Thermoelasticity of non-simple oriented materials

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Abstract—Non-simple elastic materials are materials in which the internal energy depends not only on the first but also on higher deformation gradients [1–4]. Particles of oriented materials have more than three degrees of freedom, which is equivalent to the introduction in the theory of continuous media of fields of directors [3, 5, 6] or multipolar displacements [6–8]. In the present paper, equations are derived of thermoelasticity of oriented non-simple hyper-elastic materials. The notion of a non-simple material is generalized to thermal problems, making the specific free energy depend not only on the temperature and the deformation gradients but also on the temperature gradients. The notion of oriented material is generalized to thermal problems by introducing, independently of the additional degrees of freedom, additional fields describing the temperature distribution.

1. INTRODUCTION

Notation

THE material macro- and micro-co-ordinates are denoted by X^α and $\tilde{X}^\alpha$ respectively, and the spatial macro- and micro-co-ordinates are denoted by x^i and $\tilde{x}^i$ respectively. The indices $\alpha, \beta, \ldots$ and $i, j, \ldots$ take the values 1, 2, 3. The indices denoted by Gothic letters $\mathfrak{n}, \mathfrak{b}$ can, in the entire paper, be interpreted in two ways: 1° as dead indices $\mathfrak{n}, \mathfrak{b} = \mathrm{I, II, III, IV}, \ldots$ which is equivalent to the determination of the additional (internal) degrees of freedom by means of directors [3, 5], or 2° as material indices taking values of the sequence $\alpha_{\mathrm{I}}, \alpha_{\mathrm{I}}\alpha_{\mathrm{II}}, \alpha_{\mathrm{I}}\alpha_{\mathrm{II}}\alpha_{\mathrm{III}}, \ldots$, which is equivalent to the determination of the additional degrees of freedom by means of multipolar displacements [6–8]. The indices $\Phi, \Psi, \ldots$ take the values of the sequences $\varphi_{\mathrm{I}}, \varphi_{\mathrm{I}}\varphi_{\mathrm{II}}, \ldots, \varphi_{\mathrm{I}} \ldots \varphi_{\Omega}$; $\psi_{\mathrm{I}}, \psi_{\mathrm{I}}\psi_{\mathrm{II}}, \ldots, \psi_{\mathrm{I}} \ldots \psi_{\Omega}; \ldots$. We employ the notation $(\ldots)_{,\alpha} \equiv \partial(\ldots)/\partial X^\alpha$, $(\ldots)_{,k} \equiv \partial(\ldots)/\partial x^k$.

The continuum under consideration is composed of particles **X** with material macro-co-ordinates X^α (macro-medium). The spatial macro-co-ordinates of the particle **X** are denoted by x^i. Each particle **X** is assumed, at the same time, as the origin of the so-called system of spatial 'micro-co-ordinates' $\tilde{x}^i_{(\mathbf{X})}$. It is assumed that the parametric planes $\tilde{x}^i_{(\mathbf{X})} = \text{const.}$ always coincide with the parametric planes $x^i = \text{const.}$ that is, each micro-system $\tilde{x}^i_{(\mathbf{X})}$ performs a translation motion together with the particle **X** (spatial co-ordinate systems are Cartesian systems).

In order to distinguish the inner structure of a particle of the continuum (the micro-structure of the continuum [9–11]) to each particle **X** is attached at a moment τ in the corresponding space of spatial co-ordinates $\tilde{x}^i_{(\mathbf{X})}$—a micro-medium $\mathfrak{b}_\tau(\mathbf{X})$ bounded by the surface $\partial\mathfrak{b}_\tau(\mathbf{X})$. Such a micro-medium is treated as a configuration at a time τ of an independent continuum composed of material points $\tilde{\mathbf{X}}$ with material micro-co-ordinates $\tilde{X}^\alpha$. It is assumed that these micro-media are hyper-elastic and that the thermodynamic process at every point $\tilde{\mathbf{X}}$ of each of micro-medium $\mathfrak{b}_\tau(\mathbf{X})$ is a locally reversible process.

All the properties of the macro-medium of particles $\mathbf{X}$ will be deduced from the structural properties of the micro-medium associated with each particle $\mathbf{X}$ and from the way in which these micro-media are interconnected. The densities per unit mass $\Psi(x^i, \tau)$ and the densities $\Psi_{(\mathbf{n})}(x^i, \tau)$ per unit area of a surface with normal $\mathbf{n}$ will be described in the macro-medium by means of the mean values

$$\Psi(x^i, \tau)=\frac{1}{\mathfrak{b}_\tau}\int_{\mathfrak{b}_\tau} \tilde{\Psi}(x^i, \tilde{X}^\alpha, \tau)\mathrm{d}b_\tau, \qquad \Psi_{(\mathbf{n})}(x^i, \tau)=\int_{\partial b_\tau} \frac{\tilde{\Psi}_{(\tilde{\mathbf{n}})}(x^i, \tilde{X}^\alpha, \tau)}{\Delta P(\mathbf{n}, \tilde{\mathbf{n}})}\mathrm{d}(\partial b_\tau), \tag{1.1}$$

in which $\tilde{\Psi}(x^i, \tilde{X}^\alpha, \tau)$ and $\tilde{\Psi}_{(\tilde{\mathbf{n}})}(x^i, \tilde{X}^\alpha, \tau)$ are the densities of analogous fields in $\mathfrak{b}_\tau(\mathbf{X})$ or on $\partial b_\tau(\mathbf{X})$. The function $\Delta P(\mathbf{n}, \tilde{\mathbf{n}})$ is characterized by the way of interconnexion between elements of the continuum; $\mathbf{n}$ is a unit vector normal to the section of the body and $\tilde{\mathbf{n}}$—a vector normal to $\partial b_\tau(\mathbf{X})$. For bodies with lattice structure the form of the function $\Delta P(\mathbf{n}, \tilde{\mathbf{n}})$ is given in [12].

The motion of the material point $\tilde{\mathbf{X}}$ belonging to the micro-medium $\mathfrak{b}_\tau(\mathbf{X})$ will be determined by the equation

$$z^i=x^i(X^\alpha, \tau)+x^i_n(X^\alpha, \tau)\xi^n(\tilde{X}^\alpha)+x^i_{,\Phi}(X^\alpha, \tau)\xi^\Phi(\tilde{X}^\alpha)+x^i_{n,\Phi}(X^\alpha, \tau)\xi^{n\Phi}(\tilde{X}^\alpha), \tag{1.2}$$

in which $x^i(X^\alpha, \tau)$ is the motion of the particle $\mathbf{X}$ (the motion of the spatial micro-co-ordinates system $\tilde{x}^i_{(\mathbf{X})}$) and z^i-x^i is the position of the material point $\tilde{\mathbf{X}}$ in the system $\tilde{x}^i_{(\mathbf{X})}$ of spatial micro-co-ordinates.

The temperature Θ at the time τ and at any material point $\tilde{\mathbf{X}}$ belonging to the micro-medium $\mathfrak{b}_\tau(\mathbf{X})$ will be determined by the equation

$$\Theta=\theta(X^\alpha, \tau)+\theta_n(X^\alpha, \tau)\zeta^n(\tilde{X}^\alpha)+\theta_{,\Phi}(X^\alpha, \tau)\zeta^\Phi(\tilde{X}^\alpha)+\theta_{n,\Phi}(X^\alpha, \tau)\zeta^{n\Phi}(\tilde{X}^\alpha), \tag{1.3}$$

in which $\theta(X^\alpha, \tau)$ is the 'mean' temperature of the micro-medium.

2. PRINCIPLE OF CONSERVATION OF ENERGY

The principle of conservation of energy for the region B bounded by a surface P is assumed in the familiar form

$$\int_B(\dot{\varepsilon}+\dot{\kappa})\rho\mathrm{d}B=\int_P(w_{(\mathbf{n})}+h_{(\mathbf{n})})\mathrm{d}P+\int_B(w+b)\rho\mathrm{d}B, \tag{2.1}$$

in which ε is the density of internal energy, κ—that of kinetic energy, w—density of work of body forces, h—density of heat surces, $w_{(\mathbf{n})}$—density of work of surface tractions, $h_{(\mathbf{n})}$—flux of heat through the surface P into B. In agreement with (1.1) we have ($b\equiv\mathfrak{b}_\tau(\mathbf{X})$, $\partial b\equiv\partial\mathfrak{b}_\tau(\mathbf{X})$):

$$\varepsilon\equiv\frac{1}{\rho b}\int_b\tilde{\varepsilon}\tilde{\rho}\mathrm{d}b, \qquad \rho\equiv\frac{1}{b}\int_b\tilde{\rho}\mathrm{d}b, \qquad \kappa\equiv\frac{1}{\rho b}\int_b\tfrac{1}{2}\dot{z}^k\dot{z}_k\tilde{\rho}\mathrm{d}b;$$

$$w\equiv\frac{1}{b}\int_b\tilde{f}^k\dot{z}_k\mathrm{d}b, \qquad h\equiv\frac{1}{\rho b}\int_b\tilde{h}\tilde{\rho}\mathrm{d}b;$$

$$w_{(\mathbf{n})}\equiv\int_{\partial b}\frac{\tilde{p}^k_{(\mathbf{n})}\dot{z}_k}{\Delta P(\mathbf{n}, \tilde{\mathbf{n}})}\mathrm{d}(\partial b), \qquad h_{(\mathbf{n})}\equiv\int_{\partial b}\frac{\tilde{h}_{(\mathbf{n})}}{\Delta P(\mathbf{n}, \tilde{\mathbf{n}})}\mathrm{d}(\partial b); \tag{2.2}$$

where $\tilde{\rho}$, $\tilde{\varepsilon}$, $\tilde{f}^k$, $\tilde{h}$ are mass density, internal energy density per unit mass, the body force and the supply of energy per unit mass created by distributed energy sources in the micro-medium $\mathfrak{b}_\tau(\mathbf{X})$, respectively. The density of the surface tractions and the flux of heat through the surface ∂b into the micro-medium are denoted by $\tilde{p}^k_{(\mathbf{n})}$ and $\tilde{h}_{(\mathbf{n})}$, respectively. Let us substitute in (2.2) the right-hand components of the expression (1.2). Then, the principle of conservation of energy (2.1) assumes the form

$$\int_B \dot{\varepsilon}\rho \mathrm{d}B = \int_P (\dot{x}_k p^k_{(\mathbf{n})} + \dot{x}_{kn} p^{nk}_{(\mathbf{n})} + \dot{x}_{k,\Phi} p^{\Phi k}_{(\mathbf{n})} + \dot{x}_{kn,\Phi} p^{n\Phi k}_{(\mathbf{n})} + h_{(\mathbf{n})})\mathrm{d}P$$
$$+ \int_B (\dot{x}_k \bar{f}^k + \dot{x}_{kn} \bar{f}^{nk} + \dot{x}_{k,\Phi} \bar{f}^{\Phi k} + \dot{x}_{kn,\Phi} \bar{f}^{n\Phi k} + h)\rho \mathrm{d}B, \tag{2.3}$$

in which

$$\bar{f}^k \equiv f^k (\ddot{x}^k + \ddot{x}^k_n \rho^n + \ddot{x}^k_{,\Phi}\rho^\Phi + \ddot{x}^k_{n,\Phi}\rho^{n\Phi}),$$
$$\bar{f}^{nk} \equiv f^{nk} - (\ddot{x}^k \rho^n + \ddot{x}^k_{\mathfrak{b}} \rho^{n\mathfrak{b}} + \ddot{x}^k_{,\Phi}\rho^{n/\Phi} + \ddot{x}^k_{\mathfrak{b},\Phi}\rho^{n/\mathfrak{b}\Phi}),$$
$$\bar{f}^{\Phi k} = f^{\Phi k} - (\ddot{x}^k \rho^\Phi + \ddot{x}^k_{\mathfrak{b}} \rho^{\mathfrak{b}/\Phi} + \ddot{x}^k_{,\Psi}\rho^{\Phi\Psi} + \ddot{x}^k_{\mathfrak{b},\Psi}\rho^{\Phi/\mathfrak{b}\Psi}),$$
$$\bar{f}^{n\Phi k} = f^{n\Phi k} - (\ddot{x}^k \rho^{n\Phi} + \ddot{x}^k_{\mathfrak{b}} \rho^{n\Phi/\mathfrak{b}} + \ddot{x}^k_{,\Psi}\rho^{n\Phi/\Psi} + \ddot{x}^k_{\mathfrak{b},\Psi}\rho^{n\Phi/\mathfrak{b}\Psi}), \tag{2.4}$$

with the notation

$$p^k_{(\mathbf{n})} \equiv \int_{\partial b} \frac{\tilde{p}^k_{(\mathbf{n})}}{\Delta P(\mathbf{n}, \tilde{\mathbf{n}})}\mathrm{d}(\partial b), \qquad p^{nk}_{(\mathbf{n})} = \int_{\partial b} \frac{\tilde{p}^k_{(\mathbf{n})}\zeta^n}{\Delta P(\mathbf{n}, \tilde{\mathbf{n}})}\mathrm{d}(\partial b), \qquad p^{\Phi k}_{(\mathbf{n})} \equiv \int_{\partial b} \frac{\tilde{p}^k_{(\mathbf{n})}\zeta^\Phi}{\Delta P(\mathbf{n}, \tilde{\mathbf{n}})}\mathrm{d}(\partial b),$$
$$p^{n\Phi k}_{(\mathbf{n})} \equiv \int_{\partial b} \frac{\tilde{p}^k_{(\mathbf{n})}\zeta^{n\Phi}}{\Delta P(\mathbf{n}, \tilde{\mathbf{n}})}\mathrm{d}(\partial b); \qquad f^k \equiv \frac{1}{\rho b}\int_b \tilde{f}^k \tilde{\rho}\mathrm{d}b, \qquad f^{nk} \equiv \frac{1}{\rho b}\int_b \tilde{f}^k \xi^n \tilde{\rho}\mathrm{d}b,$$
$$f^{\Phi k} \equiv \frac{1}{\rho b}\int_b \tilde{f}^k \xi^\Phi \tilde{\rho}\mathrm{d}b, \qquad {}^{n\Phi k} \equiv \frac{1}{\rho b}\int_b \tilde{f}^k \xi^{n\Phi} \tilde{\rho}\mathrm{d}b; \qquad \rho^n \equiv \frac{1}{\rho b}\int_b \xi^n \tilde{\rho}\mathrm{d}b,$$
$$\rho^\Phi \equiv \frac{1}{\rho b}\int_b \xi^\Phi \tilde{\rho}\mathrm{d}b, \qquad \rho^{n\Phi} \equiv \frac{1}{\rho b}\int_b \xi^{n\Phi} \tilde{\rho}\mathrm{d}b, \qquad \rho^{n\mathfrak{b}} \equiv \frac{1}{\rho b}\int_b \xi^n \xi^{\mathfrak{b}} \tilde{\rho}\mathrm{d}b,$$
$$\rho^{\mathfrak{b}/\Phi} \equiv \frac{1}{\rho b}\int_b \xi^{\mathfrak{b}} \xi^\Phi \tilde{\rho}\mathrm{d}b, \qquad \rho^{n/\mathfrak{b}\Phi} \equiv \frac{1}{\rho b}\int_b \xi^n \xi^{\mathfrak{b}\Phi} \tilde{\rho}\mathrm{d}b, \cdots \text{etc.} \tag{2.5}$$

The quantities defined by the equations (2.5) are surface tractions and hyper-tractions, body forces and body hyper-forces and densities of mass distribution in the non-simple oriented body.

Making use of the invariance conditions of the internal energy under a rigid motion of the deformed body, we obtain, after some simple transformations, the equations of motion

$$p^{lk}_{,l} + \bar{f}^k = 0,$$
$$(p^{mn[k}x^{l]}_n + p^{m\Phi[k}x^{l]}_{,\Phi} + p^{mn\Phi[k}x^{l]}_{n,\Phi})_{,m} + p^{[lk]} + \bar{f}^{n[k}x^{l]}_n + \bar{f}^{\Phi[k}x^{l]}_{,\Phi} + \bar{f}^{n\Phi[k}x^{l]}_{n,\Phi} = 0, \tag{2.6}$$

and the boundary conditions

$$p^k_{(\mathbf{n})} - p^{lk}n_l = 0,$$
$$(p^{nk}_{(\mathbf{n})} - p^{lnk}n_l)\dot{x}_{kn} + (p^{\Phi k}_{(\mathbf{n})} - p^{l\Phi k}n_l)\dot{x}_{k,\Phi} + (p^{n\Phi k}_{(\mathbf{n})} - p^{ln\Phi k}n_l)\dot{x}_{kn,\Phi} + h_{(\mathbf{n})} - h^l n_l = 0, \tag{2.7}$$

where p^{lk} is the stress tensor, p^{lnk}, $p^{l\Phi k}$, $p^{ln\Phi k}$ are the hyper-stress tensors. Making use of (2.7) the principle of conservation of energy, (2.3) can be reduced to the form

$$\rho\dot{\varepsilon}=(p^{mk}X^{\alpha}_{,m}+p^{m\alpha k}_{,m}+\mathcal{J}^{\alpha k})\overline{\dot{x_{k,\alpha}}}+(p^{m\alpha\Phi k}_{,m}+p^{m\Phi k}X^{\alpha}_{,m}+\mathcal{J}^{\alpha\Phi k})\overline{\dot{x_{k,\alpha\Phi}}}$$

$$+(p^{mnk}X^{\alpha}_{,m}+p^{mn\alpha k}_{,m}+\mathcal{J}^{n\alpha k})\overline{\dot{x_{kn,\alpha}}}+(p^{mn\Phi k}X^{\alpha}_{,m}+p^{mn\alpha\Phi k}_{,m}+\mathcal{J}^{n\alpha\Phi k})\overline{\dot{x_{kn,\alpha\Phi}}}$$

$$+(p^{lnk}_{,l}+\mathcal{J}^{nk})\dot{x}_{kn}+h^{l}_{,l}+h\,, \tag{2.8}$$

in which

$$p^{m\alpha\Phi k}_{,m}+\mathcal{J}^{\alpha\Phi k}\equiv 0,\qquad p^{mn\alpha\Phi k}_{,m}+\mathcal{J}^{n\alpha\Phi k}\equiv 0\quad\text{for}\quad \Phi=\varphi_1\varphi_2\ldots\varphi_\Omega, \tag{2.9}$$

should be assumed. The equations (2.3)–(2.9) are field equations of the non-simple oriented body.

3. PRODUCTION OF ENTROPY

In agreement with the model of micro-medium $\mathfrak{b}_\tau(\mathbf{X})$ (see, for example, theIntroduction), the differential equation of the entropy in the micro-medium has the familiar form

$$\tilde{\rho}\tilde{h}+\frac{\partial\tilde{h}^{i}}{\partial\tilde{x}^{i}}=\tilde{\rho}\Theta\dot{\tilde{\eta}}, \tag{3.1}$$

where $\tilde{\eta}$ is the specific entropy, $\tilde{h}$ is the supply of energy per unit mass created by distributed energy sources in the micro-medium and $\tilde{h}^i$ is the flux of heat in the micro-medium. On substituting in (3.1) the right-hand side of the equations (1.3) and integrating over the region $\mathfrak{b}_\tau(\mathbf{X})$ of the micro-medium, we obtain

$$\rho\theta\dot{\eta}=h^{i}_{,i}+\rho h-\rho(\theta_n\dot{\eta}^n+\theta_{,\Xi}\dot{\eta}^{\Xi}+\theta_{n,\Xi}\dot{\eta}^{n\Xi}), \tag{3.2}$$

where

$$h^{i}_{,i}\equiv\frac{1}{b}\int_b\frac{\partial\tilde{h}^{i}}{\partial\tilde{x}^{i}}\mathrm{d}b=\frac{1}{b}\int_{\partial b}\tilde{h}^{i}\tilde{n}_i\mathrm{d}(\partial b),\qquad \eta\equiv\frac{1}{\rho b}\int_b\tilde{\eta}\tilde{\rho}\mathrm{d}b,$$

$$\eta_n\equiv\frac{1}{\rho b}\int_b\tilde{\eta}\zeta^n\tilde{\rho}\mathrm{d}b,\qquad \eta^{\Xi}\equiv\frac{1}{\rho b}\int_b\tilde{\eta}\zeta^{\Xi}\tilde{\rho}\mathrm{d}b,\qquad \eta^{n\Xi}\equiv\frac{1}{\rho b}\int_b\tilde{\eta}\zeta^{n\Xi}\tilde{\rho}\mathrm{d}b. \tag{3.3}$$

Equation (3.2) is a differential equation for the specific entropy η in a non-simple oriented body. On dividing (3.2) by θ and integrating over the region B bounded by the surface P, we obtain, after some simple transformations

$$\int_B\dot{\eta}\rho\mathrm{d}B-\int_B\frac{h\rho}{\theta}\mathrm{d}B-\int_P\frac{h^i n_i}{\theta}\mathrm{d}P+\int_B\frac{\theta_n\mu^n+\theta_{,\Xi}\mu^{\Xi}+\theta_{n,\Xi}\mu^{n\Xi}}{\theta}\mathrm{d}B$$

$$=\int_B\left[\frac{h^i\theta_{,i}}{\theta^2}+\frac{1}{b}\int_b\frac{\tilde{h}^i}{b\Theta^2}\frac{\partial\Theta}{\partial\tilde{x}^i}\mathrm{d}b\right]\mathrm{d}B, \tag{3.4}$$

where

$$\mu^{\Xi}\equiv\frac{1}{\rho b}\left[\int_b \frac{\tilde{h}\tilde{\rho}}{\Theta}\zeta^{\Xi}\mathrm{d}b+\int_{\partial b}\frac{\tilde{h}^i\tilde{n}_i}{\Theta}\zeta^{\Xi}\mathrm{d}(\partial b)\right],$$

$$\mu^{n}\equiv\frac{1}{\rho b}\left[\int_b \frac{\tilde{\rho}\tilde{h}}{\Theta}\zeta^{n}\mathrm{d}b+\int_{\partial b}\frac{\tilde{h}^i\tilde{n}_i}{\Theta}\zeta^{n}\mathrm{d}(\partial b)\right],$$

$$\mu^{n\Xi}\equiv\frac{1}{\rho b}\left[\int_b \frac{\tilde{\rho}\tilde{h}}{\Theta}\zeta^{n\Xi}\mathrm{d}b+\int_{\partial b}\frac{\tilde{h}^i\tilde{n}_i}{\Theta}\zeta^{n\Xi}\mathrm{d}(\partial b)\right]. \tag{3.5}$$

The volume integrals in (3.5) depend on the distribution of the entropy sources $\tilde{\rho}\tilde{h}/\Theta$ in the micro-medium $\mathfrak{b}_\tau(\mathbf{X})$. The surface integrals in (3.5) depend on the flux of entropy $\tilde{h}^i\tilde{n}_i/\Theta$ through the surface $\partial b_\tau(\mathbf{X})$ into the micro-medium. The right-hand member of (3.4), which can also be represented in the form

$$\int_B\left[\frac{h^i\theta_{,i}}{\theta^2}+\frac{\theta_n(\mu^n-\dot{\eta}^n)}{\theta}+\frac{\theta_{,\Xi}(\mu^{\Xi}-\dot{\eta}^{\Xi})}{\theta}+\frac{\theta_{n,\Xi}(\mu^{n\Xi}-\dot{\eta}^{n\Xi})}{\theta}\right]\mathrm{d}B \tag{3.6}$$

is the entropy increase due to the heat flow in the region B of the body. In agreement with the principle which states that in adiabatic process the global entropy cannot decrease, the expression (3.6) is always non-negative. Hence the conclusion

$$h^i=h^i\left(\frac{\theta_{,i}}{\theta^2},\ \frac{\theta_n}{\theta},\ \frac{\theta_{,\Xi}}{\theta},\ \frac{\theta_{n,\Xi}}{\theta}\right) \tag{3.7}$$

constituting the law of heat flow in the non-simple oriented body. The right-hand member of (3.4) being non-negative, we have also

$$\int_B \dot{\eta}\rho\mathrm{d}B-\int_B\frac{h\rho}{\theta}\mathrm{d}B-\int_P\frac{h^in_i}{\theta}dP+\int_B\frac{\theta_n\mu^n+\theta_{,\Xi}\mu^{\Xi}+\theta_{n,\Xi}\mu^{n\Xi}}{\theta}\mathrm{d}B\geqslant 0.$$

The above inequality may be called the Clausius–Duhem inequality for the non-simple oriented body.

4. FREE ENERGY

The free energy of the non-simple oriented body is defined by

$$\varphi\equiv\varepsilon-\theta\eta-\theta_n\eta^n-\theta_{,\Xi}\eta^{\Xi}-\theta_{n,\Xi}\eta^{n\Xi}. \tag{4.1}$$

On eliminating from (4.1), (2.8) and (3.2) the internal energy density ε and the divergence $h^i{}_{,i}$, we obtain

$$\begin{aligned}\rho\dot{\varphi}=&(p^{lk}X^{\alpha}_{,l}+p^{l\alpha k}_{,l}+f^{\alpha k})\dot{\overline{x_{k,\alpha}}}+(p^{l\Phi k}X^{\alpha}_{,l}+p^{l\alpha\Phi k}_{,l}+f^{\alpha\Phi k})\dot{\overline{x_{k,\alpha\Phi}}}\\&+(p^{lnk}X^{\alpha}_{,l}+p^{ln\alpha k}_{,l}+f^{\alpha nk})\dot{\overline{x_{kn,\alpha}}}+(p^{ln\Phi k}X^{\alpha}_{,l}+p^{ln\alpha\Phi k}_{,l}+f^{n\alpha\Phi k})\dot{\overline{x_{kn,\alpha\Phi}}}\\&+(p^{lnk}_{,l}+f^{nk})\dot{x}_{kn}-\rho\eta\dot{\theta}-\rho\eta^n\dot{\theta}_n-\rho\eta^{\Xi}\dot{\theta}_{,\Xi}-\rho\eta^{n\Xi}\dot{\theta}_{n,\Xi}.\end{aligned} \tag{4.2}$$

The form of (4.2) suggests that the equation of state is of the form

$$\varphi=\varphi(x_{k,\alpha};\quad x_{k,\alpha\Phi};\quad x_{kn,\alpha};\quad x_{kn,\alpha\Phi};\quad x_{kn};\quad \theta;\quad \theta_{n};\quad \theta_{,\Xi};\quad \theta_{n,\Xi}). \tag{4.3}$$

The mechanical parameters of state must appear in the following combinations [3]

$$\begin{aligned} &C_{\alpha\beta}=x^{k}{}_{,\alpha}x_{k,\beta}, && C_{\alpha\beta\Phi}=x^{k}{}_{,\alpha}\,x_{k,\beta\Phi}; && \\ &D_{\alpha n}=x^{k}{}_{,\alpha}\,x_{kn}, && D_{\alpha n\beta}=x^{k}{}_{,\alpha}\,x_{kn,\beta}, && D_{\alpha n\beta\Phi}=x^{k}{}_{,\alpha}\,x_{kn,\beta\Phi} \end{aligned} \tag{4.4}$$

and therefore

$$\varphi=\varphi(C_{\alpha\beta};\quad C_{\alpha\beta\Phi};\quad D_{\alpha n};\quad D_{\alpha n\beta};\quad D_{\alpha n\beta\Phi};\quad \theta;\quad \theta_{n};\quad \theta_{,\Xi};\quad \theta_{n,\Xi}). \tag{4.5}$$

Making use of (4.2), (4.3) and (4.4) we finally obtain the following equations of state

$$p^{lk}X^{\alpha}_{,l}+p^{l\alpha k}_{,l}+\bar{f}^{\alpha k}=\rho\left(2\frac{\partial\varphi}{\partial C_{\alpha\beta}}x^{k}_{,\beta}+\frac{\partial\varphi}{\partial C_{\alpha\beta\Phi}}x^{k}_{,\beta\Phi}+\frac{\partial\varphi}{\partial D_{\alpha n}}x^{k}_{n}+\frac{\partial\varphi}{\partial D_{\alpha n\beta}}x^{k}_{n,\beta}+\frac{\partial\varphi}{\partial D_{\alpha n\beta\Phi}}x^{k}_{n,\beta\Phi}\right),$$

$$p^{l\Phi k}X^{\alpha}_{,l}+p^{l\alpha\Phi k}_{,l}+\bar{f}^{\alpha\Phi k}=\rho\frac{\partial\varphi}{\partial C_{\beta\alpha\Phi}}x^{k}_{,\beta},$$

$$p^{lnk}X^{\alpha}_{,l}+p^{l\alpha nk}_{,l}+\bar{f}^{\alpha nk}=\rho\frac{\partial\varphi}{\partial D_{\beta n\alpha}}x^{k}_{,\beta},$$

$$p^{ln\Phi k}X^{\alpha}_{,l}+p^{ln\alpha\Phi k}_{,l}+\bar{f}^{n\alpha\Phi k}=\rho\frac{\partial\varphi}{\partial D_{\beta n\alpha\Phi}}x_{,\beta}{}^{k}$$

$$p^{lnk}_{,l}+\bar{f}^{nk}=\rho\frac{\partial\varphi}{\partial D_{\alpha n}}x^{k}_{,\alpha} \tag{4.6}$$

and

$$\eta=-\frac{\partial\varphi}{\partial\theta},\qquad \eta^{n}=-\frac{\partial\varphi}{\partial\theta_{n}},\qquad \eta^{\Xi}=-\frac{\partial\varphi}{\partial\theta_{,\Xi}},\qquad \eta^{n\Xi}=-\frac{\partial\varphi}{\partial\theta_{n,\Xi}}. \tag{4.7}$$

The equations (4.6) [similarly to (4.2)] have been derived with the notation of (2.9).

5. HEAT CONDUCTION

The equation of the heat conduction will be obtained by substituting the right-hand members of equations (4.7) in (3.2) and making use of the law of heat flow (3.7). Let us denote by ε_K the mechanical parameters of state taken in some order

$$\varepsilon_K=C_{\alpha\beta},\ C_{\alpha\beta\Phi},\ D_{\alpha n},\ D_{\alpha n\beta},\ D_{\alpha n\beta\Phi} \tag{5.1}$$

and by θ_M the 'thermal' parameters taken also in some order

$$\theta_M=\theta,\ \theta_n,\ \theta_{,\Xi},\ \theta_{n,\Xi}. \tag{5.2}$$

Then, the equation of the heat conduction is obtained in the following form

$$h^{i}_{,i}+\rho c^{M}\dot{\theta}_{M}+\rho\frac{\partial^{2}\varphi}{\partial\theta_{M}\partial\varepsilon_{K}}\theta_{M}\dot{\varepsilon}_{K}+\rho h=0, \tag{5.3}$$

where

$$c^{M}\equiv\frac{\partial^{2}\varphi}{\partial\theta_{M}\partial\theta_{N}}\theta_{N}.$$

The equations of motion (2.6), the equations of state (4.6) and the equation of the heat conduction (5.3) with the law (3.7) constitute the fundamental set of equations of thermoelasticity of a non-simple oriented body. In agreement with the remarks made in the Introduction, the quantities $x^k_{\mathfrak{n}}$ can be interpreted either as multipolar Green–Rivlin displacements (then $\mathfrak{n}=\alpha_{\mathrm{I}}, \alpha_{\mathrm{I}}\alpha_{\mathrm{II}}, \alpha_{\mathrm{I}}\alpha_{\mathrm{II}}\alpha_{\mathrm{III}}, \ldots$) or as Toupin directors (then $\mathfrak{n}$, $\mathfrak{b}$ are dead indices taking the values I, II, III, ...) which shows the equivalence of the two approaches. The same applies to the quantities $\theta_{\mathfrak{n}}$, which can be interpreted either as scalar fields $\theta_{\mathrm{I}}, \theta_{\mathrm{II}}, \ldots$ or tensor fields $\theta_{\alpha\mathrm{I}}, \theta_{\alpha\mathrm{II}}, \cdots$.

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Résumé—Les matériaux élastiques, non-simples, sont des matériaux dans lesquels l'énergie interne dépend de gradients de déformation, non seulement du premier ordre, mais aussi d'un ordre plus élevé [1–4]. Les particules de matériaux orientés possèdent plus de trois degrés de liberté, ce qui entraine l'introduction, dans la théorie des milieux continus, de champs de direction [3, 5, 6] ou de déplacements multi-polaires [6–8]. Dans la présente étude, on a établi les équations de thermoélasticité pour des matériaux hyper-élastiques, orientés, non-simples. La notion de matériau non-simple a été étendue aux problèmes thermiques en précisant que l'énergie specifique libre ne dépend pas seulement de la température et des gradients de déformation, mais aussi des gradients de température. La notion de matériau orienté est étendue aux problèmes thermiques en introduisant, indépendemment des degrés de liberté additionnels, des champs additionnels relatifs à la distribution de la température.

Zusammenfassung—Nichteinfache, elastische Stoffe sind Stoffe, deren innere Energie nicht nur von den ersten, sondern auch von den höheren Deformationsgradienten abhängt [1–4]. Die Partikel orientierter Stoffe besitzen mehr als drei Freiheitsgrade, und dies bedeutet, dass in die Theorie der kontinuierlichen Medien Direktorenfelder [3, 5, 6] oder multipolare Versetzungen [6–8] eingeführt werden müssen. In der vorliegenden Arbeit werden für orientierte, nichteinfache, hyperelastische Stoffe thermoelastische Gleichungen abgeleitet. Das Konzept eines nichteinfachen Stoffes wird auf thermische Probleme verallgemeinert und die spezifische freie Energie nicht nur von der Temperatur und den Deformationsgradienten, sondern auch von den Temperaturgradienten abhängig gemacht. Das Konzept eines orientierten Stoffes wird dadurch auf thermische Probleme verallgemeinert, indem, ganz unabhängig von den zusätzlichen Freiheitsgraden, weitere Felder eingeführt werden, welche die Temperaturverteilung beschreiben.

Sommario—I materiali elastici non semplici sono dei materiali in cui l'energia interna dipende dalla prima curva di deformazione, sí, ma anche da quelle piú alte [1–4]. Le particelle di materiali orientati hanno piú di tre gradi di libertà, il che equivale all'introduzione nella teoria di mezzi continui di campi di direttrici [3, 5, 6] o spostamenti multipolari [6–8]. In questa monografia si derivano equazioni di termoelasticità di materiali iperelastici non semplici orientati. La nozione di un materiale non semplice è generalizzata in problemi termici, facendo dipendere l'energia libera specifica non solo dalle curve della temperatura e della deformazione ma anche dalle curve di temperatura. La nozione del materiale orientato è generalizzata in problemi termici introducendo, indipendentemente dal grado addizionale di libertà, campi aggiuntivi che descrivono la distribuzione o ripartizione della temperatura.

Абстракт—Непростые упругие материалы это материалы, в которых внутренняя энергия зависит не только от перворазрядных, но и от более высоких градиентов деформаций [1–4]. Частицы ориентированных материалов обладают более чем тремя степенями свободы, что равнозначно введению в теорию полей сплошных сред указателей направления (директоров) [3, 5, 6] илм многополярных перемещений [6–8]. В настоящей статье выведены уравнения по термоупругости ориентированных, непростых, сверхупругих материалов. Понятие непростого материала обобщено на термические проблемы, вводя зависимость удельной свободной энергии не только от температуры и градиентов деформации, но также от градиентов температуры. Понятие ориентированного материала обобщено на термические проблемы путем введения, независимо от добавочных степеней свободы, также добавочных полей, отображающих распределение температур.