and $\Gamma_1$ is finite then CSP($\Gamma_1$) is solvable in time $O(f(n + |D|))$ if CSP($\Gamma_2$) is solvable in time $O(f(n))$ (where $|D|$ is the domain). The point is that all the variables that are frozen to the same domain element can be replaced by a single variable, and hence, at most $|D|$ extra variables need to be introduced when eliminating the existential quantifiers.

In this paper we focus exclusively on the Boolean domain. As expected, there is a connection between frozen partial co-clones and partial polymorphisms that we sketch (for the Boolean domain) in Section 2. In Sections 3 and 4 we determine large portions of the frozen (Boolean) partial co-clone lattice which indeed is significantly simpler than the partial co-clone lattice.

## 2 Frozen partial co-clones and partial polymorphisms

In this section we give some definitions and preliminary remarks before sketching the connection between frozen (Boolean) partial co-clones and frozen (Boolean) partial polymorphisms. If $\varphi$ is a formula, then $Vars(\varphi)$ denotes the set of variables occurring in it, and $M(\varphi)$ denotes the set of all assignments to $Vars(\varphi)$ which satisfy $\varphi$ (i.e., the models of $\varphi$). The relations $\{0\}$ and $\{1\}$ are denoted by $F$ and $T$, respectively. Atomic formulas are sometimes written in prefix notation (e.g., $R(x_1, \ldots, x_n)$), or infix notation (e.g., $x_1 \neq x_2$), depending on the context. Given a function $f$, $dom(f)$ denotes the domain of $f$ (i.e., the set of tuples $t_i$ for which $f(t_i)$ is defined), and $f$ is a subfunction of $g$ if $dom(f) \subseteq dom(g)$ and $f(t_i) = g(t_i)$ for all $t_i \in dom(f)$.

**Definition 3 (frozen variable)** Let $\varphi$ be a formula and let $x \in Vars(\varphi)$. Then $x$ is said to be frozen in $\varphi$ if $\varphi \models T(x)$ or $\varphi \models F(x)$. In other words, $x$ is frozen in $\varphi$ if it is assigned the same value by all its models.

**Definition 4 (frozen implementation)** Let $\Gamma$ be a set of relations and $R$ an $n$-ary relation. Then $\Gamma$ freelyzingly implement $R$ if there is a p.p. definition $R(x_1, \ldots, x_n) \equiv \exists x \varphi$ such that $\varphi$ is a conjunction of atomic formulas over $\Gamma \cup \{\varnothing\}, Vars(\varphi) \subseteq X \cup \{x_1, \ldots, x_n\}$, and every variable in $X$ is frozen in $\varphi$.

Note that frozen implementations are slightly less general than so-called faithful implementations [5, page 34]. But the latter are not suitable for our purposes since they blow up instance sizes.

**Definition 5 (frozen partial co-clone)** Let $\Gamma$ be a set of relations. The frozen partial co-clone generated by $\Gamma$, written
where $R$ with $\langle \Gamma \rangle$ it is easy to see that frozen implementations compose to relation in co-clones ordered by set inclusion form a lattice.

**Proof:** It is sufficient to prove that if $\Gamma$ is a frozen partial co-clone such that $d \in \{0, 1\}$ is determined in $\Gamma$, then any $f \in pPol(\Gamma)$ is a subfunction of a $g \in pPol(\Gamma)$ such that $d = (d, d, \ldots, d) \in dom(g)$ and $g(d, d, \ldots, d) = d$. Note that there can be no $f \in pPol(\Gamma)$ such that $f(d, \ldots, d) \neq d$ since this contradicts the fact (observed in Proposition 8) that $(\{d\}) \in \Gamma$.

Let $f \in pPol(\Gamma)$ be a k-ary function with $\text{dom}(f) = \{t_1, \ldots, t_j\}$ such that the k-tuple $d = (d, \ldots, d)$ is not in $\text{dom}(f)$. We define the function $f_d$ with $\text{dom}(f_d) = \text{dom}(f) \cup \{d\}$ such that $f_d(d) = d$ with the goal of showing that $f_d \in pPol(\Gamma)$. Assume to the contrary that there is a relation $R_d \in \Gamma$ such that $R_d$ is not preserved by $f_d$. This means that there are $k$ (not necessarily distinct) tuples $t_1, \ldots, t_k \in R_d$ such that $(f_d(t_1[1], \ldots, t_1[k]), \ldots, f_d(t_k[1], \ldots, t_k[k])) \notin R_d$ where $m$ is the arity of $R_d$. By the definition of $f_d$ and the fact that $f \in pPol(\Gamma)$ we know that at least one of the tuples $(t_1[j], \ldots, t_k[j]) = d$. We can without loss of generality assume an $R_d$ such that $(t_1[1], \ldots, t_1[k]) = d$ and all other $(t_1[j], \ldots, t_k[j]) \neq d$.

Since $\{d\} \in \Gamma$ we have that $R(y_1, \ldots, y_{m-1}) \equiv \exists x R_d(x, y_1, \ldots, y_{m-1}) \cap (x = d)$ is in $\Gamma$ and thus is preserved by $f$. As a consequence, we have $(f(t_1[2], \ldots, t_k[2]), \ldots, f(t_1[m], \ldots, t_k[m])) \in R$ and hence, $(d, f(t_1[2], \ldots, t_k[2]), \ldots, f(t_1[m], \ldots, t_k[m])) = (f_d(t_1[1], \ldots, t_k[1]), \ldots, f_d(t_1[m], \ldots, t_k[m])) \in R_d$ and we have a contradiction. Thus, $f_d \in pPol(\Gamma)$. \hfill \qed

**Proposition 8** Let $\Gamma$ be a set of relations. Then $d \in \{0, 1\}$ is determined in $\Gamma$ if and only if $\{d\} \in \langle \Gamma \rangle_{fr}$.

**Proof:** Without loss of generality assume $d = 1$. If 1 is determined in $\Gamma$, then there is a $\Gamma$-formula $\varphi$ and a variable $x_T \in \text{Vars}(\varphi)$ such that $\exists x \varphi \models T(x_T)$ and thus $\varphi \models T(x_T)$. Let $y$ be a model of $\varphi$ with a maximum number of variables being assigned 1. Identify all variables in $\varphi$ that are assigned 1 by $m$ to $x_T$, resulting in $\varphi'$. If $\text{Vars}(\varphi') = \{x_T\}$, then $T(x_T) \equiv \varphi'$ and thus $T \in \langle \Gamma \rangle_{fr}$. Otherwise, there is a variable $x_F \in \text{Vars}(\varphi') \setminus \{x_T\}$. Identify all variables in $\text{Vars}(\varphi') \setminus \{x_T\}$ to $x_F$, resulting in $\varphi''$. Then, $T(x_T) \equiv \exists x_F \varphi''$, and $T \in \langle \Gamma \rangle_{fr}$ since $x_F$ is frozen in $\varphi''$.

The converse follows directly from the definition of determined constants. \hfill \qed

**Definition 9** (frozen partial polymorphisms) A k-ary (partial) function $f \in pPol(\Gamma)$ is said to be frozen if it is defined on every k-tuple (of length k) for which $d$ is determined in $\Gamma$ and $f(d, d, \ldots, d)$ is determined in $\Gamma$. If $f \in pPol(\Gamma)$ and $f(d, d, \ldots, d) = d$ then $f \in f_{frPol}(\Gamma)$. Let $\Gamma$ be a frozen partial co-clone, then $pPol(\Gamma) = \langle \Gamma \rangle_{fr}$.

**Lemma 10** Let $\Gamma$ be a frozen partial co-clone, then any k-ary (partial) function $f \in pPol(\Gamma)$ is a subfunction of a k-ary (partial) function $g \in f_{frPol}(\Gamma)$.
and \( f \in frPol(\Gamma_2) \), so \( f \in pPol(\mathcal{M}(\varphi)) \). Moreover, as a frozen polymorphism of \( \Gamma_2 \), \( f \) is defined on all columns of \( \mathcal{M}(\varphi) \) corresponding to variables \( X \) (Proposition 8), and finally \( f \in pPol(R) \).

Now we continue with the other direction. If \( \langle \Gamma \rangle_{fr} \not\subseteq \langle \Gamma_2 \rangle_{fr} \), then since obviously \( \langle \Gamma \rangle_{fr} \subseteq \langle \Gamma \rangle_{fr} \) for any language \( \Gamma \), we have \( \langle \langle \Gamma \rangle_{fr} \rangle_p \not\subseteq \langle \langle \Gamma_2 \rangle_{fr} \rangle_p \). Hence, by Theorem 2 there exists a (partial) function \( f' \) such that \( f' \in pPol(\langle \Gamma_2 \rangle_{fr}) \) and \( f' \notin pPol(\langle \Gamma \rangle_{fr}) \). By Lemma 10, we have that \( f' \) is a subfunction of a (partial) function \( f \in frPol(\langle \Gamma_2 \rangle_{fr}) \). It is clear that \( f \notin frPol(\langle \Gamma \rangle_{fr}) \) since not even the subfunction \( f' \) is in \( pPol(\langle \langle \Gamma \rangle \rangle_{fr}) \). Hence, by Proposition 11, we get \( f \in frPol(\Gamma_2) \) and \( f \notin frPol(\Gamma_1) \).

Corollary 13 Let \( \Gamma \) be a set of relations. Then \( Inv(frPol(\Gamma)) = \langle \Gamma \rangle_{fr} \).

3 Co-clones covered by a single frozen partial co-clone

In this section we study Boolean co-clones \( C \) such that for any set of relations \( \Gamma \) with \( \langle \Gamma \rangle = C \) we have \( C = \langle \Gamma \rangle_{fr} \). We say that such a co-clone is covered by a single frozen partial co-clone. We show that a large number of co-clones are covered by a single frozen partial co-clone, and hence a large part of the lattice of frozen partial co-clones is identical to the corresponding part of the lattice of co-clones. The lattice of Boolean co-clones is visualised in Figure 1, where the co-clones colored grey are covered by a single frozen partial co-clone. For explanations of the notation\(^1\) used in the lattice, we refer the reader to [3, 4].

We define the most relevant co-clones here. A majority operation is a ternary operation \( maj \) satisfying \( maj(x, x, y) = maj(x, y, x) = maj(y, x, x) = x \) for all \( x, y \in \{0, 1\} \). Similarly a minority operation is a ternary operation \( \text{minor} \) satisfying \( \text{minor}(x, x, y) = \text{minor}(x, y, x) = \text{minor}(y, x, x) = y \) for all \( x, y \in \{0, 1\} \). The binary operations \( \text{max} \) and \( \text{min} \) return the maximum and minimum of their arguments, respectively. The co-clones \( ID_2, ID_1, \) and \( IM_2 \) are defined as follows: \( ID_2 = Inv(\{maj\}) \), \( ID_1 = Inv(\{maj, minor\}) \), and \( IM_2 = Inv(\{max, min\}) \).

In [10] it is proved that there are 25 partial co-clones\(^2\) \( pC \) such that \( pC \subseteq IM_2 = Inv(\{max, min\}) \), and there are 33 partial co-clones \( pC \) such that \( pC \subseteq ID_1 = Inv(\{maj, minor\}) \). Since all co-clones \( C \) such that \( C \subseteq IM_2 \) or \( C \subseteq ID_1 \) are covered by a single frozen partial co-clone, there are only 8 frozen partial co-clones \( fC \subseteq IM_2 \) and 6 frozen partial co-clones \( fC \subseteq ID_1 \). This suggests that the lattice of frozen partial co-clones is significantly less complex than the partial co-clone lattice. Nevertheless, as expected the frozen partial co-clone lattice is more complex than the ordinary co-clone lattice. In particular we show in Section 4 that the co-clone \( ID_2 \) splits into 13 frozen partial co-clones. Moreover, it seems that none of the white co-clones in Figure 1 is covered by a single frozen partial co-clone.

The covering proofs make heavy use of the results in [6] which for every Boolean co-clone \( C \) gives a set of relations \( \Gamma \) such that \( \langle \Gamma \rangle_p = C \). In particular, it is shown in [6] that \( ID_1 = \langle \{\#, T, F\} \rangle_p \). The covering proofs are numerous so due to space constraints we can only present a single illustrative case.

Proposition 14 \( ID_1 = Inv(\{maj, minor\}) = \langle \{\#, T, F\} \rangle_p \) is covered by a single frozen partial co-clone.

Proof: Let \( \Gamma \subseteq ID_1 \) with \( \Gamma \not\subseteq ID \) and \( \Gamma \not\subseteq IR_2 \) (i.e., \( \langle \Gamma \rangle = ID_1 \)), where \( ID = \langle \# \rangle \) and \( IR_2 = \langle \{F, T\} \rangle \). First note that since \( \langle \Gamma \rangle = ID_1 = \langle \{\#, T, F\} \rangle \), we trivially have that \( F \) and \( T \) are determined in \( \Gamma \). Thus, according to Proposition 8 we have \( \{T, F\} \subseteq \langle \Gamma \rangle_{fr} \).

Now, since \( \Gamma \not\subseteq IR_2 \) there is a relation \( R \in \Gamma \) such that

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\(^1\)The notation used is different from Post’s notation, but is now standard in the Boolean CSP area.

\(^2\)The results in [10] are presented in terms of partial clones.
Then there is $\varphi \equiv R$ of the form:

$$\bigwedge_{i \in I} (x_i \neq y_i) \land \bigwedge_{j \in J} F(x_j) \land \bigwedge_{k \in K} T(x_k)$$

Assume no $j$ and no $k$ is in $I$, which can be ensured by
propagating unary constraints. From $R \notin IR_2$ we know
that $R$ is nonempty and $I \neq \emptyset$. Let $m \in R$, that is, a model
of $\varphi$, and let $P = \{x_i \mid i \in I, m(x_i) = 1\} \cup \{y_i \mid i \in
I, m(y_i) = 1\}$ and $N = \{x_i \mid i \in I, m(x_i) = 0\} \cup \{y_i \mid
i \in I, m(y_i) = 0\}$. Because $I \neq \emptyset$ we have at least one
{$\neq$}-constraint and so, $P, N \neq \emptyset$. Moreover, obviously
every {$\neq$}-constraint in $\varphi$ is between a variable in $P$ and
one in $N$. Now identify all the variables in $P$ to a single
variable $p$ and all those in $N$ to $n$. Then clearly the resulting
formula is logically equivalent to $(p \neq n) \land \bigwedge_{j \in J} F(x_j) \land
\bigwedge_{k \in K} T(x_k)$, and we get a frozen implementation of $\neq$ by
existentially quantifying over every $x_j$ and $x_k$ ($j \in J, k \in K$).

Finally, $\{\neq, T, F\} \subseteq (\Gamma)_f$ and so, $(\Gamma)_f = ID_1$. \qed

Theorem 15 Each co-clone colored grey in Figure 1 is covered by a single frozen partial co-clone.

4 Structure of ID

We begin by introducing the basic relations and the 13 frozen partial co-clones in $ID_2 = Inv\{(maj)\}$ (i.e., the frozen partial co-clones $(\Gamma)_f$ such that $(\Gamma)_f \subseteq ID_2$ and $(\Gamma) = ID_2$). We then prove that these 13 frozen partial co-clones cover $ID_2$ (i.e., $(\Gamma)_f$ equals one of these 13 frozen partial co-clones for any $\Gamma$ such that $(\Gamma) = ID_2$). Finally, we prove that these 13 frozen partial co-clones are all distinct. We remark that the lattice of partial co-clones in $ID_2$ has not yet been classified [10]. Hence, the results in this section can also be seen as a step towards such a classification.

Definition 16 (relations in $ID_2$) $R_p^n$ is the relation defined by $(x_1 \lor x_2)$ (2 stands for binary and $p$ for positive). Similarly, $R_3^n$ is the relation defined by $(\neg x_1 \lor \neg x_2)$, $R_2^n$ is the relation defined by $(\neg x_1 \lor x_2)$, $R_1^n$ is the relation defined by $(x_1 \lor \neg x_2)$, $R_0^n$ is the relation defined by $(\neg x_1 \lor \neg x_2)$, $R_2^n$ is the relation defined by $(\neg x_1 \lor \neg x_2) \land (x_1 \neq x_3)$ and $R_3^n$ that defined by $(\neg x_1 \lor \neg x_2) \land (x_1 \neq x_3)$, finally, $R_3^n$ is the relation defined by $(x_1 \lor x_2) \land (x_1 \neq x_3) \land (x_2 \neq x_4)$.

Definition 17 [frozen partial co-clones in $ID_2$] We define the following frozen partial co-clones:

- $\Gamma_3^p = (R_3^p)_f$,
- $\Gamma_3^n = (R_3^n)_f$, $\Gamma_3^t = (R_3^t)_f$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Frozeon partial co-clones in $ID_2$.}
\end{figure}

The inclusion structure among these frozen partial co-clones is given in Figure 2. Most of the inclusions are obvious. The main difficulty is to show that they cover all of $ID_2$. Due to space constraints and the fact that all the covering proofs are quite similar, we only present one of them. Note that the proof is similar in spirit to the constructions in [19].

Proposition 18 $(\Gamma_i^p)$ Let $R$ be a relation in $\Gamma_i^p \setminus ID_1$. Then $R$ freely implements $R_4$.

Proof: Given a relation $R \equiv \mathcal{M}(\varphi)$, then $R_4(x_{i_1}, \ldots, x_{i_d})$ denotes the projection of $R$ onto the coordinates corresponding to the variables $\{x_{i_1}, \ldots, x_{i_d}\}$ in $\varphi$.

From $R \in \Gamma_i^p$ it follows that $R$ has a p.p. definition of the form $\exists X, \bigwedge_{i \in I} R^p_i(x_{i_1}, \ldots, x_{i_d})$ where the variables in

\footnote{The mnemonics are: subscripts represent the arities of the relations in the basis, and superscripts represent the nature of these relations, in the same order ($p$ stands for positive, etc.).}
X are frozen. Write $R_i$ for $R_i|_{\{x_{i1},\ldots,x_{i4}\}}$ and $R^p_i(X_i)$ for $R^p_i(x_{i1},\ldots,x_{i4})$ to see that there is an $i_0 \in I$ such that $R_i = R^p_i(X_i)$ and all $x_i0j$'s are pairwise different.

Assume to the contrary that for all $i$, $R_i \neq R^p_i(X_i)$. By construction it follows $R_i \subset R^p_i(X_i)$ (since at least the $\{R^p_i\}$-constraint acts on $x_{i1},\ldots,x_{i4}$). But a case study on the tuples in $R^p_i(X_i) \setminus R_i$ shows that this entails $R_i \in ID_1$. Since from the definition of $R$ it follows $R \equiv \exists x, \bigwedge_{i \in I} R_i$, we get $R \in ID_1$, a contradiction.

Now consider the relation obtained from $R$ by applying the following transformations maximally while preserving $R_{i_0} = R^p_{i_0}(X_{i_0}) = \{0110, 1001, 1100\}$:

1. identify $x'$ to $x$ for some $x \in \{x_{i1},\ldots,x_{i4}\}$, $x' \notin \{x_{i1},\ldots,x_{i4}\}$; e.g., if $R_i|_{\{x_{i1},\ldots,x_{i4},x'\}} = \{01100, 01101, 10011, 11001\}$, identify $x'$ to $x_{i1}$.  
2. freeze $x' \notin \{x_{i1},\ldots,x_{i4}\}$ to 0 or 1 (using $F$ or $T$ with Proposition 8) and existentially quantify over it, e.g., if $R_i|_{\{x_{i1},\ldots,x_{i4},x'\}} = \{01100, 01101, 10010, 11000\}$, freeze $x'$ to 0 and existentially quantify over it.

When none of these operations can be applied any more, it is easily verified that all remaining $x' \notin \{x_{i1},\ldots,x_{i4}\}$ satisfy $R_i|_{\{x_{i1},\ldots,x_{i4},x'\}} = \{01100, 10011, 11000\}$ or $R_i|_{\{x_{i1},\ldots,x_{i4},x'\}} = \{01100, 10010, 11001\}$. But this is a contradiction, since then $R_i|_{\{x_{i1},\ldots,x_{i4},x'\}}$ is not closed under ternary majorities, and so $R \notin ID_2$. Thus the transformations end with $R = R^p_{i_0}(X_{i_0})$ (no other $x_j$ can be left).

To conclude, from $R$ we freezing implemented $R^p_i$, and we are done.

**Theorem 19** Given any set of relations $\Gamma$ such that $\langle \Gamma \rangle = ID_2$, then $\langle \Gamma \rangle_{fr}$ equals one of the 13 frozen partial co-clones in Definition 17.

It is easy to prove that the 13 frozen partial co-clones in Definition 17 are all distinct. Given two sets of relations $\Gamma_1$ and $\Gamma_2$ such that $\langle \Gamma_1 \rangle = \langle \Gamma_2 \rangle = ID_2$, then to prove that $\langle \Gamma_1 \rangle_{fr} \neq \langle \Gamma_2 \rangle_{fr}$ it is sufficient (according to Theorem 12) to show that $frPol(\Gamma_2) \neq frPol(\Gamma_1)$.

The operations that we use to separate the different frozen partial co-clones are all ternary minority operations that are undefined on certain tuples. We denote these minority operations by $mu(t_1,\ldots,t_n)$ where $t_1,\ldots,t_n$ are the tuples on which the minority operation is undefined. For example, $\Gamma^p_\lambda \neq \Gamma^p_3$ since $mu(010,001) \in frPol(R^p_3) = frPol(\Gamma^p_3)$ but $mu(010,001) \notin frPol(R^p_3)$, and $\Gamma^p_3 \neq \Gamma^p_2$ since $mu(100,010,001) \in frPol(R^p_3) = frPol(\Gamma^p_3)$ but $mu(100,010,001) \notin frPol(R^p_3)$.

**Proposition 20** The 13 frozen partial co-clones in Definition 17 are all distinct.

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**References**