A central limit theorem for the Euler characteristic of a Gaussian excursion set

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Abstract: We study the Euler characteristic of an excursion set of a stationary isotropic Gaussian random field $X : \Omega \times \mathbb{R}^d \to \mathbb{R}$. Let us fix a level $u \in \mathbb{R}$ and let us consider the excursion set above u, $A(T, u) = \{t \in T : X(t) \geq u\}$ where T is a bounded cube $\subset \mathbb{R}^d$. The aim of this paper is to establish a central limit theorem for the Euler characteristic of $A(T, u)$ as T grows to $\mathbb{R}^d$, as conjectured by R. Adler more than ten years ago [3].

The required assumption on X is C^3 regularity of the trajectories, non degeneracy of the Gaussian vector $X(t)$ and derivatives at any fixed point $t \in \mathbb{R}^d$ as well as integrability on $\mathbb{R}^d$ of the covariance function and its derivatives. The fact that X is C^3 is stronger than Geman's assumption traditionally used in dimension one. Nevertheless, our result extends what is known in dimension one to higher dimension. In that case, the Euler characteristic of $A(T, u)$ equals the number of up-crossings of X at level u, plus eventually one if X is above u at the left bound of the interval T.

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Introduction

The Euler characteristic, also called Euler-Poincaré index, is one of the additive functionals that can be defined on the collection of all compact sets of $\mathbb{R}^d$. It describes (a part of) the topology. In dimension one, the Euler characteristic is the number of disjoint intervals constituting the compact set. Intuitively, in dimension two, the Euler characteristic equals the number of connected components minus the number of "holes" in the compact set. In dimension three, it equals the number of connected components minus the number of "handles" plus the number of "interior hollows".

We are interested in the Euler characteristic of an excursion set $A(T,u) = \{t \in T : X(t) \geq u\}$ for a real valued smooth stationary isotropic Gaussian field $X = \{X(t); t \in \mathbb{R}^d\}$, a bounded closed cube $T \subset \mathbb{R}^d$ and a level u. We denote it as $\chi(X,T,u)$ for a while. One should consider $\chi(X,T,u)$ as an extension in dimension greater than one of the very precious one-dimensional tool constituted by $U^X(T,u)$, the number of up-crossings at level u of X on the interval T. In 2000, Adler [3] conjectured that $\chi(X,T,u)$ satisfies a central limit theorem (CLT) as T grows to $\mathbb{R}^d$. We prove it in the present paper. In dimension one, a CLT result for $U^X(T,u)$ can be found in [12] Chapter 10.

Beyond the Euler characteristic, the study of the excursion sets and the level functionals of a stationary field is a very popular theme. Many authors were and still are interested in this domain as proved by the successful books of Adler & Taylor [5] and Azaïs & Wshebor [12], or the recent papers [24, 13, 9, 30] among others. On the one hand, the description of the excursion sets appears very powerful to characterize the field X. For instance since the first of Adler's books [2] one knows that the expectation of $\chi(X,T,u)$ is a good approximation for the probability of the maximum of X on T to be greater than u. Also the line integral with respect to the level curve at any level u provides information on the anisotropy property of X (see [14] and [19]). On the other hand, at least in the Gaussian case, accurate methods such as the theory of crossings can be used to get explicit values for level functionals (see the seminal work of Slud [29] and also the paper of Kratz and León [20]).

In the present paper, three types of tools are mixed. The first one is the Hermite expansion. It allows the L^2 expansion of $\chi(X,T,u)$ into stochastic integrals with respect to Hermite polynomials. We use it to prove the finiteness of the asymptotic variance. Our second tool is the Stein method, recently revisited in [28, 27, 25] for instance. With this tool, the CLT for $\chi(X,T,u)$ turns out to be a consequence of the asymptotic normality of each stochastic integral. This point is nothing but a continuous parameter version of the celebrated Breuer-Major theorem (see [8]). Precisely, we follow Nourdin et al. [25] but with some modifications motivated by the fact that our process has parameter in $\mathbb{R}^d$ instead of $\mathbb{Z}$. The last tool is more a toolbox: differential calculus in dimension > 1. Actually we have to consider the random vectorial field $\mathbf{X} = (\nabla X, \nabla^2 X, X)$, where $\nabla X(t)$ denotes the gradient vector of X at point t and $\nabla^2 X(t)$ denotes the $d(d+1)/2$-dimensional vector whose coordinates are equal to the second derivates of X at point t.

Our study for establishing a CLT for level functionals has many precursors in the literature. The first one that we can cite is Adler's work [1] using the Euler characteristic of an excursion set to build a spectral moment estimator for two-dimensional Gaussian fields. Following this direction, we have in mind statistical outcomes of our result. They could serve various fields of application such as brain exploration or representation of the universe following [33] or the nice introduction of the forthcoming book [6], as well as worn surfaces or more generally rough surfaces as proposed in [7, 31]. Our result could be used to get the asymptotic distribution of the statistic under the null hypothesis in a test of normality. Furthermore, it should also give a functional CLT for $u \mapsto \chi(X, T, u)$ as in [24, 31] where similar questions are studied. We also have in mind extensions to non-Gaussian or to non-stationary fields, starting from the recent results [24, 13, 15, 7].

Hypothesis on X.

Throughout the paper we deal with a centered stationary isotropic Gaussian field $X = \{X(t) : t \in \mathbb{R}^d\}$ such that $\mathrm{Var}(X(0)) = 1$. We also assume that almost every realization of X is of class C^3 on $\mathbb{R}^d$. This last hypothesis should certainly be weakened, but we use it in this form to make the computations as fluent as possible.
We write X_i and X_{ij} the derivatives of X of first and second order,
- $X'(t) = \nabla X(t)$ the d-dimensional vector $(X_i(t))_{1\leq i\leq d}$,
- $X"(t)$ the $d \times d$ Hessian matrix $(X_{ij}(t))_{1\leq i,j\leq d}$,
- $\nabla^2 X(t)$ the $d(d+1)/2$-dimensional vector $(X_{ij}(t))_{1\leq i\leq j\leq d}$.

Futhermore, we consider the Gaussian vectorial field $\mathbf{X} = (\nabla X, \nabla^2 X, X)$ with values in $\mathbb{R}^D$, $D = d + d(d+1)/2 + 1$.

Assumption (A1): *for any $t \in \mathbb{R}^d$, the covariance matrix of the random vector $\mathbf{X}(t)$ has full rank D.*

It is well know that for any fixed t, $X(t)$ and $\nabla X(t)$ are independent, as well as $\nabla X(t)$ and $\nabla^2 X(t)$. This yields that $\Sigma^{\mathbf{X}}$, the covariance matrix of $\mathbf{X}(t)$, is block-diagonal with one block of size d and the other one of size $d(d+1)/2+1$. Denoting by r the covariance function of X, $r(t) = \mathrm{Cov}(X(0), X(t))$, the regularity assumption on X implies that $r \in C^6(\mathbb{R}^d)$. For any multidimensional index $\mathbf{m} = (i_1, \ldots, i_k)$ with $1 \leq k \leq 6$ and $1 \leq i_j \leq d$, we write $\frac{\partial^{\mathbf{m}} r}{\partial t^{\mathbf{m}}}(t) = \frac{\partial^k r}{\partial t_{i_1} \ldots \partial t_{i_k}}(t) = r^{(k)}_{i_1 \cdots i_k}(t)$. Moreover, since X is isotropic, for any fixed t and any $1 \leq i \neq j \leq d$, $X_i(t)$ and $X_j(t)$ are independent and there exists a real number $\lambda \geq 0$ such that the first diagonal block of $\Sigma^{\mathbf{X}}$, namely the opposite of the Hessian matrix of r at point 0, equals λI_d. Assumption **(A1)** implies that $\lambda > 0$.

Since the field X and the level u will be fixed almost everywhere in the rest of the paper, we drop the dependence in our notations and, from now on, we write $\chi(T)$ instead of $\chi(X, T, u)$.

Outline of the article.

We are interested in the asymptotic as the cube T tends to $\mathbb{R}^d$, so we start our study without taking into account what happens on the boundary of T. Hence, in Section 1 instead of considering $\chi(T)$, the Euler characteristic of the excursion above u, we consider $\varphi(T)$, a modified quantity inspired by [5] Lemma 11.7.1, which we call modified Euler characteristic of the excursion above u. Roughly speaking, by applying Morse's theorem, both notions coincide on the interior of T. The precise definition of $\varphi(T)$ is given in Section 1.2 whereas the definition of $\chi(T)$ stands in Section 2.3.

Section 1 is devoted to the study of the L^2 properties of $\varphi(T)$. In Section 1.2, as a by product, we first establish some results on the second moment of $N^{\nabla X}(T,v) = \{t \in T : \nabla X(t) = v\}$ which are of interest for their own (see Proposition 1.1). Next, we state that the usual Kac's counting approximation formula for the number of roots of a vector field (named as "Metatheorem" in [5], Th.11.2.3) not only holds almost surely, but also in $L^2(\Omega)$. Taking the limit of the Hermite expansion of this approximation yields the expansion of $\varphi(T)$ in Section 1.3.

Section 2 deals with the main result, namely Theorem 2.6, which gives a CLT for $\chi(T)$ as $T \nearrow \mathbb{R}^d$. We first solve this question for $\varphi(T)$ in Section 2.2, after establishing in Section 2.1 that the asymptotic variance of $|T|^{-1/2}\varphi(T)$ is finite, where $|T|$ stands for the volume of T. Let us mention that it seems impossible to get an explicit formula for the asymptotic variance. We give an explicit lower bound as well as some tricks to compute the coefficients in the Appendix section. The asymptotic normality of $\varphi(T)$ is obtained through a Breuer-Major type argument. We prove it in our setting, in other words for a Gaussian process indexed by a d-dimensional continuous parameter (see Proposition 2.4). In Section 2.3 we use the theory of Morse to transfer the CLT from $\varphi(T)$ to $\chi(T)$.

Two technical proofs have been postponed in the Appendix section. The proof of Proposition 1.1 includes differential calculus and sharp estimates. The proof of Lemma 2.2 deals with specific Gaussian calculus, some of them inspired from the computation of $\mathbb{E}\varphi(T)$ in [5], Chapters 11.6 and 11.7. It shows how tricky the computations is as soon as one wants to obtain an explicit formula in this domain.

1. L^2 properties of $\varphi(T)$

1.1. L^2 approximation of $\varphi(T)$

Let $T = \Pi_{1\leq i\leq d}[a_i, b_i]$ be a bounded rectangle in $\mathbb{R}^d$ and let us consider the excursion set $A(T,u) = \{t \in T : X(t) \geq u\}$. As we shall see later on in this section and in Section 2.3, the study of the Euler characteristic of $A(T,u)$ yields to consider the number of stationary points of X within the excursion set $A(T,u)$. Therefore, as an auxiliary tool, we start with a result concerning the number of roots of ∇X.

We introduce the number of points in T where the vectorial random field ∇X reaches the value $v \in \mathbb{R}^d$. For $v = 0 \in \mathbb{R}^d$, it is nothing but the number of stationary points of X in T. For any $v \in \mathbb{R}^d$, we denote $N^{\nabla X}(T,v)$ the aforementioned random variable

$$N^{\nabla X}(T,v) = \#\{t \in T : \nabla X(t) = v\}.$$

Let us also define the following approximation sequence

$$N_{\varepsilon}^{\nabla X}(T,v)=\int_{T}|\det(X"(t))|\,\delta_{\varepsilon}(\nabla X(t)-v)dt\,,$$

where $\delta_{\varepsilon}=(2\varepsilon)^{-d}1_{[-\varepsilon,\varepsilon]^{d}}$. It is well known ([5] Th.11.2.3) that $N_{\varepsilon}^{\nabla X}(T,v)\underset{\varepsilon\to 0}{\to} N^{\nabla X}(T,v)$ almost surely and that $N^{\nabla X}(T,v)$ belongs to $L^{1}(\Omega)$. The next proposition states that it also belongs to $L^{2}(\Omega)$ and that the convergence also holds in $L^{2}(\Omega)$, assuming the random field X is sufficiently smooth.

Proposition 1.1 *Let X be a stationary isotropic Gaussian field with trajectories of class C^{3} that satisfies Assumption* **(A1)**. *Then*

1. *for any $v\in\mathbb{R}^{d}$, $N^{\nabla X}(T,v)\in L^{2}(\Omega)$,*
2. *$v\mapsto\mathbb{E}[(N^{\nabla X}(T,v))^{2}]$ is a continuous function on $\mathbb{R}^{d}$,*
3. *for any $v\in\mathbb{R}^{d}$, $N_{\varepsilon}^{\nabla X}(T,v)\underset{\varepsilon\to 0}{\to}N^{\nabla X}(T,v)$ in $L^{2}(\Omega)$.*

Proof. See the appendix section. □

Remark. In dimension $d=1$, the fact that the number of up-crossings of X at level u, namely $U^{X}(T,u)$, can be approximated in $L^{2}(\Omega)$ has already been established (see for instance Th.10.10 in [12]). The usual condition for this result to hold is $\int_{|t|\leq 1}\Theta(t)/|t|\,dt<+\infty$ where $\Theta(t)=r''(t)+\lambda$, known as Geman's assumption. It is weaker than assuming X has C^{3}-trajectories and turns out to be a necessary condition for $U^{X}(T,u)$ to belong to $L^{2}(\Omega)$ (see [21]).
In dimension $d>1$, since a long time, papers have been devoted to the study of higher moments of the number of roots of a stationary Gaussian field. In particular, thirty years ago, Elizarov gave a sufficient condition in [16]: the one dimensional integrals $\int_{0}^{\delta}(r_{11ii}^{(4)}(\tau,0,\ldots,0)-r_{11ii}^{(4)}(0,0,\ldots,0))\,\tau^{-1}\,d\tau$ have to be finite for all $i=1,\ldots,d$. Since in our present case, r is C^{6} in a neighborhood of 0 and since $r^{(5)}(0)=0$, the required condition is fulfilled. Despite this partial answer and another specific one in [23], we could not find any statement similar to Proposition 1.1. Actually our proof is inspired from [10] and [11].

We now give the definition of $\varphi(T)$, the modified Euler characteristic of the excursion set $A(T,u)$, by prescribing

$$\varphi(T)=\sum_{k=0}^{d}(-1)^{k}\mu_{k}(T)\,,\quad\text{where}$$

$$\mu_{k}(T)=\#\{t\in T:\ X(t)\geq u,\ \nabla X(t)=0,\ \mathrm{index}(X"(t))=d-k\}.$$

Here the "index" stands for the number of negative eigenvalues.

Proposition 1.2 *Let X be a stationary isotropic Gaussian field with trajectories of class C^{3} and satisfying Assumption* **(A1)**. *Then*

1. *$\varphi(T)\in L^{2}(\Omega)$*

2. *the following convergence holds almost surely and in $L^2(\Omega)$*

$$\varphi(T) = \lim_{\varepsilon \to 0^+} (-1)^d \int_T \det(X"(t)) 1_{[u,\infty)}(X(t))\, \delta_\varepsilon(\nabla X(t)) dt\,.$$

Proof. Let us introduce

$$\varphi(\varepsilon, T) = (-1)^d \int_T \det(X"(t)) 1_{[u,\infty)}(X(t))\, \delta_\varepsilon(\nabla X(t)) dt\ .$$

The almost sure convergence in the second point is contained in Theorem 11.2.3 of [5], so we only prove the first point and the convergence in $L^2(\Omega)$.
It is obvious that $|\varphi(\varepsilon, T)| \leq N_\varepsilon^{\nabla X}(T, 0)$ and Proposition 1.1 implies the convergence of $\mathbb{E}[(N_\varepsilon^{\nabla X}(T, 0))^2]$ towards $\mathbb{E}[(N^{\nabla X}(T, 0))^2]$. Then, the dominated convergence theorem allows us to conclude that

$$\mathbb{E}[\varphi^2(\varepsilon, T)] \to \mathbb{E}[\varphi^2(T)] \leq \mathbb{E}[(N^{\nabla X}(T, 0))^2] < +\infty.$$

We obtain as a bonus that $\varphi(T) \in L^2(\Omega)$. Furthermore, the above convergence of the $L^2(\Omega)$-norms and the a.s. convergence of $\varphi(\varepsilon, T)$ imply the $L^2(\Omega)$ convergence. $\square$

Remark: The first study in dimension $d > 1$ on the finiteness of the second order moment of the Euler characteristic can be found in the 40 years old article of Adler and Hasofer [4]. Much more recently, the fact that $\varphi(T)$ belongs to $L^2(\Omega)$ has been implicitly established in [32] under convenient assumptions.

1.2. Hermite type expansion of $\varphi(T)$

In the following, we use the Hermite polynomials $(H_n)_{n\in\mathbb{N}}$ defined by $H_n(x) = (-1)^n\, e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2/2})$. They provide an orthogonal basis of $L^2(\mathbb{R}, \phi(x)dx)$ where ϕ denotes the standard Gaussian density on $\mathbb{R}$. We also denote by ϕ_m the standard Gaussian density on $\mathbb{R}^m$.

In order to get an expansion of $\varphi(T)$ as stochastic integrals with respect to Hermite polynomials, as a first step, we establish the expansion of $\varphi(\varepsilon, T)$.
We identify any symmetric matrix of size $d \times d$ with the $d(d+1)/2$-dimensional vector containing the coefficients on and above the diagonal and write $\tilde{\det}$ the associated determinant map.
Let us recall that $D = d + d(d+1)/2 + 1$. We consider the map G_ε defined on $\mathbb{R}^D$ by

$$(x, y, z) \in \mathbb{R}^d \times \mathbb{R}^{d(d+1)/2} \times \mathbb{R} \,\mapsto\, G_\varepsilon(x, y, z) = \delta_\varepsilon(x)\, \tilde{\det}(y)\, 1_{[u,\infty)}(z)\,,$$

and the map f_u defined on $\mathbb{R}^{d(d+1)/2+1}$ by

$$(y, z) \in \mathbb{R}^{d(d+1)/2} \times \mathbb{R} \,\mapsto\, f_u(y, z) = \tilde{\det}(y)\, 1_{[u,\infty)}(z)\,.$$

On the other hand, we recall that $\Sigma^{\mathbf{X}}$ stands for the covariance matrix of the D-dimensional Gaussian vector

$$\mathbf{X}(t) = (\nabla X(t), \nabla^2 X(t), X(t))\,.$$

We choose Λ a $D \times D$ matrix such that $\Lambda\,{}^t\Lambda = \Sigma^{\mathbf{X}}$, where ${}^t\Lambda$ denotes the transpose of Λ. We can thus write, for any fixed $t \in \mathbb{R}^d$, $\mathbf{X}(t) = \Lambda Y(t)$ with $Y(t)$ a D-dimensional standard Gaussian vector. Since the matrix $\Sigma^{\mathbf{X}}$ is block diagonal with blocks of respective dimensions d and $d(d+1)/2+1$, Λ also factorizes into $\begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}$ with blocks of the same size as those of $\Sigma^{\mathbf{X}}$. Furthermore, we recall that X is isotropic and thus $\Lambda_1 = \sqrt{\lambda}\, I_d$ where $\lambda = -r^{(2)}_{ii}(0)$ for any $i = 1, \ldots, d$. For $y = (\underline{y}, \bar{y}) \in \mathbb{R}^D = \mathbb{R}^d \times \mathbb{R}^{D-d}$, we define

$$\widetilde{G}_\varepsilon(y) = G_\varepsilon(\Lambda y) = \delta_\varepsilon(\Lambda_1 \underline{y}) f_u(\Lambda_2 \bar{y}) = \delta_\varepsilon \circ \Lambda_1(\underline{y})\, f_u \circ \Lambda_2(\bar{y}). \tag{1}$$

Since the map $\widetilde{G}_\varepsilon$ clearly belongs to $L^2(\mathbb{R}^D, \phi_D(y)dy)$ the following expansion converges in this space

$$\widetilde{G}_\varepsilon(y) \;=\; \sum_{q=0}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}^D;\, |\mathbf{n}|=q} c(\widetilde{G}_\varepsilon, \mathbf{n}) \widetilde{H_{\mathbf{n}}}(y)\,,$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_D)$, $|\mathbf{n}| = n_1+n_2+\cdots+n_D$ and $\widetilde{H_{\mathbf{n}}}(y) = \prod_{1\le j\le D} H_{n_j}(y_j)$. The factorization wich appears in (1) induces a factorization of the Hermite coefficient into

$$c(\widetilde{G}_\varepsilon, \mathbf{n}) = c(\delta_\varepsilon \circ \Lambda_1, \underline{n})\, c(f_u \circ \Lambda_2, \bar{n})\,,$$

where for $\mathbf{n} = (\underline{n}, \bar{n}) \in \mathbb{N}^d \times \mathbb{N}^{D-d}$, the Hermite coefficients of the maps $\delta_\varepsilon \circ \Lambda_1$ and $f_u \circ \Lambda_2$ are given by

$$c(\delta_\varepsilon \circ \Lambda_1, \underline{n}) \;=\; \frac{1}{\underline{n}!} \int_{\mathbb{R}^d} \delta_\varepsilon(\sqrt{\lambda} y) \widetilde{H_{\underline{n}}}(y) \phi_d(y)\, dy \tag{2}$$

$$c(f_u \circ \Lambda_2, \bar{n}) \;=\; \frac{1}{\bar{n}!} \int_{\mathbb{R}^{D-d}} f_u(\Lambda_2 z) \widetilde{H_{\bar{n}}}(z) \phi_{D-d}(z)\, dz\,. \tag{3}$$

Writing $\varphi(\varepsilon, T)$ as $\varphi(\varepsilon, T) = \int_T \widetilde{G}_\varepsilon(Y(t))dt$ yields the following expansion

$$\varphi(\varepsilon, T) = (-1)^d \sum_{q=0}^{\infty} \sum_{\substack{\mathbf{n}=(\underline{n},\bar{n}) \\ |\mathbf{n}|=q}} c(\delta_\varepsilon \circ \Lambda_1, \underline{n})\, c(f_u \circ \Lambda_2, \bar{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\, dt. \tag{4}$$

We will take the limit as ε goes to 0 in (4) to obtain the expansion of $\varphi(T)$. We first compute the limit in (2):

$$c(\delta_\varepsilon \circ \Lambda_1, \underline{n}) \;\underset{\varepsilon\to 0}{\longrightarrow}\; \frac{1}{\underline{n}!} (2\pi\lambda)^{-d/2}\, \widetilde{H_{\underline{n}}}(\underline{0}) := d(\underline{n})\,. \tag{5}$$

In what follows, we introduce

$$a(\mathbf{n}) = d(\underline{n})\, c(f_u \circ \Lambda_2, \bar{n}) \;\text{ for }\; \mathbf{n} = (\underline{n}, \bar{n}) \in \mathbb{N}^d \times \mathbb{N}^{D-d}\,. \tag{6}$$

Proposition 1.3 *Let X be a stationary isotropic Gaussian field with C^3 trajectories that satisfies Assumption* **(A1)***. The following expansion holds in $L^2(\Omega)$*

$$\varphi(T) = (-1)^d \sum_{q=0}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\,dt.$$

Note that, according to Mehler's Formula (see [12] Lemma 10.7), if $|\mathbf{n}| \neq |\mathbf{m}|$ then $\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(s)), \widetilde{H_{\mathbf{m}}}(Y(t))) = 0$. The above expansion turns out to be orthogonal in $L^2(\Omega)$.

Proof. Let us take the formal limit of the right ha,d side of (4) and define the random variable

$$\eta(T) = (-1)^d \sum_{q=0}^{\infty} \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\,dt.$$

The first step consists in proving that $\eta(T)$ belongs to $L^2(\Omega)$. Let Q be a positive integer. Let us denote by π^Q the projection onto the first Q chaos in $L^2(\Omega)$ and by π_Q the projection onto the remaining ones, so that

$$\begin{aligned}
\pi^Q(\eta(T)) &= \sum_{q=0}^{Q} \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\,dt \\
\pi_Q(\eta(T)) &= \eta(T) - \pi^Q(\eta(T))\,.
\end{aligned}$$

Using the orthogonality relations between the Hermite polynomials, we have

$$\begin{aligned}
\mathbb{E}[\pi^Q(\eta(T))^2] &= \sum_{q=0}^{Q} \mathbb{E}[\,(\sum_{|\mathbf{n}|=q} a(\mathbf{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\,dt)^2\,] \\
&\leq \sum_{q=0}^{Q} \lim_{\varepsilon\to 0} \mathbb{E}[\,(\sum_{\substack{\mathbf{n}=(\underline{n},\bar{n})\\ |\mathbf{n}|=q}} c(\delta_\varepsilon \circ \Lambda_1, \underline{n})\, c(f_u \circ \Lambda_2, \bar{n}) \int_T \widetilde{H_{\mathbf{n}}}(Y(t))\,dt)^2] \\
&\leq \lim_{\varepsilon\to 0} \mathbb{E}[\varphi(\varepsilon, T)^2] = \mathbb{E}[\varphi(T)^2] \;<\; \infty,
\end{aligned}$$

where we have used Fatou's lemma in the second line, and (4) and Proposition 1.2 in the last one. Thus $\mathbb{E}[\eta(T)^2] < \infty$.
It remains to show that $\varphi(T) = \eta(T)$ in $L^2(\Omega)$. In the next lines we write $||\cdot||_2$ for the norm in $L^2(\Omega)$. Then, we have

$$\begin{aligned}
&||\varphi(T) - \eta(T)||_2 \\
\leq\;& ||\pi_Q(\varphi(T) - \eta(T))||_2 + ||\pi^Q(\varphi(T) - \varphi(\varepsilon, T))||_2 + ||\pi^Q(\varphi(\varepsilon, T) - \eta(T))||_2 \\
\leq\;& ||\pi_Q(\varphi(T))||_2 + ||\pi_Q(\eta(T))||_2 + ||\varphi(T) - \varphi(\varepsilon, T)||_2 + ||\pi^Q(\varphi(\varepsilon, T) - \eta(T))||_2\,.
\end{aligned}$$

The first two terms tend to 0 as $Q \to \infty$ because both functions $\varphi(T)$ and $\eta(T)$ belong to $L^2(\Omega)$, the third one tends to zero as $\varepsilon \to 0$ due to Prop.1.2, and the last term tends to zero as $\varepsilon \to 0$ for fixed Q due to (4), (5), (6) and the definition of $\eta(T)$. Hence by taking the limit as $\varepsilon \to 0$ and $Q \to \infty$, we get $\varepsilon(T) = \eta(T)$. $\square$

2. Central limit theorem for $\chi(T)$

In this section we will prove our main result, which consists in a central limit theorem for the Euler characteristic $\chi(T)$ of the excursion set $A(T, u)$ when T grows to $\mathbb{R}^d$. We will first concentrate on $\varphi(T)$ and will be interested in the asymptotics of $\zeta(T) = \frac{\varphi(T)-\mathbb{E}\varphi(T)}{|T|^{1/2}}$. To make it precise, we assume that the compact rectangle T has the following shape $T = [-N, N]^d$ with N a positive integer, and we let N go to infinity. We will prove that the random variable

$$\zeta([-N, N]^d)) = \frac{\varphi([-N, N]^d) - \mathbb{E}\varphi([-N, N]^d)}{(2N)^{d/2}}$$

converges in distribution to a centered Gaussian variable.

We need to introduce the following assumption.
Assumption (A2): Denoting $\psi(t) = \max\{\left|\frac{\partial^{\mathbf{m}} r}{\partial t^{\mathbf{m}}}(t)\right|; \mathbf{m} \in \{1, \dots, d\}^k, 0 \le k \le 4\}$,

$$\psi(t) \to 0 \text{ when } ||t|| \to +\infty\ ,\ \psi \in L^1(\mathbb{R}^d) \text{ and } \int_{\mathbb{R}^d} r(t)dt > 0\,.$$

Note that **(A2)** implies that $r \in L^q(\mathbb{R}^d)$ for all $q \ge 1$ and hence that X admits a spectral density f_X that is continuous. Moreover $f_X(0) = \int_{\mathbb{R}^d} r(t)dt > 0$.

2.1. Asymptotic variance of $\varphi(T)$

We start with a crucial result, which states that the variance of $\zeta([-N, N]^d))$ has a finite limit as N goes to infinity. As expected, the asymptotic variance depends on the level u.

Proposition 2.1 *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories and satisfying Assumptions* **(A1)-(A2)**. *For any level u,*

$$\mathrm{Var}\,(\zeta([-N, N]^d)) \underset{N\to+\infty}{\to} V(u) \text{ with } V(u) < +\infty.$$

Moreover, $V(u) \ge f_X(0)\,\lambda^d\,H_d(u)^2\,\phi(u)^2$.

Proof.
Starting from the Hermite type expansion of $\varphi([-N, N]^d)$ in $L^2(\Omega)$ as is given in Proposition 1.3, and using the orthogonality, we obtain

$$\mathrm{Var}\,(\zeta([-N, N]^d)) = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n})a(\mathbf{m})\,R^N(\mathbf{n},\mathbf{m}) \qquad (7)$$

with

$$\begin{aligned} R^N(\mathbf{n},\mathbf{m}) &= (2N)^{-d}\int_{[-N,N]^d}\int_{[-N,N]^d}\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(s)),\widetilde{H_{\mathbf{m}}}(Y(t)))\,dsdt \\ &= \int_{[-2N,2N]^d}\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(0)),\widetilde{H_{\mathbf{m}}}(Y(v)))\prod_{1\le k\le d}(1-\frac{|v_k|}{2N})\,dv. \end{aligned}$$

A slight generalization of Mehler's formula (see Lemma 10.7 in [12]) allows us to write for any $\mathbf{n},\mathbf{m}\in\mathbb{N}^D$ such that $|\mathbf{n}|=|\mathbf{m}|$,

$$\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(0)),\widetilde{H_{\mathbf{m}}}(Y(v))) = \sum_{\substack{d_{ij}\ge 0\\ \sum_i d_{ij}=n_j\,;\,\sum_j d_{ij}=m_i}} \mathbf{n}!\,\mathbf{m}!\prod_{1\le i,j\le D}\frac{(\Gamma^Y_{ij}(v))^{d_{ij}}}{(d_{ij})!}, \tag{8}$$

where $\Gamma^Y_{ij}(v)=\mathrm{Cov}(Y_i(0),Y_j(v))$. Since $\Gamma^Y(v)=\Lambda^{-1}\Gamma^{\mathbf{X}}(v)^t(\Lambda^{-1})$ with $\Gamma^{\mathbf{X}}$ the covariance function of $(\nabla X,\nabla^2X,X)$, we have for any $v\in\mathbb{R}^d$,

$$\sup_{1\le i,j\le D}|\Gamma^Y_{ij}(v)|\le K\,\psi(v),$$

where ψ has been introduced in Assumption **(A2)** and K is some positive constant. Hence, for $|\mathbf{n}|=|\mathbf{m}|=q$,

$$\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(0)),\widetilde{H_{\mathbf{m}}}(Y(v)))\ \le\ K'\,\psi^q(v),$$

with some positive constant K'. By Assumption **(A2)**, $\psi\in L^q(\mathbb{R}^d)$, so we can apply the dominated convergence theorem and get

$$R^N(\mathbf{n},\mathbf{m})\underset{N\to\infty}{\longrightarrow}R(\mathbf{n},\mathbf{m})=\int_{\mathbb{R}^d}\mathrm{Cov}(\widetilde{H_{\mathbf{n}}}(Y(0)),\widetilde{H_{\mathbf{m}}}(Y(v)))\,dv. \tag{9}$$

According to (7), we write $\mathrm{Var}\,(\zeta([-N,N]^d))=\sum_{q=1}^\infty V_q^N$ and we know that

$$V_q^N\underset{N\to\infty}{\longrightarrow}V_q:=\sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D\\ |\mathbf{n}|=|\mathbf{m}|=q}}a(\mathbf{n})a(\mathbf{m})\,R(\mathbf{n},\mathbf{m}). \tag{10}$$

Note that for any q, $V_q^N\ge 0$ and so $V_q\ge 0$. We will establish that

$$\sup_N\sum_{q=Q+1}^\infty V_q^N\underset{Q\to\infty}{\longrightarrow}0. \tag{11}$$

Using Fatou's lemma, it will prove that the series $V=\sum_{q=1}^\infty V_q$ is convergent and that $\mathrm{Var}\,(\zeta([-N,N]^d))$ tends to V. The first step of Proposition 2.1 will thus be achieved.
Let us remark that (11) is equivalent to $\mathrm{Var}\left(\pi_Q(\zeta([-N,N]^d))\right)\underset{Q\to\infty}{\longrightarrow}0$ uniformly with respect to N, where π_Q is the projection onto the terms of order $>Q$.

Let $\mathbf{s} \in \mathbb{R}^d$ and set $\theta_{\mathbf{s}}$ the shift operator associated with the field X, i.e. $\theta_{\mathbf{s}} X_\cdot = X_{\mathbf{s}+\cdot}$. Introducing the set of indices

$$\mathcal{I}_N = [-N, N)^d \cap \mathbb{Z}^d ,$$

we can write

$$\zeta([-N,N]^d) = \zeta([-N,N)^d) = (2N)^{-d/2} \sum_{\mathbf{s}\in\mathcal{I}_N} \theta_{\mathbf{s}} \circ \zeta([0,1)^d).$$

Then, denoting by $V_{N,Q}$ the variance $\mathrm{Var}\left(\pi_Q(\zeta([-N,N]^d))\right)$ and using the stationarity of X, we obtain

$$V_{N,Q} = (2N)^{-d} \sum_{\mathbf{s}\in\mathcal{I}_{2N}} \alpha_{\mathbf{s}}(N)\, \mathbb{E}\left(\pi_Q(\zeta([0,1)^d))\, \pi_Q(\theta_{\mathbf{s}} \circ (\zeta([0,1)^d)))\right),$$

where $\alpha_{\mathbf{s}}(N)$ denotes the cardinal of $\{\mathbf{t} \in \mathcal{I}_N : \mathbf{t} - \mathbf{s} \in \mathcal{I}_N\}$, which is certainly less than $(2N)^d$.

Let us choose a such that $\psi(\mathbf{s}) \le \rho < 1/K$ for $||\mathbf{s}||_\infty \ge a$. We split $V_{N,Q}$ into $V^1_{N,Q} + V^2_{N,Q}$ where in $V^1_{N,Q}$ the sum runs for the indices $\mathbf{s} \in \{\mathbf{s} \in \mathcal{I}_{2N} : ||\mathbf{s}||_\infty < a+1\}$ and in $V^2_{N,Q}$ for $\{\mathbf{s} \in \mathcal{I}_{2N} : ||\mathbf{s}||_\infty \ge a+1\}$.

At first, it holds for $2N > a+1$,

$$|V^1_{N,Q}| \le (2N)^{-d}\,(2a+2)^d\,(2N)^d\, \mathbb{E}\left(\pi_Q(\zeta([0,1)^d))^2\right)$$

which goes to 0 as Q goes to ∞ uniformly with respect to N.

Next, for any $\mathbf{s} \in \mathcal{I}_{2N}$ such that $||\mathbf{s}||_\infty \ge a+1$, we write

$$\begin{aligned}&\mathbb{E}\left(\pi_Q(\zeta([0,1)^d))\, \pi_Q(\theta_{\mathbf{s}} \circ (\zeta([0,1)^d)))\right)\\ &\qquad = \sum_{q=Q+1}^{\infty} \int_{[0,1]^d}\int_{[0,1]^d} \mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s}+u))]dtdu, \qquad (12)\end{aligned}$$

where

$$\mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s}+u))] = \mathbb{E}[\sum_{\mathbf{n}\in\mathbb{N}^D;|\mathbf{n}|=q} a(\mathbf{n})\widetilde{H}_{\mathbf{n}}(Y(t)) \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n})\widetilde{H}_{\mathbf{n}}(Y(\mathbf{s}+u))].$$

Arcones inequality ([8] Lemma 1) implies that

$$|\mathbb{E}[F_q(Y(t))F_q(Y(\mathbf{s}+u))]| \le K^q\, \psi^q(\mathbf{s}+u-t) \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n})^2\mathbf{n}!.$$

Let us remark that the series $\sum_{\mathbf{n}\in\mathbb{N}^D} a(\mathbf{n})^2\mathbf{n}!$ diverges so that we have to handle it carefully in what follows. Recall that equation (6) writes as $a(\mathbf{n}) = d(\underline{n})\, c(f_u \circ \Lambda_2, \overline{n})$ with $d(\underline{n})$ given by (5). In Imkeller et al. [18] it is shown that $\sup_x |\frac{H_l(x)}{\sqrt{l!}}\,\varphi(x)| \le C$ for a universal constant C. This yields $d^2(\underline{n})\underline{n}! \le C^d$ and hence

$$\sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n})^2\mathbf{n}! \le C^d\, q^d \sum_{|\overline{n}|\le q} c(f_u\circ\Lambda_2,\overline{n})^2\overline{n}! \le C^d\, q^d\, ||f_u\circ\Lambda_2||^2.$$

Therefore the absolute value of (12) can be bounded by

$$C^d\,||f_u\circ\Lambda_2||^2\sum_{q=Q+1}^{\infty}q^d\,K^q\int_{[0,1]^d}\int_{[0,1]^d}\psi^q(\mathbf{s}+u-t)\,dudt.$$

Hence

$$\begin{aligned}|V^2_{N,Q}| &\le C^d\,||f_u\circ\Lambda_2||^2\sum_{q=Q+1}^{\infty}q^d\,K^q\,\rho^{q-1}\\ &\quad\times\sum_{\mathbf{s}\in\mathcal{I}_{2N}\,;\,||\mathbf{s}||_\infty\ge a+1}\int_{[0,1]^d}\int_{[0,1]^d}\psi(\mathbf{s}+u-t)\,dudt,\end{aligned}$$

where we have used that for any $||\mathbf{s}||_\infty\ge a+1$ and $u,t\in[0,1]^d$, $\psi(\mathbf{s}+u-t)\le\rho$. On the one hand, since $\rho<1/K$, $\sum_{q=Q+1}^{\infty}q^dK^q\rho^{q-1}$ is the tail of a convergent series. On the other hand,

$$\begin{aligned}&\sum_{\mathbf{s}\in\mathcal{I}_{2N}\,;\,||\mathbf{s}||_\infty\ge a+1}\int_{[0,1]^d}\int_{[0,1]^d}\psi(\mathbf{s}+u-t)\,dudt\\ &\le\sum_{\mathbf{s}\in\mathcal{I}_{2N}}\int_{[-1,1]^d}\psi(\mathbf{s}+u)\,du\le 2\int_{\mathbb{R}^d}\psi(u)\,du<+\infty.\end{aligned}$$

Hence $\sup_N|V^2_{N,Q}|$ goes to 0 as Q goes to infinity and we have proved that $\mathrm{Var}\zeta([-N,N]^d)$ tends to

$$V=\sum_{q=1}^{\infty}\sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D\\|\mathbf{n}|=|\mathbf{m}|=q}}a(\mathbf{n})a(\mathbf{m})\,R(\mathbf{n},\mathbf{m})<+\infty\,. \tag{13}$$

The first assertion of Proposition 2.1 being established, it remains to prove that $V\ge f_X(0)\,\lambda^d\,H_d(u)^2\,\phi(u)^2$.
Actually, in the sum (13), each q-term is non-negative so that V is greater than the $q=1$ term. The next lemma, which is proved in the Appendix, allows us to conclude for the lower bound of V. $\square$

Lemma 2.2 *Let us denote by V_1 the term corresponding to $q=1$ in the sum* (13)*. Then*

$$V_1=V_1(u)=f_X(0)\,\lambda^d\,H_d(u)^2\,\phi(u)^2\,.$$

If one is interested in getting an explicit value for the asymptotic variance V, it is sufficient to combine equation (13) with equations (3),(5),(6) for $a(\mathbf{n})$ and equations (8),(9) for $R(\mathbf{n},\mathbf{m})$. Nevertheless, it seems very difficult to simplify the final expression. The previous lemma is a successfull attempt of simplification of the first term of the series (13). The tricky computations that we used in the proof, partially inspired by [5] Section 11.7, could be extended to the next terms in the series. Another attempt to get explicit formulae can be found in the working paper [17] that is concerned with dimension $d=2$.

2.2. Central Limit Theorem for $\varphi(T)$

Theorem 2.3 *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories and satisfying Assumptions* **(A1)-(A2)**. *As $N \nearrow +\infty$,*

$$\zeta([-N,N]^d) = \frac{\varphi([-N,N]^d) - \mathbb{E}\varphi([-N,N]^d)}{(2N)^{d/2}}$$

converges in distribution to a centered Gaussian variable with finite variance V given by (13).

Proof. By the proof of Proposition 2.1 (see (11)), we already know that

$$\sup_N \operatorname{Var}\left(\pi_Q(\zeta([-N,N]^d))\right) \underset{Q\to\infty}{\longrightarrow} 0\ .$$

So $\pi_Q(\zeta([-N,N]^d)) \to 0$ in $L^2(\Omega)$ when $N \to \infty$ and $Q \to \infty$ in this order. Hence, in order to establish the CLT for $\zeta([-N,N]^d)$, it is enough to show, for a fixed Q as N goes to infinity, the asymptotic normality of the sequence

$$\begin{aligned}\pi^Q(\zeta([-N,N]^d)) &= \zeta([-N,N]^d) - \pi_Q(\zeta([-N,N]^d)) \\ &= \frac{1}{(2N)^{d/2}} \int_{[-N,N]^d} \sum_{q=1}^{Q} G_q(Y(t))\,dt, \end{aligned} \tag{14}$$

where we have defined $G_q(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n})\widetilde{H_{\mathbf{n}}}(\mathbf{x})$. Note that Proposition 2.1 states that the asymptotic variance of $\pi^Q(\zeta([-N,N]^d))$ is finite. Then, the result follows from the classical Breuer-Major theorem (see for instance Arcones's paper [8]), although in this theorem the parameter set is $\mathbb{Z}$ whereas in our setting the parameter set is $\mathbb{R}^d$. □

For completeness, we give a statement and a proof of the Breuer-Major theorem, both adapted to our setting, namely Proposition 2.4 below. Our proof follows very closely the proof of the CLT in Nourdin et al. [25]. The main tool is the expansion into the Wiener-Itô chaos. Standard references for this matter are Nualart's [26] and Major's [22] books for instance.

Proposition 2.4 *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories that satisfies Assumptions* **(A1)-(A2)**. *For any fixed positive integer Q, as $N \nearrow +\infty$, $\pi^Q(\zeta([-N,N]^d))$ converges in distribution to a centered Gaussian variable with finite variance $\sigma_Q^2 = \sum_{q=1}^{Q} V_q$, where the V_q's are introduced in* (10).

Proof.
We will write $\pi^Q(\zeta([-N,N]^d))$ as a sum of multiple Wiener-Itô integrals of order q ($1 \le q \le Q$). With this goal in mind, we remember that for any $t \in \mathbb{R}^d$, $Y(t)$ is a D-dimensional standard Gaussian vector given by $Y(t) = \Lambda^{-1}\mathbf{X}(t)$,

where $\mathbf{X}$ is the D-dimensional vector field $\mathbf{X} = (\nabla X, \nabla^2 X, X)$. Assumption **(A2)** yields that X admits a spectral density f_X and so the following spectral representation holds

$$X(t) = \int_{\mathbb{R}^d} e^{i<t,\lambda>} \sqrt{f(\lambda)}\, dW(\lambda),\ t \in \mathbb{R}^d\,,$$

where W is a complex Brownian measure on $\mathbb{R}^d$.
With any $\lambda = (\lambda_1, \ldots, \lambda_d)$ in $\mathbb{R}^d$ we associate the vector $\nu(\lambda)$ in $\mathbb{C}^D$, defined by

$$\nu(\lambda) = ((i\lambda_j)_{1\le j\le d}, (-\lambda_j\lambda_k)_{1\le j\le k\le d}, 1)\ ,$$

so that we can write the next D-dimensional spectral representation

$$Y(t) = \Lambda^{-1}\mathbf{X}(t) = \int_{\mathbb{R}^d} e^{i<t,\lambda>} \sqrt{f(\lambda)} \left(\Lambda^{-1}\nu(\lambda)\right)\, dW(\lambda),\ t \in \mathbb{R}^d.$$

In what follows, for any $t \in \mathbb{R}^d$ and any $j = 1, \ldots, D$, we denote by $\varphi_{t,j}$ the square integrable map on $\mathbb{R}^d$ such that

$$Y_j(t) = \int_{\mathbb{R}^d} \varphi_{t,j}(\lambda) dW(\lambda)\,.$$

Let us remark that $(\varphi_{t,j})_{1\le j\le D}$ is an orthogonal system in $L^2(\mathbb{R}^d)$ since $Y(t)$ is a standard Gaussian vector. Then Itô's formula for multiple Wiener-Itô integrals ([22] Th.4.3) allows us to write for any $\mathbf{n} = (n_1, \ldots, n_D) \in \mathbb{N}^D$ such that $|\mathbf{n}| = n_1 + \cdots + n_D = q$,

$$\begin{aligned}
\widetilde{H}_{\mathbf{n}}(Y(t)) &= \prod_{1\le j\le D} H_{n_j}(Y_j(t)) \\
&= \int_{\mathbb{R}^{dq}} \left(\varphi_{t,1}^{\otimes n_1} \otimes \cdots \otimes \varphi_{t,D}^{\otimes n_D}\right)(\lambda_1, \ldots, \lambda_q) dW(\lambda_1) \ldots dW(\lambda_q) \\
&= I_q(\varphi_{t,1}^{\otimes n_1} \otimes \cdots \otimes \varphi_{t,D}^{\otimes n_D})\,, \qquad (15)
\end{aligned}$$

where I_q stands for the Wiener-Itô integral of order q. We shall use the following property of I_q,

$$\forall h \in L^2((\mathbb{R}^d)^q)\, s.t. \forall x \in (\mathbb{R}^d)^q,\ h(-x) = \bar{h}(x),\ I_q(h) = I_q(\mathrm{Sym}(h))\,,$$

where $\mathrm{Sym}(h)$ stands for the symmetrization of h,

$$\mathrm{Sym}(h)(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma\in S_q} h(x_{\sigma(1)}, \ldots, x_{\sigma(q)})\,,\ x_1, \ldots, x_q \in \mathbb{R}^d$$

with S_q the group of all permutations of $\{1, \ldots, q\}$.
For $\mathbf{n} = (n_1, \ldots, n_D) \in \mathbb{N}^D$ such that $|\mathbf{n}| = q$, we define

$$\mathcal{A}_{\mathbf{n}} = \{\mathbf{m} \in \{1, \ldots, D\}^q : \sum_{j=1}^{q} 1_{\{i\}}(m_j) = n_i\,, \forall i = 1, \ldots, D\},$$

and remark that the family $(\mathcal{A}_{\mathbf{n}})_{\mathbf{n}\in\mathbb{N}^D;|\mathbf{n}|=q}$ provides a partition of $\{1,2,\dots,D\}^q$. It allows us the following notation

$$\text{for } \mathbf{m}\in\mathcal{A}_{\mathbf{n}}\,, \quad \text{let} \quad b_{\mathbf{m}} = \frac{1}{Card\mathcal{A}_{\mathbf{n}}}\, a(\mathbf{n})\,,$$

with $Card\mathcal{A}_{\mathbf{n}}$ standing for the cardinal of $\mathcal{A}_{\mathbf{n}}$. So $\mathbf{m}\mapsto b_{\mathbf{m}}$ is symmetric on $\{1,2,\dots,D\}^q$. Let us also remark that for any $\mathbf{m}=(m_1,\dots,m_q)\in\mathcal{A}_{\mathbf{n}}$,

$$\mathrm{Sym}(\varphi_{t,1}^{\otimes n_1}\otimes\cdots\otimes\varphi_{t,D}^{\otimes n_D}) = \mathrm{Sym}(\varphi_{t,m_1}\otimes\varphi_{t,m_2}\dots\otimes\varphi_{t,m_q})$$

and so $I_q(\varphi_{t,1}^{\otimes n_1}\otimes\cdots\otimes\varphi_{t,D}^{\otimes n_D}) = I_q(\varphi_{t,m_1}\otimes\varphi_{t,m_2}\dots\otimes\varphi_{t,m_q})$. We use this last identity and (15) to compute $G_q(Y(t))$ as follows

$$\begin{aligned} G_q(Y(t)) &= \sum_{\mathbf{n}\in\mathbb{N}^D;\,|\mathbf{n}|=q} a(\mathbf{n}) I_q(\varphi_{t,1}^{\otimes n_1}\otimes\cdots\otimes\varphi_{t,D}^{\otimes n_D}) \\ &= \sum_{\mathbf{m}\in\{1,2,\dots,D\}^q} b_{\mathbf{m}} \sum_{\mathbf{m}\in\mathcal{A}_{\mathbf{n}}} I_q(\varphi_{t,m_1}\otimes\dots\otimes\varphi_{t,m_q})\,. \end{aligned}$$

Fubini's theorem for multiple Wiener integrals applied to formula (14) yields

$$\pi^Q(\zeta([-N,N]^d)) = \sum_{q=1}^{Q} I_q(g_q^N)\,, \quad \text{where}$$

$$g_q^N = \frac{1}{(2N)^{d/2}}\int_{[-N,N]^d} \sum_{\mathbf{m}\in\{1,2,\dots,D\}^q} b_{\mathbf{m}}\ \varphi_{t,m_1}\otimes\dots\otimes\varphi_{t,m_q}\, ds\,.$$

This expression corresponds with equation (4.43) in [25]. Hence, according to (4.47) of [25], if h is a twice differentiable bounded map with bounded derivatives and if Z_Q is a centered Gaussian random variable with variance equal to $\sigma^2=\sum_{q=1}^Q V_q$, then

$$\begin{aligned} &|\,\mathbb{E}[h(Z_Q)] - \mathbb{E}[h(\pi^Q(\zeta([-N,N]^d)))]\,| \\ &\qquad \le \frac{||h''||_\infty}{2}\sum_{p,q=1}^{Q} ||\delta_{pq}V_p - \frac{1}{q} < \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||_{L^2(\Omega)}, \end{aligned} \tag{16}$$

where δ_{pq} denotes the Kronecker symbol and $\mathbf{D}$ denotes the Malliavin derivative (see [25] for its definition). We are now in position of proving the CLT for $\pi^Q(\zeta([-N,N]^d))$: it is sufficient to establish that the right hand side of (16) tends to 0.

First step: it consists in considering the terms corresponding to $p=q$ and establishing

$$||V_q - \frac{1}{q}\ < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||_{L^2(\Omega)} \to 0. \tag{17}$$

Let us compute the expectation of the term within the brackets:

$$\begin{aligned}
&\mathbb{E}\left(\frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}}\right) = q!||g_q^N||^2_{\mathcal{H}^{\otimes q}}\\
=\ & \frac{q!}{(2N)^d}\int_{[-N,N]^d}\int_{[-N,N]^d}\sum_{\mathbf{m},\mathbf{l}\in\{1,2,\dots,D\}^q} b_{\mathbf{m}}b_{\mathbf{l}}\prod_{j=1}^q \Gamma^Y_{m_j\, l_j}(s_1-s_2)ds_1ds_2\\
=\ & \frac{1}{(2N)^d}\int_{[-N,N]^d}\int_{[-N,N]^d}\sum_{\substack{\mathbf{n},\mathbf{n'}\in\mathbb{N}^D\\ |\mathbf{n}|=|\mathbf{n'}|=q}} a(\mathbf{n})a(\mathbf{n'})\operatorname{Cov}(\widetilde{H_{\mathbf{n}}}(Y(s_1)),\widetilde{H_{\mathbf{n'}}}(Y(s_2)))ds_1ds_2.
\end{aligned}$$

The last line is equal to V_q^N and we already proved that $V_q^N \to V_q$. So

$$\mathbb{E}\left(\frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}}\right) \ \to\ V_q\,.$$

We now turn to the $L^2(\Omega)$ convergence. We note that

$$||V_q - \frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||^2_{L^2(\Omega)}$$

$$\leq ||\frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||^2_{L^2(\Omega)} + \left(V_q - \mathbb{E}(\frac{1}{q} < \mathbf{D}I_q(g_q^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}})\right)^2 .$$

We already know that the second term tends to 0. The fact that the first term also tends to 0 is actually included in the next step ($p = q$ case).

Second step: it consists in showing that for $q \geq p$,

$$||\frac{1}{q} < \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||^2_{L^2(\Omega)} \to 0. \tag{18}$$

Formula (3.36) of [25] implies

$$\begin{aligned}
& ||\frac{1}{q} < \mathbf{D}I_p(g_p^N), \mathbf{D}I_q(g_q^N) >_{\mathcal{H}} ||^2_{L^2(\Omega)}\\
\leq\ & p!\binom{q-1}{p-1}^2 (q-p)!\,\mathbb{E}[I_p(g_p^N)]^2 ||g_q^N \otimes_{q-p} g_q^N||_{\mathcal{H}^{\otimes 2p}}\\
+\ & \frac{p^2}{2}\sum_{l=1}^{p-1}(l-1!)^2\binom{p-1}{l-1}^2\binom{q-1}{l-1}^2(p+q-2l)!\\
& (||g_p^N \otimes_{p-l} g_p^N||_{\mathcal{H}^{\otimes 2l}} + ||g_q^N \otimes_{q-l} g_q^N||_{\mathcal{H}^{\otimes 2l}}),
\end{aligned}$$

where for $e < p$,

$$\begin{aligned}
g_p^N \otimes_e g_p^N \ =\ & \frac{1}{(2N)^d}\int_{[-N,N]^d\times[-N,N]^d}\sum_{\mathbf{m},l\in\{1,\dots,D\}^p} b_{\mathbf{m}}b_{\mathbf{l}}\prod_{j=1}^e \Gamma^Y_{m_j\, l_j}(s_1-s_2)\\
& \times \varphi_{s_1,m_{e+1}}\otimes\ldots\otimes\varphi_{s_1,m_p}\otimes\varphi_{s_2,l_{e+1}}\otimes\ldots\otimes\varphi_{s_2,l_p}ds_1ds_2\,.
\end{aligned}$$

In this form, defining $I(N) = [-N,N]^d \times [-N,N]^d \times [-N,N]^d \times [-N,N]^d$, we get

$$||g_p^N \otimes_e g_p^N||^2_{\mathcal{H}^{\otimes 2(p-e)}} \leq \big(D^p \sum_{\mathbf{m}\in\{1,\ldots,D\}^p} |b_{\mathbf{m}}|^2\big)^2 \mathcal{Z}(N)\,,$$

with

$$\mathcal{Z}(N) = \frac{1}{(2N)^{2d}} \int_{I(N)} \psi^e(s_1-s_2)\psi^e(s_3-s_4)\psi^{p-e}(s_1-s_3)\psi^{p-e}(s_2-s_4) ds_1 ds_2 ds_3 ds_4\,.$$

Moreover we have $\psi^e(s_3-s_4)\psi^{p-e}(s_1-s_3) \leq \psi^p(s_3-s_4) + \psi^p(s_1-s_3)$. Thus we can write $\mathcal{Z}(N) \leq \mathcal{Z}_1(N) + \mathcal{Z}_2(N)$ where

$$\mathcal{Z}_1(N) \leq \frac{1}{(2N)^{2d}} \int_{I(N)} \psi^e(s_1-s_2)\psi^p(s_3-s_1)\psi^{p-e}(s_2-s_4) ds_1 ds_2 ds_3 ds_4.$$

The inner integral $\int_{[-N,N]^d} \psi^p(s_3-s_1)ds_3$ is less than $\int_{\mathbb{R}^d} \psi^p(v)dv < +\infty$, and for the remaining terms

$$\begin{aligned} &\frac{1}{(2N)^{2d}} \int_{[-N,N]^d[\times -N,N]^d\times[-N,N]^d} \psi^e(s_1-s_2)\psi^{p-e}(s_2-s_4)ds_1 ds_2 ds_4 \\ &\quad \leq \frac{1}{(2N)^{2d}} \int_{[-N,N]^d[\times[-N,N]^d} \psi^e * \psi^{p-e}(s_1-s_4) ds_1 ds_4 \\ &\quad\quad \leq \frac{1}{(2N)^d} \int_{\mathbb{R}^d} \psi^e * \psi^{p-e}(s)ds \to 0. \end{aligned}$$

The term $\mathcal{Z}_2(N)$ can be treated similarly, yielding $||g_p^N \otimes_e g_p^N||^2_{\mathcal{H}^{\otimes 2(p-e)}} \to 0$. Hence (18) holds in force. Together with (17), it implies that (16) tends to zero. □

The same proof can be used to get the next result, which deals with a collection of levels. Let us emphasize that the coefficients $a(\mathbf{n})$ appearing in the asymptotic variance given by (13) do depend on the level u (see (6)). We denote them as $a(\mathbf{n},u)$ in the next theorem.

Theorem 2.5 *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories and satisfying Assumptions* **(A1)-(A2)***. For any level u, we denote*

$$\zeta([-N,N]^d,u) = \frac{\varphi([-N,N]^d,u) - \mathbb{E}\varphi([-N,N]^d,u)}{(2N)^{d/2}}\,.$$

Let $u_1,\ldots,u_K$ be K fixed levels in $\mathbb{R}$. As $N \nearrow +\infty$, the random vector

$$\big(\zeta([-N,N]^d,u_1),\ldots,\zeta([-N,N]^d,u_K)\big)$$

converges in distribution to a centered Gaussian vector with covariance matrix $(C(u_i,u_j))_{1\leq i,j\leq K}$ given by

$$C(u,v) = \sum_{q=1}^{\infty} \sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=q}} a(\mathbf{n},u)a(\mathbf{m},v)\,R(\mathbf{n},\mathbf{m})\,. \tag{19}$$

2.3. Morse's theory and Central Limit Theorem for $\chi(T)$

We follow the presentation of Adler & Taylor's book [5] Section 9.4, inspired by Morse's theorem, to give a precise definition of $\chi(T)$, the Euler characteristic of the excursion set $A(T,u)$.

We still work with $T=[-N,N]^d$ and for $\ell=0,1,\ldots,d$, we denote by $\partial_\ell T$ the collection of all the ℓ-dimensional faces of T. In particular, $\partial_d T$ only contains the interior $\overset{\circ}{T}=(-N,N)^d$. Each ℓ-dimensional face J of T is associated with a cardinal ℓ subset $\sigma(J)$ of $\{1,\ldots,d\}$ and a sequence $(\varepsilon_j)_{j\in\{1,\ldots,d\}\setminus\sigma(J)}$ in $\{-1,+1\}^{d-\ell}$ such that

$$J=\{v\in T:\,-N<v_j<N\,\text{for}\,j\in\sigma(J),\,v_j=\varepsilon_j N\,\text{for}\,j\notin\sigma(J)\}. \tag{20}$$

The Euler characteristic of $A(T,u)$ can be computed as

$$\chi(T)=\sum_{0\le\ell\le d}\ \sum_{J\in\partial_\ell T}\widetilde{\varphi}(J)\ \text{ with }\ \widetilde{\varphi}(J)=\sum_{k=0}^{\ell}(-1)^k\widetilde{\mu}_k(J)$$

where for any ℓ-dimensional face J of T

$$\begin{aligned}\widetilde{\mu}_k(J)\ \ =\ \ &\#\{v\in J:\,X(v)\ge u,\,X_j(v)=0\text{ for }j\in\sigma(J),\\ &\varepsilon_j X_j(v)>0\text{ for }j\notin\sigma(J),\ \operatorname{index}((X_{ij}(v))_{i,j\in\sigma(J)})=\ell-k\}.\end{aligned}$$

Let us remark that the above formula can be written as $\chi(T)=\sum_{0\le\ell<d}\ \sum_{J\in\partial_\ell T}\widetilde{\varphi}(J)+\widetilde{\varphi}(\overset{\circ}{T})$. Moreover, Bulinskaya Lemma (Lemma 11.2.10 of [5]) entails that with probability one there is no point t in the boundary set ∂T satisfying $\nabla X(t)=0$, then $\widetilde{\mu}_k(\overset{\circ}{T})=\mu_k(T)$. Hence, comparing with the definition of $\varphi(T)$ that is given in Section 1.2, we obtain $\widetilde{\varphi}(\overset{\circ}{T})=\varphi(T)$. Therefore, we now write

$$\chi(T)=\sum_{0\le\ell<d}\ \sum_{J\in\partial_\ell T}\widetilde{\varphi}(J)\,+\,\varphi(T). \tag{21}$$

Recall that we want to establish that $\chi(T)$ satisfies a central limit theorem. More precisely we will prove that $|T|^{-1/2}(\chi(T)-\mathbb{E}\chi(T))$ converges in distribution to a centered Gaussian random variable as T grows to $\mathbb{R}^d$. Recall also that Proposition 2.3 already provides a CLT for $\varphi(T)$. So, according to (21), there only remains to prove that for any $\ell=0,1,\ldots,d-1$ and any face J in $\partial_\ell T$, the variance of $|T|^{-1/2}\widetilde{\varphi}(J)$ tends to 0 as T grows to $\mathbb{R}^d$.

For $\ell=0$, the previous statement is obvious since $\widetilde{\varphi}(\{v\})$ is either 0 or 1 for any vertex v of T.

Let us now be concerned with $\ell\in\{1,\ldots,d-1\}$. We deal with a fixed face $J\in\partial_\ell T$ and we use (20) to introduce the following notations.

• For any $v\in\mathbb{R}^\ell$, we define $v^{(J)}=(v_1^{(J)},\ldots,v_d^{(J)})\in\mathbb{R}^d$ by

$$v_j^{(J)}=v_j\ \text{ if }\ j\in\sigma(J)\ ;\ v_j^{(J)}=\varepsilon_j N\ \text{ if }\ j\notin\sigma(J)\,.$$

• A random field $X^{(J)}$ is defined on $\mathbb{R}^\ell$ by

$$X^{(J)}(v) = X(v^{(J)}) \text{ for any } v \in \mathbb{R}^\ell .$$

It clearly inherits the properties of X so that $X^{(J)}$ is Gaussian, stationary, isotropic, centered and its trajectories are a.s. of class C^3.

The same arguments as in Section 2 apply for $X^{(J)}$ instead of X. A statement similar to Proposition 2.1 can thus be formulated:

$$\mathrm{Var}((2N)^{-\ell/2}(\widetilde{\varphi}(J) - \mathbb{E}\widetilde{\varphi}(J))) \underset{N\to+\infty}{\to} V^{(J)} < +\infty.$$

Hence the variance of $\widetilde{\varphi}(J)$ is negligible with respect to $(2N)^{-d}$ and we are finally able to state our main result.

Theorem 2.6 *Let X be a stationary isotropic Gaussian field indexed by $\mathbb{R}^d$ with C^3 trajectories that satisfies Assumptions* **(A1)-(A2)***. As $N \nearrow +\infty$,*

$$\frac{\chi([-N,N]^d) - \mathbb{E}\chi([-N,N]^d)}{(2N)^{d/2}}$$

converges in distribution to a centered Gaussian variable with finite variance V given by (13).

Appendix A: Proofs

A.1. Proof of Proposition 1.1

We recall the following Rice's formulas (see [5] Chapter 11 or [12] Chapter 6) for the first two factorial moments, in the case of a Gaussian stationary random field,

$$\mathbb{E}[N^{\nabla X}(T,v)] = |T|\,\mathbb{E}\left(|\det(X"(0))|\right)\,p_0(v) \tag{22}$$

$$\begin{aligned}\mathbb{E}[N^{\nabla X}(T,v)(N^{\nabla X}(T,v)-1)] &= \int_{T_0} |T\cap(T-s)|\,\mathbb{E}\left(|\det(X"(0))\,\det(X"(s))|\right.\\ &\qquad \left./\,\nabla X(0) = \nabla X(s) = v\right)\,p_{0,s}(v,v)ds\,, \end{aligned}\tag{23}$$

where $p_t(.)$ and $p_{t,s}(.,.)$ are the probability density functions of $\nabla X(t)$ and $(\nabla X(t), \nabla X(s))$ respectively and T_0 denotes the rectangle around 0 obtained from $T = \Pi_{1\le j\le d}[a_j, b_j]$ by prescribing $T_0 = \Pi_{1\le j\le d}[a_j - b_j, b_j - a_j]$. Note that in (23), both sides are simultaneously finite or infinite.

<u>Point *1.*</u> We shall establish that the right hand side of (23) is finite. It is clearly sufficient to focus on the behavior near 0 of the integrand.
Let us start with an upper bound for $p_{0,t}(v,v)$ for t in a neighborhood of 0. The vector $(\nabla X(0), \nabla X(t))$ is a $2d$ centered Gaussian vector. Let us denote

by $\Gamma^{\nabla X}(t)$ its covariance matrix. It can be written with blocks of size $d \times d$, $\Gamma^{\nabla X}(t) = \begin{pmatrix} \lambda\, I_d & -r"(t) \\ -r"(t) & \lambda\, I_d \end{pmatrix}$, and its determinant is given by

$$\begin{aligned} \det(\Gamma^{\nabla X}(t)) &= \det(\lambda^2 I_d - (r"(t))^2) = \det(2\lambda I_d - \Theta(t))\, \det(\Theta(t)) \\ &\sim (2\lambda)^d\, ||t||^{2d} \text{ as } ||t|| \to 0. \end{aligned}$$

Therefore, for a certain constant C,

$$\forall v \in \mathbb{R}^d\,,\; p_{0,t}(v,v) \le p_{0,t}(0,0) \le C\, ||t||^{-d}\,. \tag{24}$$

We now introduce the event $\mathcal{C}(v,t) = \{\nabla X(0) = \nabla X(t) = v\}$ and turn to the study of

$$g(v,t) := \mathbb{E}\left(|\det(X"(0))\det(X"(t))|\,/\,\mathcal{C}(v,t)\right),$$

for t and v fixed in $\mathbb{R}^d$ with $0 < ||t|| \le 1$. Using Cauchy-Schwarz inequality and the stationarity of X, we obtain

$$\begin{aligned} g(v,t)^2 &\le \mathbb{E}\left([\det(X"(0))]^2\,/\,\mathcal{C}(v,t)\right)\, \mathbb{E}\left([\det(X"(t))]^2\,/\,\mathcal{C}(v,t)\right) \\ &= h(v,t)\, h(v,-t) = h(v,t)^2\,, \end{aligned}$$

with $h(v,t) = \mathbb{E}\left(\det(X"(0)^2)\,/\,\mathcal{C}(v,t)\right)$.
We use the following result: *If A is a $d \times d$ symmetric positive matrix and if v_1 is a vector in $\mathbb{R}^d$ with norm 1, then* $\det(A) \le \langle Av_1, v_1\rangle\, \det\left((\langle Av_i, v_j\rangle)_{2\le i,j\le d}\right)$, *where $(v_1, v_2, \ldots, v_d)$ is any orthonormal basis of $\mathbb{R}^d$ containing v_1.*
We apply this inequality to the positive matrix $A = X"(0)^2$ and the vector $v_1 = t/||t||$. It yields, for a positive constant C,

$$\begin{aligned} \det(X"(0)^2) &\le \langle X"(0)^2 \frac{t}{||t||}, \frac{t}{||t||}\rangle\, \det\left((\langle X"(0)^2 v_i, v_j\rangle)_{2\le i,j\le d}\right) \\ &\le C\, ||t||^{-2}\, ||X"(0)t||^2\, ||X"(0)||^{2(d-1)}\,. \end{aligned} \tag{25}$$

Let us now introduce $Y : [0,1] \to \mathbb{R}^d$ the vectorial process defined by $Y(x) = X'(xt)$. Hence $Y(0) = \nabla X(0)$, $Y(1) = \nabla X(t)$, $Y'(0) = X"(0)t$, and for $i = 1, \ldots, d$, $Y"_i(x) =< X^{(3)}_{i.}(xt)t, t >$. Writing a Taylor formula between 0 and 1 gives $Y(1) = Y(0) + Y'(0) + \int_0^1 Y"(x)(1-x)dx$, and so

$$X"(0)t = \nabla X(t) - \nabla X(0) - \int_0^1 \langle X^{(3)}(xt)t, t\rangle (1-x)dx\,.$$

Then, under $\mathcal{C}(v,t)$, $X"(0)t = -\int_0^1 \langle X^{(3)}(xt)t, t\rangle(1-x)dx$. Introducing this identity within (25) and using Cauchy-Schwarz inequality, we get
$\mathbb{E}\left(\det(X"(0)^2)\,/\,\mathcal{C}(v,t)\right)$

$$\begin{aligned} &\le C\, ||t||^{-2}\, \mathbb{E}(||X"(0)||^{4(d-1)}\,/\,\mathcal{C}(v,t))^{1/2} \\ &\quad \times \mathbb{E}(||\int_0^1 \langle X^{(3)}(xt)t, t\rangle(1-x)dx||^4\,/\,\mathcal{C}(v,t))^{1/2} \\ &\le C\, ||t||^2\, \mathbb{E}(||X"(0)||^{4(d-1)}\,/\,\mathcal{C}(v,t))^{1/2}\, \mathbb{E}(\sup_{x\in[0,1]} ||X^{(3)}(xt)||^4\,/\,\mathcal{C}(v,t))^{1/2}\,. \end{aligned}$$

In the next lemma, we will prove that both conditional expectations are bounded. We will thus obtain

$$\mathbb{E}\left(\det(X''(0)^2)\,/\,\mathcal{C}(v,t)\right) \leq Cte\,||t||^2\,.$$

Then, thanks to (24), $\int_{||t||\leq 1} g(v,t)p_{0,t}(v,v)dt$ will be finite. The first point of Proposition 1.1 will thus be established.

Lemma A.1 *There exists a positive constant C such that for any $t \in \mathbb{R}^d$, $||t|| \leq 1$,*

$$\mathbb{E}(||X"(0)||^{4(d-1)}\,/\,\mathcal{C}(v,t)) \leq C \quad \textit{and} \quad \mathbb{E}(\sup_{x\in[0,1]} ||X^{(3)}(xt)||^4\,/\,\mathcal{C}(v,t)) \leq C\,.$$

Proof of the lemma. We start with writing a Taylor's formula around 0 for the covariance function r. Let us remind that r is assumed to be C^6. The isotropy of X allows us to write $r(t) = R(||t||)$ with $R : \mathbb{R}^+ \to \mathbb{R}$ of class C^6. We know that $R(0) = 1$, $R'(0) = R^{(3)}(0) = 0$, $R"(0) = -\lambda$ and we introduce $R^{(4)}(0) = \mu$. We use the standard notations $o(||t||)$ and $O(||t||)$ for $||t|| \to 0$. So

$$r(t) = 1 - \frac{\lambda}{2}||t||^2 + \frac{\mu}{4!}||t||^4 + o(||t||^5) \text{ as } ||t|| \to 0\,.$$

Taking the second derivatives in the above formula yields the Hessian matrix of r, namely $r"(t) = (r^{(2)}_{ij}(t))_{1\leq i,j\leq d}$ with

$$r"(t) = -\lambda I_d + \Theta(t) \text{ with } \Theta(t) = \frac{\mu}{3!}(||t||^2 I_d + (2t_it_j)_{1\leq i,j\leq d}) + o(||t||^3)\,. \quad (26)$$

Let $K = d(d+1)/2$. We consider the K-dimensional Gaussian vector $\nabla^2 X(0)$ whose coordinates are the coefficients above and on the diagonal of the symmetric matrix $X"(0)$. We write down the following K-dimensional regression system

$$\nabla^2 X(0) = A(t)\,\nabla X(0) + B(t)\,\nabla X(t) + Z(t)\,, \quad (27)$$

where $A(t)$ and $B(t)$ are two matrices of size $K \times d$ and $Z(t)$ is a K-dimensional centered Gaussian vector that is independent from $\nabla X(0)$ and $\nabla X(t)$. In that form, we have

$$\mathbb{E}(||X"(0)||^{4(d-1)}\,/\,\mathcal{C}(v,t)) = \mathbb{E}\left(||(A(t)+B(t))\,v + Z(t)||^{4(d-1)}\right)\,,$$

where for any t, $(A(t)+B(t))\,v + Z(t)$ is a Gaussian random vector with mean $(A(t)+B(t))\,v$. Its higher moments will be bounded with respect to t, $||t|| \leq 1$, as soon as its mean and its variance are also bounded.

We now compute the regression coefficients. Writing the covariances between the coordinates in (27) and using (26) allows us to write the next linear $2d$-dimensional system for any fixed k $(1 \leq k \leq K)$,

$$\begin{pmatrix} \lambda I_d & -r"(t) \\ -r"(t) & \lambda I_d \end{pmatrix} \begin{pmatrix} A(t)_{k.} \\ B(t)_{k.} \end{pmatrix} = \begin{pmatrix} 0_d \\ r^{(3)}_{k.}(t) \end{pmatrix}\,.$$

Here $\begin{pmatrix} A(t)_{k.} \\ B(t)_{k.} \end{pmatrix}$ and $\begin{pmatrix} 0_d \\ r^{(3)}_{k.}(t) \end{pmatrix}$ are considered as column vectors of size $2d$ and the first matrix on the left as a block matrix of size $2d \times 2d$. The inverse of this block matrix is given by $\begin{pmatrix} M_1(t) & M_2(t) \\ M_2(t) & M_1(t) \end{pmatrix}$ where $M_1(t)$ and $M_2(t)$ are $d \times d$ matrices such that

$$\lambda M_1(t) - r"(t) M_2(t) = I_d \;\text{ and }\; - r"(t) M_1(t) + \lambda M_2(t) = 0_d \,. \tag{28}$$

Hence, solving the system yields $A(t)_{k.} + B(t)_{k.} = (M_1(t) + M_2(t))\, r^{(3)}_{k.}(t)$. From (28), it is not difficult to derive that

$$M_1(t) + M_2(t) = \lambda^{-2}(\lambda I_d - r"(t))^{-1} = \lambda^{-2}(2\lambda I_d - \Theta(t))^{-1} \;\to\; 2\lambda^{-1} I_d \,.$$

Since $r^{(3)}$ is bounded, we get that $||(A(t)+B(t))\, v|| \leq C$ for any $t \in \mathbb{R}^d$, $||t|| \leq 1$. It remains to prove that the variance of $Z(t)$ is also bounded. The computation of the covariance in (27) yields,

$$\mathrm{Cov}(Z_k(t), Z_l(t)) = r^{(4)}_{k,l}(0) - \langle r^{(3)}_{k.}(t), M_1(t) r^{(3)}_{l.}(t) \rangle \,, 1 \leq k, l \leq K,$$

where $M_1(t)$ can be derived from (28) and (26),

$$M_1(t) = \lambda^{-1}(I_d - \lambda^{-2} r"(t)^2)^{-1} = \lambda \left((2\lambda I_d - \Theta(t))\, \Theta(t) \right)^{-1} \,.$$

Since $||r^{(3)}(t)|| = O(||t||)$ and $||\Theta(t)|| = O(||t||^2)$, we get $||\mathrm{Cov}(Z_k(t), Z_l(t))|| \leq C$ for any $t \in \mathbb{R}^d$, $||t|| \leq 1$.

We have just established the first upper bound of the lemma. For the second one, we can proceed similarly by considering the regression system for any fixed $t \neq 0$

$$\nabla^3 X(s) = C(s,t)\, \nabla X(0) + D(s,t)\, \nabla X(t) + Z^*(s,t) \,,$$

with self-understanding notations and appropriate dimensions for the vectors and matrices. Therefore, for any $x \in [0,1]$, conditionaly to $C(v,t)$, $X^{(3)}(xt)$ is nothing but the Gaussian random vector given by $(C(xt,t) + D(xt,t))\, v + Z^*(xt,t)$. Hence, if we prove that the conditional mean of $X^{(3)}(xt)$, namely $(C(xt,t) + D(xt,t))\, v$, as well as its conditional variance are uniformly bounded with respect to $x \in [0,1]$ and $||t|| \leq 1$, then any conditional moment of the supremum of $||X^{(3)}(xt)||$ for $x \in [0,1]$ will be bounded with respect to $||t|| \leq 1$. This can easily be done once the regression system is solved. Actually, we have $C(xt,t) = -\lambda\, (2\lambda I_d - \Theta(t))^{-1}\, (r^{(4)}(0) + o(||t||))$, the same holds for $D(xt,t)$ and $\mathrm{Cov}(Z^*(xt,t)) = r^{(6)}(0) - \lambda^2\, {}^t\!CC(xt,t) - \lambda^2\, {}^t\!DD(xt,t) + 2C(xt,t)\, r"(t)\, D(xt,t)$. All these quantities are bounded and the lemma is thus proved. $\square$

Point 2. According to (22), the map $v \mapsto \mathbb{E}[N^{\nabla X}(T,v)]$ is continuous. Hence, the second assertion of Proposition 1.1 holds true if we show the continuity of the second factorial moment of $N^{\nabla X}(T,.)$. Thanks to the previous point, introducing $F(v,t) = g(v,t)\, p_{0,t}(v,v)$, we know that for $\eta > 0$ there exists a $\delta > 0$ such that $\int_{||t||\leq\delta} F(v,t)dt < \eta$ uniformly with respect to v contained in any compact subset of $\mathbb{R}^d$. Moreover, let us observe that $v \mapsto F(v,t)$ is continuous for any fixed $t \neq 0$. Indeed, using a regression system similar to the one performed in (27) for both $\nabla^2 X(0)$ and $\nabla^2 X(t)$, one can write

$$g(v,t) = \left|\tilde{\det}\left((A(t)+B(t))v + Z(t)\right) \times \tilde{\det}\left((\bar{A}(t)+\bar{B}(t))v + \bar{Z}(t)\right)\right| .$$

In this form, the continuity of $g(.,t)$, and hence of $F(.,t)$, is obvious. Then,

$$\begin{aligned}|\mathbb{E}[N^{\nabla X}(T,v)(N^{\nabla X}(T,v)-1)] \quad &- \quad \mathbb{E}[N^{\nabla X}(T,v')(N^{\nabla X}(T,v')-1)]| \\ &\leq \quad 2\eta + \int_{||s||>\delta} |F(v,t) - F(v',t)|\, dt.\end{aligned}$$

The continuity follows by taking the lim sup as $v' \to v$ in the left hand side of the inequality.

<u>Point *3.*</u> We use the so called "area formula" in order to prove the third assertion. One can see [12] Prop. 6.1 for instance for a reference. It states that, if $f : \mathbb{R}^d \to \mathbb{R}$ is continuous and bounded, then

$$\int_{\mathbb{R}^d} N^{\nabla X}(T,u)\, f(u)du = \int_T |\det(X"(t))|\, f(\nabla X(t))dt\, a.s.$$

This identity can easily be extended to functions f that are almost everywhere continuous. It applies here with $f = \delta_\varepsilon(. - v)$, and thus

$$\begin{aligned}\mathbb{E}[(N^{\nabla X}(T,v))^2] \quad &\leq \quad \limsup_{\varepsilon\to 0} \mathbb{E}[(\int_T |\det(X"(t))|\, \delta_\varepsilon(\nabla X(t) - v)dt)^2] \\ &= \quad \limsup_{\varepsilon\to 0} \mathbb{E}[(\int_{\mathbb{R}^d} N^{\nabla X}(T,u)\delta_\varepsilon(u-v)du)^2] \\ &\leq \quad \limsup_{\varepsilon\to 0} \int_{\mathbb{R}^d} \mathbb{E}[(N^{\nabla X}(T,u))^2]\delta_\varepsilon(u-v)du = \mathbb{E}[(N^{\nabla X}(T,v))^2].\end{aligned}$$

In the first line we have used Fatou's lemma and the a.s. convergence of $N_\varepsilon^{\nabla X}(T,0)$ to $N^{\nabla X}(T,0)$, in the second one the area formula and finally in the third one Jensen inequality and the continuity proved before. It implies that $\mathbb{E}[(N_\varepsilon^{\nabla X}(T,v))^2]$ is finite and tends to $\mathbb{E}[(N^{\nabla X}(T,v))^2]$. Combined with the a.s. convergence, it gives the third point of Proposition 1.1. □

A.2. Proof of Lemma 2.2

Let us recall (13), which yields the following expression for V_1

$$V_1 = \sum_{\substack{\mathbf{n},\mathbf{m}\in\mathbb{N}^D \\ |\mathbf{n}|=|\mathbf{m}|=1}} a(\mathbf{n})a(\mathbf{m})\, R(\mathbf{n},\mathbf{m}). \tag{29}$$

Although the notation does not mention it explicitely, the coefficients $a(\mathbf{n})$ depend on the level u. Actually, for $\mathbf{n} = (\underline{n}, \overline{n}) \in \mathbb{N}^D = \mathbb{N}^d \times \mathbb{N}^{D-d}$,

$$a(\mathbf{n}) = a(\mathbf{n}, u) = d(\underline{n})c(f_u \circ \Lambda_2, \overline{n}) \text{ with } f_u = \tilde{\det} \otimes 1_{[u,+\infty)}$$

as given by (6) and (1).
Throughout the proof, K denotes the integer $d(d+1)/2$, so that $D - d = K + 1$. Let us recall that Λ has been introduced as any square root matrix of $\Sigma^{\mathbf{X}}$, the covariance matrix of $\mathbf{X}(0)$. From now on, without loss of generality, we choose Λ, and so Λ_2, to be lower triangular as

$$\Lambda_2 = \begin{pmatrix} L & \mathbf{0} \\ {}^t l & \alpha \end{pmatrix}, \tag{30}$$

with L a $K \times K$ lower triangular matrix, l a vector in $\mathbb{R}^K$, ${}^t l$ denotes the $1 \times K$ matrix containing the coordinates of l and $\alpha > 0$. Furthermore, the fact that $\mathrm{Cov}(X(t), X_{ii}(t)) = -\lambda \neq 0$ implies that the vector l does not vanish and the fact that $\mathrm{Var}(X(t)) = 1$ implies that $||l||^2 + \alpha^2 = 1$.
Hence, for $(y, z) \in \mathbb{R}^{D-d} = \mathbb{R}^K \times \mathbb{R}$, we have

$$f_u \circ \Lambda_2(y, z) = \tilde{\det}(Ly)\, 1_{[u,+\infty)}(\langle l, y \rangle + \alpha z)\ . \tag{31}$$

The computation of $\tilde{\det}(Ly)$ is solved in the next lemma. It states that the expansion of the map $y \in \mathbb{R}^K \mapsto \tilde{\det}(Ly)$ in the basis of Hermite polynomials on $\mathbb{R}^K$ only involves Hermite polynomials of degree exactly equal to d.

Lemma A.2 *Let L be the matrix introduced in* (30). *There exists a family of real numbers $(\beta_{\mathbf{m}})_{\mathbf{m} \in \mathbb{N}^K;\, |\mathbf{m}|=d}$ depending on L such that*

$$\forall y \in \mathbb{R}^K\, ,\ \tilde{\det}(Ly) = \sum_{\mathbf{m} \in \mathbb{N}^K;\, |\mathbf{m}|=d} \beta_{\mathbf{m}} \widetilde{H_{\mathbf{m}}}(y) = \sum_{\mathbf{m} \in \mathbb{N}^K;\, |\mathbf{m}|=d} \beta_{\mathbf{m}} y^{(\mathbf{m})}$$

where $y^{(\mathbf{m})} = \prod_{1 \le k \le K} (y_k)^{m_k}$.

Proof. The map $F : y \in \mathbb{R}^K \mapsto F(y) = \tilde{\det}(Ly)$ is a polynomial function of degree d. We first expand it in the basis of Hermite polynomials on $\mathbb{R}^K$ as follows

$$F(y) = \sum_{q=0}^{d} \sum_{\mathbf{m} \in \mathbb{N}^K;\, |\mathbf{m}|=q} \beta_{\mathbf{m}} \widetilde{H_{\mathbf{m}}}(y)\, ,$$

where the coefficients are given by

$$\beta_{\mathbf{m}} = \frac{1}{\mathbf{m}!} \int_{\mathbb{R}^K} F(y) \widetilde{H_{\mathbf{m}}}(y) \phi_K(y) dy = \frac{1}{\mathbf{m}!} (F * \phi_K^{(\mathbf{m})})(0) \ = \ \frac{1}{\mathbf{m}!} (F^{(\mathbf{m})} * \phi_K)(0)\, .$$

We shall establish that $\beta_{\mathbf{m}} = 0$ for all indices $\mathbf{m} \in \mathbb{N}^K$ such that $|\mathbf{m}| < d$. This will prove the first equality in the lemma.
Note that in the previous expression of $\beta_{\mathbf{m}}$ for $\mathbf{m} = (m_1, \dots, m_K)$ with $|\mathbf{m}| = q$,

$G^{(\mathbf{m})}$ denotes the derivative $\frac{\partial^q G}{\partial y_{m_1}\cdots\partial y_{m_K}}$ for any function G defined on $\mathbb{R}^K$.
In order to compute $F^{(\mathbf{m})}$, we write down $F(y)$ as $F(y) = \det(A(y))$, where for any $y \in \mathbb{R}^K$, $A(y)$ is a symmetric $d \times d$ matrix. The map $y \in \mathbb{R}^K \mapsto A(y)$ is linear, so that for any $1 \le i, j \le d$ we have $A(y)_{i,j} = \sum_{k=1}^K a_{ij}^k\, y_k$. Hence, denoting by $\operatorname{sgn}(\sigma)$ the sign of any permutation $\sigma \in S_d$, we get

$$F(y) = \det(A(y)) = \sum_{\sigma\in\mathcal{S}_d} \operatorname{sgn}(\sigma) \prod_{i=1}^d \left(\sum_{k=1}^d a^k_{i,\sigma(i)} y_k \right).$$

Then, for $k = 1, \ldots, K$, $\dfrac{\partial F}{\partial y_k}(y) = \displaystyle\sum_{1\le i\le d} \det(\widehat{A(y)}^{ik})$, where $\widehat{A(y)}^{ik}$ denotes the $d \times d$ matrix obtained from $A(y)$ replacing the i-th line by the line $(a^k_{i1}, \ldots, a^k_{id})$.
More generally, for any $\mathbf{m} \in \mathbb{N}^K$ with $|\mathbf{m}| < d$, $F^{(\mathbf{m})}(y)$ can be written as a sum of determinants of $d \times d$ matrices, which are obtained from $A(y)$ replacing $|\mathbf{m}|$ lines by lines equal to $(a^k_{i1}, \ldots, a^k_{id})$ with some $i \in \{1, \ldots, d\}$ and some $k \in \{1, \ldots, K\}$. Let us denote $\widehat{A(y)}^{(\mathbf{m})}$ for any such matrix.
Coming back to the computation of the coefficients $\beta_{\mathbf{m}}$, we write each $\beta_{\mathbf{m}}$ as a sum of terms which are equal to $\frac{1}{\mathbf{m}!}\int_{\mathbb{R}^K} \det(\widehat{A(y)}^{(\mathbf{m})})\phi_K(y)dy$. We will prove that they vanish for all indices $\mathbf{m}$ such that $|\mathbf{m}| < d$.
Each of the above integrals is equal to $\mathbb{E}\left(\det(\widehat{A(Z)}^{(\mathbf{m})})\right)$ with Z a $N(0, I_K)$ Gaussian vector such that the $d \times d$ matrix $X"(0)$ is equal to $X"(0) = A(Z)$. Hence each integral reduces to the computation of

$$\sum_{\sigma\in\mathcal{S}_d} \operatorname{sgn}(\sigma)\mathbb{E}\big(X_{i_1,\sigma(i_1)}X_{i_2,\sigma(i_2)}\ldots X_{i_{d-|\mathbf{m}|},\sigma(i_{d-|\mathbf{m}|})}\big)a^{k_1}_{j_1,\sigma(j_1)}\ldots a^{k_{|\mathbf{m}|}}_{j_{|\mathbf{m}|},\sigma(j_{|\mathbf{m}|})}$$

where $\{i_1, i_2, \ldots, i_{d-|\mathbf{m}|}\} \cup \{j_1, i_2, \ldots, j_{|\mathbf{m}|}\} = \{1, \ldots, d\}$. If $d - |\mathbf{m}|$ is odd, since all the Gaussian random variables $X_{i,j}$ have a symmetric distribution, then the above term is zero. Thus, let us assume that $d - |\mathbf{m}| = 2l$. Then the standard Wick's formula says that

$$\begin{aligned}&\mathbb{E}\big(X_{i_1,\sigma(i_1)}X_{i_2,\sigma(i_2)}\ldots X_{i_{d-|\mathbf{m}|},\sigma(i_{d-|\mathbf{m}|})}\big)\\ &= \sum \mathbb{E}(X_{k_1,\sigma(k_1)}X_{k_2,\sigma(k_2)})\ldots\mathbb{E}(X_{k_{2l-1},\sigma(k_{2l-1})}X_{k_{2l},\sigma(k_{2l})}),\end{aligned}$$

where the sum is taken over all the different ways of grouping the indices $i_1, i_2, \ldots, i_{d-|\mathbf{m}|}$ into l pairs denoted by $\{k_1, k_2\}, \ldots, \{k_{2l-1}, k_{2l}\}$. With any fixed permutation σ, we associate a new permutation σ' as follows:

$$\sigma'(j_k) = \sigma(j_k) \text{ for } 1 \le k \le |\mathbf{m}|$$

$$\sigma'(k_1) = \sigma(k_2),\ \sigma'(k_2) = \sigma(k_1),\ \sigma'(k_m) = \sigma(k_m) \text{ for the remaining indices.}$$

Then, $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$ because σ' is the composition of σ with a transposition.
Moreover

$$\mathbb{E}(X_{k_1,\sigma(k_1)}X_{k_2,\sigma(k_2)}) = r^{(4)}_{k_1,\sigma(k_1),k_2,\sigma(k_2)}(0) = \mathbb{E}(X_{k_1,\sigma'(k_1)}X_{k_2,\sigma'(k_2)}).$$

This implies the cancellation of all pairs of two associated permutations and hence

$$\sum_{\sigma\in\mathcal{S}_d} \operatorname{sgn}(\sigma)\mathbb{E}\big(X_{i_1,\sigma(i_1)}X_{i_2,\sigma(i_2)}\dots X_{i_{d-|\mathbf{m}|},\sigma(i_{d-|\mathbf{m}|})}\big)a^{k_1}_{j_1,\sigma(j_1)}\dots a^{k_{|\mathbf{m}|}}_{j_{|\mathbf{m}|},\sigma(j_{|\mathbf{m}|})} = 0.$$

We have proved that all the coefficients $\beta_{\mathbf{m}}$ with $|\mathbf{m}| < d$ equal 0 and so the first expansion of the lemma is established.

In order to get the second expansion, we remark that for any $x \in \mathbb{R}$ and any positive integer k, $\lim_{s\to+\infty} s^{-k}H_k(sx) = x^k$. Then, using our first expansion, we get for any $y \in \mathbb{R}^K$,

$$\sum_{\mathbf{m}\in\mathbb{N}^K;\,|\mathbf{m}|=d} \beta_{\mathbf{m}}\, y^{(\mathbf{m})} \;=\; \lim_{s\to+\infty} s^{-d}\,\tilde{\det}(s\,Ly) \;=\; \tilde{\det}(Ly).$$

Lemma A.2 is proved. $\square$

Let us come back to equation (29). Note that $|\mathbf{n}| = 1$ implies that we can write $\mathbf{n} = \mathbf{e}_i$ for one index $i = 1,\dots,D$ where $(\mathbf{e}_i)_{1\le i\le D}$ stands for the canonical basis of $\mathbb{R}^D$.
Then for $\mathbf{n} = \mathbf{e}_i$, due to the explicit form of $d(\underline{n})$ given in (5), we have

$$a(\mathbf{e}_i) = \begin{cases} 0 & \text{if} \quad 1 \le i \le d \\ (2\pi\lambda)^{-d/2}\, c(f_u\circ\Lambda_2, \overline{e_i}) & \text{if} \quad d+1 \le i \le D \end{cases}$$

Hence,

$$V_1 = \lambda^{-d}(2\pi)^{-d} \sum_{d+1\le i,j\le D} c(f_u\circ\Lambda_2,\overline{e_i})\, c(f_u\circ\Lambda_2,\overline{e_j})\, R(\mathbf{e}_i,\mathbf{e}_j),$$

where we deduce from (9) that $R(\mathbf{e}_i,\mathbf{e}_j) = \int_{\mathbb{R}^d} \operatorname{Cov}(Y_i(0),Y_j(v))\,dv$.

Remark A.3 *Denoting by $\Gamma^{\mathbf{X}}$ the covariance function of the D-dimensional Gaussian field $\mathbf{X} = (\nabla X, \nabla^2 X, X)$, for any $1 \le i,j \le D$,*

$$\int_{\mathbb{R}^d} \Gamma^{\mathbf{X}}_{i,j}(v)\,dv = (2\pi)^d\, f_X(0)\,\delta_{D,D}(i,j),$$

where δ stands for the Kronecker symbol.

Indeed, $\Gamma^{\mathbf{X}}_{D,D}(v) = \mathbb{E}(X(0)X(v)) = r(v)$, and using the spectral density of X, one can write $r(v) = \widehat{f_X}(v)$, where $\widehat{f_X}$ denotes the Fourier transform with $\widehat{f_X}(v) = \int_{\mathbb{R}^d} e^{i<v,\lambda>} f_X(\lambda)d\lambda$. Hence $\int_{\mathbb{R}^d}\Gamma^{\mathbf{X}}_{D,D}(v)\,dv = \widehat{\widehat{f_X}}(0)$ and, since f_X is supposed to be continuous, the inversion formula yields $\widehat{\widehat{f_X}}(0) = (2\pi)^d\, f_X(0)$.
For $(i,j) \ne (D,D)$, $\Gamma^{\mathbf{X}}_{i,j}$ is the covariance function between a derivative of X and another derivative of X, with at least one of both derivatives being of order at least one. So, up to a power of (-1), $\Gamma^{\mathbf{X}}_{i,j}$ equals a derivative of order

at least one of the function r. When computing the integral of $\Gamma^{\mathbf{X}}_{i,j}$ over all $\mathbb{R}^d$, one can use Fubini's theorem to pick out a well chosen integral over $\mathbb{R}$, which can be computed through a primitive along one direction of $\Gamma^{\mathbf{X}}_{i,j}$. Since r and all its derivatives tend to 0 at infinity due to assumption **(A2)**, we get $\int_{\mathbb{R}^d} \Gamma^{\mathbf{X}}_{i,j}(v)\, dv = 0$ for $(i,j) \neq (D,D)$.

We come back to the computation of $R(\mathbf{e}_i, \mathbf{e}_j) = \int_{\mathbb{R}^d} \mathrm{Cov}(Y_i(0), Y_j(v))\, dv$. Remember that the covariance function of the vector field Y is given by

$$\Gamma^Y(v) = \Lambda^{-1}\, \Gamma^{\mathbf{X}}(v)\, {}^t\Lambda^{-1}\,.$$

Using Remark A.3, we get

$$R(\mathbf{e}_i, \mathbf{e}_j) = \int_{\mathbb{R}^d} \Gamma^Y_{ij}(v)\, dv = (2\pi)^d\, f_X(0)\, (\Lambda^{-1})_{iD} (\Lambda^{-1})_{jD}\,.$$

Since Λ is lower triangular, Λ^{-1} is also lower triangular. Furthermore, (30) yields $(\Lambda^{-1})_{iD} = \alpha^{-1} \delta_{i,D}$. Then, for any $d+1 \le i,j \le D$,

$$R(\mathbf{e}_i, \mathbf{e}_j) = (2\pi)^d\, f_X(0)\, \alpha^{-2}\, \delta_{D,D}(i,j)$$

and therefore

$$V_1 = f_X(0)\, \lambda^{-d}\, \alpha^{-2}\, c(f_u \circ \Lambda_2, \overline{e_D})^2\,. \tag{32}$$

According to (3) and (31) we get

$$\begin{aligned} c(f_u \circ \Lambda_2, \overline{e_D}) &= \int_{\mathbb{R}^K \times \mathbb{R}} \tilde{\det}(Ly)\, 1_{[u,+\infty)}(\langle l, y\rangle + \alpha z)\, z\, \phi_K(y)\, \phi(z)\, dy\, dz \\ &= \int_{\mathbb{R}^K} \tilde{\det}(Ly)\, \phi(\frac{1}{\alpha}(u - \langle l, y\rangle))\, \phi_K(y)\, dy\ , \end{aligned}$$

where we have used $z\phi(z) = -\phi'(z)\,,\ \forall z \in \mathbb{R}$.

Hence, using Lemma A.2, we have $c(f_u \circ \Lambda_2, \overline{e_D}) = \sum_{|\mathbf{m}|=d} \beta_{\mathbf{m}}\, I_{\mathbf{m}}$ where we introduce the next integral

$$\begin{aligned} I_{\mathbf{m}} &:= \int_{\mathbb{R}^K} \widetilde{H_{\mathbf{m}}}(y)\, \phi(\frac{1}{\alpha}(u - \langle l, y\rangle))\, \phi_K(y)\, dy \\ &= (-1)^d \int_{\mathbb{R}^K} \phi(\frac{1}{\alpha}(u - \langle l, y\rangle))\, \phi_K^{(\mathbf{m})}(y)\, dy \\ &= (-1)^d (\varphi_0 * \phi_K^{(\mathbf{m})})(ul^*) \ = \ (-1)^d\, (\varphi_0 * \phi_K)^{(\mathbf{m})}\, (ul^*)\,. \end{aligned}$$

In the previous lines, φ_0 denotes the map $y \in \mathbb{R}^K \mapsto \varphi_0(y) = \phi(\frac{1}{\alpha}\langle l, y\rangle)$ and l^* is any vector in $\mathbb{R}^K$ such that $\langle l, l^*\rangle = 1$.

Remark A.4 *For any* $y \in \mathbb{R}^K$, $(\varphi_0 * \phi_K)(y) = \alpha\, \phi(\langle l, y\rangle)\,.$

Indeed, the map $x \in \mathbb{R} \mapsto \frac{1}{\alpha}\int_{\mathbb{R}^K} \phi(\frac{1}{\alpha}(x - \langle l, z\rangle))\,\phi_K(z)\,dz$ is the probability density function of a random variable $Z = \alpha N + \langle l, G\rangle$ where N is a standard Gaussian random variable and G is a standard Gaussian vector of dimension K independent of N. But Z is clearly Gaussian, centered, with variance $\alpha^2 + ||l||^2 = 1$.

Using the previous remark, we get for any $y \in \mathbb{R}^K$,

$$(\varphi_0 * \phi_K)^{(\mathbf{m})}(y) = \alpha\, l^{(\mathbf{m})}\,\phi^{(d)}(\langle l, y\rangle) = (-1)^d\,\alpha\, l^{(\mathbf{m})}\,H_d(\langle l, y\rangle)\,\phi(\langle l, y\rangle)\,.$$

Coming back to the computation of $I_{\mathbf{m}}$, we get $I_{\mathbf{m}} = \alpha\, l^{(\mathbf{m})}\,H_d(u)\,\phi(u)$, since $\langle l, ul^*\rangle = u$. Then, the desired Hermite coefficient can be written in the following form

$$c(f_u \circ \Lambda_2, \overline{e_D}) = \alpha\,\Big(\sum_{|\mathbf{m}|=d} \beta_{\mathbf{m}}\, l^{(\mathbf{m})}\Big)\,H_d(u)\,\phi(u) = \alpha\,\tilde{\det}(Ll)\,H_d(u)\,\phi(u)\,,$$

where we have used Lemma A.2 to compute the sum inside the parenthesis.
It remains to compute $\tilde{\det}(Ll) = \det(A(l))$. We write out the coordinates of the $K+1$-dimensional Gaussian vector $(\nabla^2 X, X)$ in the following order $((X_{ij})_{1\le i<j\le d}\ ,\ (X_{ii})_{1\le i\le d}\ ,\ X)$, so that the lower triangular matrix L can be written as $L = \begin{pmatrix} L^{(1)} & 0 \\ L^{(3)} & L^{(2)} \end{pmatrix}$ and the vector $l = \begin{pmatrix} l^{(1)} \\ l^{(2)} \end{pmatrix}$, where the top part has length $K-d$ whereas the bottom part has length d. With these notations, $(X_{ij})_{1\le i<j\le d} = L^{(1)}\,(Y_k)_{d+1\le k\le K-d}$. Since $\mathrm{Cov}(X, X_{ij}) = 0$ for all $i<j$, we deduce that $l^{(1)}$ vanishes and since $\mathrm{Cov}(X, X_{ii}) = r_{ii}(0) = -\lambda$ for any $i = 1, \dots, d$, we deduce that all the coordinates of $L^{(2)}\,l^{(2)}$ are equal to λ. It implies that the symmetric $d\times d$ matrix $A(l)$ induced by the K dimensional vector $L\,l$ is diagonal and equal to $-\lambda\, I_d$. Hence, $\tilde{\det}(Ll) = \det(A(l)) = (-\lambda)^d$ and we finally obtain

$$c(f_u \circ \Lambda_2, \overline{e_D}) = \alpha\,(-\lambda)^d\,H_d(u)\,\phi(u)\,.$$

Thanks to (32), we have $V_1 = f_X(0)\,\lambda^d\,H_d(u)^2\,\phi(u)^2$ and Lemma 2.2 is proved. □

References

[1] Adler R.J. *A spectral moment problem in two dimension.* Biometrika, 64, 367-373 (1977).

[2] Adler R.J. *The Geometry of Random Fields.* Wiley Series in Probability and Mathematical Statistics, Wiley, Chichester, UK (1981).

[3] Adler R.J. *On excursion sets, tube formulas and maxima of random fields.* Ann. Appl. Probab., vol 10, 1-74 (2000).

[4] Adler R.J., Hasofer A.M., *Level crossings for random fields*, Ann. Probab.,vol 4, 1-12 (1976).

[5] Adler R.J., Taylor J.E. *Random Fields and Geometry.* Springer Monographs in Mathematics. Springer (2007).

[6] Adler R.J., Taylor J.E., Worsley, K.J. *Applications of random fields and geometry: foundations and case studies.* In preparation, available on R. Adler's home page (2007).

[7] Ahmad O., Pinoli J-C. *Lipschitz-Killing curvatures of the excursion sets of skew Student's t random fields.* Stochastic Models, Vol 29, 273-289 (2013).

[8] Arcones M. A. *Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors.* Ann. Probab., vol 22, 2242-2274 (1994).

[9] Auffinger A., Ben Arous G.,*Complexity of random smooth functions on the high dimensional sphere.* Ann. Probab., vol 41, 4214-4247 (2013).

[10] Azaïs J-M., Delmas C. *Asymptotic expansions for the distribution of the maximum of a Gaussian random field.* Extremes 5, 181-212 (2002).

[11] Azaïs J-M., Wschebor M. *On the distribution of the maximum of a Gaussian field with d parameters.* Ann. Appl. Probab., vol 15, 254-278 (2005).

[12] Azaïs J-M., Wschebor M. *Level Sets and Extrema of Random Processes and Fields.* Wiley (2009).

[13] Bulinski A., Spodarev E., Timmermann F. *Central limit theorem for the excursion sets volumes of weakly dependent random fields.* Bernoulli 18, 100-118 (2012).

[14] Cabaña E.M. *Affine processes: a test of isotropy based on level sets.* SIAM Journal on Applied Mathematics, 886 - 891 (1987).

[15] Cheng D., Xiao Y. *The Mean Euler Characteristic and Excursion Probability of Gaussian Random Fields with Stationary Increments.* arXiv:1211.6693 (2012).

[16] Elizarov A.I. *On the Variance of the Number of Stationary Points of a Homogeneous Gaussian Field.* Theory of Probability & Its Applications, Vol. 29, No. 3, 569-570 (1985).

[17] Estrade A., Fariñas A., León J.R. *Computation in dimension 2.* Working paper, available at http://w3.mi.parisdescartes.fr/ estradea/ (2015).

[18] Imkeller P., Perez-Abreu V., Vives J. *Chaos expansions of double intersection local time of Brownian motion in and renormalization.* Stoch. Process. Appl., vol 56, 1-34, (1995).

[19] Iribarren I. *Asymptotic behavior of the integral of a function on the level set of a mixing random field.* Probab. Math. Statistics 10, No. 1, 45-56 (1989).

[20] Kratz M., León J.R. *Central limit theorem for level functionals of stationary Gaussian processes and fields.* J. Theoret. Probab., vol 14, No 3, 639-672 (2001).

[21] Kratz M., León J.R. *On the second moment of the number of crossings by a stationary Gaussian process.* Ann. Probab., vol 34, 1601-1607 (2006).

[22] Major P. *Multiple Wiener-Ito integrals with applications to limit theorems.* Lecture Notes in Mathematics 849 Second edition. Revised version, Springer Verlag, Berlin-Heidelberg-New York, (2014).

[23] Malevich T.L., Volodina L.N. *Conditions for the Finiteness of the Moments of the Number of Zeros of a Gaussian Vector Field.* Theory of Probability & Its Applications 33:1, 50-61 (1989).

[24] Meschenmoser D., Shashkin A. *Functional central limit theorem for the volume of excursion sets generated by associated random fields.* Statist. Probab. Lett. 81, 642-646 (2011).

[25] Nourdin I., Peccati G. and Podolskij M. *Quantitative Breuer-Major theorems.* Stochastic Processes and their Applications, vol 121, 793-812 (2011).

[26] Nualart D. *The Malliavin Calculus and Related Topics.* Second Edition, Springer Verlag Berlin-Heidelberg-New York, (2006).

[27] Nualart D., Peccati G. *Central limit theorems for sequences of multiple stochastic integrals.* Ann. Probab. vol 33, 177-193. (2005).

[28] Peccati G., Tudor C.A. *Gaussian limits for vector-valued multiple stochastic integrals.* Séminaire de Probabilités XXXVIII, 247-262 (2005).

[29] Slud E. V. *MWI Representation of the Number of Curve-Crossings by a Differentiable Gaussian Process, with Applications.* Ann. Probab., vol 22, Number 3, 1355-1380 (1994).

[30] Spodarev E., Zaporozhets D. *Asymptotics of the mean Minkowski functionals of Gaussian excursions.* Preprint, available at http://www.uni-ulm.de/en/mawi/institute-of-stochastics/mitarbeiter/evgeny-spodarev/ (2013).

[31] Taheriyoun A.R. *Testing the covariance function of stationary Gaussian random fields.* Statist. Probab. Lett. 82, 606-613 (2012).

[32] Taheriyoun A.R., Shafie K., Jafari Jozani M. *A note on the higher moments of the Euler characteristic of the excursion sets of random fields.* Statist. Probab. Lett. 79, 1074-1082 (2009).

[33] Taylor J.E., Worsley K.J. *Random fields of multivariate test statistics, with application to shape analysis.* Ann. Statist., vol 36, 1-27 (2008).