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Acoustics of a Porous Medium Saturated by a Bubbly Fluid Undergoing Phase Change

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Abstract. The objective of this work is the derivation of the wave equations for describing acoustics in a deformable porous medium saturated by a bubbly fluid, when capillary, thermal and phase change effects are accounted for. This is performed using an homogenisation technique: the macroscopic model is obtained by upscaling the bubble-scale and the pore-scale descriptions. For convenience a bubbly fluid near the bubble point, in the bulk of which a small perturbation can generate small bubbles is considered. Although the derived macroscopic wave equations are similar in their structure to Biot's equations that describe wave propagation in saturated porous media, important differences appear as a result of the presence of bubbles. In effect, gas–liquid phase change considerably decreases the apparent rigidity of the bubbly fluid, and consequently decreases the wave velocity in the porous medium. Moreover, this phenomenon is amplified for very small bubbles, for which the apparent rigidity of the bubbly fluid can be negative. The influence of the bubbly liquid apparent rigidity on the wave velocity and attenuation is highlighted on an illustrative example: it is shown that they strongly differ from wave velocities and attenuations in porous media saturated by a liquid or by a gas.

Key words: acoustics, bubbly fluid, phase change, homogenisation.

Nomenclature

A	Lamé constant.
$\mathcal{A}$	effective compressibility of the fluid mixture.
c	wave velocity.
$\mathbf{c}$	elasticity tensor.
C_p	heat capacity.
d	diffusivity.
$\mathcal{D}$	rate of deformation tensor.
$\mathbf{E}$	deformation tensor.
H	thermal memory function.
$\mathbf{I}$	identity tensor.
j	mass flux of vaporization-condensation.
k	thermal conductivity.

$\mathbf{K}$	dynamic permeability tensor.
K_c	capillary membrane rigidity.
K_l, K_v	bulk modulus of the liquid, the vapor, respectively.
$\tilde{K}$	bulk rigidity.
l, l', l''	characteristic lengths of the bubble, the pore and the wavelength, respectively.
l^{therm}	thermal boundary layer thickness.
L^{φ}	vaporization enthalpy.
N	Lamé constant.
$\mathcal{N}$	dimensionless number.
$\mathbf{n}$	unit normal to the bubble surface.
p, P	pressure.
p_c	capillary pressure.
Q	quality factor.
Q_l, Q_v	dimensionless numbers.
r	bubble radius perturbation.
R	bubble radius.
R_l, R_v	dimensionless numbers.
$\mathcal{R}$	perfect gas constant.
S	dimensionless number.
T	temperature.
T^c	characteristic temperature.
u	displacement.
v	velocity.
$\mathcal{W}$	Weber number.
x, y, z	dimensionless space variables for the sample, the pores and the bubbles, respectively.
X	physical space variable.

Greek Letters

$\boldsymbol{\alpha}$	elastic tensor.
β	volume concentration of vapor in the bubbly liquid.
γ	elastic coefficient.
Γ	bubble surface.
Δ	Laplace operator.
ε	scale ratio.
θ	bulk deformation.
κ	wave number.
λ	wavelength.
μ	dynamic viscosity.
μ^{eff}	effective dynamic viscosity.
ν	kinematic viscosity.
π	effective bubbly-fluid pressure.
ρ	density.
ρ^c	characteristic density.
$\tilde{\rho}$	bulk density.
σ	surface tension.
Σ	dimensionless number.
ϕ	porosity.
$\boldsymbol{\sigma}$	stress.
ω	pulsation.

$\omega*$	dimensionless pulsation.
Ω, Ω'	periods at the bubble and the pore scales, respectively.
Ω_l, Ω_v	domains of the period occupied by the liquid and the vapor, respectively.

1. Introduction

The acoustics of fluid-saturated porous media is of interest in oil exploration, geophysics, foundation engineering, and many other fields. The investigation of wave propagation in fluid-saturated elastic porous media was pioneered by Frenkel (1944); Beranek (1947); Zwicker and Kosten (1949) and Morse (1952) and then rigorously developed by Biot (1956). Biot (1962), who used a phenomenological approach, that is, a reasoning on the basis of the physics at the macroscopic scale, for deriving the mathematical model. It is a well-known fact that the presence of gas bubbles within the fluid greatly alters the velocity and the attenuation of acoustic waves, for example, see (Bedford and Stern, 1983; Dontsov *et al.*, 1987; Nakoryakov *et al.*, 1989; Smeulders and Van Dongen, 1997; Herskowitz *et al.*, 1999). Mathematical models are presented in (Dontsov *et al.*, 1987; Nakoryakov *et al.*, 1989) where gas–liquid phase change, thermal damping and capillarity effects are neglected, that correspond to large-size bubbles as shown in the following, see part 4.3. Gas–liquid phase change and capillarity is also neglected in (Smeulders and Van Dongen, 1997) where gas bubble radius is larger than the pore-size. Acoustics of a bubbly fluid on its own was studied in many works, among them see, for example (van Wijngaarden, 1972; Nakoryakov *et al.*, 1993). In (Auriault and Lebaigue, 1989; Boutin and Auriault, 1993; Auriault and Boutin, 2000), it was shown that the effective bulk modulus of the mixture is strongly dependent on phase change, thermal and capillary effects and also on the bubble-size.

In this paper we investigate acoustic waves in a porous medium saturated by a liquid that contain gas bubbles, when gas–liquid phase change, thermal and capillary effects are accounted for. The theory presented below applies to various bubbly liquid mixtures. For convenience the study is carried out for a fluid mixture which is near the bubble point and that consists of water and steam bubbles that are much smaller than the pore-size. Thus, three scales may be distinguished: the bubble-scale, the pore-scale and the macroscopic scale (whose characteristic length is the wavelength). We assume these three scales to be well separated: the three-phase medium can thus be seen as a continuum at the macroscopic scale.

We use an homogenisation technique, namely the homogenisation method for periodic structures for determining the macroscopic wave equations. In other words, we derive the continuous macroscopic model by upscaling the bubble-scale and the pore-scale descriptions. Whatever is the bubble-size, the derived macroscopic description has the same structure as Biot's equations, which describe wave propagation in saturated porous media. We show that the presence and the size of the bubbles strongly affect the apparent fluid rigidity, the wave velocity and the wave attenuation.

The paper is organized as follows. In Section 2 we describe the medium under consideration and we briefly introduce the homogenisation method. Then, in Section 3 we present the physical description at the bubble-scale and at the pore-scale. Section 4 is devoted to preliminary estimations and focuses in particular on the influence of bubble size. The purpose of Section 5 is the normalization of the local description presented in Section 3. It consists of a scale analysis followed by the estimation of the dimensionless numbers. Homogenisation of the dimensionless description is detailed in the Appendix. Since the bubble-size has no impact on the form of the macroscopic model, the reasoning (i.e. normalization in § 3 and homogenisation in Appendix) is carried for small-size bubbles. In Section 5 we present and discuss the derived macroscopic behaviour. We consider an illustrative example so as to investigate wave velocities and attenuations. Large-size bubbles lead to the behaviour of a porous medium saturated by a gas. When the fluid mixture consists of either small-size or medium-size bubbles, wave attenuations and velocities strongly differ from those obtained for the same porous structure but saturated by a liquid or by a gas.

2. Medium Description and Multiple Scale Homogenisation Process

We consider an elastic porous matrix saturated by an incompressible liquid that contains small vapor bubbles at finite concentration. As mentioned above, three scales may be distinguished: the bubble scale, the pore scale and the macroscopic scale. In order to fit the homogenisation method for periodic structures to a three-scale problem, we assume the medium to be doubly periodic: the bubbly liquid and the porous medium are both periodic. No specific internal geometry is at issue, the work is aimed towards deriving a general macroscopic model. At the bubble scale, consider the medium to be Ω-periodic and its characteristic length to be l. The liquid and the gas occupy the domains Ω_l and Ω_v, respectively, and their common boundary is Γ (Figure 1). For simplicity we assume that all bubbles are of same radius $R = O(l)$. Extension of the results to different bubble sizes would be straightforward. A second periodic structure exists at the pore level, whose period

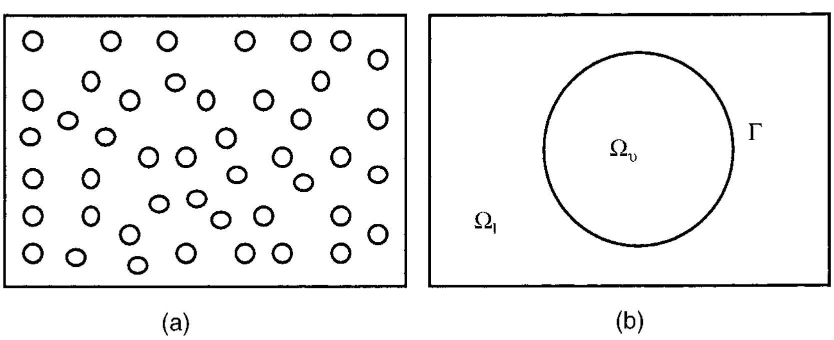

Figure 1. (a) Bubbly fluid at finite concentration, (b) periodic cell at the bubble scale.

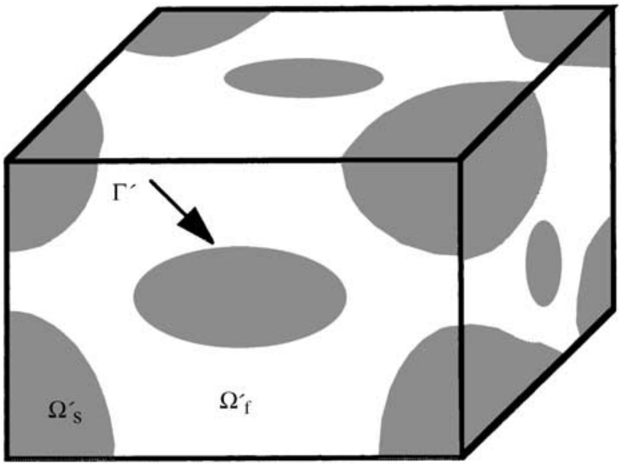

Figure 2. Period of the porous medium.

is denoted by Ω' and whose characteristic length is l'. The liquid–vapor mixture and the porous matrix occupy the domains Ω_f' and Ω_s', respectively, and their common boundary is Γ' (Figure 2). We investigate how acoustic waves propagate, when gas–liquid phase change is occurring within the fluid mixture. The wave propagation introduces a third characteristic length l'' which is related to the wave velocity c, the pulsation ω and the wavelength λ by: $l'' = \mathrm{O}(c/\omega) = \mathrm{O}(\lambda/2\pi)$.

The purpose of the present work is to derive a continuous macroscopic model that describes wave propagation within the medium. Now, the three-phase medium can be seen as a continuum when the larger scales are separated, that is when $l'' \gg l'$. This condition of separation of scales is a necessary condition for the existence of an equivalent macroscopic description. For simplicity we also assume $l' \gg l$ and we will consider the case of equal scale ratios

$$\frac{l}{l'} = \frac{l'}{l''} = \varepsilon. \tag{1}$$

The existence of three well-separated characteristic lengths for the liquid–vapor mixture entails that any physical variable in the liquid and in the vapor *a priori* depends on three dimensionless space variables x, y and z

$$x = \frac{X}{l''}, \qquad y = \frac{X}{l'}, \qquad z = \frac{X}{l}, \qquad x = \varepsilon y = \varepsilon^2 z, \tag{2}$$

where X is the physical space variable. The physics in the porous matrix is related to the pore scale and to the macroscopic scale and is therefore linked to two characteristic lengths l' and l'', only. Thus, the physical variables in the porous matrix *a priori* depend on the dimensionless variables x and y. The macroscopic description is obtained from the bubble-scale and the pore-scale descriptions by using the homogenisation method of multiple scale expansions. It consists (see

Auriault, 1991) firstly in writing the local description in a dimensionless form and then in estimating the dimensionless numbers with respect to the powers of ε. Furthermore, the method is based upon the statement that if the scales are well separated, the physical variables may be looked for in the form of asymptotic expansions in powers of ε (Sanchen-Palencia, 1980)

$$\begin{aligned}\Phi &= \Phi(x, y, z)\\ &= \sum_n \varepsilon^n \Phi^{(n)}(x, y, z), \qquad n \text{ integer}, \quad \text{in} \quad \Omega_{\rm l} \quad \text{and} \quad \Omega_{\rm v}\\ \Phi &= \Phi(x, y) = \sum_n \varepsilon^n \Phi^{(n)}(x, y), \qquad n \text{ integer}, \quad \text{in} \quad \Omega'_{\rm s}.\end{aligned} \tag{3}$$

Incorporating these asymptotic expansions in the dimensionless equations and solving the boundary-value problems arising at the successive orders of ε leads to the macroscopic description.

3. Physics at the Microscopic Scales

3.1. INTRODUCTION

An acoustic perturbation of small amplitude and constant frequency is superimposed to the initial equilibrium. Thus, any physical variable Φ is written as

$$\Phi^t_\alpha = \Phi^e_\alpha + \Phi_\alpha \exp(i\omega t), \qquad \Phi_\alpha = \mathrm{O}\,(\eta \Phi^e_\alpha), \qquad \eta \ll 1, \tag{4}$$

where the indexes t and e stand for ‘total’ and ‘equilibrium’, respectively. $\alpha = l, v, s$ stand for the liquid, the vapor and the solid, respectively. We are looking for the acoustical description of the porous medium. Therefore the displacement perturbation is much smaller than the pore size. We will consider

$$\frac{u}{R} = \mathrm{O}\,(\varepsilon),$$

where u is an estimation of the displacement. Since the amplitude of the perturbation is small, the equations that describe wave propagation are linearized. Convective terms in material time derivatives are negligible.

3.2. DESCRIPTION AT THE BUBBLE-SCALE

The bubble-scale description accounts, in each phase, for the conservation of momentum, mass and energy. Capillary and phase change effects appear in the conditions over the bubble surface, Γ. With the notations defined in (4), we have

- Momentum balance in the liquid-phase and in the vapor

$$\operatorname{div} \boldsymbol{\sigma}_\alpha = -\rho^e_\alpha \omega^2 \boldsymbol{u}_\alpha, \qquad \alpha = l, v. \tag{5}$$

$$
\begin{aligned}
\sigma_{\mathrm{l}ij} &= -p_{\mathrm{l}} I_{ij} + 2\mu_{\mathrm{l}} \mathcal{D}_{\mathrm{l}ij}, \\
\sigma_{\mathrm{v}ij} &= -p_{\mathrm{v}} I_{ij} + \mu_{\mathrm{v}} \mathcal{D}_{\mathrm{v}kk} I_{ij} + 2\mu_{\mathrm{v}} \mathcal{D}_{\mathrm{v}ij}.
\end{aligned} \tag{6}
$$

$\boldsymbol{\sigma}$ and $\boldsymbol{\mathcal{D}}$ represent the stress and the rate of deformation tensors, respectively, while p, ρ and u are the pressure, the density and the displacement and μ denotes the viscosity.

- Mass balance in the liquid-phase and in the vapor

$$
\rho_\alpha + \rho_\alpha^e \mathrm{div}\, u_\alpha = 0, \qquad \alpha = l, \mathrm{v}. \tag{7}
$$

- Vapor equation of state

We assume that the vapor is a perfect gas. Thus we have

$$
p_{\mathrm{v}}^t = \mathcal{R} \rho_{\mathrm{v}}^t T_{\mathrm{v}}^t.
$$

Under small acoustic perturbations the above equation reduces to

$$
p_{\mathrm{v}} = P_{\mathrm{v}}^e \left(\frac{\rho_{\mathrm{v}}}{\rho_{\mathrm{v}}^e} + \frac{T_{\mathrm{v}}}{T^e} \right). \tag{8}
$$

- Energy balance in the liquid-phase and in the vapor

In the vapor and in the fluid the energy balance equations are respectively written as

$$
\mathrm{div}\, (k_{\mathrm{v}}\ \mathrm{grad}\ T_{\mathrm{v}}) = i\omega(\rho_{\mathrm{v}}^e C_{p\mathrm{v}} T_{\mathrm{v}} - p_{\mathrm{v}}), \tag{9}
$$

$$
\mathrm{div}\, (k_{\mathrm{l}}\ \mathrm{grad}\ T_{\mathrm{l}}) = i\omega\rho_{\mathrm{l}}^e C_{p\mathrm{l}} T_{\mathrm{l}}, \tag{10}
$$

T, k and C_p represent the temperature, the thermal conductivity and the specific heat at constant pressure.

- Conditions over the bubble surface Γ

Capillary effects: As a result of capillary effects, normal stresses are discontinuous over Γ

$$
(\sigma_{\mathrm{v}ij} - \sigma_{\mathrm{l}ij}) n_j = p_{\mathrm{c}} n_i, \qquad p_{\mathrm{c}} = \frac{2\sigma r}{R^2}, \tag{11}
$$

$\mathbf{n}$ denotes a unit vector normal to Γ and pointing towards the liquid. p_{c} and σ represent the capillary pressure and the surface tension, while R is the bubble radius at rest and r is the perturbation of the bubble curvature.
We further assume that

$$
\int_\Gamma p_{\mathrm{c}} n_i \,\mathrm{d}S = 0, \tag{12}
$$

which guarantees the equilibrium of the bubble surface.

Continuity of temperature fields: The surface Γ is assumed to be in thermodynamic equilibrium

$$T_\mathrm{v} = T_\mathrm{l} = T_\Gamma, \tag{13}$$

and T_Γ is given by Clausius–Clapeyron's law

$$T_\Gamma = p_\mathrm{v} \frac{T^e}{L^\varphi \rho_\mathrm{v}^e}, \tag{14}$$

where L^φ is the enthalpy of vaporization.

Continuity of heat fluxes:

$$(k_\mathrm{l} \operatorname{grad} T_\mathrm{l}) \cdot \mathbf{n} = (k_\mathrm{v} \operatorname{grad} T_\mathrm{v}) \cdot \mathbf{n} + L^\varphi j \cdot \mathbf{n}. \tag{15}$$

j is the phase-change mass flux over the vapor-liquid boundary Γ.

Continuity of displacements:

$$i\,\omega\, u_\Gamma \cdot \mathbf{n} = i\,\omega\, u_\mathrm{l} \cdot \mathbf{n} - \frac{1}{\rho_\mathrm{l}^t} j \cdot \mathbf{n} = i\,\omega\, u_\mathrm{v} \cdot \mathbf{n} - \frac{1}{\rho_\mathrm{v}^t} j \cdot n. \tag{16}$$

3.3. DESCRIPTION AT THE PORE-SCALE

For the sake of simplicity, thermoelastic effects are assumed to be negligible in the solid-phase. Thus, the mass balance equation for the solid phase is not required for describing the acoustics of the phase (Miklowitz, 1980). The description consists of Navier's equations in the solid-phase and of the classical boundary conditions over the pore surface, in which subscript f denotes quantities related to the fluid mixture.

• Momentum balance in the solid-phase

$$\operatorname{div} \boldsymbol{\sigma}_\mathrm{s} = -\rho_\mathrm{s}^e \omega^2 \boldsymbol{u}_\mathrm{s}, \tag{17}$$

$$\sigma_{\mathrm{s}ij} = a_{ijk\mathrm{l}} \boldsymbol{E}_{k\mathrm{l}}. \tag{18}$$

$\boldsymbol{E}$ represents the deformation tensor.

• Conditions over the bubbly fluid/solid interface Γ'

No-slip condition:

$$u_\mathrm{f} = u_\mathrm{s}. \tag{19}$$

Continuity of normal stresses:

$$\sigma_{\mathrm{f}ij} n_j = \sigma_{\mathrm{s}ij} n_j. \tag{20}$$

4. Preliminary Estimations

4.1. MATERIALS PROPERTIES

In view of the many equations that govern the whole problem, the order-of-magnitude analysis which will be carried out in the next section, as part of the normalisation process, promises to be rather complicated. For this reason, the present section relates specifically to performing a few preliminary estimations which are mainly concerned with the physics at the bubble-scale.

Let us consider the typical values shown on Table I for the physical characteristics of the vapor, the liquid-phase, the solid-phase and the skeleton.

Table I. Physical properties of the three-phase medium

Medium	Liquid	Vapor	Solid	Skeleton
Density ρ^e [kg/m^3]	10^3	1.2	2.5×10^3	1.75×10^3
Bulk modulus K [Pa]	$2\ 10^9$	1.4×10^5	40×10^9	10×10^9
Dynamic viscosity μ [Pa.s]	10^{-3}	20×10^{-6}		
Kinematic viscosity ν [m^2/s]	10^{-6}	15×10^{-6}		
Conductivity k [W/K.m]	0.6	0.026	3	
Heat capacity C_p [J/K.kg]	4.18×10^3	10^3	0.8×10^3	
Diffusivity d [m^2/s]	1.4×10^{-7}	2.1×10^{-5}	1.5×10^{-6}	

Furthermore, the bubbly fluid is characterized by:

- the surface tension $\sigma = 0.075$ N/m,
- the enthalpy of vaporization $\mathcal{L}^{\varphi} = 22.5 \times 10^5$ J/kg,
- the volume concentration of vapor in the bubbly fluid is $\beta = \mathrm{O}(1)$, which in the context of the homogenisation technique means $\varepsilon \ll \beta \leqslant 1$. Therefore, the following analysis is also valid for $\beta \ll 1$.

The liquid-phase is supposed to be initially at rest and at atmospheric pressure $P^e \approx 10^5$ Pa. As an illustrative example, let's consider a porous medium which is subjected to the propagation of a wave of velocity $c = 4000$ m/s, and of low frequency $f = 100$ Hz (*in fine*, the macroscopic model will be valid for a wide range of frequencies). The corresponding wavelength and macroscopic characteristic length are thus $\lambda \approx 40$ m and $l'' \approx 10$ m, respectively. We further assume the characteristic pore size to be $l' \approx 1$ cm and the bubble radius to be $R \approx 0{,}01$ mm. Hence, the scale ratio is $\varepsilon = l'/l'' \approx 10^{-3}$.

We are now in position to estimate quantities with respect to ε. Since the analysis below will focus on the fluid mixture, let us evaluate the following property ratios which will condition the physics at the bubble-scale:

$$\frac{\rho_{\rm v}^e}{\rho_1^e} = \mathrm{O}\,(\varepsilon), \qquad \frac{\mu_{\rm v}}{\mu_1} = \mathrm{O}\,(\varepsilon),$$
$$\frac{k_{\rm v}}{k_1} = \mathrm{O}\,(1), \frac{C_{p\rm v}}{C_{p1}} = \mathrm{O}, (1), \qquad \frac{K_{\rm v}}{K_1} = \mathrm{O}\,(\varepsilon). \tag{21}$$

4.2. INFLUENCE OF BUBBLE-SIZE ON HEAT TRANSFER

In this part, we highlight the impact of the bubble-size upon heat transfer within the fluid mixture. The thermal boundary-layer thickness is defined by

$$l_\alpha^{\rm therm} = \sqrt{\frac{d_\alpha}{\omega}} = \sqrt{\frac{k_\alpha}{\rho_\alpha^e C_{p\alpha}\omega}}, \qquad \alpha = l, v.$$

From heat balance equations in the vapor (9) and in the liquid-phase (10), we define the following dimensionless numbers:

$$\mathcal{N}_\alpha = \frac{|\ \mathrm{div}\ (k_\alpha\ \mathrm{grad}\ T_\alpha)\ |}{|\ i\,\omega\, C_{p\alpha} T_\alpha\ |} = \mathrm{O}\left(\frac{l_\alpha^{\rm therm}}{R}\right)^2, \qquad \alpha = l, v,$$

which shows that the local heat transfer regimes are conditioned by the order of the ratios

$$\frac{l_\alpha^{\rm therm}}{R}, \qquad \alpha = l, v.$$

From the property-ratio estimations (21), it turns out that

$$\frac{\mathcal{N}_{\rm v}}{\mathcal{N}_1} = \mathrm{O}\left(\frac{k_{\rm v}}{k_1} \times \frac{C_{p1}}{C_{p\rm v}} \times \frac{\rho_1^e}{\rho_{\rm v}^e}\right) = \mathrm{O}\,(\varepsilon^{-1}), \qquad \frac{l_{\rm v}^{\rm therm}}{l_1^{\rm therm}} = \mathrm{O}\,(\varepsilon^{-1/2}). \tag{22}$$

From these estimations, we define three characteristic bubble-sizes from the order of the ratio of the thermal layer thickness in the vapor to the bubble radius:

- Large bubble-size: $\frac{l_{\rm v}^{\rm therm}}{R} = \mathrm{O}\,(\varepsilon^{1/2})$, $\mathcal{N}_{\rm v} = \mathrm{O}\,(\varepsilon)$, $\mathcal{N}_1 = \mathrm{O}\,(\varepsilon^2)$. The thermal layer thickness in the vapor is lower than the bubble size. Heat transfer is negligible outside the boundary-layer and the process is quasi-adiabatic.
- Medium bubble-size: $\frac{l_{\rm v}^{\rm therm}}{R} = \mathrm{O}\,(1)$, $\mathcal{N}_{\rm v} = \mathrm{O}\,(1)$, $\mathcal{N}_1 = \mathrm{O}\,(\varepsilon)$. The layer thickness is of order of the bubble radius. Heat transfer occurs in the entire bubble volume. The heat transfer regime is transient.
- Small bubble-size: $\frac{l_{\rm v}^{\rm therm}}{R} = \mathrm{O}\,(\varepsilon^{-1/2})$, $\mathcal{N}_{\rm v} = \mathrm{O}\,(\varepsilon^{-1})$, $\mathcal{N}_1 = \mathrm{O}(1)$. The layer thickness is greater than the bubble-size. The process is isothermal.

We have shown that the bubble-size has an impact upon heat transfer process. As a result, it also affects phase change and capillary effects. The fact that the bubble-size has an influence on capillary effects is obvious from the expression of the capillary rigidity

$$K_c = \frac{2\sigma}{3R}.$$

In effect, it shows that the lower the bubble radius, the higher the capillary effects.

As shown in (Auriault and Boutin, 2000), the combined thermal, capillary and phase change effects are highlighted via the estimation of the apparent bulk modulus of the bubbly fluid, $\tilde{K}_f$. Below, we summarize the derivation of this estimation, which is detailed in (Auriault and Boutin, 2000).

Since both pressures p_l and p_v are of same order, the order of magnitude of the effective bulk modulus is given by

$$\tilde{K}_f = \mathrm{O}\left(-\frac{p_l}{\theta}\right),$$

where θ is the dilatation of the fluid mixture.

Noticing that

$$\theta = \beta\theta_v, \qquad p_l = \mathrm{O}\left(p_v + \frac{2\sigma r}{R^2}\right), \qquad r = \frac{R\theta_v}{3},$$

we get

$$\tilde{K}_f = \mathrm{O}\left(-\frac{p_v}{\beta\theta_v} - \frac{2\sigma}{3\beta R}\right) = \mathrm{O}\left(-\frac{p_v}{\beta\theta_v} - \frac{K_c}{\beta}\right). \tag{23}$$

An estimation of θ_v which would reveal thermal and phase change effects is therefore required. For this purpose, let decompose θ_v as follows:

$$\theta_v = \theta_{pT} + \theta_\varphi, \tag{24}$$

where θ_{pT} represents the dilatation in the absence of phase change, whereas θ_φ is the dilatation due to phase change.

θ_{pT} is estimated from the equation of state (8)

$$\theta_{pT} = \mathrm{O}\left(-\frac{p_v}{p_v^e} + \frac{T_v}{T^e}\right) = \mathrm{O}\left(-\frac{p_v}{p_v^e} + \frac{T_\Gamma}{T^e}\right).$$

Using Clapeyron's law (14), we get

$$\theta_{pT} = \mathrm{O}\left(-\frac{p_v}{p_v^e}\left(1 - \frac{p_v^e}{\rho_v^e L^\varphi}\right)\right). \tag{25}$$

θ_φ can be expressed in terms of the phase change mass flux, $\int_\Gamma j \cdot \mathbf{n}\, \mathrm{d}S$, as follows:

$$\theta_\varphi = \frac{\Delta\Omega_\varphi}{\Omega_\mathrm{v}} = -\frac{1}{\omega\ \rho_\mathrm{v}^e \Omega_\mathrm{v}} \int_\Gamma j \cdot \mathbf{n}\, \mathrm{d}S. \tag{26}$$

Noticing that the vapor heat flux term in Equation (15) is negligible, see estimation (22), we get

$$L^\varphi j \cdot \mathbf{n} = -k_\mathrm{l} \text{ grad } T_\mathrm{l} \cdot \mathbf{n}.$$

Integrating this equation over the bubble interface, Γ, and using the divergence theorem leads to an expression for the vaporization/condensation mass flux term

$$\begin{aligned} L^\varphi \int_\Gamma j \cdot \mathbf{n}\, \mathrm{d}S &= -\int_\Gamma k_\mathrm{l} \text{ grad } T_\mathrm{l} \cdot \mathbf{n} \\ &= -\int_{\Omega_\mathrm{l}} \text{div } (\lambda_\mathrm{l} \text{ grad } T_\mathrm{l})\, \mathrm{d}\Omega \\ &= -i\,\omega\,\rho_\mathrm{l}^e\, C_{p\mathrm{l}} \int_{\Omega_\mathrm{l}} T_\mathrm{l}\, \mathrm{d}\Omega. \end{aligned} \tag{27}$$

Now, T_l is determined by solving the boundary-value problem defined by equation (10) and boundary condition (13). It can be shown (Auriault, 1983; Auriault and Boutin, 2000) that the solution to this boundary-value problem is written as

$$T_\mathrm{l} = \frac{p_\mathrm{v} T^e}{L^\phi\ \rho_\mathrm{v}^e}(1 - h(y)), \tag{28}$$

in which h is a complex-valued function of the dimensionless pulsation $\omega^* = \omega/\omega^{\text{therm}}$, and where $\omega^{\text{therm}} = d_\mathrm{l}/R^2$. It can be shown that (see Auriault, 1983; Auriault and Boutin, 2000):

$$\lim_{\omega^* \to 0} h(y) = 0, \qquad \lim_{\omega^* \to \infty} h(y) = 1.$$

As a consequence, h is such that:

- $|1 - h| \ll 1$ for large-size bubbles,
- $|h| = \mathrm{O}(1)$ for medium-size bubbles,
- $|h| \ll 1$ for small-size bubbles.

Thus, the expression for the flux (27) becomes

$$\begin{aligned} L^\varphi \int_\Gamma j \cdot \mathbf{n}\, \mathrm{d}S &= -i\,\omega\,\rho_\mathrm{l}^e C_{p\mathrm{l}} \int_{\Omega_\mathrm{l}} T_\mathrm{l}\, \mathrm{d}\Omega \\ &= -i\,\omega\,\rho_\mathrm{l}^e C_{p\mathrm{l}} \frac{p_\mathrm{v} T^e \Omega_\mathrm{l}}{L^\varphi \rho_\mathrm{v}^e}(1 - H), \end{aligned} \tag{29}$$

in which the complex-valued thermal memory function H is defined by

$$H = \frac{1}{\Omega_\mathrm{l}} \int_{\Omega_\mathrm{l}} h\, \mathrm{d}\Omega, \tag{30}$$

and where:

- $|1-H| \ll 1$ for large-size bubbles,
- $|H| = \mathrm{O}(1)$ for medium-size bubbles,
- $|H| \ll 1$ for small-size bubbles.

The vaporization-condensation mass flux term is therefore estimated as:

$$\int_\Gamma j \cdot \mathbf{n}\,\mathrm{d}S = \mathrm{O}(\mathcal{C}\,\omega\,\Omega_\mathrm{l}\,\rho_\mathrm{v}^e(1-H)), \tag{31}$$

where

$$\mathcal{C} = \frac{\rho_\mathrm{l}^e C_{p\mathrm{l}} T^e P^e}{(\rho_\mathrm{v}^e L^\varphi)^2} \approx 20. \tag{32}$$

From (26) and (31), we get the estimation for θ_φ

$$\theta_\varphi = \mathrm{O}\left(-\frac{1-\beta}{\beta}\,\mathcal{C}\,\frac{p_\mathrm{v}}{p_\mathrm{l}}(1-H)\right). \tag{33}$$

From (25) and (33), we finally get the estimation of θ_v

$$\theta_\mathrm{v} = \mathrm{O}\left(-\frac{p_\mathrm{v}}{P^e}\left(\frac{P^e}{p_\mathrm{v}^e} - \frac{P^e}{\rho_\mathrm{v}^e L^\varphi} + \frac{1-\beta}{\beta}\,\mathcal{C}\,(1-H)\right)\right). \tag{34}$$

Therefore, according to (23), the estimation of $\tilde{K}_\mathrm{f}$ is given by

$$\tilde{K}_\mathrm{f} = \mathrm{O}\left(\frac{P^e}{\beta(\frac{P^e}{P^e+2\sigma/R} - \frac{P^e}{L^\varphi\rho_\mathrm{v}^e} + \frac{1-\beta}{\beta}\mathcal{C}(1-H))} - \frac{2\sigma}{3\beta R}\right). \tag{35}$$

Note that $\tilde{K}_\mathrm{f}$ is complex-valued. From Equation (35), three different regimes are obtained with respect to the bubble-size:

• For large-size bubbles ($l_\mathrm{v}^\mathrm{therm} \ll R$), we get

$$|1-H| \ll 1, \qquad \tilde{K}_\mathrm{f} = \mathrm{O}\left(\frac{P^e}{\beta}\right)$$

Capillary and phase change effects are negligible.

• For medium-size bubbles ($l_\mathrm{v}^\mathrm{therm} = \mathrm{O}(R)$), we get

$$|1-H| = \mathrm{O}(1), \qquad \tilde{K}_\mathrm{f} = \mathrm{O}(\frac{P^e}{(1-\beta)\mathcal{C}(1-H)}) \ll P^e.$$

Phase change effects are dominant.

• For small-size bubbles ($l_\mathrm{v}^\mathrm{therm} \gg R$), we get

$$|H| \ll 1, \qquad \tilde{K}_\mathrm{f} = \mathrm{O}\left(-\frac{2\sigma}{3\beta R}\right) = \mathrm{O}(-K_\mathrm{c}),$$

which shows that the apparent bulk modulus of the fluid mixture is negative, (Auriault and Lebaigue, 1989).

Thus, the characteristic bubble-size has a strong impact upon the apparent bulk modulus. We can therefore expect that the wave velocity and the wave attenuation be conditioned by the bubble-size.

4.4. CASE UNDER CONSIDERATION

It can be shown that fluid mixtures of small-size, medium-size and large-size bubbles lead to similar macroscopic equations. The differences between the behaviours rely on the form and the properties of the effective coefficients. As for the case of large-size bubbles, it is of lower interest as it leads to the behaviour of a porous medium saturated by a gas.

In the analysis below, we will consider the case of small-size bubbles, which according to the above estimations is characterized by

$$\mathcal{N}_{\mathrm{v}} = \left(\frac{l_{\mathrm{v}}^{\mathrm{therm}}}{R}\right)^2 = \mathrm{O}(\varepsilon^{-1}), \qquad \mathcal{N}_{\mathrm{l}} = \left(\frac{l_{\mathrm{l}}^{\mathrm{therm}}}{R}\right)^2 = \mathrm{O}(1),$$
$$|H| \ll 1, \qquad \tilde{K}_{\mathrm{f}} = \mathrm{O}\left(-\frac{2\sigma}{3\beta R}\right) = \mathrm{O}(-K_{\mathrm{c}}). \tag{36}$$

It can further be shown that

$$\frac{\tilde{K}_{\mathrm{f}}}{\tilde{K}_m} \ll 1,$$

which proves that the bulk deformation mainly occurs within the fluid. As a result, the fluid mixture dilatation is such that

$$\theta = \beta\theta_{\mathrm{v}} = \mathrm{O}\left(\frac{u}{\lambda}\right) = \mathrm{O}\left(\frac{u}{l''}\right),$$

from which we deduce

$$\theta_{\mathrm{v}} = \mathrm{O}\left(\frac{u}{l''}\right). \tag{37}$$

Since $\theta_{\mathrm{v}} = \frac{3r}{R}$, we get

$$\frac{r}{u} = \mathrm{O}\,(\varepsilon^2), \tag{38}$$

which shows that the movement of the bubble interface is a rigid displacement at the first two orders.

The purpose of the next section is thus the normalization of the equations in the case of small-size bubbles and equal scale ratios. The homogenisation procedure for small-size bubbles is detailed in appendix A. The derived macroscopic behaviour is presented in Section 6 and the differences between the behaviours for small-size bubbles and for medium-size bubbles will then be outlined.

5. Normalization

5.1. INTRODUCTION

The purpose of the present section is to normalize the local description (5)–(20) in the case of small-size bubbles. For each equation, the procedure consists in: (i) defining the dimensionless numbers that emerge from the equation; (ii) estimating these numbers with respect to ε; (iii) deducing the corresponding dimensionless writing of the equation.

The estimation of the dimensionless numbers requires the choice of a characteristic length. Let choose l for normalizing the equations at the bubble scale and l' for normalizing the equations at the pore scale. Once they are normalized the equations are written in terms of dimensionless quantities. For ease of formulation, we will keep the same notations for the dimensional quantities and their dimensionless counterparts.

5.2. NORMALIZATION OF THE BUBBLE-SCALE DESCRIPTION

- Momentum balance in the liquid-phase

Equations (5) and (6) can be expressed as follows:

$$-\operatorname{grad} p_\alpha + i\omega\mu_\alpha(\operatorname{grad}\operatorname{div} u_\alpha + \Delta u_\alpha) = -\rho_\alpha^e\omega^2 u_\alpha, \qquad \alpha = l, v. \tag{39}$$

From Equation (39) emerge the following dimensionless numbers:

$$Q_\alpha = \frac{|\operatorname{grad} p_\alpha|}{|\rho_\alpha^e\omega^2 u_\alpha|}, \qquad R_\alpha = \frac{|\mu_\alpha \Delta u_\alpha|}{|\rho_\alpha^e\omega^2 u_\alpha|}, \qquad \alpha = l, v.$$

Since we are using l for normalizing the equations at the bubble scale, the numbers to be estimated are thus the following:

$$\begin{aligned} Q_{\alpha_l} &= \mathrm{O}\left(\frac{\delta p/l}{\rho_\alpha^e\omega^2 u_\alpha}\right) = \mathrm{O}\left(\frac{\delta p/l}{\rho_\alpha^e\omega v_\alpha}\right), \qquad \alpha = l, v, \\ R_{\alpha_l} &= \mathrm{O}\left(\frac{\mu_\alpha\omega u_\alpha/l^2}{\rho_\alpha^e\omega^2 u_\alpha}\right) = \mathrm{O}\left(\frac{\mu_\alpha v_\alpha/l^2}{\rho_\alpha^e\omega v_\alpha}\right), \qquad \alpha = l, v. \end{aligned}$$

The wave propagation is driven by a macroscopic pressure gradient $\mathrm{O}(\delta p/l")$, where δp is the macroscopic pressure drop amplitude. In the liquid phase, this macroscopic pressure gradient term is balanced by the inertial term in (39)

$$\frac{\delta p}{l"} = \mathrm{O}(\rho_1^e\omega v_1), \tag{40}$$

from which we deduce

$$\frac{\delta p}{l} = \mathrm{O}(\varepsilon^{-2}\rho_\mathrm{l}^e \omega v_\mathrm{l}), \qquad Q_{l_\mathrm{l}} = \mathrm{O}(\varepsilon^{-2}). \tag{41}$$

Viscous effects are quite low in the present problem. In order to preserve generality of the investigation we wish to take these effects into account at the pore scale: this is of no consequence upon the final result (Levy, 1979; Auriault, 1980). Assuming that the mixture viscosity is of same order as the liquid viscosity, the condition for which viscous effects will appear at the pore scale is

$$\mu_\mathrm{l}\frac{v_\mathrm{l}}{l'^2} = \mathrm{O}(\rho_\mathrm{l}^e \omega v_\mathrm{l}), \tag{42}$$

which entails

$$\mu_\mathrm{l}\frac{v_\mathrm{l}}{l^2} = \mathrm{O}(\varepsilon^{-2}\rho_\mathrm{l}^e \omega v_\mathrm{l}), \qquad R_{l_\mathrm{l}} = \mathrm{O}(\varepsilon^{-2}). \tag{43}$$

Hence, the dimensionless momentum balance in the liquid-phase is written as

$$-\mathrm{grad}\ p_\mathrm{l} + i\omega\mu_\mathrm{l}(\mathrm{grad\ div}\ u_\mathrm{l} + \Delta u_\mathrm{l}) = -\varepsilon^2\rho_\mathrm{l}^e\omega^2 u_\mathrm{l} \tag{44}$$

or

$$\mathrm{div}\ \boldsymbol{\sigma}_\mathrm{l} = -\varepsilon^2\rho_\mathrm{l}^e\omega^2 u_\mathrm{l}, \tag{45}$$

where

$$\sigma_{\mathrm{l}ij} = -p_\mathrm{l} I_{ij} + 2\mu_\mathrm{l}\mathcal{D}_{\mathrm{l}ij}. \tag{46}$$

• Momentum balance in the vapor

The dimensionless momentum balance equation in the vapor is deduced from the order-of-magnitude relationships obtained for the liquid-phase (41), (43) and for the property ratios (21). We obtain

$$-\ \mathrm{grad}\ p_\mathrm{v} + \varepsilon i\omega\mu_\mathrm{v}(\mathrm{grad\ div}\ u_\mathrm{v} + \Delta u_\mathrm{v}) = -\varepsilon^3\rho_\mathrm{v}^e\omega^2 u_\mathrm{v}, \tag{47}$$

which can also be written as

$$\mathrm{div}\ \boldsymbol{\sigma}_\mathrm{v} = -\varepsilon^3\rho_\mathrm{v}^e\omega^2 u_\mathrm{v}, \tag{48}$$

in which

$$\sigma_{\mathrm{v}ij} = -p_\mathrm{v} I_{ij} + 2\varepsilon\mu_\mathrm{v}\mathcal{D}_{\mathrm{v}ij}. \tag{49}$$

• Mass balance in the vapor

From the mass balance equation in the vapor arises the following dimensionless number:

$$S_{\mathrm{v_l}} = \frac{\rho_\mathrm{v}}{\rho_\mathrm{v}^e u/l} = \mathrm{O}\left(\varepsilon^{-1}\frac{\rho_\mathrm{v}}{\rho_\mathrm{v}^e}\right).$$

The mass balance equation in the vapor actually states that

$$\frac{\rho_v}{\rho_v^e} = O(\operatorname{div} u_v) = O(\theta_v),$$

which according to (37) leads to

$$\frac{\rho_v}{\rho_v^e} = O\left(\frac{u}{l''}\right) = O(\varepsilon^3).$$

Therefore we get

$$S_{v_1} = O(\varepsilon^2),$$

and the dimensionless form of the equation is

$$\varepsilon^2 \rho_v + \rho_v^e \operatorname{div} u_v = 0. \tag{50}$$

- Mass balance in the liquid-phase

In the liquid-phase, the number to be estimated is

$$S_{l_1} = \frac{\rho_l}{\rho_l^e u/l} = O\left(\varepsilon^{-1}\frac{\rho_l}{\rho_l^e}\right).$$

The mass balance equation in the liquid-phase means that

$$\frac{\rho_l}{\rho_l^e} = O(\operatorname{div} u_l) = O(\theta_l).$$

Now, noticing that

$$\frac{\theta_v}{\theta_l} = O\left(\frac{K_l}{K_v}\right) = O(\varepsilon^{-1}),$$

we obtain

$$\frac{\rho_l}{\rho_l^e} = O(\theta_l) = O(\varepsilon\theta_v) = O(\varepsilon^4), \qquad S_{l_1} = O(\varepsilon^3).$$

The corresponding dimensionless equation is thus written as follows:

$$\varepsilon^3 \rho_l + \rho_l^e \operatorname{div} u_l = 0. \tag{51}$$

- Vapor equation of state

As shown in (Boutin and Auriault, 1993), the dimensionless equation of state reads

$$p_v = P_v^e\left(\frac{\rho_v}{\rho_v^e} + \varepsilon\frac{T_v}{T^e}\right). \tag{52}$$

- Energy balance in the vapor

From Equation (9) arise two dimensionless numbers

$$M_{\mathrm{v}} = \frac{p_{\mathrm{v}}}{\rho_{\mathrm{v}}^{e} C_{\mathrm{pv}} T_{\mathrm{v}}}, \qquad N_{\mathrm{v}} = \frac{k_{\mathrm{v}}}{\rho_{\mathrm{v}}^{e} C_{\mathrm{pv}} \omega}.$$

We have defined the small-size bubble case by (see (36))

$$N_{\mathrm{v}} = \mathrm{O}(\varepsilon^{-1}).$$

Further assuming that $M_{\mathrm{v}} = \mathrm{O}(1)$, we obtain the following dimensionless form for the equation:

$$\varepsilon^{-1} \operatorname{div}\,(k_{\mathrm{v}} \operatorname{grad} T_{\mathrm{v}}) = i\omega(\rho_{\mathrm{v}}^{e} C_{\mathrm{pv}} T_{\mathrm{v}} - p_{\mathrm{v}}). \tag{53}$$

• Energy balance in the liquid-phase

The only dimensionless number that emerges from Equation (10) is N_{l}, which according to (36) is O(1). Therefore, the dimensionless energy balance equation in the liquid-phase is

$$\operatorname{div}\,(k_{\alpha} \operatorname{grad} T_{\mathrm{l}}) = i\omega\rho_{\mathrm{l}}^{e} C_{p\mathrm{l}} T_{\mathrm{l}}. \tag{54}$$

• Conditions over the bubble surface Γ

Capillary effects: Capillary effects are characterized by the Weber's number which can be defined by

$$\mathcal{W} = \frac{p_{\mathrm{c}}}{p_{\mathrm{l}}} = \mathrm{O}\left(\frac{2\sigma r/R^2}{\tilde{K}_{\mathrm{f}} \frac{u}{l''}}\right).$$

Now, according to (36), we have

$$\mathcal{W} = \mathrm{O}\left(\frac{2\sigma r/R^2}{\frac{2\sigma}{3\beta R}\frac{u}{l''}}\right) = \mathrm{O}\left(3\,\frac{r}{u}\,\frac{l''}{R}\right) = \mathrm{O}(1).$$

Thus, the dimensionless capillary law is

$$(\sigma_{\mathrm{v}ij} - \sigma_{\mathrm{l}ij})n_j = p_{\mathrm{c}} n_i, \qquad p_{\mathrm{c}} = \frac{2\sigma r}{R^2}, \qquad \int_{\Gamma} p_{\mathrm{c}}\, n_i \,\mathrm{d}S = 0. \tag{55}$$

Continuity of temperature fields: When placed in a dimensionless form, the continuity of temperature fields (13) simply reads

$$T_{\mathrm{v}} = T_{\mathrm{l}} = T_{\Gamma}. \tag{56}$$

Continuity of heat fluxes: As shown in (Auriault and Boutin, 2000), the mass flux $j \cdot \mathbf{n}$ is mainly generated by the heat flux in the liquid. Thus,

$$\mid (k_{\mathrm{l}} \operatorname{grad} T_{\mathrm{l}}) \cdot \mathbf{n} \mid = \mathrm{O} \mid L^{\varphi} j \cdot \mathbf{n} \mid .$$

Furthermore, according to (21) and (22), we have

$$\frac{|\ (k_{\mathrm{l}}\ \mathrm{grad}\ T_{\mathrm{l}})\cdot\mathbf{n}\ |}{|\ (k_{\mathrm{v}}\ \mathrm{grad}\ T_{\mathrm{v}})\cdot\mathbf{n}\ |} = \mathrm{O}\left(\frac{k_{\mathrm{l}}}{k_{\mathrm{v}}}\frac{l_{\mathrm{v}}^{\mathrm{therm}}}{l_{\mathrm{l}}^{\mathrm{therm}}}\right) = \mathrm{O}(\varepsilon^{-2}).$$

Therefore, the dimensionless heat flux balance is written as

$$(k_{\mathrm{l}}\ \mathrm{grad}\ T_{\mathrm{l}})\cdot\mathbf{n} = \varepsilon^{2}(k_{\mathrm{v}}\ \mathrm{grad}\ T_{\mathrm{v}})\cdot\mathbf{n} + L^{\varphi} j\cdot\mathbf{n}. \tag{57}$$

Continuity of displacements: As shown in (Auriault and Boutin, 2000), we have

$$\frac{j\cdot\mathbf{n}}{\rho_{\mathrm{l}}^{e}\omega u_{\mathrm{l}}} = \mathrm{O}\left(\mathcal{C}\frac{\rho_{\mathrm{v}}^{e}}{\rho_{\mathrm{l}}^{e}}\frac{l}{u_{\mathrm{l}}}\frac{p_{\mathrm{v}}^{0}}{P^{e}}\right) = \mathrm{O}(\varepsilon^{3}). \tag{58}$$

Thus, the dimensionless displacement condition on Γ is written as

$$i\omega u_{\Gamma i}\, n_i = i\omega u_{\mathrm{l}i}, \qquad n_i - \varepsilon^{3}\frac{j_i\, n_i}{\rho_{\mathrm{l}}^{t}} = i\omega u_{\mathrm{v}i}, \qquad n_i - \varepsilon^{2}\frac{j_i\, n_i}{\rho_{\mathrm{v}}^{t}}. \tag{59}$$

5.3. NORMALIZATION OF THE PORE SCALE DESCRIPTION

• Momentum balance in the solid-phase

From the momentum balance equation in the solid-phase arises the following dimensionless number:

$$P_{\mathrm{s}} = \frac{|\ \mathrm{div}\ \boldsymbol{\sigma}_{\mathrm{s}}\ |}{|\ \rho_{\mathrm{s}}^{e}\omega^{2}u_{\mathrm{s}}\ |}.$$

Since l' is used for normalizing the pore scale description, the number to be estimated is

$$P_{\mathrm{s_l}} = \frac{a/l'}{\rho_{\mathrm{s}}^{e}\omega^{2}l'},$$

where a is a characteristic value of the elastic coefficients. Now, the wave velocity, c, is such that

$$c = \mathrm{O}\left(\sqrt{\frac{a}{\rho_{\mathrm{s}}^{e}}}\right) = \mathrm{O}\left(\frac{\lambda\omega}{2\pi}\right) = \mathrm{O}(l''\omega),$$

from which we get

$$P_{\mathrm{s_l}} = \mathrm{O}\left(\frac{l''^{2}}{l'^{2}}\right) = \mathrm{O}(\varepsilon^{-2}).$$

Hence, the dimensionless momentum balance in the solid-phase is expressed as follows

$$\mathrm{div}\ \boldsymbol{\sigma}_{\mathrm{s}} = -\varepsilon^{2}\rho_{\mathrm{s}}^{e}\omega^{2}u_{\mathrm{s}}, \tag{60}$$

$$\sigma_{\mathrm{s}ij} = a_{ijkl}\boldsymbol{E}_{kl}. \tag{61}$$

• Conditions over the bubbly fluid/solid interface Γ'

No-slip condition: The dimensionless displacement condition over Γ' is merely

$$u_{\mathrm{f}} = u_{\mathrm{s}}. \tag{62}$$

Continuity of normal stresses: The dimensionless number introduced by the condition of continuity of stresses is the ratio

$$\Sigma = \mathrm{O}\left(\frac{\sigma_{\mathrm{s}}}{\sigma_{\mathrm{f}}}\right).$$

The solid and the mixture isotropic stress perturbations are of same order of magnitude at the macroscopic level, which in terms of order of magnitude is expressed as

$$\Sigma_{l''} = \frac{au}{P_{\mathrm{f}}l''} = \mathrm{O}(1),$$

from which we deduce

$$\Sigma_{l'} = \frac{au}{P_{\mathrm{f}}l'} = \mathrm{O}(\varepsilon^{-1}).$$

Therefore, the dimensionless form of the continuity of stresses is

$$\varepsilon\sigma_{\mathrm{f}ij}n_j = \sigma_{\mathrm{s}ij}n_j. \tag{63}$$

5.4. HOMOGENISATION PROCEDURE

In the above normalized equations all quantities are dimensionless, including the divergence, gradient and Laplace operators. In particular, the gradient operator, grad, is equal to

- $\mathrm{grad}_z + \varepsilon\,\mathrm{grad}_y + \varepsilon^2\mathrm{grad}_x$

in the equations at the bubble scale since l has been used for normalising, (64)

- $\mathrm{grad}_y + \varepsilon\,\mathrm{grad}_x$

in the equations at the pore scale since l' has been used for normalising.

The method consists now in introducing the asymptotic expansions (3) in the set of dimensionless equations. Solving the boundary-value problems arising at the successive orders of ε at the bubble-scale and then at the pore scale leads to the macroscopic description.

The upscaling procedure is detailed in the Appendix. In the next section we present and comment the derived macroscopic behaviour.

6. Derived Macroscopic Behaviour

We obtain the following set of macroscopic equations for describing wave propagation in a porous structure saturated by a bubbly fluid:

$$\frac{\partial \sigma_{Tij}}{\partial x_j} = -\omega^2(\langle \rho_s \rangle_{\Omega'} u_{si} + (1-\beta)\rho_l \langle u_{fi} \rangle_{\Omega'}), \tag{A.37}$$

$$\sigma_{Tij} = c_{ijkh} E_{xkh}(u_s) - \alpha_{ij}\pi, \tag{A.38}$$

$$\frac{\partial(\langle u_{fi} \rangle_{\Omega'} - \phi u_{si})}{\partial x_i} = -\alpha_{ij} E_{xij}(u_s) - \gamma\pi - \phi \mathcal{A}\pi, \tag{A.39}$$

$$\langle u_{fi} \rangle_{\Omega'} - \phi u_{si} = -\frac{K_{ij}}{i\omega}\left(\frac{\partial \pi}{\partial x_j} - \omega^2(1-\beta)\rho_l u_{si}\right), \tag{A.40}$$

in which σ_{Tij} is the total stress tensor and is defined by Equation (A.38), c_{ijkh}, is the bulk elastic tensor of the empty porous matrix, α_{ij} and γ are effective elastic coefficients, K_{ij} is the dynamic permeability and is complex-valued and ω-dependent, π is the effective bubbly-fluid pressure, ϕ is the porosity and $\mathcal{A} = 1/\tilde{K}_f$ is the complex-valued apparent compressibility of the bubbly fluid. It can be shown that we would get the same macroscopic description for $l'/l'' \ll l/l'$, which proves that this model remains valid for a wide range of ω.

The macroscopic description (A.37)–(A.40) has the classical structure of the law that describes wave propagation in saturated porous media (Auriault, 1980). In effect, when $\mathcal{A} = 0$, we retrieve the behaviour of a porous medium saturated by an incompressible fluid. The wave velocity in a porous matrix saturated by a compressible fluid (e.g. a gas) is related to the real part of $\mathcal{A}$, $Re(\mathcal{A}) > 0$. For a bubbly fluid, not only is the apparent compressibility affected by the presence of bubbles in the liquid, but it is also conditioned by the characteristic bubble size. In effect, for medium-size bubbles $Re(\mathcal{A})$ is much greater than the value obtained for a gas saturating the medium, whereas for small-size bubbles $\mathcal{A}$ becomes real and negative. For large-size bubbles, the value of $Re(\mathcal{A})$ is comparable with the value obtained for a gas-saturated medium.

From the above comments, we can expect that wave velocity and attenuation be affected by the presence of bubbles within the liquid. The propagation of the dilatational waves should therefore be influenced by the presence of small-size or medium-size bubbles. In order to investigate these effects, let us consider for ease of formulation, the case of homogeneous and isotropic matrix, solid and permeability

$$\langle \rho_s \rangle_{\Omega'} = (1-\phi)\rho_s, \qquad c_{ijkl} = A I_{ij} I_{kl} + N(I_{ik} I_{jl} + I_{il} I_{jk}),$$
$$K_{ij} = K I_{ij}, \qquad \alpha_{ij} = \alpha I_{ij},$$

where A and N are the Lamé constants (Biot notations).

We also simplify the notation by letting $\phi u_{\mathrm{f}} = \langle u_{\mathrm{f}i} \rangle_{\Omega'}$, $\theta_{\mathrm{s}} = E_{xii}(u_{\mathrm{s}})$ and $\theta_{\mathrm{f}} = E_{xii}(u_{\mathrm{f}})$. The macroscopic model becomes

$$N\Delta u_{\mathrm{s}} + \operatorname{grad}[(A+N)\theta_{\mathrm{s}}] - \alpha \operatorname{grad} \pi = -\omega^2((1-\phi)\rho_{\mathrm{s}} u_{\mathrm{s}} + \phi(1-\beta)\rho_{\mathrm{l}} u_{\mathrm{f}}), \quad (65)$$

$$\phi\theta_{\mathrm{f}} + (\alpha-\phi)\theta_{\mathrm{s}} = -(\phi\mathcal{A} + \gamma)\pi, \quad (66)$$

$$u_{\mathrm{f}} - u_{\mathrm{s}} = -\frac{K}{i\omega\phi}(\operatorname{grad} \pi - \omega^2(1-\beta)\rho_{\mathrm{l}} u_{\mathrm{s}}). \quad (67)$$

Rotational waves equations are obtained by applying the curl operator to Equations (65) and (67). Introducing

$$\operatorname{curl} u_{\mathrm{s}} = \omega, \qquad \operatorname{curl} u_{\mathrm{f}} = \Omega,$$

we get

$$N\Delta\omega = -\omega^2((1-\phi)\rho_{\mathrm{s}}\omega + \phi(1-\beta)\rho_{\mathrm{l}}\Omega),$$

$$\Omega - \omega = \frac{K}{i\omega\phi}\omega^2(1-\beta)\rho_{\mathrm{l}}\omega.$$

Eliminating Ω between these two latter equations yields the wave equation

$$N\Delta\omega = -\omega^2\left[(1-\phi)\rho_{\mathrm{s}} + \phi(1-\beta)\rho_{\mathrm{l}} + \frac{K}{i\omega}\omega^2(1-\beta)^2\rho_{\mathrm{l}}^2\right]\omega. \quad (68)$$

As expected, the propagation of rotational waves is not affected by the presence of bubbles. In effect, this equation is identical to that for porous media saturated by an incompressible fluid.

Dilatational wave equations are obtained by applying the divergence operator to Equations (65) and (67) and the Laplace operator to equation (66)

$$(A+2N)\Delta\theta_{\mathrm{s}} - \alpha\Delta\pi = -\omega^2((1-\phi)\rho_{\mathrm{s}}\theta_{\mathrm{s}} + \phi(1-\beta)\rho_{\mathrm{l}}\theta_{\mathrm{f}}),$$

$$\phi\Delta\theta_{\mathrm{f}} + (\alpha-\phi)\Delta\theta_{\mathrm{s}} = -(\phi\mathcal{A}+\gamma)\Delta\pi,$$

$$\theta_{\mathrm{f}} - \theta_{\mathrm{s}} = -\frac{K}{i\omega\phi}(\Delta\pi - \omega^2(1-\beta)\rho_{\mathrm{l}}\theta_{\mathrm{s}}).$$

Eliminating $\Delta\pi$ between these three equations gives the wave equations for θ_{s} and θ_{f}

$$(A+2N)\Delta\theta_{\mathrm{s}} = a_{11}\theta_{\mathrm{s}} + a_{12}\theta_{\mathrm{f}},$$

$$\phi\Delta\theta_{\mathrm{f}} + (\alpha-\phi)\Delta\theta_{\mathrm{s}} = a_{21}\theta_{\mathrm{s}} + a_{22}\theta_{\mathrm{f}},$$

with

$$a_{11} = \frac{i\omega\phi\alpha}{K} - \omega^2(1-\phi)\rho_{\mathrm{s}} + \omega^2\alpha(1-\beta)\rho_{\mathrm{l}},$$

$$a_{12} = -\frac{i\omega\phi\alpha}{K} - \omega^2\phi(1-\beta)\rho_{\mathrm{l}},$$

$$a_{21} = -(\phi\mathcal{A}+\gamma)\left(\frac{i\omega\phi}{K} + \omega^2(1-\beta)\rho_{\mathrm{l}}\right), \qquad a_{22} = (\phi\mathcal{A}+\gamma)\frac{i\omega\phi}{K}.$$

Let us look for θ_s and θ_f in the form

$$\theta_s = C_1 e^{i\omega/c}, \qquad \theta_f = C_2 e^{i\omega/c},$$

where c is the complex velocity. We obtain

$$\begin{aligned}
&-\omega^2(A+2N)C_1 = c^2 a_{11} C_1 + c^2 a_{12} C_2,\\
&-\omega^2 \phi C_2 - \omega^2(\alpha - \phi)C_1 = c^2 a_{21} C_1 + c^2 a_{22} C_2.
\end{aligned}$$

Eliminating C_1 and C_2 yields an equation for c

$$\begin{aligned}
&c^4(a_{11}a_{22} - a_{12}a_{21}) + c^2\omega^2(\phi a_{11} + (A+2N)a_{22} -\\
&\quad - (\alpha - \phi)a_{12}) + \omega^4(A+2N)\phi = 0
\end{aligned} \tag{69}$$

As usual in Biot's theory, two dilatational waves are obtained from Equation (69), namely waves P_1 and P_2.

In order to highlight the impact of the presence of the bubbles upon the velocities and the attenuations of waves P_1 and P_2, let us solve the above equation in a particular case. We consider the simple ideal case of a bundle of capillary tubes for representing the porous structure. Thus, the pores are circular cylinders of radius a. The properties of the porous medium and of the bubbly fluid are those shown in Table I. The bubble concentration is $\beta = 0,2$ and the effective viscosity of the bubbly fluid is supposed to be isotropic and equal to the liquid viscosity: $\mu^{\text{eff}} = \mu$. Coefficients α and γ are calculated from the formulas in (Auriault and Sanchez-Palencia, 1977). The thermal memory function H is the function G is in (Allard *et al.*, 1993), see (A2.8) in this paper. It can be put in the form

$$H = \frac{i\omega^*}{i\omega^* + \sqrt{F_t^2 + i\omega^*}},$$

in which F_t is a shape factor ($F_t = 2$ for cylindrical pores), and $\omega^* = \omega/\omega_t$ is a dimensionless frequency, where the characteristic frequency ω_t is given by

$$\omega_t = \frac{d_l}{\Lambda_t^2}, \qquad \Lambda_t = \frac{\Omega_l}{\Gamma} = \frac{1-\beta}{3\beta}R.$$

The attenuation is measured by means of the inverse quality factor, Q^{-1}, where the quality factor Q is the ratio of the imaginary part κ_i of the wave number $\kappa = 2\pi/\lambda$ to its real part κ_r

$$Q = \frac{\kappa_i}{\kappa_r}.$$

The velocities and attenuations of waves P_1 and P_2 are plotted in Figures 3 through 6 in the cases of medium-size bubbles (water with vapor bubbles and phase change) and small-size bubbles. They are compared with the velocities and attenuations of waves P_1 and P_2 in the same porous matrix but saturated by water and by steam.

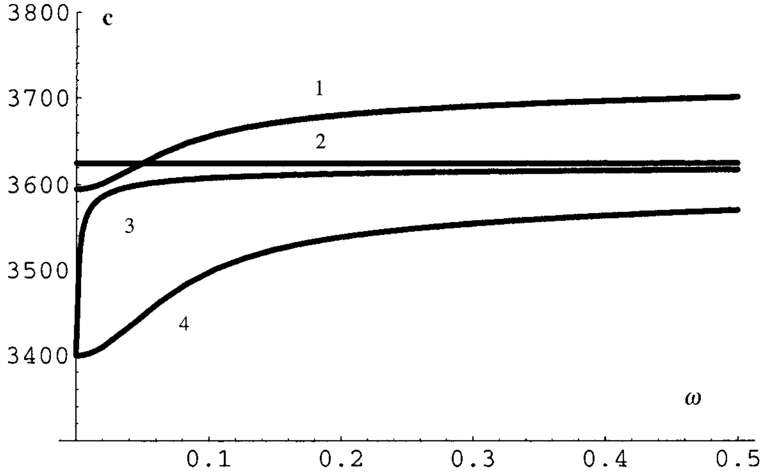

Figure 3. P_1 wave velocity in the porous medium saturated by: 1. water; 2. water vapor; 3. water with vapor bubbles and phase change, medium-size bubbles; 4. water with small vapor bubbles.

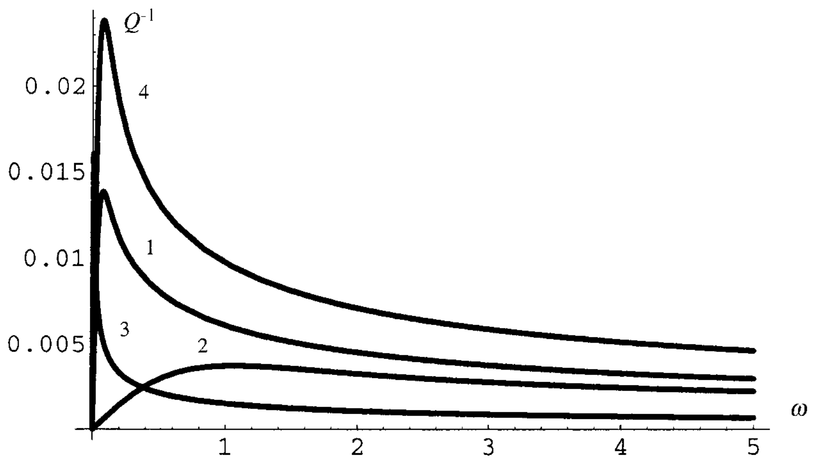

Figure 4. P_1 wave attenuation Q^{-1} in the porous medium saturated by: 1. water; 2. water vapor ($100\,Q^{-1}$); 3. water with vapor bubbles and phase change, medium-size bubbles; 4. water with small vapor bubbles.

As shown in these figures, the behaviour of the porous medium saturated by the bubbly fluid strongly differs from the behaviours of the porous medium saturated by water or by steam. Particularly, the velocity of wave P_1 is considerably affected by the presence of bubbles (Figure 3). Gas–liquid phase change causes an important decrease of the velocity at low frequencies in the case of medium-size bubbles. This can be seen by comparing curve 3 to a curve intermediate to curves 1 and 2. Such an intermediate curve would describe the behaviour of a bubbly fluid without gas–liquid phase change. This decrease is even more pronounced for P_1

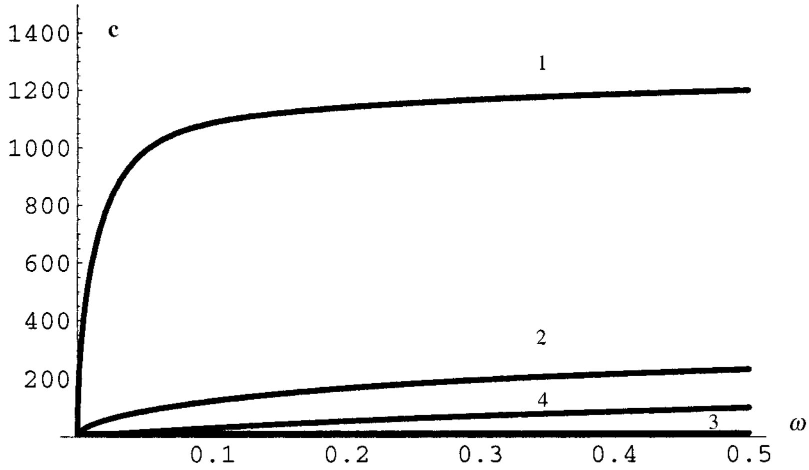

Figure 5. P_2 wave velocity in the porous medium saturated by: 1. water; 2. water vapor; 3. water with vapor bubbles and phase change, medium-size bubbles; 4. water with small vapor bubbles.

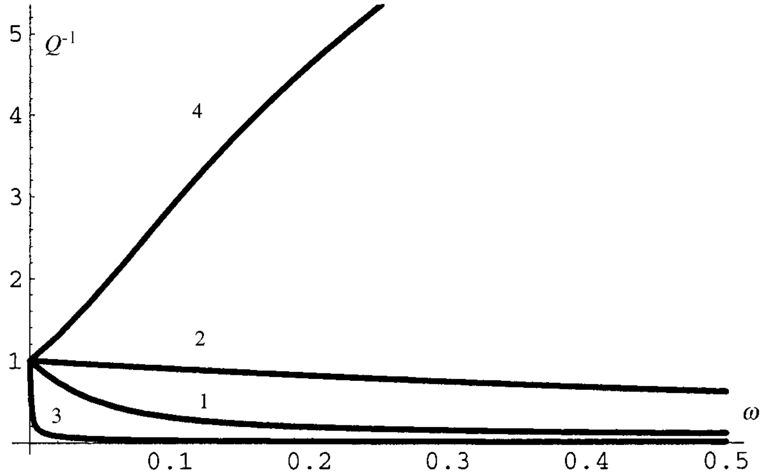

Figure 6. P_2 wave attenuation Q^{-1} in the porous medium saturated by: 1. water; 2. water vapor; 3. water with vapor bubbles and phase change, medium-size bubbles; 4. water with small vapor bubbles.

wave velocity in the case of small-size bubbles, which is caused by the negative apparent rigidity of the bubbly fluid.

7. Conclusion

We have used the homogenisation technique so as to derive the wave equations for acoustics in an elastic porous medium saturated by a bubbly fluid, when capillary, thermal and phase change effects are taken into account. Since the bubble and the pore scales are separated the bubbly fluid is seen at the pore scale as an effective fluid with effective viscosity and bulk modulus. The analysis shows that the effective bulk modulus $\tilde{K}_{\mathrm{f}}$ is strongly affected by phase change and capillary effects. This is of particular interest since $\tilde{K}_{\mathrm{f}}$, together with the solid bulk modulus are the main parameters for characterizing the wave propagation.

For very small bubbles, which appear near the bubble point, $\tilde{K}_{\mathrm{f}}$ can become negative. As shown in a former study (Auriault and Lebaigue, 1989), this leads to unstable wave propagation when considering the bubbly fluid on its own. The above results show that when the same fluid is saturating a porous medium, wave propagation becomes stable. This is due to the fact that the solid bulk modulus is positive and, in general, its value is quite large.

The presence of a bubbly fluid in the porous medium results in significant differences in wave velocity and attenuation when comparing with velocities and attenuations obtained for a porous medium saturated by a pure liquid or by a gas. Note that dilatational wave P_1, which is of particular interest in oil exploration and geophysics is considerably affected by the presence of bubbles.

Appendix A. Homogenisation in the Case of Small-size Bubbles

A.1. PERTURBATION EQUATIONS

Substituting expansions (3) into the set of dimensionless equations and invoking the differentiation rule (64), we get the set of perturbation equations

- In the liquid-phase

$$\frac{\partial \sigma_{\mathrm{l}ij}^{(0)}}{\partial z_j} = 0, \tag{A.1}$$

$$\frac{\partial \sigma_{\mathrm{l}ij}^{(1)}}{\partial z_j} + \frac{\partial \sigma_{\mathrm{l}ij}^{(0)}}{\partial y_j} = 0, \tag{A.2}$$

$$\frac{\partial \sigma_{\mathrm{l}ij}^{(2)}}{\partial z_j} + \frac{\partial \sigma_{\mathrm{l}ij}^{(1)}}{\partial y_j} + \frac{\partial \sigma_{\mathrm{l}ij}^{(0)}}{\partial x_j} = -\rho_l^e \omega^2 u_{\mathrm{l}i}^{(0)}, \tag{A.3}$$

$$\vdots$$

$$\sigma_{\mathrm{l}ij}^{(0)} = -p_{\mathrm{l}}^{(0)} I_{ij} + 2\mu i\omega\, E_{zij}(u_{\mathrm{l}}^{(0)}), \tag{A.4}$$

$$\sigma_{\mathrm{l}ij}^{(1)} = -p_{\mathrm{l}}^{(1)} I_{ij} + 2\mu i\omega (E_{zij}(u_{\mathrm{l}}^{(1)}) + E_{yij}(u_{\mathrm{l}}^{(0)})), \tag{A.5}$$

$$\sigma_{\mathrm{l}ij}^{(2)} = -p_{\mathrm{l}}^{(2)} I_{ij} + 2\mu i\omega (E_{zij}(u_{\mathrm{l}}^{(2)}) + E_{yij}(u_{\mathrm{l}}^{(1)}) + E_{xij}(u_{\mathrm{l}}^{(0)})), \tag{A.6}$$

$$\vdots$$

$$\frac{\partial u_{li}^{(0)}}{\partial z_i} = 0, \tag{A.7}$$

$$\frac{\partial u_{li}^{(1)}}{\partial z_i} + \frac{\partial u_{li}^{(0)}}{\partial y_i} = 0, \tag{A.8}$$

$$\frac{\partial u_{li}^{(2)}}{\partial z_i} + \frac{\partial u_{li}^{(1)}}{\partial y_i} + \frac{\partial u_{li}^{(0)}}{\partial x_i} = 0. \tag{A.9}$$

$$\cdots$$

- In the vapor

$$\frac{\partial \sigma_{vij}^{(0)}}{\partial z_j} = 0, \tag{A.10}$$

$$\frac{\partial \sigma_{vij}^{(1)}}{\partial z_j} + \frac{\partial \sigma_{vij}^{(0)}}{\partial y_j} = 0, \tag{A.11}$$

$$\frac{\partial \sigma_{vij}^{(2)}}{\partial z_j} + \frac{\partial \sigma_{vij}^{(1)}}{\partial y_j} + \frac{\partial \sigma_{vij}^{(0)}}{\partial x_j} = 0, \tag{A.12}$$

$$\vdots$$

$$\sigma_{vij}^{(0)} = -p_v^{(0)} I_{ij}, \tag{A.13}$$

$$\sigma_{vij}^{(1)} = -p_v^{(1)} I_{ij} + 2\mu i\omega\, E_{zij}(u_v^{(0)}), \tag{A.14}$$

$$\sigma_{vij}^{(2)} = -p_v^{(2)} I_{ij} + 2\mu i\omega (E_{zij}(u_v^{(1)}) + E_{yij}(u_v^{(0)})), \tag{A.15}$$

$$\vdots$$

$$\frac{\partial u_{vi}^{(0)}}{\partial z_i} = 0, \tag{A.16}$$

$$\frac{\partial u_{vi}^{(1)}}{\partial z_i} + \frac{\partial u_{vi}^{(0)}}{\partial y_i} = 0, \tag{A.17}$$

$$\frac{\partial u_{vi}^{(2)}}{\partial z_i} + \frac{\partial u_{vi}^{(1)}}{\partial y_i} + \frac{\partial u_{vi}^{(0)}}{\partial x_i} = -\frac{\rho_v^{(0)}}{\rho_v^e}, \tag{A.18}$$

$$\vdots$$

$$p_{\mathrm{v}}^{(0)} = p_{\mathrm{v}}^{e} \frac{\rho_{\mathrm{v}}^{(0)}}{\rho_{\mathrm{v}}^{e}}, \tag{A.19}$$

$$\vdots$$

• On Γ

$$(\sigma_{\mathrm{v}ij}^{(n)} - \sigma_{\mathrm{l}ij}^{(n)}) n_j = p_{\mathrm{c}}^{(n)} n_i, \qquad n = 0, 1, 2 \cdots , \tag{A.20}$$

$$\int_{\Gamma} p_{\mathrm{c}}^{(n)} \mathrm{d}\Gamma = 0, \qquad n = 0, 1, 2 \cdots , \tag{A.21}$$

$$u_{\mathrm{l}i}^{(0)} = u_{\mathrm{v}i}^{(0)}, \tag{A.22}$$

$$u_{\mathrm{l}i}^{(1)} = u_{\mathrm{v}i}^{(1)}, \tag{A.23}$$

$$u_{\Gamma i}^{(2)} n_i = u_{\mathrm{l}i}^{(2)} n_i = u_{\mathrm{v}i}^{(2)} n_i - \frac{j_i^{(0)} n_i}{i\omega \rho_{\mathrm{v}}^{e}}, \tag{A.24}$$

$$\vdots$$

$$L^{\varphi} j_i^{(0)} n_i = k_{\mathrm{l}} \frac{\partial T_{\mathrm{l}}^{(0)}}{\partial z_i} n_i, \tag{A.25}$$

$$\vdots$$

$$T_{\mathrm{l}}^{(0)} = T_{\mathrm{v}}^{(0)} = T_{\Gamma}^{(0)} = \frac{p_{\mathrm{v}}^{(0)T^e}}{L^{\varphi} \rho_{\mathrm{v}}^{e}}, \tag{A.26}$$

$$\vdots$$

• On Γ'

$$u_{\mathrm{l}i}^{(n)} = u_{\mathrm{s}i}^{(n)}, \qquad n = 0, 1, 2 \cdots , \tag{A.27}$$

$$\sigma_{\mathrm{l}ij}^{(n)} n_j = \sigma_{\mathrm{s}ij}^{(n)} n_j, \qquad n = 0, 1, 2 \cdots . \tag{A.28}$$

The analysis below requires the definition of the following volume averages:

$$\langle \phi_\alpha \rangle_{\Omega} = \frac{1}{\Omega} \int_{\Omega_\alpha} \phi_\alpha \mathrm{d}z, \qquad \alpha = l, v.$$

$$\langle \phi_\alpha \rangle_{\Omega'} = \frac{1}{\Omega'} \int_{\Omega'_\alpha} \phi_\alpha \mathrm{d}y, \qquad \alpha = f, s.$$

The first step consists in homogenising the bubble-scale description so as to derive the behaviour of the fluid mixture at the pore scale.

From (A.10) and (A.13), we get

$$p_{\mathrm{v}}^{(0)} = p_{\mathrm{v}}^{(0)}(x, y).$$

Now, as mentioned in paragraph 4.4, the movement of the bubble interface is a rigid displacement at the first two orders. Thus, the interface velocity, $\mathbf{v}_\Gamma = i\omega u_\Gamma$ is such that

$$v_{\Gamma i}^{(n)} = w_{\Gamma i}^{(n)}(x, y) + v_{\Gamma i}^{\prime(n)}(z), \qquad v_{\Gamma i}^{\prime(n)} n_i = 0, \qquad n = 0, 1. \tag{A.29}$$

Let us now extract from (A.1), (A.4), (A.7) and (A.29) at the first order, the boundary-value problem in Ω_{l} that determines $p_{\mathrm{l}}^{(0)}$ and $u_{\mathrm{l}}^{(0)}$

$$-\frac{\partial p_{\mathrm{l}}^{(0)}}{\partial z_i} + \mu_{\mathrm{l}} i\omega \frac{\partial^2 u_{\mathrm{l}i}^{(0)}}{\partial z_j \partial z_j} = 0,$$

$$\frac{\partial u_{\mathrm{l}i}^{(0)}}{\partial z_i} = 0,$$

$$u_{\mathrm{l}i}^{(0)} = w_{\mathrm{l}i}^{(0)}(x, y) + v_{\mathrm{l}i}^{\prime(0)}(z), \qquad v_{\mathrm{l}i}^{\prime(0)} n_i = 0 \quad \text{on} \quad \Gamma,$$

$u_{\mathrm{l}}^{(0)}$ and $p_{\mathrm{l}}^{(0)}$ are Ω-periodic. A similar boundary value problem was solved in (Boutin and Auriault, 1993). It turns out that the first order pressure and displacement in the liquid are independent of $\mathbf{z}$

$$u_{\mathrm{l}}^{(0)} = u_{\mathrm{l}}^{(0)}(x, y), \qquad p_{\mathrm{l}}^{(0)} = p_{\mathrm{l}}^{(0)}(x, y),$$

from which we deduce

$$p_{\mathrm{c}}^{(0)} = p_{\mathrm{c}}^{(0)}(x, y), \qquad r^{(0)} = r^{(0)}(x, y).$$

Thus, Equations (A.2) and (A.11) reduce to

$$\frac{\partial \sigma_{\mathrm{l}ij}^{(1)}}{\partial z_j} - \frac{\partial p_{\mathrm{l}}^{(0)}}{\partial y_i} = 0, \qquad \frac{\partial \sigma_{\mathrm{v}ij}^{(1)}}{\partial z_j} - \frac{\partial p_{\mathrm{v}}^{(0)}}{\partial y_i} = 0.$$

Integrating both equations over Ω_{l} and Ω_{v}, respectively, and using (A.20) and (A.22) with $n = 1$, we get

$$(1-\beta)\frac{\partial p_{\mathrm{l}}^{(0)}}{\partial y_i} + \beta \frac{\partial p_{\mathrm{v}}^{(0)}}{\partial y_i} = 0,$$

which yields

$$\pi^{(0)} = (1-\beta) p_{\mathrm{l}}^{(0)} + \beta p_{\mathrm{v}}^{(0)} = \pi^{(0)}(x). \tag{A.30}$$

This momentum balance shows that the behaviour of the fluid mixture is static at the first order of approximation.

In a given vapor bubble, $p_{\mathrm{v}}^{(1)}$ and $u_{\mathrm{v}}^{(0)}$ are the solution to the following set of equations

$$-\frac{\partial p_{\mathrm{v}}^{(0)}}{\partial y_i} - \frac{\partial p_{\mathrm{v}}^{(1)}}{\partial z_i} + \mu_{\mathrm{v}} i\omega \frac{\partial^2 u_{\mathrm{v}i}^{(0)}}{\partial z_j \partial z_j} = 0,$$

$$\frac{\partial u_{\mathrm{v}i}^{(0)}}{\partial z_i} = 0, \qquad u_{\mathrm{v}i}^{(0)} = u_{\mathrm{l}i}^{(0)}(x, y), \quad \text{on} \quad \Gamma.$$

It can easily be shown that $u_{\mathrm{v}}^{(0)}$ is $\mathbf{z}$-independent. Thus, we have

$$u_{\mathrm{v}}^{(0)} = u_{\mathrm{l}}^{(0)} = u_{\mathrm{f}}^{(0)}(x, y).$$

Let us now consider mass balance Equations (A.8) and (A.17)

$$\frac{\partial u_{\mathrm{l}i}^{(1)}}{\partial z_i} + \frac{\partial u_{\mathrm{f}i}^{(0)}}{\partial y_i} = 0, \qquad \frac{\partial u_{\mathrm{v}i}^{(1)}}{\partial z_i} + \frac{\partial u_{\mathrm{f}i}^{(0)}}{\partial y_i} = 0.$$

Integrating both equations over Ω_{l} and Ω_{v}, respectively, and using (A.23), we obtain

$$\frac{\partial u_{\mathrm{f}i}^{(0)}}{\partial y_i} = 0. \tag{A.31}$$

As a consequence, we get

$$\frac{\partial u_{\mathrm{l}i}^{(1)}}{\partial z_i} = 0, \qquad \frac{\partial u_{\mathrm{v}i}^{(1)}}{\partial z_i} = 0.$$

Let us now determine $u_{\mathrm{l}}^{(1)}$, which is given by the following boundary-value problem:

$$-\frac{\partial p_{\mathrm{l}}^{(0)}}{\partial y_i} - \frac{\partial p_{\mathrm{l}}^{(1)}}{\partial z_i} + 2\mu_{\mathrm{l}} i\omega \frac{\partial}{\partial z_j}(E_{yij}(u_{\mathrm{f}}^{(0)}) + E_{zij}(u_{\mathrm{l}}^{(1)})) = 0,$$

$$\frac{\partial u_{\mathrm{l}i}^{(1)}}{\partial z_i} = 0,$$

$$u_{\mathrm{l}i}^{(1)} = w_{\mathrm{l}i}^{(1)}(x, y) + v_{\mathrm{l}i}^{\prime(1)}(\mathbf{z}), \qquad v_{\mathrm{l}i}^{\prime(1)} n_i = 0 \quad \text{on} \quad \Gamma,$$

$u_{\mathrm{l}}^{(1)}$ and $p_{\mathrm{l}}^{(1)}$ are Ω-periodic. $u_{\mathrm{l}}^{(1)}$ appears to a linear vectorial function of $E_{yij}(u_{\mathrm{f}}^{(0)})$ and $\frac{\partial p_{\mathrm{l}}^{(0)}}{\partial y_i}$. However, when $E_{yij}(u_{\mathrm{f}}^{(0)}) = 0$, $u_{\mathrm{l}}^{(1)}$ becomes independent of $\frac{\partial p_{\mathrm{l}}^{(0)}}{\partial y_i}$. That yields $u_{\mathrm{l}}^{(1)}$ in the form

$$u_{\mathrm{l}i}^{(1)} = w_{\mathrm{l}i}^{(1)}(x, y) + \tau_{ijk}(\mathbf{z}) E_{yjk}(u_{\mathrm{f}}^{(0)}).$$

Thus, the momentum balance Equations (A.3) and (A.12) become

$$\frac{\partial \sigma^{(2)}_{1ij}}{\partial z_j} + \frac{\partial \sigma^{(1)}_{1ij}}{\partial y_j} - \frac{\partial p^{(0)}_1}{\partial x_j} = -\rho^e_1 \omega^2 u^{(0)}_{fi},$$

$$\frac{\partial \sigma^{(2)}_{vij}}{\partial z_j} - \frac{\partial p^{(1)}_v}{\partial y_j} - \frac{\partial p^{(0)}_v}{\partial x_j} = 0.$$

Integrating both equations over their respective domain of definition and using (A.20) and (A.21) with $n = 2$, we get

$$-\frac{\partial \pi^{(0)}}{\partial x_i} - \frac{\partial \pi^{(1)}}{\partial y_i} + 2\mu^{\mathrm{eff}}_{ijkl} i\omega \frac{\partial E_{yij}(u^{(0)}_f)}{\partial y_j} = -(1-\beta)\rho^e_1 \omega^2 u^{(0)}_{fi}, \tag{A.32}$$

where

$$\pi^{(0)}(x) = (1-\beta) p^{(0)}_1 + \beta p^{(0)}_v, \tag{A.33}$$

$$\pi^{(1)}(x, y) = \langle p^{(1)}_1 \rangle_\Omega + \langle p^{(1)}_v \rangle_\Omega,$$

and μ^{eff} is the effective viscosity of the mixture

$$\mu^{\mathrm{eff}}_{ijkl} = \mu_1 \langle (I_{ik} I_{jl} + E_{zij}(\boldsymbol{\tau}_{.kl})) \rangle_\Omega.$$

As in Auriault and Boutin (2000), at the two first orders we have

$$u^{(n)}_f = \frac{1}{\Omega_1} \int_{\Omega_1} u^{(n)}_1 \mathrm{d}\Omega = \frac{1}{\Omega_v} \int_{\Omega_v} u^{(n)}_v \mathrm{d}\Omega, \qquad n = 0, 1.$$

The equation of state of the vapor can be written as

$$\frac{\rho^{(0)}_v}{\rho^e_v} = \frac{p^{(0)}_v}{P^e},$$

and the vaporization-condensation flux is given by

$$L^\varphi \int_\Gamma j^{(0)} \cdot \mathbf{n}_l \mathrm{d}\Gamma = i\omega \Omega_l \rho^e_l C_{pl} T^{(0)}_\Gamma.$$

Note that

$$u^{(2)}_\Gamma \cdot \mathbf{n}_v = r^{(0)} = p^{(0)}_c \frac{R^2}{2\sigma},$$

$$\frac{1}{\Omega_v} \int_\Gamma u^{(2)}_\Gamma \cdot \mathbf{n}_v \mathrm{d}\Gamma = \frac{p^{(0)}_c}{K_c}, \qquad K_c = \frac{2\sigma}{3R}.$$

Integrating (A.9) over Ω_l, using the divergence theorem and the first equality in (A.24) yields

$$\frac{\partial \langle u_i^{(1)} \rangle_\Omega}{\partial y_i} + \frac{\partial u_{fi}^{(0)}}{\partial x_i} = \frac{\beta}{1-\beta} \frac{p_l^{(0)} - p_v^{(0)}}{K_c}. \tag{A.34}$$

Integrating now (A.18) over Ω_v, using the above relations for the density and the vaporization-condensation flux through Γ gives the volume gas balance at the pore scale

$$\frac{\partial \langle u_i^{(1)} \rangle_\Omega}{\partial y_i} + \frac{\partial u_{fi}^{(0)}}{\partial x_i} = -\frac{p_v^{(0)}}{P^e} - \frac{p_l^{(0)} - p_v^{(0)}}{K_c} + \frac{1-\beta}{\beta} \frac{\rho_l^e C_{pl} T_\Gamma^{(0)}}{\rho_v^e L^\varphi}, \tag{A.35}$$

with

$$T_\Gamma^{(0)} = \frac{p_v^{(0)} T^e}{L^\varphi \rho_v^e}.$$

Finally, eliminating $p_l^{(0)}$ and $p_v^{(0)}$ between (A.33), (A.34) and (A.35) gives

$$\frac{\partial \langle u_i^{(1)} \rangle_\Omega}{\partial y_i} + \frac{\partial u_{fi}^{(0)}}{\partial x_i} = -\mathcal{A}\pi^{(0)}, \tag{A.36}$$

with

$$\mathcal{A} = \left[\frac{1}{\mathcal{B}} - K_c \frac{(1-\beta)^2}{\beta}\right]^{-1},$$

$$\mathcal{B} = \beta \frac{1}{P^e} - (1-\beta) \frac{\rho_l^e C_{pl} T^e}{\rho_v^{e2} L^{\varphi 2}}.$$

A.3. MACROSCOPIC DESCRIPTION OF THE POROUS MEDIUM

Now, homogenisation of the pore scale description yields the macroscopic description.

Equations (A.30), (A.31), (A.32) and (A.36) represent the first-order behaviour of a compressible anisotropic Newtonian fluid of pressure π, compressibility $1/\tilde{K} = \mathcal{A}$ and of viscosity μ_{ijkl}^{eff}. Note that in the case of small-size bubbles, $\mathcal{A}$ is real valued. When normalized by using l' as the characteristic length, the description of the fluid mixture is written as follows:

$$\frac{\partial \sigma_{fij}}{\partial y_j} = -\varepsilon \rho_l^e \omega^2 u_{fi},$$

$$\sigma_{\mathrm{f}ij} = -\pi I_{ij} + 2\varepsilon\mu_{ijkl}^{\mathrm{eff}} i\omega\, E_{yij}(u_{\mathrm{f}}),$$

$$\frac{\partial u_{\mathrm{f}i}}{\partial y_i} = -\varepsilon \mathcal{A}\,\pi.$$

The above equations together with (60)–(63), constitute the classical local description of wave propagation in a porous medium saturated by a compressible fluid of pressure π. The homogenisation process is the same as in Levy, (1979) and Auriault, (1980) where an incompressible fluid is considered. In the present problem, as in Auriault *et al.*, (1990) where a quasi-static flow is investigated, the compressibility appears in the macroscopic mass balance (last term in (A.36)).

At the first order of approximation, we obtain the following macroscopic description:

$$\frac{\partial \sigma_{Tij}}{\partial x_j} = -\omega^2(\langle\rho_{\mathrm{s}}\rangle_{\Omega'} u_{\mathrm{s}i} + (1-\beta)\rho_{\mathrm{l}}\langle u_{\mathrm{f}i}\rangle_{\Omega'}), \tag{A.37}$$

$$\sigma_{Tij} = c_{ijkh}\, E_{xkh}(u_{\mathrm{s}}) - \alpha_{ij}\pi, \tag{A.38}$$

$$\frac{\partial(\langle u_{\mathrm{f}i}\rangle_{\Omega'} - \phi u_{\mathrm{s}i})}{\partial x_i} = -\alpha_{ij} E_{xij}(u_{\mathrm{s}}) - \gamma\pi - \phi\mathcal{A}\,\pi, \tag{A.39}$$

$$\langle u_{\mathrm{f}i}\rangle_{\Omega'} - \phi u_{\mathrm{s}i} = -\frac{K_{ij}}{i\omega}\left(\frac{\partial\pi}{\partial x_j} - \omega^2(1-\beta)\rho_{\mathrm{l}} u_{\mathrm{s}i}\right). \tag{A.40}$$

In the above equation, **K** is the complex-valued and ω-dependent acoustical permeability tensor. ϕ represents the porosity, while **c** is the bulk elastic tensor of the empty porous matrix and symbols $\boldsymbol{\alpha}$ and γ represent elastic coefficients.

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