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To cite this version:

HAL Id: hal-00935880
https://hal.archives-ouvertes.fr/hal-00935880v2
Submitted on 2 Feb 2015

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CASTELNUOVO-MUMFORD REGULARITY AND
SEGRE-VERONESE TRANSFORM

MARCEL MORALES AND NGUYEN THI DUNG

Abstract. In this paper we give a nice formula for the Castelnuovo-Mumford regularity of the Segre product of modules, under some suitable hypotheses. This extends recent results of David A. Cox, and Evgeny Materov (2009).

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1. Introduction

Segre and Veronese embeddings of projective variety plays a key role in Algebraic Geometry. From the algebraic point of view $K$-algebras and their free resolutions are important fields of research. One of the important numerical invariants of $S$-modules is the Castelnuovo-Mumford regularity. Several approaches to study the Castelnuovo-Mumford regularity of Segre Veronese embeddings was given. The first one was done in [B-M]. S.Goto and K. Watanabe have studied the local cohomology modules of Veronese and Segre transform of graded modules. Motivated by the paper of David A. Cox, and Evgeny Materov (2009), where is computed the Castelnuovo-Mumford regularity of the Segre Veronese embedding, we extend their result and compute the Castelnuovo-Mumford regularity of the Segre product of modules under an hypothesis on the persistence of the local cohomology modules; note that this

2010 Mathematics Subject Classification: Primary: 13D40, Secondary 14M25, 13C14, 14M05.

Key words and phrases: Segre-Veronese, Castelnuovo-Mumford regularity, Cohen-Macaulay, postulation number.

This work was supported partially by VIASM, Hanoi, Vietnam.
hypothesis is true for Cohen-Macaulay modules. In [G-W] it is given a formula for the local cohomology of a Segre product of modules with depth \( \geq 2 \). We can easily extends it to modules with depth \( \geq 1 \). By definition the Castelnuovo-Mumford regularity is the maximum of the ”real” regularities of some Segre product of local cohomology modules. In section 2, we define the ”virtual” regularities and we prove that even if the ”real” regularities and the ”virtual” regularities are different, taking the maximum over the set of all ”virtual” regularities gives the Castelnuovo-Mumford regularity. This allows us to give nice formulas for Castelnuovo-Mumford regularity of the Segre product of Cohen-Macaulay modules. Some cases of the above results were obtained by studying Hilbert-Poincaré series in [MD1]. In section 3, we study when our hypothesis are true for Segre products of Stanley Reisner rings.

2. Segre transform, Local cohomology and Castelnuovo-Mumford regularity

Let \( S \) be a polynomial ring over a field \( K \), in a finite number of variables. We suppose that \( S \) is graded by the standard graduation \( S = \bigoplus_{i \geq 0} S_i \). Let \( m = \bigoplus_{i \geq 1} S_i \) be the maximal irrelevant ideal. Let \( M \) be a finitely generated graded \( S \)-module, \( M = \bigoplus_{l \geq \sigma} M_l \), with \( \sigma \in \mathbb{Z} \) and \( M_\sigma \neq 0 \).

Remark 2.1. We choose to take as base ring polynomials rings, because of the paper [G-W], but in fact by using [B-S], all our results will be true over standard Noetherian graded rings \( R = R_0[R_1] \), where \( R_0 \) is a local ring with infinite residue field. We can assume without loss of generality that the field \( K \) is infinite.

Let \( M \) be a finitely generated graded \( S \)-module, the local cohomology modules are graded, so we can define \( \text{End}(H^i_m(M)) = \max \{ \beta \in \mathbb{Z} \mid (H^i_m(M))_\beta \neq 0 \} \), \( r_j(M) = \text{End}(H^j_m(M)) + j \) and \( \text{reg}(M) = \max r_j(M) \) the Castelnuovo-Mumford regularity of \( M \).

Let \( S_1, S_2 \) be two polynomial rings on two disjoint sets of variables, for \( i = 1, 2 \), \( M_i \) be a graded \( S_i \)-module. The Segre product \( M_1 \otimes M_2 \) is defined by \( \bigoplus_{n \in \mathbb{Z}} (M_1)_n \otimes (M_2)_n \). Note that \( M_1 \otimes M_2 \) is a \( S_1 \otimes S_2 \)-module. By using Künneth formula for global cohomology (see [G-W][Proposition (4.1.5)]and Remark 4.1.7), [SV][Section 0.2]), we can extend [G-W][Proposition (4.1.5)]:

Theorem 2.2. Let \( S_1, S_2 \) be two polynomial rings on two disjoint sets of variables, for \( i = 1, 2 \), \( M_i \) be a finitely generated graded \( S_i \)-module. Let \( m \) be the maximal irrelevant ideal of \( S_1 \otimes S_2 \). Assume that \( \dim M_1 \geq 1, \dim M_2 \geq 1, \) and \( \text{depth} M_i \geq \min(2, \dim M_i) \) for \( i = 1, 2 \). Then for all integers \( j, k, l \geq 1 \)
\[
H^k_m(M_1 \otimes M_2) \simeq (M_1 \otimes H^l_m(M_2)) \oplus (H^k_m(M_1) \otimes M_2) \oplus \bigoplus_{j = k + l - 1} (H^k_m(M_1) \otimes H^l_m(M_2)).
\]
Proof. The case $\dim M_1, \dim M_2 \geq 2$ is [G-W][Proposition (4.1.5)]. Suppose that $M_1$ is Cohen-Macaulay of dimension 1 and $\text{depth} M_2 \geq 1$. From [G-W][Remark 4.1.7] we get two exact sequences (for the notations we refer to [G-W]):

$$
0 \rightarrow M_1 \rightarrow M_1^0 \rightarrow H_{m_1}(M_1) \rightarrow 0
$$

$$
0 \rightarrow H_{m_1}(M_1) \otimes M_2 \rightarrow H_{m}(M_1 \otimes M_2) \rightarrow M_1^0 \otimes H_{m_2}(M_2) \rightarrow 0
$$

from the first one we get the exact sequence:

$$
0 \rightarrow M_1 \otimes H_{m_2}(M_2) \rightarrow M_1^0 \otimes H_{m_2}(M_2) \rightarrow H_{m_1}(M_1) \otimes H_{m_2}(M_2) \rightarrow 0,
$$

Our claim follows from these two exact sequences. $\square$

Note that if $M_1, M_2$ are Cohen-Macaulay modules of dimension 1, then $M_1 \otimes M_2$ is a Cohen-Macaulay module of dimension 1, and $\text{reg}(M_1 \otimes M_2) = \max(\text{reg}(M_1), \text{reg}(M_2))$.

**Definition 2.3.** From now on, we consider $s$-polynomial rings (with disjoint set of variables) $S_1, \ldots, S_s$, graded, with irrelevant ideals $m_1, \ldots, m_s$. For $i = 1, \ldots, s$, let $M_i$ be a finitely generated graded $S_i$-module such that $\dim M_i \geq 1$ and $\text{depth} M_i \geq \min(2, \dim M_i)$. (Without loss of generality we can assume that at most one module has dimension one). We set the following notations:

- $d_i = \dim M_i$; $\sigma_i = \min\{l \in \mathbb{Z} : (M_i)_l \neq 0\}$;
- $r_{i,j} := \text{End}(H_{m_i}^j(M_i)) + j$;
- $C := \{0, \ldots, d_1\} \times \{0, \ldots, d_2\} \times \ldots \times \{0, \ldots, d_s\}$,
- for $u \in C$, $\text{Supp} u = \{i \in \{1, \ldots, s\} \mid u_i \neq 0\}$,
- $E_{i,j} = \begin{cases} H_{m_i}^j(M_i) & \text{if } j > 0 \\ M_i & \text{if } j = 0 \end{cases}$;
- for $u \in C$, $E_u = \bigotimes_{i=1}^s E_{i,u_i}$, $\tilde{E}_u = \bigotimes_{i \in \text{Supp} u} E_{i,u_i}$.

We can state the following corollary of Theorem 2.2.

**Corollary 2.4.** With the notations introduced in 2.3, for $j \geq 1$, we have

$$
H_{m_i}^j(M_1 \otimes \ldots \otimes M_s) \simeq \bigoplus_{u \in C, s.t. \sum_{l \in \text{Supp} u} (u_l - 1) = j - 1} E_u,
$$

where $m$ is the irrelevant maximal ideal of $S_1 \otimes \ldots \otimes S_s$.

Hence we have that

$$
\text{reg}(M_1 \otimes \ldots \otimes M_s) = \max\{1 + \sum_{u \in C} (u_l - 1) + \text{End}(E_u)\}.
$$
In order to compute \( \text{reg}(M_1 \otimes \ldots \otimes M_s) \), we have to know when \( E_u \neq 0 \), and in this case find \( \text{End}(E_u) \). In a concrete example, if we know all local cohomology modules, it is possible to compute the regularity of the Segre product, but to give a formula in general is impossible. Our purpose is to give a formula in terms of the regularity data \( r_{i,u} \), looking for the optimal hypothesis.

We say that a graded \( S \)-module \( N = \oplus_{t \in \mathbb{Z}} N_t \), not necessarily finitely generated has no gaps if there is no integers \( i < j < k \) such that

\[ N_i \neq 0; N_k \neq 0; N_j = 0. \]

**Example 2.5.** Let \( M \) be a finitely generated graded \( S \)-module with \( \text{depth}M \geq 1 \), \( M = \oplus_{\sigma \geq \sigma} M_{\sigma} \), where \( \sigma \in \mathbb{Z} \) and \( M_{\sigma} \neq 0 \). Since the field \( K \) is infinite, there exists \( x \in S_1 \), a nonzero divisor of \( M \). The multiplication by \( x \) defines an injective map \( M_i \to M_{i+1} \), hence \( M \) has no gaps, and for all \( l \geq \sigma \), we have \( M_l \neq 0 \).

**Assumption 2.6.** From now on, we assume: for any \( i = 1, \ldots, s \);
\[ \dim M_i \geq 1 \text{ and } \text{depth}M_i \geq \min(2, \dim M_i). \]
For any \( \text{depth}M_i \leq j \leq \dim M_i \), if \( H^j_{m_i}(M_i) \neq 0 \) then \( H^j_{m_i}(M_i) \) has no gaps, and \( (H^j_{m_i}(M_i))_k \neq 0 \) for infinitely many \( k \). (Without loss of generality we can assume that at most one module has dimension one).

Note that our assumption is true if all the modules \( M_i \) are Cohen-Macaulay.

We introduce another piece of notation:

- For \( u \in C \), \( \Gamma_u = 1 + \sum_{t \in \text{Supp} u} (u - 1) + \text{End}(E_u) \).
- For \( u \in C \), \( \gamma_u = 1 + \sum_{t \in \text{Supp} u} (u - 1) + \min_{t \in \text{Supp} u} (\text{End}(H^u_{m_i}(M_i))) \).
- \( C_1 := \{ u \in C; E_u \neq 0 \}, C_2 := \{ u \in C; \tilde{E}_u \neq 0 \} \).

The following Lemma follows immediately from the definitions and the Assumption 2.6.

**Lemma 2.7.** With the notations introduced in 2.3, and the Assumption 2.6. Let \( \epsilon_1, \ldots, \epsilon_s \) be the canonical basis of \( \mathbb{Z}^s \). For any \( u \in C \):

1. \( \tilde{E}_u \neq 0 \) if and only if \( H^{u}_{m_i}(M_i) \neq 0 \) for all \( i \in \text{Supp} u \).
2. \( \text{End} \tilde{E}_u = \min_{i \in \text{Supp} u} (\text{End}(H^{u}_{m_i}(M_i))) \).
3. If \( E_u \neq 0 \) then \( \text{End} E_u = \text{End} \tilde{E}_u \), that is \( \Gamma_u = \gamma_u \). If \( u \) has full support then \( E_u \neq 0 \).
4. For any \( k \notin \text{Supp} u \), \( \lambda_k \in \mathbb{N}^s, \lambda_k \leq d_k, \) such that \( H^{\lambda_k}_{m_k}(M_k) \neq 0 \), we have

\[ \gamma_u + \lambda_k \epsilon_k = \min(\gamma_u + \lambda_k - 1, \text{End}(H^{\lambda_k}_{m_k}(M_k)) + \lambda_k + \sum_{t \in \text{Supp} u} (u - 1)). \]
We can state our main result:

**Theorem 2.8.** With the notations introduced in 2.3, We have:

$$\text{reg}(M_1 \otimes \cdots \otimes M_s) \leq \max \left\{ 1 + \sum_{l \in \text{Supp}u} (u_l - 1) + \min_{i \in \text{Supp}u} (\text{End}(H^u_{m_i}(M_i))) \right\}.$$ 

Moreover the Assumption 2.6 implies the equality.

**Proof.** Note that

$$\text{reg}(M_1 \otimes \cdots \otimes M_s) = (\Gamma_u|E_u \neq 0) \leq \max (\gamma_u|u \in C_2).$$

Hence the inequality is trivial. The equality will follows if we prove that

$$\max (\Gamma_u|E_u \neq 0) = \max (\gamma_u|u \in C_2).$$

By the above Lemma if $E_u \neq 0$ then $\Gamma_u = \gamma_u$. We suppose that $E_u = 0$. For any $i = 1, \ldots, s$ let $\delta_i$ be an integer such that $\text{reg}(M_i) = \text{End}(\delta_i) + \delta_i$. We will prove the following statement

(*) There exists $n \notin \text{Supp}u$, such that $\gamma_u \leq \gamma_{u+\delta_n \epsilon_n}.$

If $E_{u+\delta_n \epsilon_n} \neq 0$ then $\gamma_u \leq \gamma_{u+\delta_n \epsilon_n} = \Gamma_{u+\delta_n \epsilon_n}$, otherwise we repeat the argument. This process ends since the local cohomology module $E_{u+\sum_{l \in \text{Supp}u} \delta_l}$ is non zero by the Assumption 2.6. Hence there exist some set $J \subset \{1, \ldots, n\} \setminus \text{Supp}u$ such that $E_u + \sum_{n \in J} \delta_n \epsilon_n \neq 0$ and $\gamma_u \leq \gamma_u + \sum_{n \in J} \delta_n \epsilon_n = \Gamma_u + \sum_{n \in J} \delta_n \epsilon_n$ the claim is true.

Now we prove (*): we have $E_u = 0 \iff \text{End}(\bigotimes_{i \notin \text{Supp}u} H^u_{m_i}(M_i)) < \max_{j \notin \text{Supp}u} \sigma_j.$

Let $n \notin \text{Supp}u$ such that $\max_{j \notin \text{Supp}u} \sigma_j = \sigma_n.$ Thus the condition $E_u = 0$ is equivalent to

$$\min_{i \in \text{Supp}u} (\text{End}(H^u_{m_i}(M_i))) < \sigma_n.$$

But $\sigma_n \leq \text{reg}(M_n) = \text{End}(\delta_n) + \delta_n$ by [B-S, Theorem 15.3.1]. So

$$\min_{i \in \text{Supp}u} (\text{End}(H^u_{m_i}(M_i))) \leq \text{End}(\delta_n) + \delta_n - 1.$$

It implies that

$$\gamma_u = 1 + \sum_{l \in \text{Supp}u} (u_l - 1) + \min_{i \in \text{Supp}u} (\text{End}(H^u_{m_i}(M_i))) \leq \text{End}(\delta_n) + \delta_n + \sum_{l \in \text{Supp}u} (u_l - 1).$$

On the other hand, since $\text{depth}(M_n) \geq 1$, then $\delta_n \geq 1$, and we have trivially that $\gamma_u \leq \gamma_u + (\delta_n - 1)$ which implies that

$$\gamma_u \leq \min(\gamma_u + \delta_n - 1, \text{End}(\delta_n) + \delta_n + \sum_{l \in \text{Supp}u} (u_l - 1)) = \gamma_u + \delta_n \epsilon_n.$$

The last equality follows from Lemma 2.7. \qed
In order to apply our results to Segre Veronese transform, we recall the following Proposition from [MD2]:

**Proposition 2.9.** If $M$ is a Cohen-Macaulay module of dimension $d$, then

$$\text{reg} M[\tau]^{<n>} = d - \lceil \frac{d - \text{reg} M + \tau}{n} \rceil.$$ 

Hence we have the following consequence:

**Theorem 2.10.** Let $S_1, \ldots, S_s$ be graded polynomial rings on disjoint sets of variables. For all $i = 1, \ldots, s$, let $M_i$ be a graded finitely generated $S_i$-Cohen-Macaulay module with $\dim M_i \geq 1$. (Without loss of generality we can assume that at most one module has dimension one). Let $d_i = \dim M_i, b_i = d_i - 1 \geq 1, \alpha_i = d_i - \text{reg}(M_i)$, where $\text{reg}(M_i)$ is the Castelnuovo-Mumford regularity of $M_i$. Then

1. $\text{reg}(M_1 \otimes \ldots \otimes M_s) = \max \{1 + \sum_{u \in \text{C}_2} b_u - \max_{l \in \text{Supp}} \{\alpha_l\}\}$.

2. For $n_i \in \mathbb{N}$, let $M_i[\tau_i]^{<n_i>}$ be the shifted $n_i$-Veronese transform of $M_i$, then

$$\text{reg}(M_1[\tau_1]^{<n_1>} \otimes \ldots \otimes M_s[\tau_s]^{<n_s>}) = \max \{1 + \sum_{u \in \text{C}_2} b_u - \max_{l \in \text{Supp}} \{\left\lceil \frac{\alpha_l + \tau_i}{n_l} \right\rceil\}\}.$$

As a Corollary we generalize one of the main results of [C-M][Theorem 1.4]

**Corollary 2.11.** ([C-M][Theorem 1.4]) For $i = 1, \ldots, s$, let $S_i$ be graded polynomial rings on disjoint sets of variables, with $\dim S_i \geq 1$. (Without loss of generality we can assume that at most one ring has dimension one). Let $m_i, n_i \in \mathbb{Z}$, and $S_i[m_i]^{<n_i>}$ be the $n_i$-Veronese transform of $S_i[m_i]$, then

$$\text{reg}(S_1[m_1]^{<n_1>} \otimes \ldots \otimes S_s[m_s]^{<n_s>}) = \max \{1 + \sum_{u \in \text{C}_2} b_u - \max_{l \in \text{Supp}} \{\left\lceil \frac{b_l + m_i + 1}{n_l} \right\rceil\}\}.$$

### 2.1. Local Cohomology Modules without Gaps

Let $M$ be a finitely generated graded $S$-module. We recall the local duality’s theorem (see [S]):

We have an isomorphism:

$$H^i_m(M) \simeq \text{Hom}_S(\text{Ext}^{n-i}_S(M, S), E(S/m)).$$

We denote by $D^i(M)$ the finitely generated graded $S$-module $\text{Ext}^{n-i}_S(M, S)$. The following Lemma follows immediately from the local duality’s theorem and the Example 2.5.

**Lemma 2.12.**

- If $\text{depth}(D^i(M)) \geq 1$ then $H^i_m(M)$ has no gaps and for all $l \leq \text{End}(H^i_m(M))$, we have $(H^i_m(M))_l \neq 0$.
- Let $M$ be a finitely generated graded $S$-module with $\text{depth} M \geq 1$ and $\dim M = d$. It is known that $\text{depth} D^d(M) \geq \min\{d, 2\}$. Hence the top local cohomology $H^d_m(M)$ has no gaps and for all $l \leq \text{End}(H^d_m(M))$, we have $(H^d_m(M))_l \neq 0$. 
It follows from [M1] that if $A$ is a standard graded simplicial toric ring of dimension $d$, and depth$A = d - 1$, then $D^{d - 1}(A)$ is a Cohen-Macaulay Module of dimension $d - 2$, so if $d \geq 3$, the module $H^{d - 1}_m(A)$ has no gaps, and for all $l \leq \operatorname{End}(H^{d - 1}_m(M))$, we have $(H^{d - 1}_m(M))_l \neq 0$.

3. Square free monomial ideals

Let $\Delta$ be a simplicial complex with support on $n$ vertices, labeled by the set $[n] = \{1, ..., n\}$, $S := K[x_1, ..., x_n]$ be a polynomial ring, $I_\Delta \subset S$ be the Stanley-Reisner ideal associated to $\Delta$, that is $I_\Delta = (x_F/F \notin \Delta)$. The Alexander dual of $\Delta$ is the simplicial complex $\Delta^*$ defined by: $F \subset [n]$ is a face of $\Delta^*$ if and only if $[n] \setminus F \notin \Delta$. The following theorem is a well known consequence of Hochster’s Theorems (see for example [Sb][Proposition 3.8]):

**Proposition 3.1.** Let $K[\Delta] := S/I_\Delta$ be the Stanley-Reisner ring associated to $\Delta$, $a = (a_1, ..., a_n) \in \mathbb{Z}$, where for all $i$, $a_i \leq 0$. Let $F = \operatorname{Supp}(a) \subset [n]$ and $|F| := \operatorname{Card}(F)$ then:

$$\dim_K(H^i_m(K[\Delta]))_a = \beta_{i+1-|F|,|n\setminus F|}(K[\Delta^*]).$$

We get the following Corollary:

**Corollary 3.2.** [Sb][Lemma 3.9]

1. \( \dim_K(H^i_m(K[\Delta]))_0 = \beta_{i+1,n}(K[\Delta^*]) \),

2. For any integer \( j > 0 \):

$$\dim_K(H^i_m(K[\Delta]))_{-j} = \sum_{h=1}^{\min(j,n)} \binom{n}{h} \binom{h+j-1}{j} \beta_{i+1-h,n-h}(K[\Delta^*]).$$

**Corollary 3.3.** Let $i$ be an integer such that $(H^i_m(K[\Delta])) \neq 0$, define $k_i$ as the smallest $0 \leq h \leq n$ such that $\beta_{i+1-h,n-h}(K[\Delta^*]) \neq 0$. If $k_i \geq 1$ then $H^i_m(K[\Delta])$ has no gaps and $(H^i_m(K[\Delta]))_k \neq 0$ for all $k \leq \operatorname{end}(H^i_m(K[\Delta]))$.

**Proof.** The formula proved in 3.2 implies that $(H^i_m(K[\Delta]))_k \neq 0$ for all $k \leq k_i$. The claim is over. $\square$

Let remark that depth$(K[\Delta]) \geq 1$ always, so there is a big class of Stanley-Reisner rings that satisfies the Assumption 2.6. In any case if we have $s$ Stanley-Reisner rings, the Castelnuovo-Mumford regularity of their Segre product, can be read off from their Betti’s tables, by the Corollary 2.4.
References


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