The affine Lie algebra $sl\hat{\_}2()$ and a conditioned space-time Brownian motion

Manon Defosseux

THE AFFINE LIE ALGEBRA $\hat{\mathfrak{sl}}_2(\mathbb{C})$ AND A CONDITIONED SPACE-TIME BROWNIAN MOTION

MANON DEFOSSEUX

ABSTRACT. We construct a sequence of Markov processes on the set of dominant weights of the Affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$ which involves tensor product of irreducible highest weight modules of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ and show that it converges towards a Doob's space-time harmonic transformation of a space-time Brownian motion.

1. INTRODUCTION

In [2], Ph. Biane, Ph. Bougerol and N. O'Connell establish a wide extension of Pitman's theorem on Brownian motion and three dimensional Bessel process, in the framework of representation theory of semi-simple complex Lie algebras. In this framework the representation of the Bessel process by a functional of a standard Brownian motion $(B_t)_{t\geq 0}$ on $\mathbb{R}$,

$$(B_t - 2 \inf_{0\leq s\leq t} B_s, t \geq 0),$$

appears to be the continuous counterpart of a similar result which holds for a random walk on the set of integral weights of $\mathfrak{sl}_2(\mathbb{C})$ and a path transformation connected with the Littelmann paths model for semi-simple complex Lie algebras (see for instance [7] for a description of this model).

In [6], C. Lecouvey, E. Lesigne and M. Peigné consider the case when $\mathfrak{g}$ is a Kac-Moody algebra and develop some aspects of [2] in that framework. In particular, they focus on some Markov chains on the Weyl chamber of a Kac-Moody algebra, which are obtained in a similar way as in [2], except that the reference measure can't be the uniform measure when the dimension of the Kac-Moody algebra is infinite. Let us say briefly how the Markov chains are obtained for a Kac-Moody algebra $\mathfrak{g}$. As in the finite dimensional case, for a dominant integral weight λ of $\mathfrak{g}$ one defines the character of the irreducible highest-weight representation $V(\lambda)$ of $\mathfrak{g}$ with highest weight λ, as a formal series

$$\mathrm{ch}_\lambda = \sum_\mu \dim(V(\lambda)_\mu) e^\mu,$$

where $V(\lambda)_\mu$ is the weight space of $V(\lambda)$ corresponding to the weight μ. Actually for every h in a subset of the Cartan subalgebra which doesn't depend on λ the series $\sum_\mu \dim(V(\lambda)_\mu)e^{\langle \mu,h\rangle}$ is absolutely convergente. For two dominant weights ω and λ, the following decomposition

$$\mathrm{ch}_\omega \mathrm{ch}_\lambda = \sum_{\beta\in P_+} m_\lambda(\beta)\mathrm{ch}_\beta,$$

where $m_\lambda(\beta)$ is the multiplicity of the module with highest weight β in the decomposition of $V(\omega)\otimes V(\lambda)$, allows to define a transition probability q_ω on the set of dominant weights, letting for β and λ two dominant weights of $\mathfrak{g}$,

$$q_\omega(\lambda,\beta)=\frac{\mathrm{ch}_\beta(h)}{\mathrm{ch}_\lambda(h)\mathrm{ch}_\omega(h)}m_\lambda(\beta),$$

where h is chosen in the region of convergence of the characters. It is a natural question to ask if there exists a sequence $(h_n)_{n\geq 0}$ of elements of $\mathfrak{h}$ such that the corresponding sequence of Markov chains converges towards a continuous process and what the limit is.

In this paper, we consider the case when $\mathfrak{g}$ is the Kac-Moody algebra of type $A_1^{(1)}$ and ω is its fundamental weight Λ_0. There is no reason to think that the results are not true in a more general context but this case presents the advantage that explicit computations can easily be done. We show that the sequence of Markov chains, with a proper normalization, converges, for a particular sequence of $(h_n)_{n\geq 0}$, towards a Doob's space-time harmonic transformation of a space-time Brownian motion killed on the boundary of a time-dependent domain. This process is related to the heat equation

$$\frac{1}{2}\Delta+\frac{\partial}{\partial t}=0,$$

in a time-dependent domain, with Dirichlet boundary conditions and the theta functions play a crucial role in the construction. One can find an extensive literature devoted to the relationship between Brownian motion and the heat equation. One can see for instance [4] for an introduction and [3] for a review of various problems specifically related to time-dependent boundaries.

The paper is organized as follows. Basic definitions and notations related to representation theory of the affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$ are given in section 2. We define in section 3 random walks on the set of integral weights of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ and Markov chains on the set of its dominant integral weights, considering tensor products of irreducible highest weight representations of $\hat{\mathfrak{sl}}_2(\mathbb{C})$. In section 4, for any positive real numbers x and u such that $x<u$, we define a space-time Brownian motion $(t+u,B_t)_{t\geq 0}$ starting from (u,x), conditioned to remain in the domain

$$D=\{(r,z)\in\mathbb{R}\times\mathbb{R}:0\leq z\leq r\}.$$

For this we introduce a space-time harmonic function remaining positive on D which appears naturally considering the limit of a sequence of characters of $\hat{\mathfrak{sl}}_2(\mathbb{C})$. We prove in section 5 that this conditioned space-time Brownian motion is the limit of a sequence of Markov processes constructed in section 3. We show in section 6 how it is related to a Brownian motion conditoned - in Doob's sense - to remain in an interval.

2. The affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$

We consider the affine Lie algebra $\hat{\mathfrak{sl}}_2(\mathbb{C})$ associated to the generalized Cartan matrix

$$A=\begin{pmatrix}2&-2\\-2&2\end{pmatrix}.$$

The reader is invited to refer to [5] for a detailed description of this object. Let $\mathfrak{h}$ be a Cartan subalgebra of $\hat{\mathfrak{sl}}_2(\mathbb{C})$. We denote by $S=\{\alpha_0,\alpha_1\}$ the set of simple roots and by $\{\alpha_0^\vee,\alpha_1^\vee\}$ the set of simple coroots. Let Λ_0 be a fundamental weight such that $\langle \Lambda_0, \alpha_i^\vee \rangle = \delta_{i0}$, $i \in \{0,1\}$, and $\{\alpha_0, \alpha_1, \Lambda_0\}$ is a basis of $\mathfrak{h}^*$. We denote by $\mathfrak{h}_\mathbb{R}$ the subset of $\mathfrak{h}$ defined by

$$\mathfrak{h}_\mathbb{R} = \{x \in \mathfrak{h} : \langle \Lambda_0, x \rangle \in \mathbb{R}, \text{ and } \langle \alpha_i, x \rangle \in \mathbb{R}, i \in \{0,1\}\}.$$

Let $\delta = \alpha_0 + \alpha_1$ be the so-called null root. We denote by P (resp. P_+) the set of integral (resp. dominant) weights defined by

$$P = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z},\, i = 0,1\},$$

$$(\text{resp. } P_+ = \{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \geq 0,\, i = 0,1\}).$$

The Cartan subalgebra $\mathfrak{h}$ is equipped with a non degenerate symmetric bilinear form $(.,.)$ defined below, which identifies $\mathfrak{h}$ and $\mathfrak{h}^*$, through the linear isomorphism

$$\begin{aligned} \nu : \mathfrak{h} &\to \mathfrak{h}^*, \\ h &\mapsto (h,.). \end{aligned}$$

We still denote by $(.,.)$ the induced non degenerate symmetric bilinear form on $\mathfrak{h}^*$. It is defined on $\mathfrak{h}^*$ by

$$\begin{cases} (\Lambda_0, \alpha_1) & = 0 \\ (\Lambda_0, \Lambda_0) & = 0 \\ (\delta, \alpha_1) & = 0 \\ (\Lambda_0, \delta) & = 1 \\ (\alpha_1, \alpha_1) & = 2. \end{cases}$$

The level of an integral weight $\lambda \in P$, is defined as the integer (δ, λ). For $k \in \mathbb{N}$, we denote by P_k the set integral weights of level k. It is defined by

$$P_k = \{\lambda \in P : (\delta, \lambda) = k\}.$$

That is, an integral weight of level k can be written

$$k\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where $x \in \mathbb{Z}$, $y \in \mathbb{C}$, and a dominant weight of level k can be written

$$k\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where $x \in \{0, \dots, k\}$, $y \in \mathbb{C}$. Recall the following important property : all weights of an highest weight irreducible representation of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ have the same level.

Notation. For $\lambda \in \mathfrak{h}^*$, the projection of λ on $vect\{\Lambda_0, \alpha_1\}$, denoted $\bar{\lambda}$, is defined by $\bar{\lambda} = x\Lambda_0 + y\alpha_1$, when $\lambda = x\Lambda_0 + y\alpha_1 + z\delta$, $x, y, z \in \mathbb{C}$.

Characters. For $\lambda \in P_+$, we denote by ch_λ the character of the irreducible highest-weight module $V(\lambda)$ of $\hat{\mathfrak{sl}}_2(\mathbb{C})$ with highest weight λ. That is

$$\mathrm{ch}_\lambda(h) = \sum_{\mu \in P} \dim(V(\lambda)_\mu) e^{\langle \mu, h \rangle}, \quad h \in \mathfrak{h},$$

where $V(\lambda)_\mu$ is the weight space of $V(\lambda)$ corresponding to the weight μ. The above series converges absolutely for every $h \in \mathfrak{h}$ such that $\mathrm{Re}\langle \delta, h \rangle > 0$ (see chapter 11 of [5]). For $\beta \in \mathfrak{h}^*$, we write $\mathrm{ch}_\lambda(\beta)$ for $\mathrm{ch}_\lambda(\nu^{-1}(\beta))$. We have

$$\mathrm{ch}_\lambda(\beta) = \sum_{\mu \in P} \dim(V(\lambda)_\mu) e^{(\mu, \beta)}, \quad \beta \in \mathfrak{h}^*.$$

The Weyl character's formula states that

$$\mathrm{ch}_\lambda(.) = \frac{\sum_{w\in W}\det(w)e^{(w(\lambda+\rho),.)}}{\sum_{w\in W}\det(w)e^{(w(\rho),.)}},$$

where $\rho = 2\Lambda_0 + \frac{1}{2}\alpha_1$ and W is the group of linear transformations of $\mathfrak{h}^*$ generated by the reflections s_{α_0} and s_{α_1} defined by

$$s_{\alpha_i}(x) = x - 2\frac{(\alpha_i, x)}{(\alpha_i,\alpha_i)}\alpha_i,\ x\in\mathfrak{h}^*,\ i\in\{0,1\}.$$

As proved for instance in chapter 6 of [5], the affine Weyl group W is the semi-direct product $T \ltimes W_0$ where W_0 is the Weyl group generated by s_{α_1} and T is the group of transformations t_k, $k\in\mathbb{Z}$, defined by

$$t_k(\lambda) = \lambda + k(\lambda,\delta)\alpha_1 - (k(\lambda,\alpha_1) + k^2(\lambda,\delta))\delta,\ \ \lambda\in\mathfrak{h}^*.$$

Thus for $a\in\mathbb{R}^*$, $y\in\mathbb{R}_+^*$, and a dominant weight λ of level $n\in\mathbb{N}^*$, such that $\lambda = n\Lambda_0 + \frac{1}{2}x\alpha_1$, the Weyl character formula becomes

$$\mathrm{ch}_\lambda(ia\alpha_1 + y\Lambda_0) = \frac{\sum_{k\in\mathbb{Z}}\sin(a(x+1)+2ak(n+2))e^{-y(k(x+1)+k^2(n+2))}}{\sum_{k\in\mathbb{Z}}\sin(a+8ak)e^{-y(k+4k^2)}}. \tag{1}$$

Letting a goes to zero in the previous identity, one also obtains that

$$\mathrm{ch}_\lambda(y\Lambda_0) = \frac{\sum_{k\in\mathbb{Z}}(x+1+2k(n+2))e^{-y(k(x+1)+k^2(n+2))}}{\sum_{k\in\mathbb{Z}}(1+8k)e^{-y(k+4k^2)}}, \tag{2}$$

for every $y\in\mathbb{R}_+^*$.

3. Markov chains on the sets of integral or dominant weights

Let us choose for this section a dominant weight $\omega\in P_+$ and $h\in\mathfrak{h}_\mathbb{R}^*$ such that $(\delta,h)>0$.

Random walks on P. We define a probability measure μ_ω on P letting

$$\mu_\omega(\beta) = \frac{\dim(V(\omega)_\beta)}{\mathrm{ch}_\omega(h)}e^{\langle\beta,h\rangle},\quad \beta\in P. \tag{3}$$

If $(X(n), n\geq 0)$ is a random walk on P whose increments are distributed according to μ_ω, it is important for our purpose to keep in mind that the function

$$x\in\mathbb{R}\mapsto [\frac{\mathrm{ch}_\omega(i\frac{x}{2}\alpha_1+h)}{\mathrm{ch}_\omega(h)}]^n,$$

is the Fourier transform of the projection of $X(n)$ on $\mathbb{R}\alpha_1$.

Markov chains on P_+. Let us consider for $\lambda\in P_+$ the following decomposition

$$\mathrm{ch}_\omega\mathrm{ch}_\lambda = \sum_{\beta\in P_+} m_\lambda(\beta)\mathrm{ch}_\beta,$$

where $m_\lambda(\beta)$ is the multiplicity of the module with highest weight β in the decomposition of $V(\omega)\otimes V(\lambda)$, leads to the definition a transition probability q_ω on P_+ given by

$$q_\omega(\lambda,\beta) = \frac{\mathrm{ch}_\beta(h)}{\mathrm{ch}_\lambda(h)\mathrm{ch}_\omega(h)}m_\lambda(\beta),\quad \beta\in P_+. \tag{4}$$

Let us notice that if $(\Lambda(n), n \geq 0)$ is a Markov process starting from $\lambda_0 \in P_+$, with transition probabilities q_ω then

$$\mathbb{E}(\frac{\mathrm{ch}_{\Lambda(n)}(ix+h)}{ch_{\Lambda(n)}(h)}) = \frac{\mathrm{ch}_{\lambda_0}(ix+h)}{\mathrm{ch}_{\lambda_0}(h)}[\frac{\mathrm{ch}_\omega(ix\alpha_1+h)}{ch_\omega(h)}]^n,$$

for every $x \in \mathbb{R}$. If λ_1 and λ_2 are two dominant weights such that $\lambda_1 = \lambda_2\,(mod\,\delta)$ then the irreducible modules $V(\lambda_1)$ and $V(\lambda_2)$ are isomorphic. Thus if we consider the random process $(\bar\Lambda(n), n \geq 0)$, where $\bar\Lambda(n)$ is the projection of $\Lambda(n)$ on $vect\{\Lambda_0, \alpha_1\}$, then $(\bar\Lambda(n), n \geq 1)$ is a Markov process satisfying

$$\mathbb{E}(\frac{\mathrm{ch}_{\bar\Lambda(n)}(ix\alpha_1+h)}{\mathrm{ch}_{\bar\Lambda(n)}(h)}) = \frac{\mathrm{ch}_{\bar\lambda_0}(ix\alpha_1+h)}{\mathrm{ch}_{\bar\lambda_0}(h)}[\frac{\mathrm{ch}_\omega(ix\alpha_1+h)}{ch_\omega(h)}]^n, \tag{5}$$

for every $x \in \mathbb{R}$, where $\bar\lambda_0$ is the projection of λ_0 on $vect\{\Lambda_0, \alpha_1\}$. More generally, for $n, m \in \mathbb{N}$, one gets

(6)

$$\mathbb{E}(\frac{\mathrm{ch}_{\bar\Lambda(n+m)}(ix\alpha_1+h)}{\mathrm{ch}_{\bar\Lambda(n+m)}(h)}|\bar\Lambda(k),\, 0 \leq k \leq m) = \frac{\mathrm{ch}_{\bar\Lambda(m)}(ix\alpha_1+h)}{\mathrm{ch}_{\bar\Lambda(m)}(h)}[\frac{\mathrm{ch}_\omega(ix\alpha_1+h)}{\mathrm{ch}_\omega(h)}]^n,$$

for every $x \in \mathbb{R}$. Let us notice that if ω is a dominant weight of level k, and λ_0 a dominant weight of level k_0, then $\bar\Lambda(n)$ and $\Lambda(n)$ are dominant weights of level $nk + k_0$, for every $n \in \mathbb{N}$.

4. A conditioned space-time Brownian motion

A class of space-time harmonic functions. Considering the asymptotic of the previous characters, one obtains an interesting class of space-time harmonic functions involving the Jacobi's theta function θ defined by

$$\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z},$$

for z and τ two complex numbers, τ being in the upper half-plane. This is not surprising as the characters of affine Lie algebras are themself a linear combination of theta functions (see [5]). For $a \in \mathbb{R}^*$, $x \in [0,t]$, if $(\lambda_n)_n$ is a sequence of dominant weights such that

$$\begin{aligned}\lambda_n &= [nt]\Lambda_0 + \frac{1}{2}(\lambda_n, \alpha_1)\alpha_1 \\ &\underset{n\to\infty}{\sim} nt\Lambda_0 + nx\frac{1}{2}\alpha_1,\end{aligned}$$

then the sum

$$\sum_{k\in\mathbb{Z}} \sin(\frac{a}{n}((\lambda_n,\alpha_1)+1) + 2\frac{a}{n}k([nt]+2))e^{-\frac{2}{n}(k((\lambda_n,\alpha_1)+1)+k^2([nt]+2))},$$

which is the numerator of $\mathrm{ch}_{\lambda_n}(i\frac{a}{n}\alpha_1 + \frac{2}{n}\Lambda_0)$ in the right-hand side of identity (1), converges, when n goes to infinity, towards

$$\sum_{k\in\mathbb{Z}} \sin(ax + 2kat)e^{-2(kx+k^2t)}.$$

Definition 4.1. *For $a \in \mathbb{R}^*$, we define a function ϕ_a on $\mathbb{R}\times\mathbb{R}_+^*$ letting*

$$\phi_a(x,t) = \frac{\pi}{\mathrm{sh}(a\pi)}\sum_{k\in\mathbb{Z}} \sin(ax+2kat)e^{-2(kx+k^2t)},\ (x,t)\in\mathbb{R}\times\mathbb{R}_+^*.$$

Similarly, considering the asymptotic of the numerator of $\mathrm{ch}_{\lambda_n}(\frac{2}{n}\Lambda_0)$ in (2) leads naturally to the following definition.

Definition 4.2. *We define a function ϕ_0 on $\mathbb{R}\times\mathbb{R}_+^*$ letting*

$$\phi_0(x,t)=\sum_{k\in\mathbb{Z}}(x+2kt)e^{-2(kx+k^2t)},\ (x,t)\in\mathbb{R}\times\mathbb{R}_+^*.$$

Let us notice that $\lim_{a\to 0}\phi_a=\phi_0$ and

$$\phi_{\frac{n\pi}{t}}(x,t)=\frac{\pi}{\mathrm{sh}(\frac{n\pi^2}{t})}\sin(n\pi\frac{x}{t})\sum_{k\in\mathbb{Z}}e^{-2(kx+k^2t)},$$

for every $n\in\mathbb{N}^*$, $(x,t)\in\mathbb{R}\times\mathbb{R}^*$.

Proposition 4.3. *For $a\in\mathbb{R}$, the function*

$$(x,t)\in\mathbb{R}\times\mathbb{R}_+^*\mapsto e^{\frac{a^2}{2}t}\phi_a(x,t),$$

is a space-time harmonic function, i.e. ϕ_a satisfies

$$(\frac{1}{2}\frac{\partial^2}{\partial x^2}+\frac{\partial}{\partial t})\phi_a=-\frac{a^2}{2}\phi_a.$$

Moreover ϕ_a satisfies the following boundary conditions

$$\forall t\in\mathbb{R}_+^*,\ \begin{cases}\phi_a(0,t) &=0\\ \phi_a(t,t) &=0.\end{cases}$$

Proof. Actually, each summand of the sum in the definition of $e^{\frac{a^2}{2}t}\phi_a$ is a space-time harmonic function because for any $k\in\mathbb{Z}$, one has

$$e^{i(ax+2kat)-2(kx+k^2t)+\frac{a^2}{2}t}=e^{(ia-2k)x-\frac{1}{2}(ia-2k)^2t}.$$

The first boundary condition follows from the change of variable $k\mapsto -k$, whereas the last one follows from the change of variable $k\mapsto -1-k$. $\square$

Some properties of the functions ϕ_a, $a\in\mathbb{R}$.

Lemma 4.4.

$$\sum_{k\in\mathbb{Z}}\sin(ax+2kta)e^{-2(kx+k^2t)}=e^{\frac{x^2}{2t}-a^2\frac{t}{2}}\sum_{k\in\mathbb{Z}}\sqrt{\frac{\pi}{2t}}e^{-\frac{1}{2t}k^2\pi^2}\,\mathrm{sh}(k\pi a)\sin(k\frac{\pi}{t}x)$$

$$\sum_{k\in\mathbb{Z}}(x+2kt)e^{-2(kx+k^2t)}=e^{\frac{x^2}{2t}}\sum_{k\in\mathbb{Z}}\sqrt{\frac{\pi}{2t}}e^{-\frac{1}{2t}k^2\pi^2}k\pi\sin(k\frac{\pi}{t}x)$$

Proof. As $\sin(ax+2kta)e^{-2(kx+k^2t)}=e^{\frac{x^2}{2t}}\sin(a(x+2kt))e^{-\frac{1}{2t}(x+2kt)^2}$, the first identity follows from a Poisson summation formula, which is obtained computing the Fourier coefficients of the $2t$-periodic function $x\mapsto e^{-\frac{x^2}{2t}}\phi_a(x,t)$. The second identity follows similarly from the identity $(x+2kt)e^{-2(kx+k^2t)}=e^{\frac{x^2}{2t}}(x+2kt)e^{-\frac{1}{2t}(x+2kt)^2}$. Let us notice that the second identity can also be derived from the well known Jacobi's theta function identity

$$\frac{1}{\sqrt{\pi t}}\sum_{n\in\mathbb{Z}}e^{-\frac{1}{t}(n+x)^2}=\sum_{n\in\mathbb{Z}}\cos(2n\pi x)e^{-n^2\pi^2t},\tag{7}$$

which is valid for $x \in \mathbb{R}$, $t \in \mathbb{R}_+^*$, and which is also a particular case of the Poisson summation formula (see [1]). Considering the partial derivative with respect to x of the left and the right hand sides in identity (7) leads to the identity

$$\frac{1}{\sqrt{\pi t}} \sum_{n \in \mathbb{Z}} \frac{1}{t}(n+x) e^{-\frac{1}{t}(n+x)^2} = \sum_{n \in \mathbb{Z}} n\pi \sin(2n\pi x) e^{-n^2\pi^2 t}, \tag{8}$$

for $x \in \mathbb{R}$, $t \in \mathbb{R}_+^*$. As

$$\phi_0(x,t) = \sum_{n \in \mathbb{Z}} 2t\left(\frac{x}{2t} + n\right) e^{-2t(n+\frac{x}{2t})^2 + \frac{x^2}{2t}},$$

one obtains the second replacing respectively t and x by $\frac{1}{2t}$ and $\frac{x}{2t}$ in (8). □

Lemma 4.5. *Let $t \in \mathbb{R}_+^*$, and $x \in]0,t[$. If $(\lambda_n)_n$ is a sequence of dominant weights such that*

$$\lambda_n \sim nt\Lambda_0 + nx\frac{1}{2}\alpha,$$

then

$$\lim_{n \to \infty} \frac{ch_{\lambda_n}(\frac{ia}{n}\alpha_1 + \frac{2}{n}\Lambda_0)}{ch_{\lambda_n}(\frac{2}{n}\Lambda_0)} = \frac{\phi_a(x,t)}{\phi_0(x,t)}.$$

Proof. Lemma 4.4 implies that

$$\lim_{n \to \infty} \frac{\sum_{k \in \mathbb{Z}} \sin(\frac{a}{n}(1+8k)) e^{-\frac{2}{n}(k+4k^2)}}{\frac{1}{n}\sum_{k \in \mathbb{Z}} (1+8k) e^{-\frac{2}{n}(k+4k^2)}} = \frac{\mathrm{sh}(a\pi)}{\pi}.$$

Thus the lemma follows from identities (1) and (2). □

Proposition 4.6. *Let $a \in \mathbb{R}^*$, and $t \in \mathbb{R}_+^*$. Then*

1. *The function $\phi_0(.,t)$ is C^∞ on $[0,t]$,*
2. *the function $\frac{\phi_a(.,t)}{\phi_0(.,t)}$ is bounded on $[0,t]$,*
3. *$\forall x \in]0,t[$, $\phi_0(x,t) \neq 0$,*
4. *the function $\phi_0(.,t)$ doesn't change of sign on $[0,t]$.*

Proof. The first property follows immediately from a dominated convergence theorem. As for any $r \in \mathbb{R}$ and $y \in \mathbb{R}_+^*$, $\frac{\mathrm{ch}_\lambda(ir\alpha_1 + y\Lambda_0)}{\mathrm{ch}_\lambda(y\Lambda_0)}$ is a Fourier transform of a probability measure, it is bounded by 1. Previous lemma implies that

$$\forall x \in]0,t[, \quad \left|\frac{\phi_a(x,t)}{\phi_0(x,t)}\right| \leq 1.$$

As the function $\frac{\phi_a(.,t)}{\phi_0(.,t)}$ is easily shown to be continuous on $[0,t]$, the second property follows. For the third property, we notice that the function $\phi_{\frac{\pi}{t}}$ is defined by

$$\phi_{\frac{\pi}{t}}(x,t) = 2t \sin\left(\frac{x}{t}\pi\right) \sum_{k \in \mathbb{Z}} e^{-2(kx + k^2 t)}, \ (x,t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Thus for $x \in [0,t]$,

$$\phi_{\frac{\pi}{t}}(x,t) = 0 \Leftrightarrow x \in \{0,t\}.$$

The function $\frac{\phi_{\frac{\pi}{t}}(.,t)}{\phi_0(.,t)}$ being bounded on $[0,t]$, the third property follows. The fourth one is an immediate consequence of the first and the third ones. □

Let us enounce a very classical result on Fourier series that will be used to prove proposition 4.8

Lemma 4.7. *Let t be a positive real number and $f:[0,t]\to\mathbb{R}$ be a function such that $f(0)=f(t)=0$, which is C^3 on $[0,t]$. Then letting $c_n=\frac{1}{t}\int_0^t \sin(\frac{z}{t}n\pi)f(z)\,dz$, $n\in\mathbb{N}$, the series $\sum_n nc_n$ converges absolutely and*

$$f(x)=\sum_{n=1}^{+\infty}c_n\sin(\frac{x}{t}n\pi),$$

for every $x\in[0,t]$.

Proposition 4.8. *Let t be a positive real number and μ be a probability measure on $[0,t]$. Then μ is characterized by the quantities*

$$\int_0^t \frac{\phi_{n\pi/t}(x,t)}{\phi_0(x,t)}\,\mu(dx),\ n\in\mathbb{N}.$$

Proof. For $t\in\mathbb{R}_+^*$, $x\in\mathbb{R}$, we let

$$e(x,t)=\sum_{k\in\mathbb{Z}}e^{-2(kx+k^2t)}.$$

Let u be a C^3 function on $[0,t]$. We first notice that the function $\frac{u(.)\phi_0(.,t)}{e(.,t)}$ satisfies the condition of lemma 4.7. We let for $n\in\mathbb{N}^*$,

$$c_n=\frac{1}{t}\int_0^t\frac{u(x)\phi_0(x,t)}{e(x,t)}\sin(\frac{x}{t}n\pi)\,dx.$$

One has

$$\begin{aligned}\int_0^t u(x)\,\mu(dx)&=\int_0^t u(x)\frac{\phi_0(x,t)}{e(x,t)}\frac{e(x,t)}{\phi_0(x,t)}\,\mu(dx)\\&=\int_0^t\sum_{n=1}^{+\infty}c_n\sin(\frac{x}{t}n\pi)\frac{e(x,t)}{\phi_0(x,t)}\,\mu(dx)\end{aligned}$$

Using the two identities of lemma 4.4 one obtains

$$\begin{aligned}\int_0^t u(x)\,\mu(dx)&=\int_0^t\sum_{n=1}^{+\infty}c_n\frac{\sum_{k\in\mathbb{Z}}e^{-\frac{\pi^2}{2t}k^2}\sin((n+k)\pi\frac{x}{t})}{\sum_{k\in\mathbb{Z}}e^{-\frac{\pi^2}{2t}k^2}k\pi\pi\sin(k\pi\frac{x}{t})}\,\mu(dx)\\&=\sum_{n=1}^{+\infty}c_n\int_0^t\frac{\sum_{k\in\mathbb{Z}}e^{-\frac{\pi^2}{2t}k^2}\sin((n+k)\pi\frac{x}{t})}{\sum_{k\in\mathbb{Z}}e^{-\frac{\pi^2}{2t}k^2}k\pi\pi\sin(k\pi\frac{x}{t})}\,\mu(dx)\end{aligned}$$

where the last identity follows from the fact that the series $\sum nc_n$ is absolutely convergent. Thus

$$\int_0^t u(x)\,\mu(dx)=\sum_{n=1}^{+\infty}\frac{c_n}{2}\frac{\mathrm{sh}(n\frac{\pi^2}{t})}{\pi}\int_0^t\frac{\phi_{n\pi/t}(x,t)}{\phi_0(x,t)}\,\mu(dx).$$

□

A conditioned space-time Brownian motion. Let us denote by C the fundamental Weyl chamber defined by

$$C = \{x \in \mathfrak{h}^* : (x, \alpha_i) \geq 0, i \in \{0, 1\}\}.$$

That is, an element of C can be written

$$t\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where $t \in \mathbb{R}_+, x \in [0, t], y \in \mathbb{C}$. For $x \in \mathbb{R}$, we denote by $\mathbb{W}_x$ the Wiener measure on the set $C(\mathbb{R}_+)$ of real valued continuous functions on $\mathbb{R}_+$, under which the coordinate process $(X_t, t \geq 0)$ is a Brownian motion starting from x, and denote the natural filtration of $(X_t)_{t \geq 0}$ by $(\mathcal{F}_t)_{t \geq 0}$. One considers the stopping times T_u, $u \in \mathbb{R}_+$, defined by

$$T_u = \inf\{t \geq 0 : X_t = 0 \text{ or } X_t = t + u\}.$$

Proposition 4.6 ensures that $\phi_0(X_s, s + u)$ doesn't change of sign whenever $s \in [0, T_u]$. Let x be a positive real number such that $u > x$. One has $\mathbb{W}_x(T_u > 0) = 1$. The function ϕ_0 being space-time harmonic, the process $\phi_0(X_t, t + u)$ is a local martingale. Actually, each summand of the sum is a local martingale, for which the quadratic variation is easily shown to be integrable, so that, each summand is a true Martingale. As their sum converges absolutely in L_2 norm, one obtains that $\phi_0(X_t, t + u), t \geq 0$, is a true martingale. As $\phi(X_{T_u}, T_u + u) = 0$, one defines a measure $\mathbb{Q}_{x,u}$ on $C(\mathbb{R}_+)$ letting

$$\mathbb{Q}_{x,u}(A) = \mathbb{E}_x(\frac{\phi_0(X_t, t + u)}{\phi_0(x, u)} 1_{\{T_u > t\} \cap A}),\ A \in \mathcal{F}_t.$$

Let r and s be two positive real numbers such that $r < t$. Using that

$$T_u = T_{u+r} \circ \theta_r + r \ \text{ on } \{T_u \geq r\},$$

where θ_r the shift operator defined by

$$\forall t \in \mathbb{R}_+^*,\ X_t \circ \theta_r = X_{t+r},$$

one easily proves that $(X_t, t \geq 0)$ is an inhomogeneous Markov process under $\mathbb{Q}_{x,u}$ satisfying

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_{x,u}}(f(X_{t+r})|\mathcal{F}_r) &= \mathbb{E}_{X_r}(\frac{\phi_0(X_t, t + r + u)}{\phi_0(X_r, r + u)} f(X_t) 1_{T_{u+r} > t}) \\ &= \mathbb{E}_{\mathbb{Q}_{X_r, r+u}}(f(X_t)), \end{aligned} \tag{9}$$

for any real valued measurable bounded function f.

Proposition 4.9. *For $r, t, u \in \mathbb{R}_+^*$, $x \in]0, u[$, and $a \in \mathbb{R}$, one has*

$$\mathbb{Q}_{x,u}(\frac{\phi_a(X_t, t + u)}{\phi_0(X_t, t + u)}) = \frac{\phi_a(x, u)}{\phi_0(x, u)} e^{-\frac{a^2}{2}t}. \tag{10}$$

and

$$E_{\mathbb{Q}_{x,u}}(\frac{\phi_a(X_{t+r}, t + r + u)}{\phi_0(X_{t+r}, t + r + u)}|\mathcal{F}_r) = \frac{\phi_a(X_r, r + u)}{\phi_0(X_r, r + u)} e^{-\frac{a^2}{2}t}. \tag{11}$$

Proof. One proves as previously that

$$(e^{\frac{a^2}{2}t}\phi_a(X_t,t+u),t\geq 0)$$

is a true martingale, which implies that

$$\mathbb{W}_x(\phi_a(X_t,t+u)e^{\frac{a^2}{2}t}1_{T_u<t})=0,$$

and identity (10). The second identity follows, using (9). □

5. The conditioned Brownian motion and the Markov chains on the set of dominant weights

Let us focus on the Markov chains defined in section 3 when $\omega=\Lambda_0$. We recall that the weights occurring in $V(\Lambda_0)$ are

$$\Lambda_0+k\alpha_1-(k^2+s)\delta,\ k\in\mathbb{Z},\ s\in\mathbb{N},$$

with respective multiplicities $p(s)$, the number of partitions of s (see for instance chapter 9 in [8]). If we consider, for $h=\frac{1}{2}(h_1\alpha_1+h_2\Lambda_0)$, with $h_1\in\mathbb{R},h_2\in\mathbb{R}_+^*$, the associated probability measure μ_{Λ_0} defined by (3) and the associated random walk $(X(n),n\geq 0)$, then its projection on $\mathbb{Z}\alpha_1$ is a random walk with increments distributed according to a probability measure $\bar{\bar{\mu}}_{\Lambda_0}$ defined by

$$\bar{\bar{\mu}}_{\Lambda_0}(k)=C_he^{kh_1-\frac{h_2}{2}k^2},\ k\in\mathbb{Z},$$

where C_h is a normalizing constant depending on h.

The main theorem. For $n\in\mathbb{N}^*$, we consider a random walk $(X_k^n,k\geq 0)$ starting from 0, whose increments are distributed according to probability measure μ_{Λ_0} associated to $h=\frac{2}{n}\Lambda_0$. If we denote by $(\bar{\bar{X}}_k^n,k\geq 0)$ its projection on $\mathbb{Z}\alpha_1$, standard method shows that the sequence of processes $(\frac{2}{n}\bar{\bar{X}}_{[nt]}^n,t\geq 0)$ converges towards a standard Brownian motion on $\mathbb{R}$ when n goes to infinity.
Let x and u be two positive numbers such that $x<u$. For $n\in\mathbb{N}^*$, we consider a Markov process $(\Lambda_k^n,k\geq 0)$ starting from $[nu]\Lambda_0+[xn]\frac{1}{2}\alpha_1$, with the transition probability q_ω defined by (4), with $\omega=\Lambda_0$ and $h=\frac{2}{n}\Lambda_0$. It is important to notice that $(\Lambda_k^n,\delta)=[nu]+k$ for every $k\in\mathbb{N}$. If $\bar{\Lambda}_k^n$ is the projection of Λ_k^n on $vect\{\Lambda_0,\alpha_1\}$ for every $k\in\mathbb{N}$ and $n\in\mathbb{N}^*$, then the following convergence holds.

Theorem 5.1. *The sequence of processes $(\frac{1}{n}\bar{\Lambda}_{[nt]}^n,t\geq 0)$ converges when n goes to infinity towards the process $((t+u)\Lambda_0+\frac{X_t}{2}\alpha_1,t\geq 0)$ under $\mathbb{Q}_{x,u}$*

Proof. Let $t\in\mathbb{R}_+^*$. We denote by μ_t^n the law of $\frac{1}{n}(\Lambda_{[nt]}^n,\alpha_1)$, for $n\in\mathbb{N}$. The probability measure μ_t^n is carried by $[0,t+u]$. The intervale $[0,t+u]$ being a compact set, the space of probability measures on $[0,t+u]$ endowed with the weak topology is also compact. Suppose that a subsequence of $(\mu_t^n)_n$ converges towards μ_t. For $\lambda\in P_m^+$, one has

$$\frac{\mathrm{ch}_\lambda(\frac{a}{n}\alpha_1+\frac{2}{n}\Lambda_0)}{\mathrm{ch}_\lambda(\frac{2}{n}\Lambda_0)}=\frac{\phi_a(\frac{1}{n}((\lambda,\alpha_1)+1),\frac{1}{n}(m+2))}{\phi_0(\frac{1}{n}((\lambda,\alpha_1)+1),\frac{1}{n}(m+2))}\frac{\phi_0(\frac{1}{n},\frac{4}{n})}{\phi_a(\frac{1}{n},\frac{4}{n})},$$

for any $a\in\mathbb{R}$, and $n\in\mathbb{N}^*$. The function $(x,t)\mapsto\frac{\phi_a(x,t+u)}{\phi_0(x,t)}$ can be shown to be uniformly continuous on $\{(x,t)\in\mathbb{R}\times[0,T]:0\leq x\leq u+t\}$ for every $T\in\mathbb{R}_+$. As

$\lim_{n\to\infty} \frac{\phi_0(\frac{1}{n},\frac{4}{n})}{\phi_a(\frac{1}{n},\frac{4}{n})} = 1$, and

$$\Big[\frac{\mathrm{ch}_{\Lambda_0}(\frac{a}{n}\alpha_1 + \frac{2}{n}\Lambda_0)}{\mathrm{ch}_{\Lambda_0}(\frac{2}{n}\Lambda_0)}\Big]^{[nt]} = \mathbb{E}(e^{ia\frac{2}{n}\bar{X}^{(n)}_{[nt]}}),$$

identity (5) implies that μ_t satisfies

$$\int_0^{t+u} \frac{\phi_a(z,t+u)}{\phi_0(z,t+u)}\,\mu_t(dz) = \frac{\phi_a(x,u)}{\phi_0(x,u)} e^{-\frac{a^2}{2}t}.$$

Proposition 4.8 implies that $(\mu_t^n)_n$ converges towards μ_t and proposition 4.9 implies that μ_t is the distribution of X_t under $\mathbb{Q}_{x,u}$. Convergence of the sequence of random processes $(\frac{1}{n}(\Lambda^n_{[nt]},\alpha_1), t \geq 0)$ - in the sense of finite dimensional distributions convergence - follows similarly from identity (6) and (11).

□

6. Brownian motion conditioned to remain in an interval

In this section we discuss the connection between the conditioned Brownian motion constructed in this paper and the Brownian motion conditioned - in the sense of Doob - to remain in an interval. The connection is not surprising when we keep in mind that the dominant term in a character of a highest weight irreducible module of an affine algebra involves the so-called asymptotic dimensions, which are related to eigenfunctions for the Laplacian on an interval (see chapter 13 of [5]). Let $u \in \mathbb{R}_+^*$. The function h defined on $[0,u]$ by

$$h(x) = \sin(\pi\frac{x}{u}),\ x \in [0,u],$$

is the Dirichlet eigenfunction on the interval $[0,u]$ corresponding to the eigenvalue $-\frac{\pi^2}{u^2}$ at the bottom of the spectrum. Brownian motion conditioned - in the sense of Doob - to remain in the interval $[0,u]$, has the Doob-transformed semi-group $(q_t)_{t\geq 0}$ defined for $t \in \mathbb{R}_+^*$ by

$$q_t(x,y) = \frac{h(y)}{h(x)} e^{\frac{\pi^2}{u^2}\frac{t}{2}} p_t^0(x,y), \quad x,y \in]0,u[,$$

where p_t^0 is the semi-group of the standard Brownian motion on $\mathbb{R}$, killed on the boundary of $[0,u]$.

For $c \in]0,1[$, one defines a space-time harmonic function $\phi_0^{(c)}$ on $\mathbb{R}\times\mathbb{R}_+^*$ letting

$$\phi_0^{(c)}(x,t) = \phi_0(cx, c^2 t),$$

for $x,t \in \mathbb{R}\times\mathbb{R}_+^*$. This function satisfies the following boundary conditions

$$\forall t \in \mathbb{R}_+^*,\ \begin{cases} \phi_0^{(c)}(0,t) & = 0 \\ \phi_0^{(c)}(ct,t) & = 0. \end{cases}$$

As in section 4, one defines for a real number x satisfying $0 < x < u$, a probability $\mathbb{Q}^{(c)}_{x,u}$ on $C(\mathbb{R}_+)$ letting

$$\mathbb{Q}^{(c)}_{x,u}(A) = \mathbb{E}_x\Big(\frac{\phi_0^{(c)}(X_t, t+\frac{u}{c})}{\phi_0^{(c)}(x,\frac{u}{c})} 1_{\{T_u^{(c)} > t\}\cap A}\Big),\ A \in \mathcal{F}_t,$$

where $T_u^{(c)} = \inf\{s \geq 0 : X_s = 0 \text{ or } X_s = cs + u\}$. Thus, under the probability measure $\mathbb{Q}_{x,u}^{(c)}$, $(t + \frac{u}{c}, X_t)_{t\geq 0}$ is a space-time Brownian motion starting from $(\frac{u}{c}, x)$, conditioned to remain in the domain

$$\{(r,z) \in \mathbb{R} \times \mathbb{R} : 0 \leq z \leq cr\}.$$

Theorem 6.1. *The probability measure $Q_{x,u}^{(c)}$ converges, when c goes to 0, towards the law of a standard Brownian starting from x, conditioned - in the sense of Doob - to remain in $[0,u]$.*

Proof. Lemma 4.4 easily implies that

$$\lim_{c\to 0} \frac{\phi_0^{(c)}(y, t + \frac{u}{c})}{\phi_0^{(c)}(x, \frac{u}{c})} = \frac{\sin(y\frac{\pi}{u})}{\sin(x\frac{\pi}{u})} e^{\frac{\pi^2}{u^2}\frac{t}{2}},$$

for every $y \in [0,u]$, $t > 0$, which implies the theorem, as the quotient inside the limit is uniformly bounded for $y \in [0, ct+u]$ and $c \in]0,1[$. □

References

[1] R. Bellman, *A brief introduction to theta functions*, Holt, Rinehart and Winston, 1961.

[2] Ph. Biane, Ph. Bougerol and N. O'Connell, *Littelmann paths and Brownian paths,* Duke Math. J., 130, no. 1, 127-167, 2005.

[3] Crank J., *Free and Moving Boundary Problems*, Clarendon Press, Oxford, 1984.

[4] Doob, J. L., *Classical Potential Theory and Its Probabilistic Counterpart,* Springer, New York.1984.

[5] V. G. Kac, *Infinite dimensional Lie algebras,* third edition, Cambridge university press, 1990.

[6] C. Lecouvey, E. Lesigne, M. Peigné, *Conditioned random walks from Kac-Moody root systems*, arXiv:1306.3082 [math.CO], 2013.

[7] P. Littelmann, *Paths and root operators in representation theory*, Annals of Mathematics 142, pp. 499–525, 1995.

[8] J. Hong and S.-J. Kang, *Introduction to Quantum groups and Crystal Bases,* American mathematical society, 2002.

Laboratoire de Mathématiques Appliquées à Paris 5, Université Paris 5, 45 rue des Saints Pères, 75270 Paris Cedex 06.

E-mail address: manon.defosseux@parisdescartes.fr