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CONVEXITY OF PARABOLIC SUBGROUPS IN ARTIN GROUPS

RUTH CHARNEY AND LUIS PARIS

Abstract. We prove that any standard parabolic subgroup of any Artin group is convex with respect to the standard generating set.

1. Introduction

Let $S$ be a finite set. A Coxeter matrix over $S$ is a square matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of $S$ satisfying $m_{s,s} = 1$ for all $s \in S$, and $m_{s,t} = m_{t,s} \in \{2, 3, 4, \ldots\} \cup \{\infty\}$ for all $s, t \in S$, $s \neq t$. The Coxeter graph which represents the above Coxeter matrix is the labelled graph $\Gamma = \Gamma(M)$ defined as follows. (1) $S$ is the set of vertices of $\Gamma$. (2) Two vertices $s, t \in S$ are joined by an edge if $m_{s,t} \geq 3$. (3) This edge is labelled by $m_{s,t}$ if $m_{s,t} \geq 4$.

The Coxeter system of $\Gamma$ is the pair $(W, S) = (W_\Gamma, S)$, where $S$ is the set of vertices of $\Gamma$, and $W$ is the group defined by the following presentation.

$$W = \langle S \mid s^2 = 1 \text{ for all } s \in S \rangle.$$ 

The group $W$ itself is called the Coxeter group of $\Gamma$.

If $a, b$ are two letters and $m$ is an integer $\geq 2$, we set

$$\Pi(a, b : m) = \begin{cases} (ab)^{m/2} & \text{if } m \text{ is even} \\ a(ba)^{m-1}/2 & \text{if } m \text{ is odd} \end{cases}$$

In other words, $\Pi(a, b : m)$ denotes the word $aba \cdots$ of length $m$. Let $\Sigma = \{\sigma_s \mid s \in S\}$ be an abstract set in one-to-one correspondence with $S$. The Artin system of $\Gamma$ is the pair $(A, \Sigma) = (A_\Gamma, \Sigma)$, where $A$ is the group defined by the following presentation.

$$A = \langle \Sigma \mid \Pi(\sigma_s, \sigma_t : m_{s,t}) = \Pi(\sigma_t, \sigma_s : m_{s,t}) \text{ for all } s, t \in S, s \neq t \text{ and } m_{s,t} \neq \infty \rangle.$$ 

The group $A$ itself is called the Artin group of $\Gamma$. Observe that the map $\Sigma \to S$, $\sigma_s \mapsto s$, induces an epimorphism $\theta : A_\Gamma \to W_\Gamma$. The kernel of $\theta$ is called the colored Artin group of $\Gamma$, and it is denoted by $CA_\Gamma$.

The Coxeter groups were introduced by Tits in his manuscript [14]. The latter is one of the main sources for the celebrated Bourbaki book “Groupes et algèbres de Lie, Chapitres IV, V et VI” [3]. They play an important role in many areas such as Lie theory, hyperbolic geometry, and, of course, group theory. Furthermore, there is a quite extensive literature on Coxeter groups. We recommend [7] for a detailed study of the subject.

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The Artin groups were also introduced by Tits, as extensions of Coxeter groups [15]. They are involved in several fields (singularities, knot theory, mapping class groups, and so on), and they have been the object of many papers in the last three decades. Most results, however, involve only special classes of Artin groups, such as spherical type Artin groups (where the corresponding Coxeter group is finite) or right-angled Artin groups (where all $m_{s,t} = 2, \infty$). Artin groups as a whole are still poorly understood. In particular, it is not known if they are torsion free, or if they have solvable word problem (see [9]). The present paper is an exception to that rule since it concerns all Artin groups.

The flagship example of an Artin group is the braid group $B_n$ on $n$ strands. It is associated to the Coxeter graph $A_{n-1}$ depicted in Figure 1, and its associated Coxeter group is the symmetric group $S_n$.

![Figure 1. The Coxeter graph $A_{n-1}$.](image)

For $T \subset S$, we denote by $W_T$ the subgroup of $W$ generated by $T$, we denote by $\Gamma_T$ the full subgraph of $\Gamma$ spanned by $T$, we set $\Sigma_T = \{\sigma_s \mid s \in T\}$, and we denote by $A_T$ the subgroup of $A$ generated by $\Sigma_T$. By [3], the pair $(W_T, T)$ is the Coxeter system of $\Gamma_T$, and by [11], $(A_T, \Sigma_T)$ is the Artin system of $\Gamma_T$. The group $A_T$ (resp. $W_T$) is called a standard parabolic subgroup of $A$ (resp. of $W$).

Let $m$ and $n$ be two positive integers such that $m \leq n$. A fundamental example of a standard parabolic subgroup is the braid group $B_m$ embedded in $B_n$ via the homomorphism which sends the standard generators of $B_m$ to the first $m-1$ standard generators of $B_n$.

Let $G$ be a group, and let $S$ be a generating set for $G$. An expression for an element $\alpha \in G$ is a word $\hat{\alpha} = s_1^{\varepsilon_1} \cdots s_{\ell}^{\varepsilon_{\ell}}$ on $S \sqcup S^{-1}$ which represents $\alpha$. The length of $\alpha$ (with respect to $S$), denoted by $\lg_S(\alpha)$, is the minimal length of an expression for $\alpha$. A geodesic for $\alpha$ is an expression of length $\lg_S(\alpha)$. Geometrically, this corresponds to a geodesic path from 1 to $\alpha$ in the Cayley graph of $(G, S)$.

Let $T$ be a subset of $S$, and let $H$ be the subgroup of $G$ generated by $T$. We say that $H$ is convex in $G$ with respect to $S$ if for all $\alpha \in H$ and all geodesic $\hat{\alpha} = s_1^{\varepsilon_1} \cdots s_{\ell}^{\varepsilon_{\ell}}$ of $\alpha$, we have $s_1, \ldots, s_{\ell} \in T$. Or equivalently, the Cayley graph of $(H, T)$ is a convex subspace of the Cayley graph of $(G, S)$. In particular, if $H$ is convex, then $\lg_S(\alpha) = \lg_T(\alpha)$ for all $\alpha \in H$, that is, $H$ is isometrically embedded in $G$.

The following is of importance in the study of Coxeter groups.

**Theorem 1.1** (Bourbaki [3]). Let $\Gamma$ be a Coxeter graph, let $(W, S)$ be its Coxeter system, and let $T$ be a subset of $S$. Then $W_T$ is a convex subgroup of $W$ with respect to $S$.

In the present paper we prove that the same result holds for Artin groups. That is:

**Theorem 1.2.** Let $\Gamma$ be a Coxeter graph, let $(A, \Sigma)$ be its Artin system, and let $T$ be a subset of $S$. Then $A_T$ is a convex subgroup of $A$ with respect to $\Sigma$. 

Our original motivation for studying the convexity of parabolic subgroups in Artin groups comes from a question asked us by Arye Juhasz. In a work in preparation [10], he provides a solution to the word problem for a certain class of Artin groups, and he proves that these groups are torsion free. His proof uses a condition on the groups that he calls the A-S condition, a form of convexity for a certain type of word. It follows immediately from Theorem 1.2 that this condition is satisfied by all Artin groups. More generally, as for Coxeter groups, the study of Artin groups often goes through the study of their (standard) parabolic subgroups. So any result on these subgroups is likely to be useful in further developments in the theory of Artin groups.

Although Theorem 1.2 may seem natural, it is a surprise for the experts. As far as we know, it was not even known for the braid group $B_m$ embedded in $B_n$, although the proof in this case is easy (see Proposition 2.1 below). Theorem 1.2 also comes as a surprise because, in general, the family of standard generators (that is, $\Sigma$) is not the best for studying combinatorial questions on Artin groups. In the most well understood case, when $A$ is of spherical type, a larger generating set is generally used. This is called the set of simple elements and we denote it by $S$. It follows from [5] that $(A_T, S_T)$ is isometrically embedded in $(A, S)$, but the image is not convex since there are $S$-geodesics for elements of $A_T$ whose terms do not belong to $S_T \cup S_T^{-1}$.

2. The proof

As promised, we start with the proof of Theorem 1.2 in the particular case of the braid group $B_m$ embedded in $B_n$. We treat this case separately for two reasons. Firstly, because some readers may want to use Theorem 1.2 in that case without necessarily learning all the background on Artin groups needed to prove our theorem. Secondly, because the proof of Theorem 1.2 is, in a sense, a (non trivial) extension of the proof of Proposition 2.1. So, the reader may want to keep in mind the proof of Proposition 2.1 when reading the proof of Theorem 1.2.

Recall that a braid on $n$ strands is an $n$-tuple $\beta = (b_1, \ldots, b_n)$ of paths, $b_i : [0, 1] \to \mathbb{R}^3$, such that

1. $b_i(0) = (i, 0, 0)$ for all $i \in \{1, \ldots, n\}$, and there is a permutation $w \in \mathfrak{S}_n$ such that $b_i(1) = (w(i), 0, 1)$ for all $i \in \{1, \ldots, n\}$;
2. $(p_3 \circ b_i)(t) = t$ for all $i \in \{1, \ldots, n\}$ and $t \in [0, 1]$, where $p_3$ denotes the projection of $\mathbb{R}^3$ on the third coordinate;
3. $b_i \cap b_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$.

The isotopy classes of braids form a group, called the braid group on $n$ strands and denoted by $B_n$. By Artin [1, 2], this group has the following presentation.

\[
B_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \text{ for } 1 \leq i \leq n-2, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i-j| \geq 2 \right\rangle.
\]

In other words, $B_n$ is the Artin group associated to the Coxeter graph $A_{n-1}$ depicted in Figure 1.
Proposition 2.1. Let \( m,n \in \mathbb{N}, m \leq n \). Then \( \mathcal{B}_m \) is a convex subgroup of \( \mathcal{B}_n \) with respect to \( \{ \sigma_1, \ldots, \sigma_{n-1} \} \).

Proof. Let \( p_{1,3} : \mathbb{R}^3 \to \mathbb{R}^2 \) denote the projection on the first and third coordinates. Let \( \beta = (b_1, \ldots, b_n) \) be a braid. We project each \( b_i \) on the plane \( \mathbb{R}^2 \) via \( p_{1,3} \) and, up to isotopy, we may suppose that these projections form only finitely many regular double crossings. As usual, we indicate in each crossing which strand goes over the other. We obtain in this way a braid diagram which represents the isotopy class of \( \beta \). Recall also that each generator \( \sigma_i \) has a “canonical” diagram with precisely one crossing.

Let \( \hat{\alpha} = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_\ell}^{\varepsilon_\ell} \) be a word which represents an element \( \alpha \in \mathcal{B}_n \). By concatenating the canonical diagrams of the \( \sigma_{i_\ell}^{\varepsilon_\ell} \)'s, we obtain a diagram of \( \alpha \) with precisely \( \ell \) crossings. Conversely, by applying standard methods, from a diagram of \( \alpha \) with \( \ell \) crossings, we can define a (non unique) word \( \hat{\alpha} = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_\ell}^{\varepsilon_\ell} \) of length \( \ell \) which represents \( \alpha \).

Let \( \alpha \in \mathcal{B}_m \), and let \( \hat{\alpha} = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_\ell}^{\varepsilon_\ell} \) be a geodesic of \( \alpha \) with respect to the generating set of \( \mathcal{B}_n \). Let \( D \) be the diagram of \( \alpha \) obtained from \( \hat{\alpha} \). By removing the strands \( b_{m+1}, \ldots, b_n \) from \( D \), we obtain another diagram \( D' \) of \( \alpha \), but now viewed as an element of \( \mathcal{B}_m \). Since no new crossings are introduced by this procedure, \( D' \) has at most \( \ell \) crossings. But \( \hat{\alpha} \) was geodesic, so the number of crossings cannot be less than \( \ell \). This means that the only strands involved in the crossings in \( D \) are \( b_1, \ldots, b_m \), and therefore \( \sigma_{i_1}, \ldots, \sigma_{i_\ell} \in \{ \sigma_1, \ldots, \sigma_{m-1} \} \). \( \square \)

We turn now to the proof of Theorem 1.2. We fix a Coxeter graph \( \Gamma \), and denote by \( (W,S) \) its Coxeter system, and by \( (A,\Sigma) \) its Artin system.

Let \( X,Y \) be two subsets of \( S \), and let \( w \) be an element of \( W \). We say that \( w \) is \( (X,Y) \)-minimal if it is of minimal length among the elements of the double-coset \( W_X wW_Y \). The following will be implicitly used throughout the paper.

Proposition 2.2 (Bourbaki [3]). Let \( X,Y \) be two subsets of \( S \), and let \( w \in W \).

1. There exists a unique \( (X,Y) \)-minimal element lying in \( W_X wW_Y \).
2. The following are equivalent.
   - \( w \) is \( (\emptyset,Y) \)-minimal,
   - \( \ell_S(ws) > \ell_S(w) \) for all \( s \in Y \),
   - \( \ell_S(wu) = \ell_S(w) + \ell_S(u) \) for all \( u \in W_Y \).
3. The following are equivalent.
   - \( w \) is \( (X,\emptyset) \)-minimal,
   - \( \ell_S(sw) > \ell_S(s) \) for all \( s \in X \),
   - \( \ell_S(uw) = \ell_S(u) + \ell_S(w) \) for all \( u \in W_X \).

Now, set \( S^f = \{ X \subset S \mid W_X \text{ is finite} \} \). Then the following can be easily proved using the previous proposition (see for instance [12, Lemma 3.2]).

Lemma 2.3. Let \( \preceq \) be the relation on \( W \times S^f \) defined as follows. Set \( (u,X) \preceq (v,Y) \) if the following three conditions hold,

1. \( X \subset Y \),
2. \( v^{-1}u \in W_Y \), and
3. \( v^{-1}u \) is \( (\emptyset,X) \)-minimal.
Then $\preceq$ is a partial order relation on $W \times S^f$.

Recall that the derived complex of a partially ordered set $(E, \preceq)$ is defined to be the abstract simplicial complex made of the finite nonempty chains of $E$. The Salvetti complex of $\Gamma$, denoted by $\text{Sal}(\Gamma)$, is defined to be the geometric realization of the derived complex of $(W \times S^f, \preceq)$. Note that the action of $W$ on $W \times S^f$ defined by $w \cdot (u, X) = (wu, X)$ leaves invariant the order relation $\preceq$, hence it induces a "natural" (free and properly discontinuous) action of $W$ on $\text{Sal}(\Gamma)$. The quotient space will be denoted by $\overline{\text{Sal}}(\Gamma) = \text{Sal}(\Gamma)/W$.

The spaces $\text{Sal}(\Gamma)$ and $\overline{\text{Sal}}(\Gamma)$ admit cellular decompositions. Recall that a finite Coxeter group $W_X$ can be realized as an orthogonal reflection group acting on $\mathbb{R}^k, k = |X|$. The Coxeter cell for $W_X$ is the convex hull of the orbit of a generic point $p$ in $\mathbb{R}^k$. Viewed as a combinatorial object, it is independent of choice of the point $p$. Its 1-skeleton is the Cayley graph of $(W_X, X)$. The cell decompositions of $\text{Sal}(\Gamma)$ and $\overline{\text{Sal}}(\Gamma)$, which we now describe, are made up of Coxeter cells for the subgroups $W_X, X \in S^f$.

For $(u, X) \in W \times S^f$, we set

$$C(u, X) = \{(v, Y) \in W \times S^f \mid (v, Y) \preceq (u, X)\},$$

and we denote by $B(u, X)$ the simplicial subcomplex of $\text{Sal}(\Gamma)$ spanned by $C(u, X)$. It is shown in [12] that $B(u, X)$ is a closed disc of dimension $|X|$ for all $(u, X) \in W \times S^f$; in fact it is naturally isomorphic to the Coxeter cell for $W_X$. The set $\{B(u, X) \mid (u, X) \in W \times S^f\}$ is a regular cellular decomposition of $\text{Sal}(\Gamma)$ (see also [13, 6]).

Observe that every $w \in W \setminus \{1\}$ sends the closed disc $B(u, X)$ homeomorphically onto $B(wu, X)$, and that $\text{int}(B(u, X)) \cap \text{int}(B(wu, X)) = \emptyset$, for all $(u, X) \in W \times S^f$. It follows that the cellular decomposition of $\text{Sal}(\Gamma)$ induces a cellular decomposition of the quotient $\overline{\text{Sal}}(\Gamma) = \text{Sal}(\Gamma)/W$. This decomposition has one cell for each $X \in S^f$, corresponding to the orbit of $B(1, X)$ under $W$. The closure of this cell will be denoted by $\overline{B}(X)$. Note that this cellular decomposition is not regular in general.

The $k$-skeletons of $\text{Sal}(\Gamma)$ and $\overline{\text{Sal}}(\Gamma)$ for $k = 0, 1, 2$ can be described as follows.

**The 0-skeleton.** Let $u \in W$. Then $C(u, \emptyset) = \{(u, \emptyset)\}$, and $B(u, \emptyset)$ is a vertex of $\text{Sal}(\Gamma)$. We denote it by $x(u)$. So, the 0-skeleton of $\text{Sal}(\Gamma)$ is the set $\{x(u) \mid u \in W\}$, which is in one-to-one correspondence with $W$. The 0-skeleton of $\overline{\text{Sal}}(\Gamma)$ is reduced to a single point, $\overline{B}(\emptyset)$, that we denote by $x_0$.

**The 1-skeleton.** Let $u \in W$ and $s \in S$. Then $C(u, \{s\}) = \{(u, \emptyset), (us, \emptyset), (u, \{s\})\}$, and $B(u, \{s\})$ is an edge of $\text{Sal}(\Gamma)$ connecting $x(u)$ to $x(us)$. We denote it by $a(u, s)$ and orient it from $x(u)$ to $x(us)$. Note that there is another edge connecting $x(u)$ to $x(us)$, namely $a(us, s)$, but this edge is oriented in the other direction (see Figure 2). Observe that the action of $W$ preserves orientations of edges. Since all edges are of this form, we see that the 1-skeleton of $\overline{\text{Sal}}(\Gamma)$ is just the Cayley graph of $(W, S)$. Descending to $\overline{\text{Sal}}(\Gamma)$, the 1-skeleton consists of a loop at $x_0$ formed by the edge $a_s = \overline{B}(\{s\})$, for each $s \in S$ (see Figure 2).

**The 2-skeleton.** Let $s, t \in S, s \neq t$. Set $X = \{s, t\}$. First, notice that $X \in S^f$ if and only if $m_{s,t} \neq \infty$. Now, assume $m_{s,t} \neq \infty$, set $m = m_{s,t}$, and take $u \in W$. Then
\( a(v, s) \quad x(v) \quad a(u, s) \quad x(u) \quad a(u, t) \quad x_0 \quad a_s \quad x_0 \)

**Figure 2.** 1-skeletons of \( \text{Sal}(\Gamma) \) and \( \overline{\text{Sal}}(\Gamma) \).

\[ \mathbb{B}(u, X) \text{ is isomorphic to the Coxeter cell for } W_X. \text{ Namely, it is a } 2m\text{-gon with vertices } \{x(uw) \mid w \in W_X\}. \text{ The boundary of } \mathbb{B}(u, X) \text{ is the loop } \]
\[ a(u, s)a(us, t) \cdots a(u\Pi(s, t : m - 1), r)a(u\Pi(t, s : m - 1), r')^{-1} \cdots a(ut, s)^{-1}a(u, t)^{-1} \]
\[ \text{where } r = t \text{ and } r' = s \text{ if } m \text{ is even, and } r = s \text{ and } r' = t \text{ if } m \text{ is odd (see Figure 3). Hence, } \]
\[ \overline{\mathbb{B}}(X) \text{ is a 2-cell whose boundary is } \]
\[ \overline{a}_s \overline{a}_t \cdots \overline{a}_r \overline{a}_{r'}^{-1} \cdots \overline{a}_s^{-1}\overline{a}_t^{-1} = \Pi(\overline{a}_s, \overline{a}_t : m)\Pi(\overline{a}_t, \overline{a}_s : m)^{-1}. \]

**Figure 3.** 2-skeletons of \( \text{Sal}(\Gamma) \) and \( \overline{\text{Sal}}(\Gamma) \).

From the above it follows that the 2-skeleton of \( \overline{\text{Sal}}(\Gamma) \) is precisely the Cayley 2-complex associated with the standard presentation of \( A \).

**Proposition 2.4.** We have \( \pi_1(\overline{\text{Sal}}(\Gamma), x_0) = A \), \( \pi_1(\text{Sal}(\Gamma), x(1)) = CA \), and the exact sequence associated to the regular covering \( \text{Sal}(\Gamma) \twoheadrightarrow \overline{\text{Sal}}(\Gamma) \) is
\[ 1 \rightarrow CA \rightarrow A \rightarrow W \rightarrow 1. \]

Now, assume we are given a subset \( T \) of \( S \), and set \( S_T^f = \{X \in S^f \mid X \subset T\} \). Observe that the embedding \( W_T \times S_T^f \hookrightarrow W \times S^f \) induces an embedding \( \iota_T : \text{Sal}(\Gamma_T) \hookrightarrow \text{Sal}(\Gamma) \). The following provides our main tool for proving Theorem 1.2.
Theorem 2.5 (Godelle, Paris [8]). The embedding $\iota_T : \text{Sal}(\Gamma_T) \hookrightarrow \text{Sal}(\Gamma)$ admits a retraction $\pi_T : \text{Sal}(\Gamma) \to \text{Sal}(\Gamma_T)$.

In the proof of Theorem 1.2, we will need the following explicit description of the map $\pi_T$. Let $(u, X) \in W \times S^f$. Write $u = u_0u_1$, where $u_0 \in W_T$ and $u_1$ is $(T, \emptyset)$-minimal. Set $X_0 = T \cap u_1Xu_1^{-1}$. Note that, since $u_1W Xu_1^{-1}$ is finite, $W X_0$ is finite, hence $X_0 \in S^f_T$. Then we set $\pi_T(u, X) = (u_0, X_0)$. It is proved in [8] that the map $\pi_T : W \times S^f \to W_T \times S^f_T$ induces a continuous map $\pi_T : \text{Sal}(\Gamma) \to \text{Sal}(\Gamma_T)$ which is a retraction of $\iota_T$.

Now, the following lemma follows from the above description.

Lemma 2.6. Let $u \in W$. Write $u = u_0u_1$, where $u_0 \in W_T$ and $u_1$ is $(T, \emptyset)$-minimal. Then

1. $\pi_T(x(u)) = x(u_0)$.
2. Let $s \in S$ and set $t = u_1su_1^{-1}$. If $t \in T$, then $\pi_T(a(u, s)) = a(u_0, t)$. If $t \notin T$, then $\pi_T(a(u, s)) = x(u_0)$.

Proof. We leave Part (1) to the reader and turn to Part (2). We take $u \in W$ and $s \in S$, and define $u_0, u_1$ and $t$ as in the Lemma. Recall that $C(u, \{s\}) = \{(u, \emptyset), (us, \emptyset), (u, \{s\})\}$, and that $a(u, s)$ is the edge of $\text{Sal}(\Gamma)$ spanned by $C(u, \{s\})$.

Assume first that $t \in T$. We have $us = u_0u_1s = u_0tu_1$, $u_0t \in W_T$ and $u_1$ is $(T, \emptyset)$-minimal, hence $\pi_T((u, \emptyset)) = (u_0, \emptyset)$, $\pi_T((us, \emptyset)) = (u_0, t)$, and $\pi_T((u, \{s\}) = (u_0, \emptyset)$. Therefore, $\pi_T(a(u, s)) = a(u_0, t)$.

Assume now that $t \notin T$. We claim that $u_1s$ is $(T, \emptyset)$-minimal. If $\lg_S(u_1s) < \lg_S(u_1)$, then this is clear since $u_1$ is $(T, \emptyset)$-minimal. Suppose that $\lg_S(u_1s) > \lg_S(u_1)$. If $u_1s$ were not $(T, \emptyset)$-minimal, then there would exist $s_0 \in T$ such that $\lg_S(s_0u_1s) < \lg_S(u_1)$ and since $u_1$ is $(T, \emptyset)$-minimal, $\lg_S(s_0u_1) > \lg_S(u_1)$. By the “folding condition” (see [4, Chap. II, Sec. 3]), this would imply that $s_0u_1s = u_1$, that is, $t = u_1su_1^{-1} = s_0 \in T$, a contradiction.

Thus, if $t \notin T$, then $us = u_0u_1s$, $u_0 \in W_T$, and $u_1s$ is $(T, \emptyset)$-minimal. It follows that $\pi_T((u, \emptyset)) = \pi_T((us, \emptyset)) = \pi_T((u, \{s\}) = (u_0, \emptyset)$, hence $\pi_T(a(u, s)) = x(u_0)$.

Proof of Theorem 1.2. Let $\tilde{\alpha} = \sigma_{s_1}^{\varepsilon_1} \cdots \sigma_{s_\ell}^{\varepsilon_\ell} \in (\Sigma \sqcup \Sigma^{-1})^*$ be a word in the alphabet $\Sigma \sqcup \Sigma^{-1}$. Set

$$\tilde{\gamma}(\tilde{\alpha}) = \bar{a}_{s_1}^{\varepsilon_1} \cdots \bar{a}_{s_\ell}^{\varepsilon_\ell}.$$ 

This is a loop in $\text{Sal}(\Gamma)$ based at $x_0$. Note that, if $\alpha$ is the element of $A$ represented by the word $\tilde{\alpha}$, then $\alpha$ is the element of $A = \pi_1(\text{Sal}(\Gamma), x_0)$ represented by the loop $\gamma(\tilde{\alpha})$.

Denote by $\gamma(\tilde{\alpha})$ the lift of $\tilde{\gamma}(\tilde{\alpha})$ in $\text{Sal}(\Gamma)$ with initial point $x(1)$. The path $\gamma(\tilde{\alpha})$ can be described as follows. For $i \in \{0, 1, \ldots, \ell\}$ we set $u_i = s_1^{\varepsilon_1} s_2^{\varepsilon_2} \cdots s_i^{\varepsilon_i} = s_1s_2 \cdots s_i \in W$. For $i \in \{1, \ldots, \ell\}$, we set $a_i = a(u_{i-1}, s_i)$ if $\varepsilon_i = 1$ and $a_i = a(u_i, s_i)$ if $\varepsilon_i = -1$. Then

$$\gamma(\tilde{\alpha}) = a_1^{\varepsilon_1} \cdots a_\ell^{\varepsilon_\ell}.$$ 

Let $i \in \{1, \ldots, \ell\}$. Write $u_i = v_iw_i$, where $v_i \in W_T$ and $w_i$ is $(T, \emptyset)$-minimal. If $\varepsilon_i = 1$, let $t_i = w_{i-1}s_iw_{i-1}^{-1}$. If $\varepsilon_i = -1$, let $t_i = w_iw_{i-1}^{-1}$. Define

$$\tau_i = \begin{cases} \sigma_{t_i}^{\varepsilon_i} & \text{if } t_i \in T \\ 1 & \text{if } t_i \notin T \end{cases}$$
\[ \tilde{\tau} = \tau_1 \tau_2 \cdots \tau_\ell \in (\Sigma_T \sqcup \Sigma_T^{-1})^*. \]

Note that \( \lg(\tilde{\tau}) \leq \lg(\tilde{\alpha}). \)

By Lemma 2.6, the projection of \( \pi_T(a_i) \) in \( \text{Sal}(\Gamma_T) \) is the edge \( \bar{a}_i \), if \( t_i \in T \), and the vertex \( x_0 \) otherwise. Thus the image of \( \pi_T(\gamma(\tilde{\alpha})) \) in \( \text{Sal}(\Gamma_T) \) is the loop corresponding to \( \tilde{\tau} \). In other words, \( \pi_T(\gamma(\tilde{\alpha})) = \gamma(\tilde{\tau}). \)

**Claim 1.** If \( \tilde{\alpha} \in (\Sigma_T \sqcup \Sigma_T^{-1})^* \), then \( \tilde{\tau} = \tilde{\alpha} \).

**Proof of Claim 1.** In this case we have \( u_i \in W_T \), hence \( v_i = u_i \) and \( w_i = 1 \) for all \( i \in \{0, 1, \ldots, \ell\} \). It follows that \( t_i = s_i \in T \), so \( \tau_i = \sigma_{\varepsilon_i}^s \) for all \( i \). Hence,
\[
\tilde{\tau} = \tau_1 \tau_2 \cdots \tau_\ell = \sigma_{\varepsilon_1}^{s_1} \sigma_{\varepsilon_2}^{s_2} \cdots \sigma_{\varepsilon_\ell}^{s_\ell} = \tilde{\alpha}.
\]

**Claim 2.** Suppose \( \tilde{\alpha} \in (\Sigma \sqcup \Sigma^{-1})^* \) represents the element \( \alpha \in A_T. \) Then \( \tilde{\tau} \in (\Sigma_T \sqcup \Sigma_T^{-1})^* \) also represents \( \alpha. \)

**Proof of Claim 2.** Choose a word \( \tilde{\alpha}' \in (\Sigma_T \sqcup \Sigma_T^{-1})^* \) which represents \( \alpha \). By construction, \( \tilde{\gamma}(\tilde{\alpha}') \) and \( \tilde{\gamma}(\tilde{\alpha}) \) represent \( \alpha \), hence they are homotopic in \( \text{Sal}(\Gamma) \) relative to \( x_0 \). So, \( \gamma(\tilde{\alpha}') \) is homotopic to \( \gamma(\tilde{\alpha}) \) relative to endpoints. It follows that \( \pi_T(\gamma(\tilde{\alpha}')) = \gamma(\tilde{\alpha}') = \gamma(\tilde{\alpha}) \) is homotopic to \( \pi_T(\gamma(\tilde{\alpha})) = \gamma(\tilde{\tau}) \) in \( \text{Sal}(\Gamma_T) \). We conclude that \( \tilde{\tau} \) and \( \tilde{\alpha}' \) represent the same element of \( A_T \), namely, \( \alpha \).

**Claim 3.** Let \( \tilde{\alpha} \in (\Sigma \sqcup \Sigma^{-1})^* \). If \( \lg(\tilde{\tau}) = \lg(\tilde{\alpha}) \), then \( \tilde{\alpha} \in (\Sigma_T \sqcup \Sigma_T^{-1})^* \).

**Proof of Claim 3.** In order to have \( \lg(\tilde{\tau}) = \lg(\tilde{\alpha}) \), we must have \( t_i \in T \) for all \( i \in \{1, \ldots, \ell\} \). We will prove by induction on \( i \) that \( s_i \) also lies in \( T \) for all \( i \).

Let \( i = 1 \). Suppose first that \( \varepsilon_1 = 1 \). Then \( u_0 = w_0 = 1 \), so \( s_1 = w_0s_1w_0^{-1} = t_1 \in T \). Suppose now that \( \varepsilon_1 = -1 \). If we had \( s_1 \not\in T \), then \( s_1 \) would be \( (T, \emptyset) \)-minimal and we would have \( u_1 = w_1 = s_1 \). In this case, \( s_1 = w_1s_1w_1^{-1} = t_1 \in T \), a contradiction. So, \( s_1 \in T \).

Now assume that \( i \geq 2 \). By induction, we have \( u_{i-1} = s_1 \cdots s_{i-1} \in W_T \). Suppose first that \( \varepsilon_i = 1 \). Since \( u_{i-1} \in W_T \), we have \( u_{i-1} = u_{i-1} \) and \( w_{i-1} = 1 \), hence \( s_i = t_i \in T \). Suppose now that \( \varepsilon_i = -1 \). If we had \( s_i \not\in T \), then we would have \( u_i = u_{i-1} \) and \( w_i = s_i \), which would imply that \( s_i = t_i \in T \), a contradiction. So, \( s_i \in T \).

We can now complete the proof of the theorem. Let \( \alpha \in A_T \), and let \( \tilde{\alpha} \in (\Sigma \sqcup \Sigma^{-1})^* \) be a geodesic for \( \alpha \). By Claim 2, \( \tilde{\tau} \) is also an expression for \( \alpha \). By construction, \( \lg(\tilde{\tau}) \leq \lg(\tilde{\alpha}) \), hence \( \lg(\tilde{\tau}) = \lg(\tilde{\alpha}) \) since \( \tilde{\alpha} \) is a geodesic. By Claim 3 we conclude that \( \tilde{\alpha} \in (\Sigma_T \sqcup \Sigma_T^{-1})^* \). □

**References**


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