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# THE AFFINE LIE ALGEBRA $\widehat{\mathfrak{sl}}_2(\mathbb{C})$ AND A CONDITIONED SPACE-TIME BROWNIAN MOTION

MANON DEFOSSEUX

ABSTRACT. We construct a sequence of Markov processes on the set of dominant weights of the Affine Lie algebra  $\widehat{\mathfrak{sl}}_2(\mathbb{C})$  which involves tensor product of irreducible highest weight modules of  $\widehat{\mathfrak{sl}}_2(\mathbb{C})$  and show that it converges towards a Doob's space-time harmonic transformation of a space-time Brownian motion.

## 1. INTRODUCTION

In [2], Ph. Biane, Ph. Bougerol and N. O'Connell establish a wide extension of Pitman's theorem on Brownian motion and three dimensional Bessel process, in the framework of representation theory of semi-simple complex Lie algebras. In this framework the representation of the Bessel process by a functional of a standard Brownian motion  $(B_t)_{t \geq 0}$  on  $\mathbb{R}$ ,

$$(B_t - 2 \inf_{0 \leq s \leq t} B_s, t \geq 0),$$

appears to be the continuous counterpart of a similar result which holds for a random walk on the set of integral weights of  $\mathfrak{sl}_2(\mathbb{C})$  and a path transformation connected with the Littelmann paths model for semi-simple complex Lie algebras (see for instance [7] for a description of this model).

In [6], C. Lecouvey, E. Lesigne and M. Peigné consider the case when  $\mathfrak{g}$  is a Kac-Moody algebra and develop some aspects of [2] in that framework. In particular, they focus on some Markov chains on the Weyl chamber of a Kac-Moody algebra, which are obtained in a similar way as in [2], except that the reference measure can't be the uniform measure when the dimension of the Kac-Moody algebra is infinite. Let us say briefly how the Markov chains are obtained for a Kac-Moody algebra  $\mathfrak{g}$ . As in the finite dimensional case, for a dominant integral weight  $\lambda$  of  $\mathfrak{g}$  one defines the character of the irreducible highest-weight representation  $V(\lambda)$  of  $\mathfrak{g}$  with highest weight  $\lambda$ , as a formal series

$$\text{ch}_\lambda = \sum_{\mu} \dim(V(\lambda)_\mu) e^\mu,$$

where  $V(\lambda)_\mu$  is the weight space of  $V(\lambda)$  corresponding to the weight  $\mu$ . Actually for every  $h$  in a subset of the Cartan subalgebra which doesn't depend on  $\lambda$  the series  $\sum_{\mu} \dim(V(\lambda)_\mu) e^{\langle \mu, h \rangle}$  is absolutely convergent. For two dominant weights  $\omega$  and  $\lambda$ , the following decomposition

$$\text{ch}_\omega \text{ch}_\lambda = \sum_{\beta \in P_+} m_\lambda(\beta) \text{ch}_\beta,$$

where  $m_\lambda(\beta)$  is the multiplicity of the module with highest weight  $\beta$  in the decomposition of  $V(\omega) \otimes V(\lambda)$ , allows to define a transition probability  $q_\omega$  on the set of

dominant weights, letting for  $\beta$  and  $\lambda$  two dominant weights of  $\mathfrak{g}$ ,

$$q_\omega(\lambda, \beta) = \frac{\text{ch}_\beta(h)}{\text{ch}_\lambda(h)\text{ch}_\omega(h)} m_\lambda(\beta),$$

where  $h$  is chosen in the region of convergence of the characters. It is a natural question to ask if there exists a sequence  $(h_n)_{n \geq 0}$  of elements of  $\mathfrak{h}$  such that the corresponding sequence of Markov chains converges towards a continuous process and what the limit is.

In this paper, we consider the case when  $\mathfrak{g}$  is the Kac-Moody algebra of type  $A_1^{(1)}$  and  $\omega$  is its fundamental weight  $\Lambda_0$ . There is no reason to think that the results are not true in a more general context but this case presents the advantage that explicit computations can easily be done. We show that the sequence of Markov chains, with a proper normalization, converges, for a particular sequence of  $(h_n)_{n \geq 0}$ , towards a Doob's space-time harmonic transformation of a space-time Brownian motion killed on the boundary of a time-dependent domain. This process is related to the heat equation

$$\frac{1}{2}\Delta + \frac{\partial}{\partial t} = 0,$$

in a time-dependent domain, with Dirichlet boundary conditions and the theta functions play a crucial role in the construction. One can find an extensive literature devoted to the relationship between Brownian motion and the heat equation. One can see for instance [4] for an introduction and [3] for a review of various problems specifically related to time-dependent boundaries.

The paper is organized as follows. Basic definitions and notations related to representation theory of the affine Lie algebra  $\hat{\mathfrak{sl}}_2(\mathbb{C})$  are given in section 2. We define in section 3 random walks on the set of integral weights of  $\hat{\mathfrak{sl}}_2(\mathbb{C})$  and Markov chains on the set of its dominant integral weights, considering tensor products of irreducible highest weight representations of  $\hat{\mathfrak{sl}}_2(\mathbb{C})$ . In section 4, for any positive real numbers  $x$  and  $u$  such that  $x < u$ , we define a space-time Brownian motion  $(t + u, B_t)_{t \geq 0}$  starting from  $(u, x)$ , conditioned to remain in the domain

$$D = \{(r, z) \in \mathbb{R} \times \mathbb{R} : 0 \leq z \leq r\}.$$

For this we introduce a space-time harmonic function remaining positive on  $D$  which appears naturally considering the limit of a sequence of characters of  $\hat{\mathfrak{sl}}_2(\mathbb{C})$ . We prove in section 5 that this conditioned space-time Brownian motion is the limit of a sequence of Markov processes constructed in section 3. We show in section 6 how it is related to a Brownian motion conditioned - in Doob's sense - to remain in an interval.

## 2. THE AFFINE LIE ALGEBRA $\hat{\mathfrak{sl}}_2(\mathbb{C})$

We consider the affine Lie algebra  $\hat{\mathfrak{sl}}_2(\mathbb{C})$  associated to the generalized Cartan matrix

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

The reader is invited to refer to [5] for a detailed description of this object. Let  $\mathfrak{h}$  be a Cartan subalgebra of  $\hat{\mathfrak{sl}}_2(\mathbb{C})$ . We denote by  $S = \{\alpha_0, \alpha_1\}$  the set of simple roots and by  $\{\alpha_0^\vee, \alpha_1^\vee\}$  the set of simple coroots. Let  $\Lambda_0$  be a fundamental weight

such that  $\langle \Lambda_0, \alpha_i^\vee \rangle = \delta_{i0}$ ,  $i \in \{0, 1\}$ , and  $\{\alpha_0, \alpha_1, \Lambda_0\}$  is a basis of  $\mathfrak{h}^*$ . We denote by  $\mathfrak{h}_{\mathbb{R}}$  the subset of  $\mathfrak{h}$  defined by

$$\mathfrak{h}_{\mathbb{R}} = \{x \in \mathfrak{h} : \langle \Lambda_0, x \rangle \in \mathbb{R}, \text{ and } \langle \alpha_i, x \rangle \in \mathbb{R}, i \in \{0, 1\}\}.$$

Let  $\delta = \alpha_0 + \alpha_1$  be the so-called null root. We denote by  $P$  (resp.  $P_+$ ) the set of integral (resp. dominant) weights defined by

$$P = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}, i = 0, 1\},$$

$$(\text{resp. } P_+ = \{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \geq 0, i = 0, 1\}).$$

The Cartan subalgebra  $\mathfrak{h}$  is equipped with a non degenerate symmetric bilinear form  $(\cdot, \cdot)$  defined below, which identifies  $\mathfrak{h}$  and  $\mathfrak{h}^*$ , through the linear isomorphism

$$\begin{aligned} \nu : \mathfrak{h} &\rightarrow \mathfrak{h}^*, \\ h &\mapsto (h, \cdot). \end{aligned}$$

We still denote by  $(\cdot, \cdot)$  the induced non degenerate symmetric bilinear form on  $\mathfrak{h}^*$ . It is defined on  $\mathfrak{h}^*$  by

$$\left\{ \begin{array}{l} (\Lambda_0, \alpha_1) = 0 \\ (\Lambda_0, \Lambda_0) = 0 \\ (\delta, \alpha_1) = 0 \\ (\Lambda_0, \delta) = 1 \\ (\alpha_1, \alpha_1) = 2. \end{array} \right.$$

The level of an integral weight  $\lambda \in P$ , is defined as the integer  $(\delta, \lambda)$ . For  $k \in \mathbb{N}$ , we denote by  $P_k$  the set integral weights of level  $k$ . It is defined by

$$P_k = \{\lambda \in P : (\delta, \lambda) = k\}.$$

That is, an integral weight of level  $k$  can be written

$$k\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where  $x \in \mathbb{Z}$ ,  $y \in \mathbb{C}$ , and a dominant weight of level  $k$  can be written

$$k\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where  $x \in \{0, \dots, k\}$ ,  $y \in \mathbb{C}$ . Recall the following important property : all weights of an highest weight irreducible representation of  $\mathfrak{sl}_2(\mathbb{C})$  have the same level.

**Notation.** For  $\lambda \in \mathfrak{h}^*$ , the projection of  $\lambda$  on  $\text{vect}\{\Lambda_0, \alpha_1\}$ , denoted  $\bar{\lambda}$ , is defined by  $\bar{\lambda} = x\Lambda_0 + y\alpha_1$ , when  $\lambda = x\Lambda_0 + y\alpha_1 + z\delta$ ,  $x, y, z \in \mathbb{C}$ .

**Characters.** For  $\lambda \in P_+$ , we denote by  $\text{ch}_\lambda$  the character of the irreducible highest-weight module  $V(\lambda)$  of  $\mathfrak{sl}_2(\mathbb{C})$  with highest weight  $\lambda$ . That is

$$\text{ch}_\lambda(h) = \sum_{\mu \in P} \dim(V(\lambda)_\mu) e^{\langle \mu, h \rangle}, \quad h \in \mathfrak{h},$$

where  $V(\lambda)_\mu$  is the weight space of  $V(\lambda)$  corresponding to the weight  $\mu$ . The above series converges absolutely for every  $h \in \mathfrak{h}$  such that  $\text{Re}\langle \delta, h \rangle > 0$  (see chapter 11 of [5]). For  $\beta \in \mathfrak{h}^*$ , we write  $\text{ch}_\lambda(\beta)$  for  $\text{ch}_\lambda(\nu^{-1}(\beta))$ . We have

$$\text{ch}_\lambda(\beta) = \sum_{\mu \in P} \dim(V(\lambda)_\mu) e^{\langle \mu, \beta \rangle}, \quad \beta \in \mathfrak{h}^*.$$

The Weyl character's formula states that

$$\text{ch}_\lambda(\cdot) = \frac{\sum_{w \in W} \det(w) e^{(w(\lambda+\rho), \cdot)}}{\sum_{w \in W} \det(w) e^{(w(\rho), \cdot)}},$$

where  $\rho = 2\Lambda_0 + \frac{1}{2}\alpha_1$  and  $W$  is the group of linear transformations of  $\mathfrak{h}^*$  generated by the reflections  $s_{\alpha_0}$  and  $s_{\alpha_1}$  defined by

$$s_{\alpha_i}(x) = x - 2 \frac{(\alpha_i, x)}{(\alpha_i, \alpha_i)} \alpha_i, \quad x \in \mathfrak{h}^*, \quad i \in \{0, 1\}.$$

As proved for instance in chapter 6 of [5], the affine Weyl group  $W$  is the semi-direct product  $T \ltimes W_0$  where  $W_0$  is the Weyl group generated by  $s_{\alpha_1}$  and  $T$  is the group of transformations  $t_k$ ,  $k \in \mathbb{Z}$ , defined by

$$t_k(\lambda) = \lambda + k(\lambda, \delta)\alpha_1 - (k(\lambda, \alpha_1) + k^2(\lambda, \delta))\delta, \quad \lambda \in \mathfrak{h}^*.$$

Thus for  $a \in \mathbb{R}^*$ ,  $y \in \mathbb{R}_+^*$ , and a dominant weight  $\lambda$  of level  $n \in \mathbb{N}^*$ , such that  $\lambda = n\Lambda_0 + \frac{1}{2}x\alpha_1$ , the Weyl character formula becomes

$$(1) \quad \text{ch}_\lambda(ia\alpha_1 + y\Lambda_0) = \frac{\sum_{k \in \mathbb{Z}} \sin(a(x+1) + 2ak(n+2)) e^{-y(k(x+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} \sin(a + 8ak) e^{-y(k+4k^2)}}.$$

Letting  $a$  goes to zero in the previous identity, one also obtains that

$$(2) \quad \text{ch}_\lambda(y\Lambda_0) = \frac{\sum_{k \in \mathbb{Z}} (x+1 + 2k(n+2)) e^{-y(k(x+1) + k^2(n+2))}}{\sum_{k \in \mathbb{Z}} (1 + 8k) e^{-y(k+4k^2)}},$$

for every  $y \in \mathbb{R}_+^*$ .

### 3. MARKOV CHAINS ON THE SETS OF INTEGRAL OR DOMINANT WEIGHTS

Let us choose for this section a dominant weight  $\omega \in P_+$  and  $h \in \mathfrak{h}_\mathbb{R}^*$  such that  $(\delta, h) > 0$ .

**Random walks on  $P$ .** We define a probability measure  $\mu_\omega$  on  $P$  letting

$$(3) \quad \mu_\omega(\beta) = \frac{\dim(V(\omega)_\beta)}{\text{ch}_\omega(h)} e^{\langle \beta, h \rangle}, \quad \beta \in P.$$

If  $(X(n), n \geq 0)$  is a random walk on  $P$  whose increments are distributed according to  $\mu_\omega$ , it is important for our purpose to keep in mind that the function

$$x \in \mathbb{R} \mapsto \left[ \frac{\text{ch}_\omega(i\frac{x}{2}\alpha_1 + h)}{\text{ch}_\omega(h)} \right]^n,$$

is the Fourier transform of the projection of  $X(n)$  on  $\mathbb{R}\alpha_1$ .

**Markov chains on  $P_+$ .** Let us consider for  $\lambda \in P_+$  the following decomposition

$$\text{ch}_\omega \text{ch}_\lambda = \sum_{\beta \in P_+} m_\lambda(\beta) \text{ch}_\beta,$$

where  $m_\lambda(\beta)$  is the multiplicity of the module with highest weight  $\beta$  in the decomposition of  $V(\omega) \otimes V(\lambda)$ , leads to the definition a transition probability  $q_\omega$  on  $P_+$  given by

$$(4) \quad q_\omega(\lambda, \beta) = \frac{\text{ch}_\beta(h)}{\text{ch}_\lambda(h) \text{ch}_\omega(h)} m_\lambda(\beta), \quad \beta \in P_+.$$

Let us notice that if  $(\Lambda(n), n \geq 0)$  is a Markov process starting from  $\lambda_0 \in P_+$ , with transition probabilities  $q_\omega$  then

$$\mathbb{E}\left(\frac{\text{ch}_{\Lambda(n)}(ix+h)}{\text{ch}_{\Lambda(n)}(h)}\right) = \frac{\text{ch}_{\lambda_0}(ix+h)}{\text{ch}_{\lambda_0}(h)} \left[\frac{\text{ch}_\omega(ix\alpha_1+h)}{\text{ch}_\omega(h)}\right]^n,$$

for every  $x \in \mathbb{R}$ . If  $\lambda_1$  and  $\lambda_2$  are two dominant weights such that  $\lambda_1 = \lambda_2 \pmod{\delta}$  then the irreducible modules  $V(\lambda_1)$  and  $V(\lambda_2)$  are isomorphic. Thus if we consider the random process  $(\bar{\Lambda}(n), n \geq 0)$ , where  $\bar{\Lambda}(n)$  is the projection of  $\Lambda(n)$  on  $\text{vect}\{\Lambda_0, \alpha_1\}$ , then  $(\bar{\Lambda}(n), n \geq 1)$  is a Markov process satisfying

$$(5) \quad \mathbb{E}\left(\frac{\text{ch}_{\bar{\Lambda}(n)}(ix\alpha_1+h)}{\text{ch}_{\bar{\Lambda}(n)}(h)}\right) = \frac{\text{ch}_{\bar{\lambda}_0}(ix\alpha_1+h)}{\text{ch}_{\bar{\lambda}_0}(h)} \left[\frac{\text{ch}_\omega(ix\alpha_1+h)}{\text{ch}_\omega(h)}\right]^n,$$

for every  $x \in \mathbb{R}$ , where  $\bar{\lambda}_0$  is the projection of  $\lambda_0$  on  $\text{vect}\{\Lambda_0, \alpha_1\}$ . More generally, for  $n, m \in \mathbb{N}$ , one gets

$$(6) \quad \mathbb{E}\left(\frac{\text{ch}_{\bar{\Lambda}(n+m)}(ix\alpha_1+h)}{\text{ch}_{\bar{\Lambda}(n+m)}(h)} \mid \bar{\Lambda}(k), 0 \leq k \leq m\right) = \frac{\text{ch}_{\bar{\Lambda}(m)}(ix\alpha_1+h)}{\text{ch}_{\bar{\Lambda}(m)}(h)} \left[\frac{\text{ch}_\omega(ix\alpha_1+h)}{\text{ch}_\omega(h)}\right]^n,$$

for every  $x \in \mathbb{R}$ . Let us notice that if  $\omega$  is a dominant weight of level  $k$ , and  $\lambda_0$  a dominant weight of level  $k_0$ , then  $\bar{\Lambda}(n)$  and  $\Lambda(n)$  are dominant weights of level  $nk + k_0$ , for every  $n \in \mathbb{N}$ .

#### 4. A CONDITIONED SPACE-TIME BROWNIAN MOTION

**A class of space-time harmonic functions.** Considering the asymptotic of the previous characters, one obtains an interesting class of space-time harmonic functions involving the Jacobi's theta function  $\theta$  defined by

$$\theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z},$$

for  $z$  and  $\tau$  two complex numbers,  $\tau$  being in the upper half-plane. This is not surprising as the characters of affine Lie algebras are themselves a linear combination of theta functions (see [5]). For  $a \in \mathbb{R}^*$ ,  $x \in [0, t]$ , if  $(\lambda_n)_n$  is a sequence of dominant weights such that

$$\begin{aligned} \lambda_n &= [nt]\Lambda_0 + \frac{1}{2}(\lambda_n, \alpha_1)\alpha_1 \\ &\underset{n \rightarrow \infty}{\sim} nt\Lambda_0 + nx\frac{1}{2}\alpha_1, \end{aligned}$$

then the sum

$$\sum_{k \in \mathbb{Z}} \sin\left(\frac{a}{n}((\lambda_n, \alpha_1) + 1) + 2\frac{a}{n}k([nt] + 2)\right) e^{-\frac{2}{n}(k((\lambda_n, \alpha_1) + 1) + k^2([nt] + 2))},$$

which is the numerator of  $\text{ch}_{\lambda_n}(i\frac{a}{n}\alpha_1 + \frac{2}{n}\Lambda_0)$  in the right-hand side of identity (1), converges, when  $n$  goes to infinity, towards

$$\sum_{k \in \mathbb{Z}} \sin(ax + 2kat) e^{-2(kx + k^2t)}.$$

**Definition 4.1.** For  $a \in \mathbb{R}^*$ , we define a function  $\phi_a$  on  $\mathbb{R} \times \mathbb{R}_+^*$  letting

$$\phi_a(x, t) = \frac{\pi}{\text{sh}(a\pi)} \sum_{k \in \mathbb{Z}} \sin(ax + 2kat) e^{-2(kx + k^2t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Similarly, considering the asymptotic of the numerator of  $\text{ch}_{\lambda_n}(\frac{2}{n}\Lambda_0)$  in (2) leads naturally to the following definition.

**Definition 4.2.** We define a function  $\phi_0$  on  $\mathbb{R} \times \mathbb{R}_+^*$  letting

$$\phi_0(x, t) = \sum_{k \in \mathbb{Z}} (x + 2kt) e^{-2(kx + k^2 t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Let us notice that  $\lim_{a \rightarrow 0} \phi_a = \phi_0$  and

$$\phi_{\frac{n\pi}{t}}(x, t) = \frac{\pi}{\text{sh}(\frac{n\pi^2}{t})} \sin(n\pi \frac{x}{t}) \sum_{k \in \mathbb{Z}} e^{-2(kx + k^2 t)},$$

for every  $n \in \mathbb{N}^*$ ,  $(x, t) \in \mathbb{R} \times \mathbb{R}^*$ .

**Proposition 4.3.** For  $a \in \mathbb{R}$ , the function

$$(x, t) \in \mathbb{R} \times \mathbb{R}_+^* \mapsto e^{\frac{a^2}{2}t} \phi_a(x, t),$$

is a space-time harmonic function, i.e.  $\phi_a$  satisfies

$$\left(\frac{1}{2} \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right) \phi_a = -\frac{a^2}{2} \phi_a.$$

Moreover  $\phi_a$  satisfies the following boundary conditions

$$\forall t \in \mathbb{R}^*, \quad \begin{cases} \phi_a(0, t) &= 0 \\ \phi_a(t, t) &= 0. \end{cases}$$

*Proof.* Actually, each summand of the sum in the definition of  $e^{\frac{a^2}{2}t} \phi_a$  is a space-time harmonic function because for any  $k \in \mathbb{Z}$ , one has

$$e^{i(ax + 2kat) - 2(kx + k^2 t) + \frac{a^2}{2}t} = e^{(ia - 2k)x - \frac{1}{2}(ia - 2k)^2 t}.$$

The first boundary condition follows from the change of variable  $k \mapsto -k$ , whereas the last one follows from the change of variable  $k \mapsto -1 - k$ .  $\square$

**Some properties of the functions  $\phi_a$ ,  $a \in \mathbb{R}$ .**

**Lemma 4.4.**

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \sin(ax + 2kta) e^{-2(kx + k^2 t)} &= e^{\frac{x^2}{2t} - a^2 \frac{t}{2}} \sum_{k \in \mathbb{Z}} \sqrt{\frac{\pi}{2t}} e^{-\frac{1}{2t} k^2 \pi^2} \text{sh}(k\pi a) \sin(k \frac{\pi}{t} x) \\ \sum_{k \in \mathbb{Z}} (x + 2kt) e^{-2(kx + k^2 t)} &= e^{\frac{x^2}{2t}} \sum_{k \in \mathbb{Z}} \sqrt{\frac{\pi}{2t}} e^{-\frac{1}{2t} k^2 \pi^2} k\pi \sin(k \frac{\pi}{t} x) \end{aligned}$$

*Proof.* As  $\sin(ax + 2kta) e^{-2(kx + k^2 t)} = e^{\frac{x^2}{2t}} \sin(a(x + 2kt)) e^{-\frac{1}{2t}(x + 2kt)^2}$ , the first identity follows from a Poisson summation formula, which is obtained computing the Fourier coefficients of the  $2t$ -periodic function  $x \mapsto e^{-\frac{x^2}{2t}} \phi_a(x, t)$ . The second identity follows similarly from the identity  $(x + 2kt) e^{-2(kx + k^2 t)} = e^{\frac{x^2}{2t}} (x + 2kt) e^{-\frac{1}{2t}(x + 2kt)^2}$ . Let us notice that the second identity can also be derived from the well known Jacobi's theta function identity

$$(7) \quad \frac{1}{\sqrt{\pi t}} \sum_{n \in \mathbb{Z}} e^{-\frac{1}{t}(n+x)^2} = \sum_{n \in \mathbb{Z}} \cos(2n\pi x) e^{-n^2 \pi^2 t},$$

which is valid for  $x \in \mathbb{R}$ ,  $t \in \mathbb{R}_+^*$ , and which is also a particular case of the Poisson summation formula (see [1]). Considering the partial derivative with respect to  $x$  of the left and the right hand sides in identity (7) leads to the identity

$$(8) \quad \frac{1}{\sqrt{\pi t}} \sum_{n \in \mathbb{Z}} \frac{1}{t} (n+x) e^{-\frac{1}{t}(n+x)^2} = \sum_{n \in \mathbb{Z}} n\pi \sin(2n\pi x) e^{-n^2\pi^2 t},$$

for  $x \in \mathbb{R}$ ,  $t \in \mathbb{R}_+^*$ . As

$$\phi_0(x, t) = \sum_{n \in \mathbb{Z}} 2t \left( \frac{x}{2t} + n \right) e^{-2t(n + \frac{x}{2t})^2 + \frac{x^2}{2t}},$$

one obtains the second replacing respectively  $t$  and  $x$  by  $\frac{1}{2t}$  and  $\frac{x}{2t}$  in (8).  $\square$

**Lemma 4.5.** *Let  $t \in \mathbb{R}_+^*$ , and  $x \in ]0, t[$ . If  $(\lambda_n)_n$  is a sequence of dominant weights such that*

$$\lambda_n \sim nt\Lambda_0 + nx\frac{1}{2}\alpha,$$

then

$$\lim_{n \rightarrow \infty} \frac{\text{ch}_{\lambda_n}(\frac{ia}{n}\alpha_1 + \frac{2}{n}\Lambda_0)}{\text{ch}_{\lambda_n}(\frac{2}{n}\Lambda_0)} = \frac{\phi_a(x, t)}{\phi_0(x, t)}.$$

*Proof.* Lemma 4.4 implies that

$$\lim_{n \rightarrow \infty} \frac{\sum_{k \in \mathbb{Z}} \sin(\frac{a}{n}(1+8k)) e^{-\frac{2}{n}(k+4k^2)}}{\frac{1}{n} \sum_{k \in \mathbb{Z}} (1+8k) e^{-\frac{2}{n}(k+4k^2)}} = \frac{\text{sh}(a\pi)}{\pi}.$$

Thus the lemma follows from identities (1) and (2).  $\square$

**Proposition 4.6.** *Let  $a \in \mathbb{R}^*$ , and  $t \in \mathbb{R}_+^*$ . Then*

- (1) *The function  $\phi_0(\cdot, t)$  is  $C^\infty$  on  $[0, t]$ ,*
- (2) *the function  $\frac{\phi_a(\cdot, t)}{\phi_0(\cdot, t)}$  is bounded on  $[0, t]$ ,*
- (3)  $\forall x \in ]0, t[, \phi_0(x, t) \neq 0$ ,
- (4) *the function  $\phi_0(\cdot, t)$  doesn't change of sign on  $[0, t]$ .*

*Proof.* The first property follows immediately from a dominated convergence theorem. As for any  $r \in \mathbb{R}$  and  $y \in \mathbb{R}_+^*$ ,  $\frac{\text{ch}_\lambda(ir\alpha_1 + y\Lambda_0)}{\text{ch}_\lambda(y\Lambda_0)}$  is a Fourier transform of a probability measure, it is bounded by 1. Previous lemma implies that

$$\forall x \in ]0, t[, \quad \left| \frac{\phi_a(x, t)}{\phi_0(x, t)} \right| \leq 1.$$

As the function  $\frac{\phi_a(\cdot, t)}{\phi_0(\cdot, t)}$  is easily shown to be continuous on  $[0, t]$ , the second property follows. For the third property, we notice that the function  $\phi_{\frac{x}{t}}$  is defined by

$$\phi_{\frac{x}{t}}(x, t) = 2t \sin\left(\frac{x}{t}\pi\right) \sum_{k \in \mathbb{Z}} e^{-2(kx + k^2 t)}, \quad (x, t) \in \mathbb{R} \times \mathbb{R}_+^*.$$

Thus for  $x \in [0, t]$ ,

$$\phi_{\frac{x}{t}}(x, t) = 0 \Leftrightarrow x \in \{0, t\}.$$

The function  $\frac{\phi_{\frac{x}{t}}(\cdot, t)}{\phi_0(\cdot, t)}$  being bounded on  $[0, t]$ , the third property follows. The fourth one is an immediate consequence of the first and the third ones.  $\square$



Let us enounce a very classical result on Fourier series that will be used to prove proposition 4.8

**Lemma 4.7.** *Let  $t$  be a positive real number and  $f : [0, t] \rightarrow \mathbb{R}$  be a function such that  $f(0) = f(t) = 0$ , which is  $C^3$  on  $[0, t]$ . Then letting  $c_n = \frac{1}{t} \int_0^t \sin(\frac{z}{t}n\pi) f(z) dz$ ,  $n \in \mathbb{N}$ , the series  $\sum_n n c_n$  converges absolutely and*

$$f(x) = \sum_{n=1}^{+\infty} c_n \sin(\frac{x}{t}n\pi),$$

for every  $x \in [0, t]$ .

**Proposition 4.8.** *Let  $t$  be a positive real number and  $\mu$  be a probability measure on  $[0, t]$ . Then  $\mu$  is characterized by the quantities*

$$\int_0^t \frac{\phi_{n\pi/t}(x, t)}{\phi_0(x, t)} \mu(dx), \quad n \in \mathbb{N}.$$

*Proof.* For  $t \in \mathbb{R}_+^*$ ,  $x \in \mathbb{R}$ , we let

$$e(x, t) = \sum_{k \in \mathbb{Z}} e^{-2(kx+k^2t)}.$$

Let  $u$  be a  $C^3$  function on  $[0, t]$ . We first notice that the function  $\frac{u(\cdot)\phi_0(\cdot, t)}{e(\cdot, t)}$  satisfies the condition of lemma 4.7. We let for  $n \in \mathbb{N}^*$ ,

$$c_n = \frac{1}{t} \int_0^t \frac{u(x)\phi_0(x, t)}{e(x, t)} \sin(\frac{x}{t}n\pi) dx.$$

One has

$$\begin{aligned} \int_0^t u(x) \mu(dx) &= \int_0^t u(x) \frac{\phi_0(x, t)}{e(x, t)} \frac{e(x, t)}{\phi_0(x, t)} \mu(dx) \\ &= \int_0^t \sum_{n=1}^{+\infty} c_n \sin(\frac{x}{t}n\pi) \frac{e(x, t)}{\phi_0(x, t)} \mu(dx) \end{aligned}$$

Using the two identities of lemma 4.4 one obtains

$$\begin{aligned} \int_0^t u(x) \mu(dx) &= \int_0^t \sum_{n=1}^{+\infty} c_n \frac{\sum_{k \in \mathbb{Z}} e^{-\frac{\pi^2}{2t}k^2} \sin((n+k)\pi\frac{x}{t})}{\sum_{k \in \mathbb{Z}} e^{-\frac{\pi^2}{2t}k^2} k\pi \sin(k\pi\frac{x}{t})} \mu(dx) \\ &= \sum_{n=1}^{+\infty} c_n \int_0^t \frac{\sum_{k \in \mathbb{Z}} e^{-\frac{\pi^2}{2t}k^2} \sin((n+k)\pi\frac{x}{t})}{\sum_{k \in \mathbb{Z}} e^{-\frac{\pi^2}{2t}k^2} k\pi \sin(k\pi\frac{x}{t})} \mu(dx) \end{aligned}$$

where the last identity follows from the fact that the series  $\sum n c_n$  is absolutely convergent. Thus

$$\int_0^t u(x) \mu(dx) = \sum_{n=1}^{+\infty} \frac{c_n}{2} \frac{\text{sh}(n\frac{\pi^2}{t})}{\pi} \int_0^t \frac{\phi_{n\pi/t}(x, t)}{\phi_0(x, t)} \mu(dx).$$

□

**A conditioned space-time Brownian motion.** Let us denote by  $C$  the fundamental Weyl chamber defined by

$$C = \{x \in \mathfrak{h}^* : (x, \alpha_i) \geq 0, i \in \{0, 1\}\}.$$

That is, an element of  $C$  can be written

$$t\Lambda_0 + \frac{x}{2}\alpha_1 + y\delta,$$

where  $t \in \mathbb{R}_+, x \in [0, t], y \in \mathbb{C}$ . For  $x \in \mathbb{R}$ , we denote by  $\mathbb{W}_x$  the Wiener measure on the set  $C(\mathbb{R}_+)$  of real valued continuous functions on  $\mathbb{R}_+$ , under which the coordinate process  $(X_t, t \geq 0)$  is a Brownian motion starting from  $x$ , and denote the natural filtration of  $(X_t)_{t \geq 0}$  by  $(\mathcal{F}_t)_{t \geq 0}$ . One considers the stopping times  $T_u, u \in \mathbb{R}_+$ , defined by

$$T_u = \inf\{t \geq 0 : X_t = 0 \text{ or } X_t = t + u\}.$$

Proposition 4.6 ensures that  $\phi_0(X_s, s + u)$  doesn't change of sign whenever  $s \in [0, T_u]$ . Let  $x$  be a positive real number such that  $u > x$ . One has  $\mathbb{W}_x(T_u > 0) = 1$ . The function  $\phi_0$  being space-time harmonic, the process  $\phi_0(X_t, t + u)$  is a local martingale. Actually, each summand of the sum is a local martingale, for which the quadratic variation is easily shown to be integrable, so that, each summand is a true Martingale. As their sum converges absolutely in  $L_2$  norm, one obtains that  $\phi_0(X_t, t + u), t \geq 0$ , is a true martingale. As  $\phi(X_{T_u}, T_u + u) = 0$ , one defines a measure  $\mathbb{Q}_{x,u}$  on  $C(\mathbb{R}_+)$  letting

$$\mathbb{Q}_{x,u}(A) = \mathbb{E}_x\left(\frac{\phi_0(X_t, t + u)}{\phi_0(x, u)} 1_{\{T_u > t\} \cap A}\right), A \in \mathcal{F}_t.$$

Let  $r$  and  $s$  be two positive real numbers such that  $r < t$ . Using that

$$T_u = T_{u+r} \circ \theta_r + r \text{ on } \{T_u \geq r\},$$

where  $\theta_r$  the shift operator defined by

$$\forall t \in \mathbb{R}_+, X_t \circ \theta_r = X_{t+r},$$

one easily proves that  $(X_t, t \geq 0)$  is an inhomogeneous Markov process under  $\mathbb{Q}_{x,u}$  satisfying

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}_{x,u}}(f(X_{t+r}) | \mathcal{F}_r) &= \mathbb{E}_{X_r}\left(\frac{\phi_0(X_t, t + r + u)}{\phi_0(X_r, r + u)} f(X_t) 1_{T_{u+r} > t}\right) \\ (9) \qquad \qquad \qquad &= \mathbb{E}_{\mathbb{Q}_{X_r, r+u}}(f(X_t)), \end{aligned}$$

for any real valued measurable bounded function  $f$ .

**Proposition 4.9.** For  $r, t, u \in \mathbb{R}_+, x \in ]0, u[$ , and  $a \in \mathbb{R}$ , one has

$$(10) \qquad \mathbb{Q}_{x,u}\left(\frac{\phi_a(X_t, t + u)}{\phi_0(X_t, t + u)}\right) = \frac{\phi_a(x, u)}{\phi_0(x, u)} e^{-\frac{a^2}{2}t}.$$

and

$$(11) \qquad \mathbb{E}_{\mathbb{Q}_{x,u}}\left(\frac{\phi_a(X_{t+r}, t + r + u)}{\phi_0(X_{t+r}, t + r + u)} | \mathcal{F}_r\right) = \frac{\phi_a(X_r, r + u)}{\phi_0(X_r, r + u)} e^{-\frac{a^2}{2}t}.$$

*Proof.* One proves as previously that

$$(e^{\frac{a}{2}t}\phi_a(X_t, t+u), t \geq 0)$$

is a true martingale, which implies that

$$\mathbb{W}_x(\phi_a(X_t, t+u)e^{\frac{a}{2}t}1_{T_u < t}) = 0,$$

and identity (10). The second identity follows, using (9).  $\square$

## 5. THE CONDITIONED BROWNIAN MOTION AND THE MARKOV CHAINS ON THE SET OF DOMINANT WEIGHTS

Let us focus on the Markov chains defined in section 3 when  $\omega = \Lambda_0$ . We recall that the weights occurring in  $V(\Lambda_0)$  are

$$\Lambda_0 + k\alpha_1 - (k^2 + s)\delta, \quad k \in \mathbb{Z}, \quad s \in \mathbb{N},$$

with respective multiplicities  $p(s)$ , the number of partitions of  $s$  (see for instance chapter 9 in [8]). If we consider, for  $h = \frac{1}{2}(h_1\alpha_1 + h_2\Lambda_0)$ , with  $h_1 \in \mathbb{R}, h_2 \in \mathbb{R}_+$ , the associated probability measure  $\mu_{\Lambda_0}$  defined by (3) and the associated random walk  $(X(n), n \geq 0)$ , then its projection on  $\mathbb{Z}\alpha_1$  is a random walk with increments distributed according to a probability measure  $\bar{\mu}_{\Lambda_0}$  defined by

$$\bar{\mu}_{\Lambda_0}(k) = C_h e^{kh_1 - \frac{h_2}{2}k^2}, \quad k \in \mathbb{Z},$$

where  $C_h$  is a normalizing constant depending on  $h$ .

**The main theorem.** For  $n \in \mathbb{N}^*$ , we consider a random walk  $(X_k^n, k \geq 0)$  starting from 0, whose increments are distributed according to probability measure  $\mu_{\Lambda_0}$  associated to  $h = \frac{2}{n}\Lambda_0$ . If we denote by  $(\bar{X}_k^n, k \geq 0)$  its projection on  $\mathbb{Z}\alpha_1$ , standard method shows that the sequence of processes  $(\frac{2}{n}\bar{X}_{[nt]}^n, t \geq 0)$  converges towards a standard Brownian motion on  $\mathbb{R}$  when  $n$  goes to infinity.

Let  $x$  and  $u$  be two positive numbers such that  $x < u$ . For  $n \in \mathbb{N}^*$ , we consider a Markov process  $(\Lambda_k^n, k \geq 0)$  starting from  $[nu]\Lambda_0 + [xn]\frac{1}{2}\alpha_1$ , with the transition probability  $q_\omega$  defined by (4), with  $\omega = \Lambda_0$  and  $h = \frac{2}{n}\Lambda_0$ . It is important to notice that  $(\Lambda_k^n, \delta) = [nu] + k$  for every  $k \in \mathbb{N}$ . If  $\bar{\Lambda}_k^n$  is the projection of  $\Lambda_k^n$  on  $\text{vect}\{\Lambda_0, \alpha_1\}$  for every  $k \in \mathbb{N}$  and  $n \in \mathbb{N}^*$ , then the following convergence holds.

**Theorem 5.1.** *The sequence of processes  $(\frac{1}{n}\bar{\Lambda}_{[nt]}^n, t \geq 0)$  converges when  $n$  goes to infinity towards the process  $((t+u)\Lambda_0 + \frac{X_t}{2}\alpha_1, t \geq 0)$  under  $\mathbb{Q}_{x,u}$*

*Proof.* Let  $t \in \mathbb{R}_+^*$ . We denote by  $\mu_t^n$  the law of  $\frac{1}{n}(\Lambda_{[nt]}^n, \alpha_1)$ , for  $n \in \mathbb{N}$ . The probability measure  $\mu_t^n$  is carried by  $[0, t+u]$ . The interval  $[0, t+u]$  being a compact set, the space of probability measures on  $[0, t+u]$  endowed with the weak topology is also compact. Suppose that a subsequence of  $(\mu_t^n)_n$  converges towards  $\mu_t$ . For  $\lambda \in P_m^+$ , one has

$$\frac{\text{ch}_\lambda(\frac{a}{n}\alpha_1 + \frac{2}{n}\Lambda_0)}{\text{ch}_\lambda(\frac{2}{n}\Lambda_0)} = \frac{\phi_a(\frac{1}{n}((\lambda, \alpha_1) + 1), \frac{1}{n}(m+2)) \phi_0(\frac{1}{n}, \frac{4}{n})}{\phi_0(\frac{1}{n}((\lambda, \alpha_1) + 1), \frac{1}{n}(m+2)) \phi_a(\frac{1}{n}, \frac{4}{n})},$$

for any  $a \in \mathbb{R}$ , and  $n \in \mathbb{N}^*$ . The function  $(x, t) \mapsto \frac{\phi_a(x, t+u)}{\phi_0(x, t)}$  can be shown to be uniformly continuous on  $\{(x, t) \in \mathbb{R} \times [0, T] : 0 \leq x \leq u+t\}$  for every  $T \in \mathbb{R}_+$ . As

$\lim_{n \rightarrow \infty} \frac{\phi_0(\frac{1}{n}, \frac{a}{n})}{\phi_a(\frac{1}{n}, \frac{a}{n})} = 1$ , and

$$\left[ \frac{\text{ch}_{\Lambda_0}(\frac{a}{n}\alpha_1 + \frac{2}{n}\Lambda_0)}{\text{ch}_{\Lambda_0}(\frac{2}{n}\Lambda_0)} \right]^{[nt]} = \mathbb{E}(e^{ia\frac{2}{n}\bar{X}_{[nt]}^{(n)}}),$$

identity (5) implies that  $\mu_t$  satisfies

$$\int_0^{t+u} \frac{\phi_a(z, t+u)}{\phi_0(z, t+u)} \mu_t(dz) = \frac{\phi_a(x, u)}{\phi_0(x, u)} e^{-\frac{a^2}{2}t}.$$

Proposition 4.8 implies that  $(\mu_t^n)_n$  converges towards  $\mu_t$  and proposition 4.9 implies that  $\mu_t$  is the distribution of  $X_t$  under  $\mathbb{Q}_{x,u}$ . Convergence of the sequence of random processes  $(\frac{1}{n}(\Lambda_{[nt]}^n, \alpha_1), t \geq 0)$  - in the sense of finite dimensional distributions convergence - follows similarly from identity (6) and (11).  $\square$

## 6. BROWNIAN MOTION CONDITIONED TO REMAIN IN AN INTERVAL

In this section we discuss the connection between the conditioned Brownian motion constructed in this paper and the Brownian motion conditioned - in the sense of Doob - to remain in an interval. The connection is not surprising when we keep in mind that the dominant term in a character of a highest weight irreducible module of an affine algebra involves the so-called asymptotic dimensions, which are related to eigenfunctions for the Laplacian on an interval (see chapter 13 of [5]). Let  $u \in \mathbb{R}_+^*$ . The function  $h$  defined on  $[0, u]$  by

$$h(x) = \sin(\pi \frac{x}{u}), \quad x \in [0, u],$$

is the Dirichlet eigenfunction on the interval  $[0, u]$  corresponding to the eigenvalue  $-\frac{\pi^2}{u^2}$  at the bottom of the spectrum. Brownian motion conditioned - in the sense of Doob - to remain in the interval  $[0, u]$ , has the Doob-transformed semi-group  $(q_t)_{t \geq 0}$  defined for  $t \in \mathbb{R}_+^*$  by

$$q_t(x, y) = \frac{h(y)}{h(x)} e^{\frac{\pi^2}{u^2} \frac{t}{2}} p_t^0(x, y), \quad x, y \in ]0, u[$$

where  $p_t^0$  is the semi-group of the standard Brownian motion on  $\mathbb{R}$ , killed on the boundary of  $[0, u]$ .

For  $c \in ]0, 1[$ , one defines a space-time harmonic function  $\phi_0^{(c)}$  on  $\mathbb{R} \times \mathbb{R}_+^*$  letting

$$\phi_0^{(c)}(x, t) = \phi_0(cx, c^2t),$$

for  $x, t \in \mathbb{R} \times \mathbb{R}_+^*$ . This function satisfies the following boundary conditions

$$\forall t \in \mathbb{R}_+^*, \quad \begin{cases} \phi_0^{(c)}(0, t) & = 0 \\ \phi_0^{(c)}(ct, t) & = 0. \end{cases}$$

As in section 4, one defines for a real number  $x$  satisfying  $0 < x < u$ , a probability  $\mathbb{Q}_{x,u}^{(c)}$  on  $C(\mathbb{R}_+)$  letting

$$\mathbb{Q}_{x,u}^{(c)}(A) = \mathbb{E}_x \left( \frac{\phi_0^{(c)}(X_t, t + \frac{u}{c})}{\phi_0^{(c)}(x, \frac{u}{c})} \mathbf{1}_{\{T_u^{(c)} > t\} \cap A} \right), \quad A \in \mathcal{F}_t,$$

where  $T_u^{(c)} = \inf\{s \geq 0 : X_s = 0 \text{ or } X_s = cs + u\}$ . Thus, under the probability measure  $\mathbb{Q}_{x,u}^{(c)}$ ,  $(t + \frac{u}{c}, X_t)_{t \geq 0}$  is a space-time Brownian motion starting from  $(\frac{u}{c}, x)$ , conditioned to remain in the domain

$$\{(r, z) \in \mathbb{R} \times \mathbb{R} : 0 \leq z \leq cr\}.$$

**Theorem 6.1.** *The probability measure  $\mathbb{Q}_{x,u}^{(c)}$  converges, when  $c$  goes to 0, towards the law of a standard Brownian starting from  $x$ , conditioned - in the sense of Doob - to remain in  $[0, u]$ .*

*Proof.* Lemma 4.4 easily implies that

$$\lim_{c \rightarrow 0} \frac{\phi_0^{(c)}(y, t + \frac{u}{c})}{\phi_0^{(c)}(x, \frac{u}{c})} = \frac{\sin(y \frac{\pi}{u})}{\sin(x \frac{\pi}{u})} e^{\frac{\pi^2}{u^2} t},$$

for every  $y \in [0, u]$ ,  $t > 0$ , which implies the theorem, as the quotient inside the limit is uniformly bounded for  $y \in [0, ct + u]$  and  $c \in ]0, 1[$ .  $\square$

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