A minmax theorem for concave-convex mappings with no regularity assumptions.

Perchet Vianney, Guillaume Vigeral

A minmax theorem for concave-convex mappings with no regularity assumptions.

Vianney Perchet* and Guillaume Vigeral†

January 10, 2014

Abstract

We prove that zero-sum games with a concave-convex payoff mapping defined on a product of convex sets have a value as soon as the payoff function is bounded and one of the set is bounded and finite dimensional. In particular, no additional regularity assumption is required, such as lower or upper semicontinuity of the function or compactness of the sets. We provide several examples that show that our assumptions are minimal.

Introduction

Classical min-max theorems (see in particular von Neumann [3], Fan [1] and Sion [4] or see Sorin [5] and the first chapter of Mertens-Sorin-Zamir [2] for a survey of zero-sum games) require, in addition to convexity (or variants of it), regularity assumptions on the mapping, such as lower or upper semi-continuity.

We consider in this short note zero-sum games with a concave-convex mapping defined on a product of convex sets. We give simple assumptions leading to existence of a value without assuming any additional regularity of the function.

Theorem 1. *Let X and Y be two nonempty convex sets and $f : X \times Y \to \mathbb{R}$ be a concave-convex mapping, i.e., $f(\cdot, y)$ is concave and $f(x, \cdot)$ is convex for every $x \in X$ and $y \in Y$. Assume that*

- *X is finite dimensional,*
- *X is bounded,*
- *$f(x, \cdot)$ is lower bounded for some x in the relative interior of X.*

*Université Paris-Diderot, LPMA, 8 place FM/13. 75013 Paris, France. email: `vianney.perchet@normalesup.org`

†Université Paris-Dauphine, CEREMADE, Place du Maréchal De Lattre de Tassigny. 75775 Paris cedex 16, France. email: `vigeral@ceremade.dauphine.fr`

Both authors are partially supported by the french Agence Nationale de la Recherche (ANR) "ANR JEUDY: ANR-10- BLAN 0112."

Then the zero-sum game on $X \times Y$ with payoff f has a value, i.e.,

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\sup_{x\in X} f(x,y).$$

Remark 2. *The assumptions of this theorem are satisfied in particular as soon as both X and Y are finite dimensional and bounded. Indeed, in that case $f(x,\cdot)$, being convex, is lower-bounded for any x*

Proof. Without loss of generality, we can assume that X has a non-empty interior in $\mathbb{R}^n$ that contains 0 and that $f(0,\cdot)$ is nonnegative. We define the following mapping λ_X from $\mathcal{S}^{n-1}$, the unit sphere of $\mathbb{R}^n$, into $\mathbb{R}_+$ by:

$$\forall p \in \mathcal{S}^{n-1}, \lambda_X(p) = \sup\Big\{\lambda \in \mathbb{R}_+ \text{ s.t. } \lambda p \in X\Big\}.$$

We claim that λ_X is bounded, continuous and bounded away from 0, i.e., $\inf_{p\in\mathcal{S}^{n-1}} \lambda_X(p) = \underline{\lambda} > 0$. The only thing to prove is the continuity; remark that since 0 belongs to $\mathring{X}$, the interior of X, then for every $\varepsilon > 0$ one has

$$(1-\varepsilon)\bar{X} \subset \mathring{X} \subset X \subset \bar{X} \text{ thus } (1-\varepsilon)\lambda_{\bar{X}} \leq \lambda_{\mathring{X}} \leq \lambda_X \leq \lambda_{\bar{X}} \text{ hence } \lambda_{\mathring{X}} = \lambda_X = \lambda_{\bar{X}},$$

where $\bar{X}$ is the closure of X. It is immediate that $\lambda_{\mathring{X}}$ is lower semi-continuous and $\lambda_{\bar{X}}$ is upper-semi continuous; this gives the continuity of λ_X.

Let $X_\varepsilon \subset X$ be an increasing sequence of compact sets containing 0 and included in the interior of X that converges to $\bar{X}$. Then the family $\lambda_\varepsilon := \lambda_{X_\varepsilon}$ is an increasing sequence of continuous mappings defined on a compact set that converges to the continuous mapping λ_X. As a consequence the convergence is uniform, and for every $\delta \in (0, \underline{\lambda})$, there exists ε such that

$$0 < \underline{\lambda} - \delta \leq \lambda_X(p) - \delta \leq \lambda_\varepsilon(p), \ \forall p \in \mathcal{S}^{n-1}. \tag{1}$$

Since X_ε is included in the interior of X, every mapping $f(\cdot, y)$ being concave is continuous on X_ε. Hence, classic minimax theorems [1, Theorem 2] yield that

$$\sup_{x\in X_\varepsilon}\inf_{y\in Y} f(x,y) = \inf_{y\in Y}\sup_{x\in X_\varepsilon} f(x,y) := v_\varepsilon. \tag{2}$$

A first direct implication of Equation (2) is that

$$\limsup_{\varepsilon\to 0} v_\varepsilon \leq \sup_{x\in X}\inf_{y\in Y} f(x,y). \tag{3}$$

A second consequence is that, for any $\eta > 0$ and ϵ there exists $y_\varepsilon \in Y$ such that $\sup_{x\in X_\varepsilon} f(x, y_\varepsilon) \leq v_\varepsilon + \eta$. Consider $x \in X$ that does not belong to X_ε, let $p_x = x/\|x\|$ and $x_\varepsilon \in X_\varepsilon$ defined by $x_\varepsilon = \lambda_\varepsilon(p_x)p_x$. Since $f(\cdot, y)$ is concave, one also has that

$$\frac{f(x,y) - f(x_\varepsilon,y)}{\|x - x_\varepsilon\|} \leq \frac{f(x_\varepsilon,y) - f(0,y)}{\|x_\varepsilon - 0\|} \leq \frac{f(x_\varepsilon,y)}{\|x_\varepsilon\|}.$$

By definition of x_ε, this immediately gives that either $f(x,y)$ is negative or

$$f(x,y) \le f(x_\varepsilon,y)\left(1+\frac{\|x\|-\lambda_\varepsilon(p_x)}{\lambda_\varepsilon(p_x)}\right) \le f(x_\varepsilon,y)\frac{\lambda_X(p_x)}{\lambda_X(p_x)-\delta} \le f(x_\varepsilon,y)\left(1+\frac{\delta}{\underline{\lambda}-\delta}\right)$$

for ε small enough, by Equation (1).

In particular, taking $y := y_\varepsilon$ and since $f(x_\varepsilon,y_\varepsilon) \le v_\varepsilon + \eta$, one gets that

$$f(x,y_\epsilon) \le (v_\varepsilon+\eta)\left(1+\frac{\delta}{\underline{\lambda}-\delta}\right),$$

since v_ε is nonnegative because $f(0,\cdot)$ is nonnegative and $0 \in X_\varepsilon$. Taking the supremum in x yields

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \sup_{x\in X} f(x,y_\varepsilon) \le (v_\varepsilon+\eta)\left(1+\frac{\delta}{\underline{\lambda}-\delta}\right).$$

Letting η, ε and then δ go to 0, we obtain

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \le \liminf_{\varepsilon\to 0} v_\varepsilon \tag{4}$$

Equations (3) and (4) give the result. □

We now prove that our three assumptions cannot be dispensed with, by the mean of the following examples.

- Assume that $X = Y = \Delta([0,1])$ and that f is given by the bilinear extension of

$$f(i,j) = \begin{cases} 0 & \text{if} \quad 0 = i < j \text{ or } 0 < j < i \\ 1 & \text{if} \quad 0 = j \le i \text{ or } 0 < i \le j \end{cases}, \qquad \forall i,j \in [0,1]. \tag{5}$$

  Then f is bounded and bilinear (and 1-Lipschitz with respect to the total variation distance). The sets X and Y are bounded, and infinite dimensional, but

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = 0 < \inf_{y\in Y}\sup_{x\in X} f(x,y) = 1.$$

- Assume that $X = Y = [1,+\infty)$ and define f by

$$f(x,y) = \begin{cases} \frac{x}{2y} & \text{if} \quad x \le y \\ 1-\frac{y}{2x} & \text{if} \quad x \ge y \end{cases} \tag{6}$$

  Then X and Y are both finite dimensional but unbounded, f is concave-convex and bounded (as well as $1/\sqrt{2}$-Lipschitz). But

$$\sup_{x\in X}\inf_{y\in Y} f(x,y) = 0 < \inf_{y\in Y}\sup_{x\in X} f(x,y) = 1.$$

– Assume that $X = [0, 1]$, $Y = \mathbb{R}_+$ and that

$$f(x, y) = \begin{cases} -xy & \text{if} \quad x > 0 \\ -y & \text{if} \quad x = 0 \end{cases} \tag{7}$$

Then X is bounded and finite dimensional, f is concave-linear and unbounded from below. But

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) = -\infty < \inf_{y \in Y} \sup_{x \in X} f(x, y) = 0.$$

Remark 3. *A consequence of these examples is that one can not weaken our assumptions even adding that f is Lipschitz.*

Remark 4. *Also recall that in the less demanding case of quasiconcave/convex mappings, regularity assumptions are necessary [4].*

References

[1] K. Fan (1953), Minimax Theorems. *Proc. Nat. Acad. Sci. USA* **39** 42–47

[2] J.-F. Mertens, S. Sorin and S. Zamir (2013), *Repeated Games.* to appear in Cambridge University Press.

[3] J. von Neumann (1928), Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen* **100** 295-320.

[4] M. Sion (1958) On general minimax theorems. *Pac. J. Math.* **8** 171–176

[5] S. Sorin (2002) *A First Course on Zero-Sum Repeated Games*, Springer, New York