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Joseph Salmon

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# Stable Recovery with Analysis Decomposable Priors

Jalal M. Fadili  
GREYC

CNRS-ENSICAEN-Univ. Caen  
Caen, France

Gabriel Peyré and Samuel Vaïter  
CEREMADE

CNRS-Univ. Paris Dauphine  
Paris, France

Charles-Alban Deledalle  
IMB

CNRS-Univ. Bordeaux 1  
Bordeaux, France

Joseph Salmon  
LTCI

CNRS-Télécom ParisTech  
Paris, France

**Abstract**—In this paper, we investigate in a unified way the structural properties of solutions to inverse problems regularized by the generic class of semi-norms defined as a decomposable norm composed with a linear operator, the so-called analysis decomposable prior. This encompasses several well-known analysis-type regularizations such as the discrete total variation, analysis group-Lasso or the nuclear norm. Our main results establish sufficient conditions under which uniqueness and stability to a bounded noise of the regularized solution are guaranteed.

## I. INTRODUCTION

**Problem statement** Suppose we observe

$$y = \Phi x_0 + w, \quad \text{where } \|w\| \leq \varepsilon,$$

where  $\Phi$  is a linear operator from  $\mathbb{R}^N$  to  $\mathbb{R}^M$  that may have a non-trivial kernel. We want to robustly recover an approximation of  $x_0$  by solving the optimization problem

$$x^* \in \underset{x \in \mathbb{R}^N}{\text{Argmin}} \frac{1}{2} \|y - \Phi x\|^2 + \lambda R(x), \quad \text{where } R(x) := \|L^* x\|_{\mathcal{A}}, \quad (1)$$

with  $L : \mathbb{R}^P \rightarrow \mathbb{R}^N$  a linear operator, and  $\|\cdot\|_{\mathcal{A}} : \mathbb{R}^P \rightarrow \mathbb{R}^+$  is a decomposable norm in the sense of [1]. Decomposable regularizers are intended to promote solutions conforming to some notion of simplicity/low complexity that complies with that of  $L^* x_0$ . This motivates the following definition of these norms.

**Definition 1.** A norm  $\|\cdot\|_{\mathcal{A}}$  is decomposable at  $\beta \in \mathbb{R}^P$  if there is a subspace  $T \subseteq \mathbb{R}^P$  and a vector  $e \in T$  such that

$$\partial \|\cdot\|_{\mathcal{A}}(\beta) = \left\{ u \in \mathbb{R}^P : \mathcal{P}_T(u) = e \quad \text{and} \quad \|\mathcal{P}_{T^\perp}(u)\|_{\mathcal{A}}^* \leq 1 \right\}$$

and for any  $z \in T^\perp$ ,  $\|z\|_{\mathcal{A}} = \sup_{v \in T^\perp, \|v\|_{\mathcal{A}}^* \leq 1} \langle v, z \rangle$ , where  $\|\cdot\|_{\mathcal{A}}^*$  is the dual norm of  $\|\cdot\|_{\mathcal{A}}$ ,  $\mathcal{P}_T$  (resp.  $\mathcal{P}_{T^\perp}$ ) is the orthogonal projector on  $T$  (resp. on its orthogonal complement  $T^\perp$ ).

Popular examples covered by decomposable regularizers are the  $\ell_1$ -norm, the  $\ell_1$ - $\ell_2$  group sparsity norm, and the nuclear norm.

**Contributions and relation to prior work** In this paper, we give sufficient conditions under which (1) admits a unique minimizer. Then we develop results guaranteeing that a stable approximation of  $x_0$  can be obtained from the noisy measurements  $y$  by solving (1), with an  $\ell_2$ -error that comes within a factor of the noise level  $\varepsilon$ . This goes beyond [1] which considered identifiability in the noiseless case, with  $L = \text{Id}$  and  $\Phi$  a Gaussian matrix.  $\ell_2$ -stability is also studied in [4] for  $L = \text{Id}$  under stronger sufficient assumptions than ours. Our results generalize the stability guarantee of [3] established when the decomposable norm is  $\ell_1$  and  $L$  is a frame. A general stability result for sublinear  $R$  is given in [2]. The stability is however measured in terms of  $R$ , and  $\ell_2$ -stability can only be obtained if  $R$  is coercive, i.e.,  $L^*$  is injective.

## II. UNIQUENESS

We first note that traditional coercivity and convexity arguments allow to show that the set of (global) minimizers of (1) is a non-empty compact set if and only if  $\ker(\Phi) \cap \ker(L^*) = \{0\}$ .

We shall now give a sufficient condition under which problem (1) admits exactly one minimizer. The following assumptions will play a pivotal role in our analysis throughout the paper.

**Assumption (SC<sub>x</sub>)** There exist  $\eta \in \mathbb{R}^M$  and  $\alpha \in \partial \|\cdot\|_{\mathcal{A}}(L^* x)$  such that the following so-called source condition is verified:

$$\Phi^* \eta = L \alpha \in \partial R(x).$$

**Assumption (INJ<sub>T</sub>)** Let  $T$  be the subspace in Definition 1 associated to  $L^* x$ .  $\Phi$  is injective on  $\ker(\mathcal{P}_{T^\perp} L^*)$ .

It is immediate to see that since  $\ker(L^*) \subseteq \ker(\mathcal{P}_{T^\perp} L^*)$ , (INJ<sub>T</sub>) implies that the set of minimizers is indeed non-empty and compact.

**Theorem 1.** For a minimizer  $x^*$  of (1), let  $T_*$  and  $e_*$  be the subspace and vector in Definition 1 associated to  $L^* x^*$ . Assume that (SC<sub>x\*</sub>) is verified with  $\|\mathcal{P}_{T_*^\perp}(\alpha)\|_{\mathcal{A}}^* < 1$ , and that (INJ<sub>T\*</sub>) holds. Then,  $x^*$  is the unique minimizer of (1).

## III. STABILITY TO NOISE

We are now ready to state our main stability result.

**Theorem 2.** Let  $T$  and  $e$  be the subspace and vector in Definition 1 associated to  $L^* x_0$ . Assume that (SC<sub>x<sub>0</sub></sub>) is verified with  $\|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^* < 1$ , and that (INJ<sub>T</sub>) holds. Then, for  $\lambda = c\varepsilon$

$$\|x^* - x_0\| \leq C\varepsilon,$$

where  $C = C_1(2 + c\|\eta\|) + C_2 \frac{(1+c\|\eta\|/2)^2}{c(1-\|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^*)}$ , and  $C_1 > 0$  and  $C_2 < 0$  are constants independent of  $\eta$  and  $\alpha$ .

In the following corollary, we provide a stronger sufficient stability condition. It will allow to construct good dual vectors  $\eta$  and  $\alpha$  that are computable, which in turn yield explicit constants in the bound. For this, suppose that (INJ<sub>T</sub>) is verified, and define  $\text{IC}(T, e) = \min_{u \in \ker(L\mathcal{P}_{T^\perp})} \|\Gamma^{[T^\perp]} e + \mathcal{P}_{T^\perp} u\|_{\mathcal{A}}$  with  $\Gamma^{[T^\perp]} = (L\mathcal{P}_{T^\perp})^+ (\Phi^* \Phi A^{[T^\perp]} - \text{Id}) L \mathcal{P}_T$  and  $A^{[T^\perp]} = U (U^* \Phi^* \Phi U)^{-1} U^*$ , and  $U$  is a matrix whose columns form a basis of  $\ker(\mathcal{P}_{T^\perp} L^*)$ . Note that  $\text{IC}(T, e)$  can be computed by solving a convex program. It also specializes to the criterion developed in [5] for the case of the  $\ell_1$  analysis prior.

**Corollary 1.** Assume that  $\text{IC}(T, e) < 1$ . Then, taking  $\eta = \Phi A^{[T^\perp]} L \mathcal{P}_T e$ , there exists  $\alpha$  such that (SC<sub>x<sub>0</sub></sub>) is satisfied. Moreover, the bound of Theorem 2 holds true substituting  $1 - \text{IC}(T, e)$  for  $1 - \|\mathcal{P}_{T^\perp}(\alpha)\|_{\mathcal{A}}^*$ .

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