# Spinors and isometric immersions of surfaces in 4-dimensional products

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# SPINORS AND ISOMETRIC IMMERSIONS OF SURFACES IN 4-DIMENSIONAL PRODUCTS

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ABSTRACT. We prove a spinorial characterization of surfaces isometrically immersed into the 4-dimensional product spaces $\mathbb{M}^3(c) \times \mathbb{R}$ and $\mathbb{M}^2(c) \times \mathbb{R}^2$, where $\mathbb{M}^n(c)$ is the $n$-dimensional real space form of curvature $c$.

## 1. INTRODUCTION

In [4], Friedrich gave a spinorial characterization of surfaces in Euclidean 3-space. Namely, he proved that the existence of a so-called generalized Killing spinor $\psi$ on surface $(M^2, g)$, that is

$$\nabla_X \psi = A(X) \cdot \psi,$$

where $A$ is a symmetric $(1,1)$-tensor, is equivalent to the Gauss and Codazzi equations and therefore to an isometric immersion of $(M^2, g)$ into $\mathbb{R}^3$ with $-2A$ as shape operator. Later on, Morel generalized in [9] this result for surfaces of the sphere $\mathbb{S}^3$ and the hyperbolic space $\mathbb{H}^3$ and we give in [12] an analogue for 3-dimensional homogeneous manifolds with 4-dimensional isometry group, as well as for surfaces into pseudo-Riemannian space forms [6] and Lorentzian products [13]. In a more recent work [2], we studied with Bayard and Lawn the spinorial characterization of surfaces into 4-dimensional space forms. A similar result was proved by Bayard for spacelike surfaces into the 4-dimensional Minkowski space [1].

In this paper, we extend this spinorial characterization for surfaces in the product spaces $\mathbb{M}^3(c) \times \mathbb{R}$ and $\mathbb{M}^2(c) \times \mathbb{R}^2$, where $\mathbb{M}^n(c)$ is the $n$-dimensional real space form of constant sectional curvature $c \neq 0$.

First we characterize immersions of surfaces into these product spaces by the existence of special spinor fields satisfying an appropriate generalized Killing-type equation, that is an equation involving the spinorial connection (see Theorem 3.1). Then, we show that this equation is equivalent to the corresponding Dirac equation with an additional condition on the norm of the spinor field (see Proposition 4.1 and Corollary 4.2).

## 2. PRELIMINARIES

In this section of preliminaries, we will first recall some basics about surfaces into the product spaces $\mathbb{M}^2(c) \times \mathbb{R}^2$ and $\mathbb{M}^3(c) \times \mathbb{R}$. In particular, we will recall the compatibility equations assuring that a surface is isometrically immersed into one of these spaces. Then, we will give some facts about restrictions of spinors on a surface into a 4-dimensional space and deduce the particular spinor fields with which we will work in the sequel.

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2.1. **Compatibilty equations.** Let $(M^2, g)$ be a Riemannian surface isometrically immersed into the product space $P = \mathbb{M}^2(c) \times \mathbb{R}^2$ or $\mathbb{M}^3(c) \times \mathbb{R}$, endowed with the product metric $\widetilde{g}$. We denote by $F$ product structure of $P$. The map $F : TP \longrightarrow TP$ is defined by $F(X_1 + X_2) = X_1 - X_2$, where $X_1$ belongs to the first factor $\big(T\mathbb{M}^2(c)$ or $T\mathbb{M}^3(c)\big)$ and $X_2$ belongs to the second factor $\big(T\mathbb{R}^2$ or $T\mathbb{R}\big)$. Obviously, $F$ satisfies

$$F^2 = Id \quad (\text{and } F \neq Id), \tag{1}$$

$$\widetilde{g}(FX, Y) = \widetilde{g}(X, FY), \tag{2}$$

$$\widetilde{\nabla} F = 0. \tag{3}$$

Moreover, we recall that the curvature of $(P, \widetilde{g})$ is given by

$$\begin{aligned} \widetilde{R}(X,Y)Z &= \frac{c}{4}\bigg[ \langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle FY, Z\rangle FX - \langle FX, Z\rangle FY \\ &\quad + \langle Y, Z\rangle FX - \langle X, Z\rangle FY + \langle Y, FZ\rangle X - \langle X, FZ\rangle Y \bigg] \end{aligned} \tag{4}$$

This product structure $F$ induces the existence of the following four operators

$$f : TM \longrightarrow TM,\ h : TM \longrightarrow NM,\ s : NM \longrightarrow TM \text{ and } t : NM \longrightarrow NM$$

defined for any $X \in TM$ and $\xi \in NM$ by

$$FX = fX + hX \quad \text{and} \quad F\xi = s\xi + t\xi. \tag{5}$$

From Equations (1) and (2), $f$ and $t$ are symmetric and we have the following relations between these four operators

$$f^2 X = X - shX, \tag{6}$$

$$t^2\xi = \xi - hs\xi, \tag{7}$$

$$fs\xi + st\xi = 0, \tag{8}$$

$$hfX + thX = 0, \tag{9}$$

$$\widetilde{g}(hX, \xi) = \widetilde{g}(X, s\xi), \tag{10}$$

for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(NM)$. Moreover, from Equation (3), we have

$$(\nabla_X f)Y = S_{hY}X + s(B(X,Y)), \tag{11}$$

$$\nabla^\perp_X(hY) - h(\nabla_X Y) = t(B(X,Y)) - B(X, fY), \tag{12}$$

$$\nabla^\perp(t\xi) - t(\nabla^\perp_X \xi) = -B(s\xi, X) - h(S_\xi X), \tag{13}$$

$$\nabla_X(s\xi) - s(\nabla^\perp_X \xi) = -f(S_\xi X) + S_{t\xi}X, \tag{14}$$

where $B : TM \times TM \longrightarrow NM$ is the second fundamental form and for any $\xi \in TM$, $S_\xi$ is the Weingarten operator associated with $\xi$ and defined by $\widetilde{g}(S_\xi X, Y) = \widetilde{g}(B(X,Y), \xi)$ for any vectors $X, Y$ tangent to $M$.
Finally, from (4), we deduce that the Gauss, Codazzi and Ricci equations are respectively given by

$$\begin{aligned} R(X,Y)Z &= \frac{c}{4}\bigg[ \langle Y, Z\rangle X - \langle X, Z\rangle Y + \langle fY, Z\rangle fX - \langle fX, Z\rangle fY \\ &\quad \langle Y, Z\rangle fX - \langle X, Z\rangle fY + \langle Y, fZ\rangle X - \langle X, fZ\rangle Y \bigg] \\ &\quad + S_{B(Y,Z)}X - S_{B(X,Z)}Y, \end{aligned} \tag{15}$$

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = \frac{c}{4}\Big[ \langle fY, Z\rangle\, hX - \langle fX, Z\rangle\, hY + \langle Y, Z\rangle\, hX - \langle X, Z\rangle\, hY \Big], \tag{16}$$

$$R^{\perp}(X,Y)\xi = \frac{c}{4}\Big[ \langle hY, \xi\rangle\, hX - \langle hX, \xi\rangle\, hY \Big] + B(S_\xi Y, X) - B(S_\xi X, Y). \tag{17}$$

Conversely, let $(M^2, g)$ a Riemannian surface endowed with a rank 2 vector bundle $E$ endowed with a metric and a compatible connection$\nabla^{\perp}$. Assume that there exist some tensors $f$, $h$, $s$, $t$ and $B$ satisfying Equations (6)-(13) (note that (14) is not required since it is the dual equation of (12)) and the Gauss-Codazzi-Ricci equations (15)-(17). Moreover we define the operator $F : TM \oplus E \longrightarrow TM \oplus E$ by relations (5). If in addition the operator $F$ satisfy that the ranks of the maps $\frac{F+Id}{2}$ and $\frac{F-Id}{2}$ are 2 and 2 (*resp.* 3 and 1), then Kowalczyk [5] and De Lira-Tojeiro-Vitório [8] proved independently that there exists an isometric immersion from $(M, g)$ into $\mathbb{M}^2(c) \times \mathbb{R}^2$ (*resp.* $\mathbb{M}^3(c) \times \mathbb{R}$) with $E$ as normal bundle, $B$ as second fundamental form and such that the product structure of $\mathbb{M}^2(c) \times \mathbb{R}^2$ (*resp.* $\mathbb{M}^3(c) \times \mathbb{R}$) coincide with $F$ over $M$. Note that this was previously proven in a more abstract way by Piccione and Tausk [11].

2.2. **Spinors on surfaces of $P$.** For details about the recalls of this section, the reader can refer to [3] for instance. Let $(M^2, g)$ be an oriented Riemannian surface, with a given spin structure, and $E$ an oriented and spin vector bundle of rank 2 on $M$. We consider the spinor bundle $\Sigma$ over $M$ twisted by $E$ and defined by

$$\Sigma = \Sigma M \otimes \Sigma E,$$

where $\Sigma M$ and $\Sigma E$ are the spinor bundles of $M$ and $E$ respectively. We endow $\Sigma$ with the spinorial connection $\nabla$ defined by

$$\nabla = \nabla^{\Sigma M} \otimes Id_{\Sigma E} + Id_{\Sigma M} \otimes \nabla^{\Sigma E}.$$

We also define the Clifford product $\cdot$ by

$$\begin{cases} X \cdot \varphi = (X \cdot_{_M} \alpha) \otimes \overline{\sigma} & \text{if } X \in \Gamma(TM) \\ X \cdot \varphi = \alpha \otimes (X \cdot_{_E} \sigma) & \text{if } X \in \Gamma(E) \end{cases}$$

for all $\varphi = \alpha \otimes \sigma \in \Sigma M \otimes \Sigma E$, where $\cdot_{_M}$ and $\cdot_{_E}$ denote the Clifford products on $\Sigma M$ and on $\Sigma E$ respectively and where $\overline{\sigma} = \sigma^+ - \sigma^-$ for the natural decomposition of $\Sigma E = \Sigma^+ E \oplus \Sigma^- E$. Here, $\Sigma^+ E$ and $\Sigma^- E$ are the eigensubbundles (for the eigenvalue 1 and $-1$) of $\Sigma E$ for the action of the normal volume element $\omega_{\perp} = i\xi_1 \cdot_E \xi_2$, where $\{\xi_1, \xi_2\}$ is a local orthonormal frame of $E$. Note that $\Sigma^+ M$ and $\Sigma^-$ are defined similarly by for the tangent volume element $\omega = ie_1 \cdot_M e_2$. We finally define the Dirac operator $D$ on $\Gamma(\Sigma)$ by

$$D\varphi = e_1 \cdot \nabla_{e_1}\varphi + e_2 \cdot \nabla_{e_2}\varphi,$$

where $\{e_1, e_2\}$ is an orthonormal basis of $TM$.

We note that $\Sigma$ is also naturally equipped with a hermitian scalar product $\langle ., . \rangle$ which is compatible to the connection $\nabla$, since so are $\Sigma M$ and $\Sigma E$, and thus also with a compatible real scalar product $\Re e\langle ., .\rangle$. We also note that the Clifford product $\cdot$ of vectors belonging to $TM \oplus E$ is antihermitian with respect to this hermitian product. Finally, we stress that the four subbundles $\Sigma^{\pm\pm} := \Sigma^{\pm} M \otimes \Sigma^{\pm} E$ are orthogonal with respect to the hermitian product. We will also consider $\Sigma^+ = \Sigma^{++} \oplus \Sigma^{--}$ and $\Sigma^- = \Sigma^{+-} \oplus \Sigma^{-+}$.Throughout the paper we will assume that the hermitian product is $\mathbb{C}-$linear w.r.t. the first entry, and $\mathbb{C}-$antilinear w.r.t. the second entry.

Now, let $(P, \widetilde{g})$ be a 4-dimensional spin manifold. It is a well-known fact that there is an identification between the spinor bundle $\Sigma P_{|M}$ of $P$ over $M$, and the spinor bundle of $M$ twisted by the normal bundle $\Sigma := \Sigma M \otimes \Sigma E$. Moreover, we have the spinorial Gauss formula: for any $\varphi \in \Gamma(\Sigma)$ and any $X \in TM$,

$$\widetilde{\nabla}_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{j=1,2} e_j \cdot B(X, e_j) \cdot \varphi \tag{18}$$

where $\widetilde{\nabla}$ is the spinorial connection of $\Sigma P$ and $\nabla$ is the spinoral connection of $\Sigma$ defined as above and $\{e_1, e_2\}$ is a local orthonormal frame of $TM$. We will also use this notation and $\{\xi_1, \xi_2\}$ for a local orthonormal frame of $E$. Here $\cdot$ is the Clifford product on $P$.
From now on, we will take $P = \mathbb{M}^2(c) \times \mathbb{R}^2$ or $\mathbb{M}^3(c) \times \mathbb{R}$. By restriction of a parallel spinor of the Euclidean space $\mathbb{R}^5$ if $c > 0$ or the Lorentzian space $\mathbb{R}^{4,1}$ if $c < 0$, we obtain on $P$ a spinor field $\varphi$ satisfying

$$\begin{cases} \widetilde{\nabla}_X \varphi = \alpha X \cdot \varphi & \text{if} \quad X \in \Gamma(T\mathbb{M}^2(c)) \quad \text{or} \quad \Gamma(T\mathbb{M}^3(c)), \\ \widetilde{\nabla}_X \varphi = 0 & \text{if} \quad X \in \Gamma(T\mathbb{R}^2) \quad \text{or} \quad \Gamma(T\mathbb{R}(c)), \end{cases}$$

with $\alpha \in \mathbb{C}$ so that $4\alpha^2 = c$. In other words, for any $X \in \Gamma(TP)$, we have

$$\widetilde{\nabla}_X \varphi = \frac{\alpha}{2}(X + FX) \cdot \varphi.$$

Hence, by the spinorial Gauss formula (18), the restriction of $\varphi$ on $M$ satisfies

$$\nabla_X \varphi = \frac{\alpha}{2}(X + fX + hX) \cdot \varphi + \eta(X) \cdot \varphi, \tag{19}$$

where $\eta(X) = -\frac{1}{2} \sum_{j=1}^{2} e_j \cdot B(e_j, X)$.

## 3. Main result

Now, we have the ingredients to state the the main result of this note.

**Theorem 3.1.** *Let $c \in \mathbb{R}$, $c \neq 0$ and $\alpha \in \mathbb{C}$ such that $4\alpha^2 = c$. Let $(M^2, g)$ be an oriented Riemannian surface and $E$ an oriented vector bundle of rank 2 over $M$ with scalar product $< \cdot, \cdot >_E$ and compatible connection $\nabla^E$. We denote by $\Sigma = \Sigma M \otimes \Sigma E$ the twisted spinor bundle. Let $B : TM \times TM \longrightarrow E$ a bilinear symmetric map and*

$$f : TM \longrightarrow TM, \ h : TM \longrightarrow E, \ s : E \longrightarrow TM \text{ and } t : E \longrightarrow E$$

*satisfying Equations* (6)-(13). *Moreover we assume that the rank of the maps $\frac{F+Id}{2}$ and $\frac{F-Id}{2}$ is 2 and 2 (resp. 3 and 1), where $F : TM \oplus E \longrightarrow TM \oplus E$ is defined from $f, h, s$ and $t$ by relations* (5). *Then, the two following statements are equivalent*

1. *There exists an isometric immersion of $(M^2, g)$ into $P = \mathbb{M}^2(c) \times \mathbb{R}^2$ (resp. $\mathbb{M}^3(c) \times \mathbb{R}$) with $E$ as normal bundle and second fundamental form $B$ such that over $M$ the product strcuture is given by $f, h, t$ and $s$.*
2. *There exists a spinor field $\varphi$ in $\Sigma$ satisfying for all $X \in \mathfrak{X}(M)$*

$$\nabla_X \varphi = \frac{\alpha}{2}(X + fX + hX) \cdot \varphi + \eta(X) \cdot \varphi,$$

*such that $\varphi^+$ and $\varphi^-$ never vanish.*

*Proof:* First, we remark that the fact that (1) implies (2) has been proved in the discussion of Section 2. The work consists in proving that (2) implies (1). The computations are in the same spirit as in [2], with some techinical difficulties due to the terms arising from the product structure. We will emphasize on these differences. We have to compute the spinorial curvature of the particular spinor $\varphi$. For this, let us compute $\mathcal{R}(e_1, e_2)\varphi$, where

$(e_1, e_2)$ is a local orthonormal frame of $TM$. We also denote by $(e_3, e_4)$ a local orthonormal frame of $E$. Then, we have

$$\begin{aligned}
\mathcal{R}(e_1,e_2)\varphi &= d^\nabla\eta(e_1,e_2)\cdot\varphi + (\eta(e_2)\cdot\eta(e_1) - \eta(e_1)\eta(e_2))\cdot\varphi \\
&\quad -\frac{\alpha}{2}\Big(\nabla_{e_2}e_1 + \nabla_{e_2}(fe_1) + \nabla^\perp_{e_2}(he_1)\Big)\cdot\varphi \\
&\quad +\frac{\alpha}{2}\Big(\nabla_{e_1}e_2 + \nabla_{e_1}(fe_2) + \nabla^\perp_{e_1}(he_2)\Big)\cdot\varphi \\
&\quad +\frac{\alpha^2}{4}(e_2+fe_2+he_2)\cdot(e_1+fe_1+he_1)\cdot\varphi \\
&\quad -\frac{\alpha^2}{4}(e_1+fe_1+he_1)\cdot(e_2+fe_2+he_2)\cdot\varphi, \\
&\quad +\frac{\alpha}{2}\Big(\eta(e_1)\cdot(e_2+fe_2+he_2) - (e_2+fe_2+he_2)\cdot\eta(e_1)\Big)\cdot\varphi \\
&\quad -\frac{\alpha}{2}\Big(\eta(e_2)\cdot(e_1+fe_1+he_1) - (e_1+fe_1+he_1)\cdot\eta(e_2)\Big)\cdot\varphi \\
&\quad -\frac{\alpha}{2}([e_1,e_2] + f[e_1,e_2] + h[e_1,e_2])\cdot\varphi
\end{aligned}$$

where we denote $d^\nabla\eta(X,Y) = \nabla_X(\eta(Y)) - \nabla_Y(\eta(X)) - \eta([X,Y])$. First, by a straightforward computation, we see that the term

$$\begin{aligned}
&+\frac{\alpha}{2}\Big(\eta(e_1)\cdot(e_2+fe_2+he_2) - (e_2+fe_2+he_2)\cdot\eta(e_1)\Big)\cdot\varphi \\
&-\frac{\alpha}{2}\Big(\eta(e_2)\cdot(e_1+fe_1+he_1) - (e_1+fe_1+he_1)\cdot\eta(e_2)\Big)\cdot\varphi
\end{aligned}$$

vanishes. Moreover, by Equations (11) and (12) and the fact that the Levi-civita is torsion-free, the term

$$\begin{aligned}
&\frac{\alpha}{2}\Big(\nabla_{e_1}e_2 + \nabla_{e_1}(fe_2) + \nabla^\perp_{e_1}(he_2)\Big)\cdot\varphi \quad -\frac{\alpha}{2}\Big(\nabla_{e_2}e_1 + \nabla_{e_2}(fe_1) + \nabla^\perp_{e_2}(he_1)\Big)\cdot\varphi \\
&-\frac{\alpha}{2}([e_1,e_2] + f[e_1,e_2] + h[e_1,e_2])\cdot\varphi
\end{aligned}$$

also vanishes. Hence, we get

$$\begin{aligned}
\mathcal{R}(e_1,e_2)\varphi &= d^\nabla\eta(e_1,e_2)\cdot\varphi + (\eta(e_2)\cdot(e_1) - \eta(e_1)\eta(e_2))\cdot\varphi \\
&\quad +\frac{\alpha^2}{2}\Big(\langle fe_1,e_2\rangle^2 - \langle fe_1,e_1\rangle\langle fe_2,e_2\rangle\Big)e_1\cdot e_2\cdot\varphi \\
&\quad +\frac{\alpha^2}{2}\Big(\langle he_2,e_3\rangle\langle fe_1,e_4\rangle - \langle he_1,e_3\rangle\langle he_2,e_4\rangle\Big)e_3\cdot e_4\cdot\varphi \\
&\quad +\frac{\alpha^2}{2}\Big(fe_2\cdot he_1 - fe_1\cdot he_2 - e_2\cdot he_1 + e_1\cdot he_2\Big)\cdot\varphi
\end{aligned}$$

But, as computed in [2], we have

$$\mathcal{R}(e_1,e_2)\varphi = -\frac{1}{2}Ke_1\cdot e_2\cdot\varphi - \frac{1}{2}K_N e_3\cdot e_4\cdot\varphi, \tag{20}$$

$$d^\nabla\eta(X,Y) = -\frac{1}{2}\sum_{j=1}^{2} e_j\cdot\Big((\overline{\nabla}_X B)(Y,e_j) - (\overline{\nabla}_Y B)(X,e_j)\Big), \tag{21}$$

where $\overline{\nabla}$ stands for the natural connection on $T^*M\otimes T^*M\otimes E$, and

$$\begin{aligned}
\eta(e_2)\cdot\eta(e_1) - \eta(e_1)\cdot\eta(e_2) &= \frac{1}{2}\big(|B(e_1,e_2)|^2 - \langle B(e_1,e_1),B(e_2,e_2)\rangle\big)e_1\cdot e_2 \\
&\quad +\frac{1}{2}\langle (S_{e_3}\circ S_{e_4} - S_{e_4}\circ S_{e_3})(e_1),e_2\rangle e_3\cdot e_4. \qquad (22)
\end{aligned}$$

Therefore, we have

$$G\cdot\varphi + R\cdot\varphi + C\cdot\varphi = 0,$$

where $G$, $R$ and $C$ are the 2-forms defined by

$$\begin{aligned} G &= \Big[K + \langle B(e_1,e_1), B(e_2,e_2)\rangle - |B(e_1,e_2)| \\ &\quad +\alpha^2\left(1 - \langle fe_1,e_2\rangle^2 + \langle fe_1,e_1\rangle\langle fe_2,e_2\rangle\right)\Big] e_1\cdot e_2, \end{aligned}$$

where $K$ is the Gauss curvature of $(M,g)$,

$$\begin{aligned} R &= \Big[K_E + \langle (S_{e_3}\circ S_{e_4} - S_{e_4}\circ S_{e_3})(e_1), e_2\rangle \\ &\quad +\alpha^2\left(\langle he_1,e_3\rangle\langle he_2,e_4\rangle - \langle he_1,e_4\rangle\langle fe_2,e_3\rangle\right)\Big] e_3\cdot e_4, \end{aligned}$$

where $K_E$ is the curvature of the bundle $E$, and

$$C = 2d^\nabla\eta(e_1,e_2) + \alpha^2\left(fe_2\cdot he_1 - fe_1\cdot he_2 + e_2\cdot he_1 - e_1\cdot he_2\right).$$

As proved in [2], if $T$ is a 2-form such that $T\cdot\varphi = 0$ with $\varphi^+$ and $\varphi^-$ nowhere vanishing, then $T = 0$. Moreover, since $G$ belongs to $\Lambda^2 M\otimes 1$, $R$ belongs to $1\otimes\Lambda^2 E$ and $C$ is of mixed type, that is, belongs to $TM\otimes E$, then each of these three parts are zero. But $G = 0$ is nothing else but

$$K + \langle B(e_1,e_1), B(e_2,e_2)\rangle - |B(e_1,e_2)| = -\frac{c}{4}\left(1 - \langle fe_1,e_2\rangle^2 + \langle fe_1,e_1\rangle\langle fe_2,e_2\rangle\right),$$

that is the Gauss equation. Similarly, $R = 0$ is equivalent to

$$K_E + \langle (S_{e_3}\circ S_{e_4} - S_{e_4}\circ S_{e_3})(e_1), e_2\rangle = -\frac{c}{4}\left(\langle he_1,e_3\rangle\langle he_2,e_4\rangle - \langle he_1,e_4\rangle\langle fe_2,e_3\rangle\right),$$

That is the Ricci equation. Finally $C = 0$, gives the Codazzi equations. Indeed, since

$$d^\nabla\eta(X,Y) = -\frac{1}{2}\sum_{j=1}^{2} e_j\cdot\Big((\overline{\nabla}_X B)(Y,e_j) - (\overline{\nabla}_Y B)(X,e_j)\Big).$$

Thus, from $C = 0$, we deduce for $j = 1, 2$

$$\begin{aligned} (\overline{\nabla}_{e_1}B)(e_2,e_j) - (\overline{\nabla}_{e_2}B)(e_1,e_j) &= \frac{c}{4}\Big[\langle fe_2,e_j\rangle he_1 - \langle fe_1,e_j\rangle he_2 \\ &\quad + \langle e_2,e_j\rangle he_1 - \langle e_1,e_j\rangle he_2\Big], \end{aligned} \tag{23}$$

which are the Coazzi equations. Since in addition, we have assumed Equations (6)-(12), by the theorem of Kowalczyk and De Lira-Tojeiro-Vitório, we get that $(M^2,g)$ is isometrically immersed into $P$ with $B$ as second fundamental form and $f$, $h$, $s$ and $t$ coming from the product structure $F$ of $P$. This concludes the proof. □

**Remark 3.2.** *Note that in the proof, we only use Equations* (11) *and* (12) *in the computations. The other Equations* (6)-(10) *and* (13)-(13) *are only needed to apply the theorem of Kowalczyk and De Lira-Tojeiro-Vitóri, as well as the hypothesis on the rank of the maps* $\frac{F+Id}{2}$ *and* $\frac{F-Id}{2}$.

## 4. The Dirac equation

Let $\varphi$ be a spinor field satisfying Equation (19), then it satisfies the following Dirac equation

$$D\varphi = \vec{H}\cdot\varphi - \frac{\alpha}{2}\Big[(2+\mathrm{tr}(f))\varphi - \beta\cdot\varphi\Big], \tag{24}$$

where $\beta$ is the 2-form defined by $\beta = \sum_{i=1,2} e_i\cdot he_i = \sum_{i,j=1}^{2} h_{ij}e_i\cdot\xi_j$, where $h_{i,j} = \langle he_i,\xi_j\rangle$.

As in [2], we will show that this equation with an appropiate condition on the norm of both $\varphi^+$ and $\varphi^-$ is equivalent to Equation (19), where the tensor $B$ is expressed in terms on the spinor field $\varphi$ and such that $\mathrm{tr}(B) = 2\vec{H}$. Moreover, from Equation (19) we deduce the following conditions on the derivatives of $|\varphi^+|^2$ and $|\varphi^-|^2$. Indeed, after decomposition onto $\Sigma^+$ and $\Sigma^-$, (19) becomes

$$\nabla_X\varphi^{\pm} = \frac{\alpha}{2}(X + fX + hX)\cdot\varphi^{\mp} + \eta(X)\cdot\varphi^{\pm}.$$

From this we deduce that

$$X(|\varphi^{\pm}|^2) = \Re e\left\langle \alpha(X + fX + hX)\cdot\varphi^{\mp}, \varphi^{\pm}\right\rangle \tag{25}$$

Now, let $\varphi$ a spinor field solution of the Dirac equation (24) with $\varphi^+$ and $\varphi^-$ nowhere vanishing and satisying the norm condition (25), we set for any vector fields $X$ and $Y$ tangent to $M$ and $\xi \in E$

(26)

$$\left\langle B^+(X,Y),\xi\right\rangle = \frac{1}{2|\varphi^+|^2}\Bigg[\frac{\alpha}{2}\left\langle (X\cdot fY + Y\cdot fX)\cdot\varphi^- + (X\cdot hY + Y\cdot hX)\cdot\varphi^-, \xi\cdot\varphi^+\right\rangle + \left\langle X\cdot\nabla_Y\varphi^+ + \alpha < X,Y > \varphi^-, \xi\cdot\varphi^+\right\rangle\Bigg],$$

and

(27)

$$\left\langle B^-(X,Y),\xi\right\rangle = \frac{1}{2|\varphi^-|^2}\Bigg[\frac{\alpha}{2}\left\langle (X\cdot jY + Y\cdot jX)\cdot\varphi^+ + (X\cdot hY + Y\cdot hX)\cdot\varphi^+, \xi\cdot\varphi^-\right\rangle + \left\langle X\cdot\nabla_Y\varphi^- + \alpha < X,Y > \varphi^+, \xi\cdot\varphi^-\right\rangle\Bigg].$$

Finally, we set $B = B^+ + B^-$. Then, we have the following

**Proposition 4.1.** *Let $\varphi \in \Gamma(\Sigma)$ satisfying the Dirac equation* (24)

$$D\varphi = \vec{H}\cdot\varphi - \frac{\alpha}{2}\Big[(2 + \mathrm{tr}(f))\varphi - \beta\cdot\varphi\Big]$$

*such that*

$$X(|\varphi^{\pm}|^2) = \Re e\left\langle \alpha(X + fX + hX)\cdot\varphi^{\mp}, \varphi^{\pm}\right\rangle$$

*then $\varphi$ is solution of Equation* (19)

$$\nabla_X\varphi = \frac{\alpha}{2}(X + fX + hX)\cdot\varphi + \eta(X)\cdot\varphi,$$

*where $\eta$ is defined by $\eta(X) = -\frac{1}{2}\sum_{j=1}^{2} e_j\cdot B(e_j,X)$.*

For the sake of clarity, the proof of this proposition will be given in the next section. Now, combining this proposition with Theorem 3.1, we get the following corollary.

**Corollary 4.2.** *Let $c \in \mathbb{R}$, $c \neq 0$ and $\alpha \in \mathbb{C}$ such that $4\alpha^2 = c$. Let $(M^2, g)$ be an oriented Riemannian surface and $E$ an oriented vector bundle of rank 2 over $M$ with scalar product $< \cdot,\cdot >_E$ and compatible connection $\nabla^E$. We denote by $\Sigma = \Sigma M \otimes \Sigma E$ the twisted spinor bundle. Let $f, h, s$ and $s$ be some maps*

$$f : TM \longrightarrow TM,\ h : TM \longrightarrow E,\ s : E \longrightarrow TM \text{ and } t : E \longrightarrow E$$

*satisfying Equations* (6)-(10). *Moreover we assume that the rank of the maps $\frac{F+Id}{2}$ and $\frac{F-Id}{2}$ are 2 and 2 (resp. 3 and 1), where $F : TM \oplus E \longrightarrow TM \oplus E$ is defined by relations* (5). *Then, the two following statements are equivalent*

(1) *There exists an isometric immersion of* $(M^2, g)$ *into* $\mathbb{M}^2(c)\times\mathbb{R}^2$ *(resp.* $\mathbb{M}^3(c)\times\mathbb{R}$*) with* $E$ *as normal bundle and mean curvature* $\vec{H}$ *such that over* $M$ *the product strcuture is given by* $f, h, t$ *and* $s$.
(2) *There exists a spinor field* $\varphi$ *in* $\Sigma$ *solution of the Dirac equation*

$$D\varphi = \vec{H}\cdot\varphi - \frac{\alpha}{2}\Big[(2+\mathrm{tr}(f))\varphi - \beta\cdot\varphi\Big]$$

*such that* $\varphi^+$ *and* $\varphi^-$ *never vanish, satisfy the norm condition* (25) *and such that the maps* $f, h, s, t$ *and the tensor* $B$ *defined by* (26) *and* (27) *satisfy relations* (11)-(13).

## 5. Proof of Proposition 4.1

First, we decompose the Dirac equation (24) on the four spinor subbundles $\Sigma^{++}$, $\Sigma^{--}$, $\Sigma^{+-}$ and $\Sigma^{-+}$. We get the following four equations

$$\left\{\begin{array}{l} D\varphi^{--} = \vec{H}\cdot\varphi^{++} - \frac{\alpha}{2}\big(2+\mathrm{tr}(f)\big)\varphi^{+-} + \frac{\alpha}{2}\beta\cdot\varphi^{-+},\\ D\varphi^{++} = \vec{H}\cdot\varphi^{--} - \frac{\alpha}{2}\big(2+\mathrm{tr}(f)\big)\varphi^{-+} + \frac{\alpha}{2}\beta\cdot\varphi^{+-},\\ D\varphi^{+-} = \vec{H}\cdot\varphi^{-+} - \frac{\alpha}{2}\big(2+\mathrm{tr}(f)\big)\varphi^{--} + \frac{\alpha}{2}\beta\cdot\varphi^{++},\\ D\varphi^{-+} = \vec{H}\cdot\varphi^{+-} - \frac{\alpha}{2}\big(2+\mathrm{tr}(f)\big)\varphi^{++} + \frac{\alpha}{2}\beta\cdot\varphi^{--}. \end{array}\right.$$

Now, we fix a point $p\in M$, and consider $e_3$ a unit vector in $E_p$ so that the mean curvature vector is given by $\vec{H} = |\vec{H}|e_3$ at $p$. We complete $e_3$ by $e_4$ to get a positively oriented and orthonormal frame of $E_p$. First, we assume that $\varphi^{--}$, $\varphi^{++}$, $\varphi^{+-}$ and $\varphi^{-+}$ do not vanish at $p$. It is easy to see that

$$\left\{e_1\cdot e_3\cdot\frac{\varphi^{--}}{|\varphi^{--}|}, e_2\cdot e_3\cdot\frac{\varphi^{--}}{|\varphi^{--}|}\right\}$$

is an orthonormal frame of $\Sigma^{++}$ for the real scalar product $\Re e\,\langle\cdot,\cdot\rangle$. Indeed, we have

$$\begin{array}{rcl} \Re e\left\langle e_1\cdot e_3\cdot\varphi^{--}, e_2\cdot e_3\cdot\varphi^{--}\right\rangle &=& \Re e\left\langle\varphi^{--}, e_3\cdot e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle\\ &=& \Re e\left(i|\varphi^{--}|^2\right) = 0. \end{array}$$

Of course, by the same argument,

$$\left\{e_1\cdot e_3\cdot\frac{\varphi^{++}}{|\varphi^{++}|}, e_2\cdot e_3\cdot\frac{\varphi^{++}}{|\varphi^{++}|}\right\},$$

$$\left\{e_1\cdot e_3\cdot\frac{\varphi^{-+}}{|\varphi^{-+}|}, e_2\cdot e_3\cdot\frac{\varphi^{-+}}{|\varphi^{-+}|}\right\},$$

$$\left\{e_1\cdot e_3\cdot\frac{\varphi^{+-}}{|\varphi^{+-}|}, e_2\cdot e_3\cdot\frac{\varphi^{+-}}{|\varphi^{+-}|}\right\}$$

are orthonormal frames of $\Sigma^{--}$, $\Sigma^{+-}$ and $\Sigma^{-+}$ respectively. We define the following bilinear forms

$$\begin{array}{l} F_{++}(X,Y) = \Re e\left\langle\nabla_X\varphi^{++}, Y\cdot e_3\cdot\varphi^{--}\right\rangle,\\ F_{--}(X,Y) = \Re e\left\langle\nabla_X\varphi^{--}, Y\cdot e_3\cdot\varphi^{++}\right\rangle,\\ F_{+-}(X,Y) = \Re e\left\langle\nabla_X\varphi^{+-}, Y\cdot e_3\cdot\varphi^{-+}\right\rangle,\\ F_{-+}(X,Y) = \Re e\left\langle\nabla_X\varphi^{-+}, Y\cdot e_3\cdot\varphi^{+-}\right\rangle, \end{array}$$

and

$$B_{++}(X,Y) = -\frac{1}{2}\Re e\left\langle\alpha(X+fX)\cdot\varphi^{-+} + \alpha hX\cdot\varphi^{+-}, Y\cdot e_3\cdot\varphi^{--}\right\rangle,$$

$$B_{--}(X,Y) = -\frac{1}{2}\Re e\left\langle\alpha(X+fX)\cdot\varphi^{+-} + \alpha hX\cdot\varphi^{-+}, Y\cdot e_3\cdot\varphi^{++}\right\rangle,$$

$$B_{+-}(X,Y) = -\frac{1}{2}\Re e\left\langle\alpha(X+fX)\cdot\varphi^{++} + \alpha hX\cdot\varphi^{--}, Y\cdot e_3\cdot\varphi^{+-}\right\rangle,$$

$$B_{-+}(X,Y) = -\frac{1}{2}\Re e\left\langle \alpha(X+fX)\cdot\varphi^{--} + \alpha hX\cdot\varphi^{++}, Y\cdot e_3\cdot\varphi^{-+}\right\rangle.$$

We have this first lemma:

**Lemma 5.1.** *We have*

(1) $\mathrm{tr}(F_{++}) = -|\vec{H}||\varphi^{--}|^2 + \frac{1}{2}\Re e\Big\langle \alpha\big(2+\mathrm{tr}(f)\big)\varphi^{-+} + \alpha\beta\cdot\varphi^{+-}, e_3\cdot\varphi^{--}\Big\rangle$,

(2) $\mathrm{tr}(F_{--}) = -|\vec{H}||\varphi^{++}|^2 + \frac{1}{2}\Re e\Big\langle \alpha\big(2+\mathrm{tr}(f)\big)\varphi^{+-} + \alpha\beta\cdot\varphi^{-+}, e_3\cdot\varphi^{++}\Big\rangle$,

(3) $\mathrm{tr}(F_{+-}) = -|\vec{H}||\varphi^{-+}|^2 + \frac{1}{2}\Re e\Big\langle \alpha\big(2+\mathrm{tr}(f)\big)\varphi^{++} + \alpha\beta\cdot\varphi^{--}, e_3\cdot\varphi^{-+}\Big\rangle$,

(4) $\mathrm{tr}(F_{-+}) = -|\vec{H}||\varphi^{+-}|^2 + \frac{1}{2}\Re e\Big\langle \alpha\big(2+\mathrm{tr}(f)\big)\varphi^{--} + \alpha\beta\cdot\varphi^{++}, e_3\cdot\varphi^{+-}\Big\rangle$,

*Proof:* We only compute the trace of $F_{++}$, the computations for the three others forms $F_{--}$, $F_{+-}$ and $F_{-+}$ are the same. We have

$$\begin{aligned}
\mathrm{tr}(F_{++}) &= F_{++}(e_1,e_1) + F_{++}(e_2,e_2)\\
&= \Re e\left\langle \nabla_{e_1}\varphi^{++}, e_1\cdot e_3\cdot\varphi^{--}\right\rangle + \Re e\left\langle \nabla_{e_2}\varphi^{++}, e_2\cdot e_3\cdot\varphi^{--}\right\rangle\\
&= -\Re e\left\langle e_1\cdot\nabla_{e_1}\varphi^{++}, e_3\cdot\varphi^{--}\right\rangle - \Re e\left\langle e_2\cdot\nabla_{e_2}\varphi^{++}, e_3\cdot\varphi^{--}\right\rangle\\
&= -\Re e\left\langle D\varphi^{++}, e_3\cdot\varphi^{--}\right\rangle
\end{aligned}$$

Since $D\varphi^{++} = \vec{H}\cdot\varphi^{--} - \frac{\alpha}{2}\big(2+\mathrm{tr}(f)\big)\varphi^{-+} + \frac{\alpha}{2}\beta\cdot\varphi^{+-}$, we get

$$\begin{aligned}
\mathrm{tr}(F_{++}) &= -\Re e\Big\langle \vec{H}\cdot\varphi^{--} - \frac{\alpha}{2}\big(2+\mathrm{tr}(f)\big)\varphi^{-+} + \frac{\alpha}{2}\beta\cdot\varphi^{+-}, e_3\cdot\varphi^{--}\Big\rangle\\
&= -\Re e\left\langle |H|e_3\cdot\varphi^{--}, e_3\cdot\varphi^{--}\right\rangle + \Re e\left\langle \frac{\alpha}{2}\big(2+\mathrm{tr}(f)\big)\varphi^{-+} - \frac{\alpha}{2}\beta\cdot\varphi^{+-}, e_3\cdot\varphi^{--}\right\rangle\\
&= -|\vec{H}||\varphi^{--}|^2 + \frac{1}{2}\Re e\Big\langle \alpha\big(2+\mathrm{tr}(f)\big)\varphi^{-+} + \alpha\beta\cdot\varphi^{+-}, e_3\cdot\varphi^{--}\Big\rangle
\end{aligned}$$

This concludes the proof. □

TNow, we have this second lemma which gives the defect of symmetry:

**Lemma 5.2.** *We have*

(1) $F_{++}(e_1,e_2) = F_{++}(e_2,e_1) - \frac{1}{2}\Re e\Big\langle \big(2+\mathrm{tr}(f)\big)\varphi^{-+} - \alpha\beta\cdot\varphi^{+-}, e_4\cdot\varphi^{--}\Big\rangle$,

(2) $F_{--}(e_1,e_2) = F_{--}(e_2,e_1) - \frac{1}{2}\Re e\Big\langle \big(2+\mathrm{tr}(f)\big)\varphi^{+-} - \alpha\beta\cdot\varphi^{-+}, e_4\cdot\varphi^{++}\Big\rangle$,

(3) $F_{+-}(e_1,e_2) = F_{+-}(e_2,e_1) + \frac{1}{2}\Re e\Big\langle \big(2+\mathrm{tr}(f)\big)\varphi^{++} - \alpha\beta\cdot\varphi^{--}, e_4\cdot\varphi^{+-}\Big\rangle$,

(4) $F_{-+}(e_1,e_2) = F_{-+}(e_2,e_1) + \frac{1}{2}\Re e\Big\langle \big(2+\mathrm{tr}(f)\big)\varphi^{--} - \alpha\beta\cdot\varphi^{++}, e_4\cdot\varphi^{-+}\Big\rangle$.

*Proof:* As for the proof of the previous lemma, we only give the details for $F_{++}$. We have

$$\begin{aligned}
F_{++}(e_1,e_2) &= \Re e\left\langle \nabla_{e_1}\varphi^{++}, e_2\cdot e_3\cdot\varphi^{--}\right\rangle\\
&= \Re e\left\langle e_1\cdot\nabla_{e_1}\varphi^{++}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle\\
&= \Re e\Big\langle \vec{H}\cdot\varphi^{--} - \frac{\alpha}{2}\big(2+\mathrm{tr}(f)\big)\varphi^{-+} + \frac{\alpha}{2}\beta\cdot\varphi^{+-} - e_2\cdot\nabla_{e_2}\varphi^{++}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\Big\rangle.
\end{aligned}$$

The first term is

$$\begin{aligned}
\Re e\left\langle \vec{H}\cdot\varphi^{--}, e_1\cdot e_2\cdot e_3\cdot\varphi^{--}\right\rangle &= -\Re e\left\langle e_3\cdot\vec{H}\cdot\varphi^{--}, e_1\cdot e_2\cdot\varphi^{--}\right\rangle\\
&= \Re e\left\langle \vec{H}\cdot e_3\cdot\varphi^{--}, i\varphi^{--}\right\rangle\\
&= -\Re e\left(i|\vec{H}||\varphi^{--}|^2\right) = 0,
\end{aligned}$$

where we have use that $ie_1 \cdot e_2 \cdot \varphi^{--} = -\varphi^{--}$, that is, $e_1 \cdot e_2 \cdot \varphi^{--} = i\varphi^{--}$ and $\vec{H} = |H|e_3$. Moreover, we have

$$\begin{aligned}
-\Re e \left\langle e_2 \cdot \nabla_{e_2}\varphi^{++}, e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \right\rangle &= \Re e \left\langle \nabla_{e_2}\varphi^{++}, e_2 \cdot e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} \right\rangle \\
&= \Re e \left\langle \nabla_{e_2}\varphi^{++}, e_1 \cdot e_3 \cdot \varphi^{--} \right\rangle \\
&= F_{++}(e_2, e_1).
\end{aligned}$$

Finally, since $\varphi^{--} \in \Sigma^+$, we have $\omega_4 \cdot \varphi^{--} = \varphi^{--}$, which implies $e_1 \cdot e_2 \cdot e_3 \cdot \varphi^{--} = -e_4 \cdot \varphi^{--}$ and we get

$$F_{++}(e_1, e_2) = F_{++}(e_2, e_1) - \frac{1}{2}\Re e\Big\langle \alpha\big(2 + \mathrm{tr}(f)\big)\varphi^{-+} - \alpha\beta \cdot \varphi^{+-}, e_4 \cdot \varphi^{--} \Big\rangle.$$

The proof is similar for the three other forms. □

By analogous computations, we also get the following lemmas. We do not give the proof which is similar to the two previous ones.

**Lemma 5.3.** *We have*

(1) $tr(B_{++}) = -\frac{1}{2}\Re e\Big\langle \alpha\big(2 + \mathrm{tr}(f)\big)\varphi^{-+} + \alpha\beta \cdot \varphi^{+-}, e_3 \cdot \varphi^{--} \Big\rangle$,
(2) $tr(B_{--}) = -\frac{1}{2}\Re e\Big\langle \alpha\big(2 + \mathrm{tr}(f)\big)\varphi^{+-} + \alpha\beta \cdot \varphi^{-+}, e_3 \cdot \varphi^{++} \Big\rangle$,
(3) $tr(B_{+-}) = -\frac{1}{2}\Re e\Big\langle \alpha\big(2 + \mathrm{tr}(f)\big)\varphi^{++} + \alpha\beta \cdot \varphi^{--}, e_3 \cdot \varphi^{-+} \Big\rangle$,
(4) $tr(B_{-+}) = -\frac{1}{2}\Re e\Big\langle \alpha\big(2 + \mathrm{tr}(f)\big)\varphi^{--} + \alpha\beta \cdot \varphi^{++}, e_3 \cdot \varphi^{+-} \Big\rangle$.

**Lemma 5.4.** *We have*

(1) $B_{++}(e_1, e_2) = B_{++}(e_2, e_1) + \frac{1}{2}\Re e\Big\langle \big(2 + \mathrm{tr}(f)\big)\varphi^{-+} - \alpha\beta \cdot \varphi^{+-}, e_4 \cdot \varphi^{--} \Big\rangle$,
(2) $B_{--}(e_1, e_2) = B_{--}(e_2, e_1) + \frac{1}{2}\Re e\Big\langle \big(2 + \mathrm{tr}(f)\big)\varphi^{+-} - \alpha\beta \cdot \varphi^{-+}, e_4 \cdot \varphi^{++} \Big\rangle$,
(3) $B_{+-}(e_1, e_2) = B_{+-}(e_2, e_1) - \frac{1}{2}\Re e\Big\langle \big(2 + \mathrm{tr}(f)\big)\varphi^{++} - \alpha\beta \cdot \varphi^{--}, e_4 \cdot \varphi^{+-} \Big\rangle$,
(4) $B_{-+}(e_1, e_2) = B_{-+}(e_2, e_1) - \frac{1}{2}\Re e\Big\langle \big(2 + \mathrm{tr}(f)\big)\varphi^{--} - \alpha\beta \cdot \varphi^{++}, e_4 \cdot \varphi^{-+} \Big\rangle$.

Now, we set

$$\begin{cases}
A_{++} := F_{++} + B_{++}, \\
A_{--} := F_{--} + B_{--}, \\
A_{+-} := F_{+-} + B_{+-}, \\
A_{-+} := F_{-+} + B_{-+},
\end{cases}$$

and

$$F_+ = \frac{A_{++}}{|\varphi^{--}|^2} - \frac{A_{--}}{|\varphi^{++}|^2} \qquad \text{and} \qquad F_- = \frac{A_{+-}}{|\varphi^{-+}|^2} - \frac{A_{-+}}{|\varphi^{+-}|^2}.$$

From the last four lemmas we deduce immediately that $F_+$ and $F_-$ are symmetric and trace-free. Moreover, by a direct computation using the conditions (25) on the norms of $\varphi^+$ and $\varphi^-$, we get the following lemma:

**Lemma 5.5.** *The symmetric operators $F^+$ and $F^-$ of $TM$ associated to the bilinear forms $F_+$ and $F_-$, defined by*

$$F^+(X) = F_+(X, e_1)e_1 + F_+(X, e_2)e_2 \quad \text{and} \quad F^-(X) = F_-(X, e_1)e_1 + F_-(X, e_2)e_2$$

*for all $X \in TM$, satisfy*

(1) $\Re e \left\langle F^+(X) \cdot e_3 \cdot \varphi^{--}, \varphi^{++} \right\rangle = 0$,
(2) $\Re e \left\langle F^-(X) \cdot e_3 \cdot \varphi^{-+}, \varphi^{+-} \right\rangle = 0$.

*Proof.* First, we have

$$A_{++}(X,Y)=\Re e\,\langle\nabla_X\varphi^{++}-\alpha(X+fX)\cdot\varphi^{-+}+\alpha hX\cdot\varphi^{+-},Y\cdot e_3\cdot\varphi^{--}\rangle,$$

Since $\left(e_1\cdot e_3\cdot\frac{\varphi^{--}}{|\varphi^{--}|},e_2\cdot e_3\cdot\frac{\varphi^{--}}{|\varphi^{--}|}\right)$ is an orthonormal frame of $\Sigma^{++}$, we have

$$\begin{aligned}&\Re e\,\left\langle\nabla_X\varphi^{++}-\frac{\alpha}{2}(X+fX)\cdot\varphi^{-+}+\frac{\alpha}{2}hX\cdot\varphi^{+-},\varphi^{++}\right\rangle\\ =\;&\frac{A_{++}}{|\varphi^{--}|^2}(X,e_1)\,\Re e\,\langle e_1\cdot e_3\cdot\varphi^{--},\varphi^{++}\rangle+\frac{A_{++}}{|\varphi^{--}|^2}(X,e_2)\,\Re e\,\langle e_2\cdot e_3\cdot\varphi^{--},\varphi^{++}\rangle.\end{aligned}$$

Similarly,

$$\begin{aligned}&\Re e\,\langle\nabla_X\varphi^{--}-\frac{\alpha}{2}(X+fX)\cdot\varphi^{+-}+\frac{\alpha}{2}hX\cdot\varphi^{-+},\varphi^{--}\rangle\\ &=\frac{A_{--}}{|\varphi^{++}|^2}(X,e_1)\,\Re e\,\langle e_1\cdot e_3\cdot\varphi^{++},\varphi^{--}\rangle+\frac{A_{--}}{|\varphi^{++}|^2}(X,e_2)\,\Re e\,\langle e_2\cdot e_3\cdot\varphi^{++},\varphi^{--}\rangle\\ &=-\frac{A_{--}}{|\varphi^{++}|^2}(X,e_1)\,\Re e\,\langle e_1\cdot e_3\cdot\varphi^{--},\varphi^{++}\rangle-\frac{A_{--}}{|\varphi^{++}|^2}(X,e_2)\,\Re e\,\langle e_2\cdot e_3\cdot\varphi^{--},\varphi^{++}\rangle.\end{aligned}$$

Summing these two formulas imply that

$$\Re e\left\langle F^+(X)\cdot e_3\cdot\varphi^{--},\varphi^{++}\right\rangle=\Re e\,\langle\nabla_X\varphi^+-\frac{\alpha}{2}(X+fX)\cdot\varphi^-+\frac{\alpha}{2}hX\cdot\varphi^-,\varphi^+\rangle.$$

By the condition (25) on the derivative of the norm of $\varphi^+$ , this last expression is zero. The proof of the second relation is similar. □

Hence, the operators $F^+$ and $F^-$ are of rank at most $\leq 1$. Since they are symmetric and trace-free, they vanish identically.

Using again that $\left(e_1\cdot e_3\cdot\frac{\varphi^{--}}{|\varphi^{--}|},e_2\cdot e_3\cdot\frac{\varphi^{--}}{|\varphi^{--}|}\right)$ is an orthonormal frame of $\Sigma^{++}$, we have

$$\nabla_X\varphi^{++}=F_{++}(X,e_1)e_1\cdot e_3\cdot\frac{\varphi^{--}}{|\varphi^{--}|}+F_{++}(X,e_2)e_2\cdot e_3\cdot\frac{\varphi^{--}}{|\varphi^{--}|}.$$

Since $F_{++}=A_{++}-B_{++}$ and denoting by $A^{++}$ and $B^{++}$ the operators of $TM$ associated to $A_{++}$ and $B_{++}$ and defined by

$$A^{++}(X)=A_{++}(X,e_1)e_1+A_{++}(X,e_2)e_2,\ B^{++}(X)=B_{++}(X,e_1)e_1+B_{++}(X,e_2)e_2,$$

we get

$$\nabla_X\varphi^{++}=\frac{1}{|\varphi^{--}|^2}\left[A^{++}(X)\cdot e_3\cdot\varphi^{--}-B^{++}(X)\cdot e_3\cdot\varphi^{--}\right]. \tag{28}$$

Similarly, we denote by $A^{--}$ and $B^{--}$ the operators of $TM$ associated to $A_{--}$ and $B_{--}$. Thus, we have

$$\nabla_X\varphi^{--}=\frac{1}{|\varphi^{++}|^2}\left[A^{--}(X)\cdot e_3\cdot\varphi^{++}-B^{--}(X)\cdot e_3\cdot\varphi^{++}\right]. \tag{29}$$

Moreover, we easily get

$$B^{++}(X)\cdot e_3\cdot\varphi^{--}=-\frac{1}{2}|\varphi^{--}|^2\Big(\alpha(X+fX)\cdot\varphi^{-+}+\alpha hX\cdot\varphi^{+-}\Big)$$

and

$$B^{--}(X)\cdot e_3\cdot\varphi^{++}=-\frac{1}{2}|\varphi^{++}|^2\Big(\alpha(X+fX)\cdot\varphi^{+-}+\alpha hX\cdot\varphi^{-+}\Big).$$

Hence,

$$\begin{aligned}\nabla_X\varphi^+\;=\;&\frac{1}{|\varphi^{--}|^2}A^{++}(X)\cdot e_3\cdot\varphi^{--}+\frac{\alpha}{2}(X+fX)\cdot\varphi^{-+}+\frac{\alpha}{2}hX\cdot\varphi^{+-}\\ &+\frac{1}{|\varphi^{++}|^2}A^{--}(X)\cdot e_3\cdot\varphi^{++}+\frac{\alpha}{2}(X+fX)\cdot\varphi^{+-}+\frac{\alpha}{2}hX\cdot\varphi^{-+}.\end{aligned}$$

Now, we set $A^+ = A^{++} + A^{--}$. From the definition of $A^{++}$ and $A^{--}$ and since $F^+ = 0$, we have $\frac{A^{++}}{|\varphi^{--}|^2} = \frac{A^{--}}{|\varphi^{++}|^2}$. Bearing in mind that $|\varphi^+|^2 = |\varphi^{++}|^2 + |\varphi^{--}|^2$, we get finally

$$\frac{A^+}{|\varphi^+|^2} = \frac{A^{++}}{|\varphi^{--}|^2} = \frac{A^{--}}{|\varphi^{++}|^2}. \tag{30}$$

So, we have

$$\nabla_X \varphi^+ = \frac{1}{|\varphi^+|^2} A^+(X) \cdot e_3 \cdot \varphi^+ + \alpha(X + fX + hX) \cdot \varphi^-. \tag{31}$$

Analogouslly, we set $A^{+-}$ and $A^{-+}$ the operators of $TM$ associated to $A_{+-}$ and $A_{-+}$, and we denote $A^- = A^{+-} + A^{-+}$. Using the fact that $F^- = 0$ we get

$$\begin{aligned}
\nabla_X \varphi^- &= \frac{1}{|\varphi^{+-}|^2} A^{-+}(X) \cdot e_3 \cdot \varphi^{+-} + \alpha(X + fX) \cdot \varphi^{++} + \alpha hX \cdot \varphi^{--} \\
&\quad + \frac{1}{|\varphi^{-+}|^2} A^{+-}(X) \cdot e_3 \cdot \varphi^{-+} + \frac{\alpha}{2}(X + fX) \cdot \varphi^{--} + \frac{\alpha}{2} hX \cdot \varphi^{++} \\
&= \frac{1}{|\varphi^-|^2} A^-(X) \cdot e_3 \cdot \varphi^- + \frac{\alpha}{2}(X + fX + hX) \cdot \varphi^+. \qquad (32)
\end{aligned}$$

We now observe that formulas (31) and (32) also hold if $\varphi^{++}$ or $\varphi^{--}$, (resp. $\varphi^{+-}$ or $\varphi^{-+}$) vanishes at $p$ : indeed, assuming for instance that $\varphi^{++}(p) = 0$, and thus that $\varphi^{--}(p) \neq 0$ since $\varphi^+(p) \neq 0$, equation (28) holds, and, from the norm condition in (25), we have

$$\Re e \left\langle \nabla_X \varphi^{--} - \frac{\alpha}{2}(X + fX) \cdot \varphi^{+-} + \frac{\alpha}{2} hX \cdot \varphi^{-+}, \varphi^{--} \right\rangle = 0.$$

Since $\left( \frac{\varphi^{--}}{|\varphi^{--}|}, i\frac{\varphi^{--}}{|\varphi^{--}|} \right)$ is an orthonormal basis of $\Sigma^{--}$, we deduce that

$$\nabla_X \varphi^{--} - \frac{\alpha}{2}(X + fX) \cdot \varphi^{+-} + \frac{\alpha}{2} hX \cdot \varphi^{-+} = i\delta(X) \frac{\varphi^{--}}{|\varphi^{--}|}$$

for some real 1-form $\delta$. Moreover, since $\varphi^{++} = 0$ at $p$, we have

$$D\varphi^{--} + \alpha(2 + \mathrm{tr}(f))\varphi^{+-} + \alpha\beta \cdot \varphi^{-+} = 0,$$

which implies

$$\big(\delta(e_1)e_1 + \delta(e_2)e_2\big) \cdot \frac{\varphi^{--}}{|\varphi^{--}|} = 0,$$

and thus that $\delta = 0$. We thus get $\nabla_X \varphi^{--} = \frac{\alpha}{2}(X + fX) \cdot \varphi^{+-} + \frac{\alpha}{2} hX \cdot \varphi^{-+}$ instead of (29), which, together with (28), easily implies (31).

Now, we set

$$\eta^+(X) = \left( \frac{1}{|\varphi^+|^2} A^+(X) \cdot e_3 \right)^+ \qquad \text{and} \qquad \eta^-(X) = \left( \frac{1}{|\varphi^-|^2} A^-(X) \cdot e_3 \right)^-$$

where, if $\sigma$ belongs to $\mathcal{C}l^0(TM \oplus E)$, we denote by $\sigma^+ := \frac{1+\omega_4}{2} \cdot \sigma$ and by $\sigma^- := \frac{1-\omega_4}{2} \cdot \sigma$ the parts of $\sigma$ acting on $\Sigma^+$ and on $\Sigma^-$ only, *i.e.*, such that

$$\sigma^+ \cdot \varphi = \sigma \cdot \varphi^+ \in \Sigma^+ \qquad \text{and} \qquad \sigma^- \cdot \varphi = \sigma \cdot \varphi^- \in \Sigma^-.$$

Setting $\eta = \eta^+ + \eta^-$ we thus get

$$\nabla_X \varphi = \eta(X) \cdot \varphi + \frac{\alpha}{2}(X + fX + hX) \cdot \varphi,$$

as claimed in Proposition 4.1.

Now, we will compute $B$ explicitely. For this, we set $A_+(X,Y) := \langle A^+(X), Y\rangle$ and $A_-(X,Y) := \langle A^-(X), Y\rangle$. Then, the form $\eta$ is given by

$$\begin{aligned}\eta(X) &= \frac{1}{2|\varphi^+|^2}\left[A_+(X,e_1)(e_1\cdot e_3 - e_2\cdot e_4) + A_+(X,e_2)(e_2\cdot e_3 + e_1\cdot e_4)\right]\\ &\quad + \frac{1}{2|\varphi^-|^2}\left[A_-(X,e_1)(e_1\cdot e_3 + e_2\cdot e_4) + A_-(X,e_2)(e_2\cdot e_3 - e_1\cdot e_4)\right]\end{aligned}$$

with

$$A_+(X,Y) = \Re e\left\langle \nabla_X\varphi^+ - \frac{\alpha}{2}(X + fX + hX)\cdot\varphi^-, Y\cdot e_3\cdot\varphi^+\right\rangle$$

and

$$A_-(X,Y) = \Re e\left\langle \nabla_X\varphi^- - \frac{\alpha}{2}(X + fX + hX)\cdot\varphi^+, Y\cdot e_3\cdot\varphi^-\right\rangle.$$

Moreover, we see easily by direct computations that for any vectors $X$ and $Y$ tangent to $M$,

$$B(X,Y) := X\cdot\eta(Y) - \eta(Y)\cdot X$$

is a vector belonging to $E$ which is such that

$$\begin{aligned}\langle B(X,Y),\xi\rangle &= \frac{1}{|\varphi^+|^2}\Re e\left\langle X\cdot\nabla_Y\varphi^+ - \alpha(X + fX + hX)\cdot Y\cdot\varphi^-, \xi\cdot\varphi^+\right\rangle\\ &\quad + \frac{1}{|\varphi^-|^2}\Re e\left\langle X\cdot\nabla_Y\varphi^- - \alpha(X + fX + hX)\cdot Y\cdot\varphi^+, \xi\cdot\varphi^-\right\rangle\end{aligned}$$

for all $\xi\in E$.

**Lemma 5.6.** *The operator $B$ defined above is symmetric in $X$ and $Y$.*

*Proof:* The proof is analogous to the symmetry of $A_{++}$ proven above and uses the Dirac equations

$$D\varphi^+ = \vec{H}\cdot\varphi^+ - \alpha\Big[(2 + \mathrm{tr}(f))\varphi -^- \beta\cdot\varphi^-\Big]$$

and

$$D\varphi^- = \vec{H}\cdot\varphi^- - \alpha\Big[(2 + \mathrm{tr}(f))\varphi^+ - \beta\cdot\varphi^+\Big].$$

□

Now, computing

$$\langle B(X,Y),\xi\rangle = \frac{1}{2}\left(\langle B(X,Y),\xi\rangle + \langle B(Y,X),\xi\rangle\right)$$

we finally obtain that $B$ is given in the discussion of Section 4.

Since $B(e_j, X) = e_j\cdot\eta(X) - \eta(X)\cdot e_j$, we obtain

$$\sum_{j=1,2} e_j\cdot B(e_j,X) = -2\eta(X) - \sum_{j=1,2} e_j\cdot\eta(X)\cdot e_j. \tag{33}$$

Writing $\eta(X)$ in the form $\sum_{k=1,2} e_k\cdot\eta_k$ for some vectors $\eta_k$ belonging to $E$, we easily get that $\sum_{j=1,2} e_j\cdot\eta(X)\cdot e_j = 0$. Indeed, we have

$$\begin{aligned}\sum_{j=1,2} e_j\cdot\eta(X)\cdot e_j &= \sum_{j=1,2} e_j\cdot\left(\sum_{k=1,2} e_k\cdot\eta_k\right)\cdot e_j\\ &= e_1\cdot(e_1\cdot\eta_1 + e_2\cdot\eta_2)\cdot e_1 + e_2\cdot(e_1\cdot\eta_1 + e_2\cdot\eta_2)\cdot e_2\\ &= -\eta_1\cdot e_1 - e_2\cdot\eta_2 - e_1\cdot\eta_1 - \eta_2\cdot e_2\\ &= e_1\cdot\eta_1 + \eta_2\cdot e_2 - e_1\cdot\eta_1 - \eta_2\cdot e_2\\ &= 0\end{aligned}$$

Thus, from (33), we get

$$\eta(X) = -\frac{1}{2}\sum_{j=1,2} e_j \cdot B(e_j, X).$$

The last claim in Proposition 4.1 is now proved. □

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