Properties of Barabanov norms and extremal trajectories associated with continuous-time linear switched systems

Moussa Gaye, Yacine Chitour, Paolo Mason

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***Abstract*— Consider continuous-time linear switched systems on $\mathbb{R}^n$ associated with compact convex sets of matrices. When the system is irreducible and the largest Lyapunov exponent is equal to zero, a Barabanov norm always exists. This paper deals with two sets of issues: (a) properties of Barabanov norms such as uniqueness up to homogeneity and strict convexity; (b) asymptotic behaviour of the extremal solutions of the system. Regarding Issue (a), we provide partial answers and propose two open problems motivated by appropriate examples. As for Issue (b), we establish, when $n = 3$, a Poincaré-Bendixson theorem under a regularity assumption on the set of matrices defining the system.**

I. Introduction

We consider the linear switched system

$$\dot{x}(t) = A(t)x(t), \tag{1}$$

where $x \in \mathbb{R}^n$ and $A(\cdot)$ is any measurable function taking values on a compact and convex subset $\mathcal{M}$ of $\mathbb{R}^{n\times n}$ (the set of $n \times n$ real matrices) also called switching law. Associated with System (1), we define its largest Lyapunov exponent as

$$\rho(\mathcal{M}) := \sup\left(\limsup_{t\to+\infty} \frac{1}{t} \log \|x(t)\|\right), \tag{2}$$

where the supremum is taken over the set of solutions of (1) associated with any non-zero initial value and any switching law. It is well-known that the stability properties of (1) only depend on the sign of $\rho(\mathcal{M})$. Indeed, System (1) is asymptotically stable (i.e., there exist $\alpha > 0$ and $\beta > 0$ such that $\|x(t)\| \leq \alpha \exp(-\beta t)\|x(0)\|$ for every $t \geq 0$ and for every solution $x(\cdot)$ of System (1)) if and only if $\rho(\mathcal{M}) < 0$. On the other hand, (1) admits a solution which goes to infinity exponentially fast if and only if $\rho(\mathcal{M}) > 0$. Finally when $\rho(\mathcal{M}) = 0$ then either every solution of (1) starting on a bounded set remains uniformly bounded (in that case, System (1) is irreducible, see Definition 1 below) or System (1) admits a solution going to infinity. The notion of Joint Spectral Radius plays an analogous role on the description of the stability properties of discrete-time switched systems (cf. [9] and references therein).

Let us consider the subset $\mathcal{M}' := \{A - \rho(\mathcal{M})Id : A \in \mathcal{M}\}$ of $\mathbb{R}^{n\times n}$, where Id denotes the identity matrix of $\mathbb{R}^{n\times n}$ and the corresponding continuous-time switched system. Notice that trajectories associated with $\mathcal{M}$ and trajectories associated with $\mathcal{M}'$ only differ at time t by a scalar factor $e^{\rho(\mathcal{M})t}$ and thus, in order to understand the global behaviour of trajectories of System (1), one can always assume that $\rho(\mathcal{M}) = 0$, by eventually replacing $\mathcal{M}$ with $\mathcal{M}'$. Thus, this paper only deals with the case $\rho(\mathcal{M}) = 0$.

The most basic tool to analyze trajectories of System (1) is the concept of Barabanov norm (see [2], [13] and Definition 2 below), which is well defined for irreducible sets of matrices. In that case recall that the value of the Barabanov norm decreases along trajectories of (1) and, starting from every point $x \in \mathbb{R}^n$, there exists a trajectory of (1) along which a Barabanov norm is constant. Such a trajectory is called an *extremal trajectory* of System (1).

Characterizing the points where a Barabanov norm v is differentiable is a natural structural question. In general, we can only infer from the fact that v is a norm, that it is differentiable almost everywhere on its level sets. We will provide a sufficient condition for differentiability of v at a point $x \in \mathbb{R}^n$ in terms of the extremal trajectories reaching x.

Another interesting issue is that of the uniqueness of Barabanov norms up to homogeneity (i.e., for every Barabanov norms $v_1(\cdot)$ and $v_2(\cdot)$ there exists $\mu > 0$ such that $v_1(\cdot) = \mu v_2(\cdot)$). For discrete-time linear switched systems, the uniqueness of the Barabanov norms has been recently addressed (cf. [11], [12] and references therein) but up to our knowledge, no uniqueness result regarding continuous-time linear switched systems has been given until now. We provide a sufficient condition for uniqueness of the Barabanov norm up to homogeneity involving the ω-limit set of extremal trajectories. We also propose an open problem which is motivated by an example of a two-dimensional continuous-time linear switched system where one has an infinite number of Barabanov norms.

In the second part of the paper, we characterize the extremal trajectories by using the Pontryagin maximum principle. Similar results have been obtained in [2], [10]. We use the previous result to address another geometric issue associated with a Barabanov norm, namely that of the strict convexity of the corresponding unit ball. This seems to be a rather delicate task in the general case. As suggested by a simple example of a linear switched system admitting a Barabanov norm which is not strictly convex, the minimal requirement in order to guarantee strict convexity appears to be the assumption that $\mathcal{M}$ is made by non-singular matrices. We show that, under this assumption, the intersection of the Barabanov unit sphere with any hyperplane has empty relative interior, which implies in particular the strict convexity

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in the case $n = 2$. Another contribution is to show that under certain regularity condition on $\mathcal{M}$, the Barabanov balls are strictly convex.

The last part of the paper is devoted to the analysis of the asymptotic behaviour of the extremal solutions of System (1). Our first obtained result consists in a characterization of the extremal solutions of (1) by using the structure of linear differential inclusions. The second one is a Poincaré-Bendixson theorem for extremal solutions of System (1), namely, every extremal solution of System (1) tends to a periodic extremal solution of System (1). This result is obtained by making a regularity assumption on the set of matrices $\mathcal{M}$ (Condition G) which is slightly weaker than the analogous Condition C considered in [3].

The structure of the paper goes as follows. In Section II, we recall basic definitions of Barabanov norms, as well as some open questions regarding uniqueness up to homogeneity of the Barabanov norm and strict convexity of its unit ball. We provide our partial result on uniqueness up to homogeneity in Section III and, in Section IV, we provide a characterization of extremal trajectories (similar to that of [2]) by using the Pontryagin maximum principle. In Section V, we collect some results regarding the strict convexity of a Barabanov ball and state our Poincaré-Bendixson result in Section VI. Most of the proofs are only sketched and will appear in a complete form in a forthcoming paper by the authors.

II. Barabanov norms

In this section, we collect well known results and some open problems for which we provide partial answers in the following sections.

Definition 1: We say that System (1) is *reducible* if there exists a proper subspace of $\mathbb{R}^n$ invariant with respect to every matrix $A \in \mathcal{M}$. Otherwise, System (1) is said to be *irreducible*.

We define the function $v(\cdot)$ on $\mathbb{R}^n$ as

$$v(y) := \sup_{x(\cdot):x(0)=y} \left(\limsup_{t \to +\infty} \|x(t)\| \right), \tag{3}$$

where the supremum is taken over all solutions of (1) starting at $y \in \mathbb{R}^n$. By $\|\cdot\|$, we mean the Euclidean norm on $\mathbb{R}^n$.

From [2], we have the following fundamental result.

Theorem 1 ([2]): Assume that $\rho(\mathcal{M}) = 0$. Then the function $v(\cdot)$ defined in (3) is a norm on $\mathbb{R}^n$ with the following properties:

1. For every solution $x(\cdot)$ of (1) we have that $v(x(t)) \leq v(x(0))$ for every $t \geq 0$,
2. For every $y \in \mathbb{R}^n$, there exists a solution $x(\cdot)$ of (1) starting at y such that $v(x(t)) = v(x(0))$ for every $t \geq 0$.

Definition 2: A norm on $\mathbb{R}^n$ satisfying Condition 1. and Condition 2. of Theorem 1 is called a *Barabanov norm.*

A solution $x(\cdot)$ of (1) is said to be v-*extremal* (or simply *extremal* whenever the choice of the Barabanov norm is clear) if $v(x(t)) = v(x(0))$ for every $t \geq 0$.

In this paper we will be concerned with the study of properties of Barabanov norms and extremal trajectories. Thus we will always assume that

$$\mathcal{M} \text{ is irreducible and } \rho(\mathcal{M}) = 0.$$

However, as stressed in the introduction, the study of extremal trajectories in the case $\rho(\mathcal{M}) = 0$ turns out to be useful for the analysis of the dynamics in the general case. Note that in the case in which $\rho(\mathcal{M}) = 0$ and $\mathcal{M}$ is reducible the system could even be unstable. A description of such instability phenomena has been studied in [7].

Since under the previous assumptions and according to the Theorem 1 a Barabanov norm always exists, our first question, for which a partial answer will be given in Section III, is the following.

Open problem 1: ***Under which conditions, the Barabanov norm is unique up to homogeneity, i.e., for every Barabanov norm $w(\cdot)$ there exists $\lambda > 0$ such that $w(\cdot) = \lambda v(\cdot)$ where $v(\cdot)$ is the Barabanov norm defined in (3)?***

An important problem is also to understand under which hypotheses on $\mathcal{M}$ the supremum in (3) is attained, for every $y \in \mathbb{R}^n$, by a solution $x(\cdot)$ of System (1). If this is the case, that solution would necessarily be an extremal solution of System (1). Hence the analysis of the Barabanov norm defined in (3) would only depend on the asymptotic behaviour of extremal solutions of System (1). However one can note that this problem is not trivial as shown by the following example, for which the supremum in (3) is not attained.

Example 1. Suppose that $\mathcal{M} := conv\{A, B\}$ with

$$A = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} \alpha & 3 \\ -0.6 & 0.7 \end{pmatrix},$$

where $\alpha \sim 0.8896$ is such that $\rho(\mathcal{M}) = 0$. It is easy to see that the closed curve constructed in Figure 1 by gluing together four trajectories of the system is the level set V of a Barabanov norm, according to Definition 2. It is easy to see that the extremal trajectories of the system tend either to $(1, 0)^T$ or to $(-1, 0)^T$. On the other hand it is possible to construct trajectories of the system starting from V, turning around the origin an infinite number of times and staying arbitrarily close to V. One deduces that the Barabanov norm on V defined in (3) is equal to the maximum of the Euclidean norm on V, which is strictly bigger than 1.

Note that the matrix A in the previous example is singular. It is actually possible to see, by using for instance the results of [1], [5], that for $n = 2$, whenever $\mathcal{M} := conv\{A, B\}$, A, B are non-singular and $\rho(\mathcal{M}) = 0$, the supremum is always attained. This justifies the following question.

Open problem 2: ***Assume that $\mathcal{M}$ is made of non-singular $\mathbb{R}^{n \times n}$ matrices. Is it true that for every $y \in \mathbb{R}^n$ the supremum in (3) is achieved?***

Also, one of the features of Example 1 is that the Barabanov unit ball (or, equivalently, the Barabanov norm) is not strictly convex since the Barabanov unit sphere contains segments. This may come from the fact that the matrix A is singular. Hence we ask the following question, for which partial answers are collected in Section V.

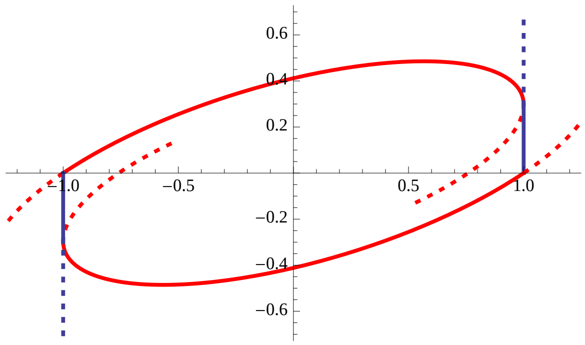

Fig. 1. Example where the supremum in (3) is not attained and the Barabanov norm is not strictly convex.

Open problem 3: *Assume that $\mathcal{M}$ is made by non-singular $\mathbb{R}^{n\times n}$ matrices. Is it true that the Barabanov balls are strictly convex?*

Regularity of Barabanov norms is also an interesting and natural issue. In the last part of this section, we give a result of independent interest on the differentiability of the Barabanov norm defined in (3).

Notation 1: We use S and B to denote the Barabanov unit sphere $\{x \in \mathbb{R}^n : v(x) = 1\}$ and the Barabanov unit ball $\{x \in \mathbb{R}^n : v(x) \leq 1\}$, respectively, where v is defined in Eq. (3).

For $x_0 \in S$ we define the subset $\mathcal{V}_{x_0}$ of $\mathbb{R}^n$ by

$$\mathcal{V}_{x_0} := \left\{ l \in \mathbb{R}^n : \begin{array}{c} \exists t_0 > 0, \exists x(\cdot) \text{ extremal with} \\ x(t_0) = x_0, \{t_j\}_{j\geq 1} \text{ s.t. } \lim_{j\to+\infty} t_j = t_0^- \\ \text{and } l = \lim_{j\to+\infty} \frac{x(t_j)-x_0}{t_j - t_0}. \end{array} \right\}. \quad (4)$$

Remark 1: The set $\mathcal{V}_{x_0}$ can be empty since Theorem 1 does not guarantee the existence of extremal trajectories extremal trajectory reaching x^0.

Notation 2: Let $u, w, x \in \mathbb{R}^n$. We denote $u^T w$ and $\partial v(x)$ the usual scalar product of u and w and the subdifferential of $v(\cdot)$ at x respectively.

The next proposition provides a sufficient condition for the differentiability of $v(\cdot)$ at x_0.

Proposition 1: Assume that $\mathcal{V}_{x_0}^{t_0}$ is not empty and it contains $(n-1)$ linearly independent elements. Then $v(\cdot)$ is differentiable at x_0.

Sketch of the proof: It is not difficult to show that $\mathcal{V}_{x_0}^{t_0} \subset \mathcal{M}x_0$. From this fact one can show that if $l_0 \in \partial v(x_0)$ and $y \in \mathcal{V}_{x_0}^{t_0}$ then $l_0^T y = 0$. Therefore, under the assumptions of Proposition 1, if l_1 and l_2 are elements of $\partial v(x_0)$ then $l_1 = l_2$. Hence, the differentiability of $v(\cdot)$ at x_0 is proved. ■

Notice that in the particular case $n = 2$ the previous result states that $v(\cdot)$ is differentiable at any point reached by an extremal trajectory. For $n = 3$ differentiability at x_0 is instead guaranteed if there are two extremal trajectories reaching x_0 from two different directions.

III. Uniqueness of Barabanov norms

From the definition of a Barabanov norm, it is clear that if $v(\cdot)$ is a Barabanov norm, then $\lambda v(\cdot)$ is also a Barabanov norm. Therefore, uniqueness of Barabanov norms can only hold up to homogeneity. In this section we provide conditions under which System (1) admits a unique Barabanov norm, up to homogeneity.

In order to state these conditions, we need to consider the union of all possible ω-limits of extremal trajectories on S,

$$\Omega := \bigcup_{\{x(\cdot):x(t)\in S\}} \omega(x(\cdot)). \quad (5)$$

Theorem 2: Assume that there exists a dense subset $\hat{\Omega}$ of Ω such that for every $z_1, z_2 \in \hat{\Omega}$ one can find an integer $N > 0$ and extremal trajectories $x_1(\cdot), \ldots, x_N(\cdot)$ with $z_1 \in \omega(x_1(\cdot)), z_2 \in \omega(x_N(\cdot))$ and $\omega(x_i(\cdot)) \cap \omega(x_{i+1}(\cdot)) \neq \emptyset$ for $i = 1, \ldots, N-1$. Then the Barabanov norm is unique, up to homogeneity.

Sketch of the proof: Let $v_1(\cdot)$ and $v_2(\cdot)$ be two Barabanov norms for System (1). Without loss of generality we identify $v_1(\cdot)$ with the Barabanov norm $v(\cdot)$ defined by (3). We define

$$\bar{\lambda} := \inf\{\lambda > 0 : {v_1}^{-1}(1) \subset {v_2}^{-1}([0, \lambda])\}. \quad (6)$$

Notice first that $\bar{\lambda}$ is well defined, it is a minimum and ${v_1}^{-1}(1) \cap {v_2}^{-1}(\bar{\lambda})$ is non-empty. Then consider $x_0 \in {v_1}^{-1}(1) \cap {v_2}^{-1}(\bar{\lambda})$ and $\hat{x}(\cdot)$ a v_2-extremal starting at x_0. One therefore gets that $\omega(\hat{x}(\cdot)) \subset {v_1}^{-1}(1) \cap {v_2}^{-1}(\bar{\lambda})$. Moreover, under the hypotheses of Theorem 2, $v_2(\cdot)$ is constant on Ω and its value is $\bar{\lambda}$. Hence one can prove that $v_1^{-1}(1) = v_2^{-1}(\bar{\lambda})$, which concludes the proof. ■

From the previous result one gets the following consequence.

Corollary 1: Assume that there exists a finite set of extremal trajectories $x_1(\cdot), \cdots, x_N(\cdot)$ on S such that $\Omega = \cup_{i=1,\cdots,N}\omega(x_i(\cdot))$ and Ω is connected. Then the Barabanov norm is unique, up to homogeneity.

Remark 2: The assumptions of the previous result are verified for instance when the set Ω is formed by a unique limit cycle (but this is not the only case, as shown in the example below).

Example 2. As in [3, Example 1], let $\mathcal{M} := conv\{A_1, A_2\}$

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The system is stable but not asymptotically stable (it admits $\|x\|^2$ as a Lyapunov function and there are periodic trajectories) and irreducible. Thus the Barabanov norm in (3) is well defined. Note that the Barabanov sphere S must contains the two cycles defined by $x_1^2 + x_2^2 = 1,\ x_3 = 0$ and $x_1^2 + x_3^2 = 1,\ x_2 = 0$.

Now let us see that the Barabanov norm is unique up to homogeneity. We claim that the ω-limit of any extremal trajectory is contained in the union of the cycles defined above. Since such cycles coincide with the intersection of the sphere S with the planes $x_3 = 0$ and $x_2 = 0$ it is enough to show that $\min\{|x_2(t)|, |x_3(t)|\}$ converges to 0 as t goes to infinity. Indeed, let $V(x) := \|x\|^2$. Then a simple computation leads to $\dot{V}(x(t)) \leq -\min\{x_2(t)^2, x_3(t)^2\}$ and, since V is positive definite, $F(t) := \int_0^t \min\{x_2(\tau)^2, x_3(\tau)^2\}d\tau$ is a (monotone) bounded function. Since $x(t) \in B$ for every $t \geq 0$ then $F'(t) = \min\{x_2(t)^2, x_3(t)^2\}$ is uniformly continuous. Hence by Barbalat's lemma we have that $\lim_{t\to\infty} \min\{x_2(t)^2, x_3(t)^2\} = 0$, which proves the claim. The hypotheses of Theorem 2 are then satisfied.

Remark 3: An adaptation of the above result can be made under the weaker assumption that the union of all possible ω-limits of extremal trajectories on a Barabanov sphere is a union $\Omega \cup -\Omega$ where Ω is a connected set satisfying the assumptions of the theorem. However up to now we do not know any example satisfying this generalized assumption.

Open problem 4: *Is it possible to weaken the assumption of Theorem 2, at least when $n = 3$, by just asking that Ω is connected?*

Besides the cases studied in this section, the uniqueness of the Barabanov norm for continuous-time switched systems remains an open question. The main difficulty is given by the fact that Barabanov norms are usually not computable, especially for systems of dimension larger than two. For $n = 2$ a simple example of non-uniqueness of the Barabanov norm is the following.

Example 3: Let $\mathcal{M} := conv\{A_1, A_2, A_3\}$ with

$$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix},\ A_3 = \begin{pmatrix} -\alpha & 1 \\ -1 & -\alpha \end{pmatrix}.$$

Then it is easy to see that, taking $\alpha \geq 1$, any norm $v_\beta(x) := \max\{|x_1|, \beta|x_2|\}$ with $\beta \in [\frac{1}{\alpha}, \alpha]$ is a Barabanov norm of the system. Moreover, one can show that the Barabanov norm defined in Eq. (3) is equal to v_1 and the corresponding ω-limit set defined in Eq. (5) reduces to the four points $(-1, 0)$, $(0, 1)$, $(1, 0)$ and $(0, -1)$, which is clearly disconnected. Note that $v_\beta(\cdot)$ is a Barabanov norm even for the system corresponding to $conv\{A_1, A_2\}$, which is reducible.

IV. The adjoint system

For simplicity of notations, in the following sections we will deal with the Barabanov norm $v(\cdot)$ defined in (3), although the results do not depend on the choice of the Barabanov norm. In this section we characterize extremal trajectories by means of the Pontryagin maximum principle (similar results can be found for instance in [2], [10]). Given a measurable function $A(\cdot)$ taking values in $\mathcal{M}$, we define the adjoint system associated with (1) as

$$\dot{l}(t) = -A^T(t)l(t). \tag{7}$$

We use $A^T(t)$ to denote the matrix transpose of $A(t)$.

Theorem 3: Let $x(\cdot)$ be an extremal solution of (1) associated with $\hat{A}(\cdot)$, $T \geq 0$ and $\hat{l} \in \partial v(x(T))$. Then there exists a non-zero solution $l(\cdot)$ of (7) associated with $\hat{A}(\cdot)$ such that for every $t \in [0, T]$ the following statements hold true:

$$l(t) \in \partial v(x(t)), \quad l(T) = \hat{l}, \tag{8}$$

$$\max_{A\in\mathcal{M}} l^T(t)Ax(t) = l^T(t)\hat{A}(t)x(t) = 0. \tag{9}$$

Proof: Let $x(\cdot)$ be as in the statement of the theorem and fix $T \geq 0$ and $\hat{l} \in \partial v(x(T))$. We consider the following optimal control problem in Mayer form (see for instance [6])

$$\max \hat{l}^T z(\tau), \tag{10}$$

among trajectories $z(\cdot)$ of (1) satisfying $z(0) = x_0$ and with free final time $\tau \geq 0$. Then, the pair $(x(\cdot), \hat{A}(\cdot))$ is an optimal solution of Problem (10). Indeed, let $z(\cdot)$ be a solution of (1) defined in $[0, \tau]$ such that $z(0) = x_0$. Since $\hat{l} \in \partial v(x(T))$, then $v(z(\tau)) - v(x(T)) \geq \hat{l}^T(z(\tau) - x(T))$. Since $v(z(\tau)) \leq v(z(0)) = v(x_0) = v(x(T))$, one gets $\hat{l}^T z(\tau) \leq \hat{l}^T x(T)$.

Consider the hamiltonian family $h_A(z, l) = l^T A z$ where $(z, l) \in \mathbb{R}^n \times \mathbb{R}^n$ and $A \in \mathcal{M}$. Then the Pontryagin maximum principle ensures the existence of a nonzero Lipschitz map $l(\cdot) : [0, T] \to \mathbb{R}^n$ satisfying the following properties:

1) $\dot{l}(t) = -\frac{\partial h_{\hat{A}(t)}}{\partial z}(x(t), l(t)) = -\hat{A}^T(t)l(t)$,
2) $l^T(t)\hat{A}(t)x(t) = \max_{A\in\mathcal{M}} l^T(t)Ax(t) = 0,\ \forall t \in [0, T]$,
3) $\hat{l} := \nabla\varphi(z(T)) = l(T)$ where $\varphi(z) = \hat{l}^T z$.

To conclude the proof of Theorem 3, it is enough to show that $l(t) \in \partial v(x(t))$ for all $t \in [0, T]$. Indeed fix $t \in [0, T]$, $x \in \mathbb{R}^n$ and let $y(\cdot)$ be a solution of System (1) such that $\dot{y}(\tau) = \hat{A}(\tau)y(\tau)$ with initial data $y(t) = x$. Then one has

$$\begin{aligned} v(x) - v(x(t)) &\geq v(y(T)) - v(x(t)) = v(y(T)) - v(x(T)), \\ &\geq l^T(T)(y(T) - x(T)) = l^T(t)(y(t) - x(t)), \end{aligned}$$

since $v(y(T)) \leq v(y(t)) = v(x)$, $l(T) \in \partial v(x(T))$ and the function $l^T(t)(y(t) - x(t))$ is constant on $[0, T]$. Hence, $v(x) - v(x(t)) \geq l^T(t)(x - x(t))$. Since $t \in [0, T]$ and $x \in \mathbb{R}^n$ are arbitrary, one deduces that $l(t) \in \partial v(x(t))$ for all $t \in [0, T]$. This proves Theorem 3. ■

As a simple consequence of the previous theorem one obtains the following result valid for all positive time.

Theorem 4: For every extremal solution $x(\cdot)$ of (1) associated with a switching law $A(\cdot)$, there exists a nonzero solution $l(\cdot)$ of (7) associated with $A(\cdot)$ such that for $t \geq 0$,

$$l(t) \in \partial v(x(t)), \tag{11}$$

$$\max_{A\in\mathcal{M}} l^T(t)Ax(t) = l^T(t)A(t)x(t) = 0. \tag{12}$$

We now introduce an assumption on the set of matrices that will be useful in the following results. Note that Barabanov introduced in [3] a similar but slightly stronger condition (denoted as *Condition C*) and showed some examples for which the condition is verified.

Condition G: For every non-zeros $x_0 \in \mathbb{R}^n$ and $l_0 \in \mathbb{R}^n$, the solution $\big(x(\cdot), l(\cdot)\big)$ of (1)-(7) such that $\dot{x}(t) = A(t)x(t)$, $\dot{l}(t) = -A^T(t)l(t)$, $\big(x(0), l(0)\big) = (x_0, l_0)$ for some $A(\cdot)$ and satisfying $\max_{A\in\mathcal{M}} l^T(t)Ax(t) = l^T(t)A(t)x(t)$ for every $t \geq 0$ is unique.

Remark 4: Unlike Condition C in [3], we do not ask here that the uniqueness property in Condition G is also valid when $\mathcal{M}$ is replaced by $\mathcal{M}^T := \{A^T : A \in \mathcal{M}\}$.

The following results are corollaries of Theorem 4.

Proposition 2: Let $x(\cdot)$ be an extremal solution of (1) starting at some point of differentiability x_0 of $v(\cdot)$. Then the following results hold true:

1) The norm $v(\cdot)$ is differentiable at $x(t)$ for every $t \geq 0$,
2) The solution $l(\cdot)$ of (7) satisfying Conditions (11)-(12) of Theorem 4 is unique and $l(t) = \nabla v(x(t))$ for $t \geq 0$,
3) Moreover, if Condition G holds true, then $x(\cdot)$ is the unique extremal solution of (1) starting at x_0.

Definition 3: We say that two extremal solutions $x_1(\cdot)$ and $x_2(\cdot)$ of System (1) intersect each other if there exist $t_1, t_2 > 0$ and $\epsilon > 0$ such that $x_1(t_1) = x_2(t_2)$ and $x_1(s_1) \neq x_2(s_2)$ for every $s_1 \in [t_1 - \epsilon, t_1)$ and $s_2 \in [t_2 - \epsilon, t_2)$.

Proposition 3: Assume that $n = 3$ and every matrix of $\mathcal{M}$ is non-singular. If two extremal solutions $x_1(\cdot)$ and $x_2(\cdot)$ of (1) intersect each other at some $z \in \mathbb{R}^3$, then $v(\cdot)$ is differentiable at z. If Condition G holds, one has forwards uniqueness for extremal trajectories starting from z.

Note that the previous result is not a direct consequence of Propositions 1 and 2 since here we are not asking that the trajectories $x_1(\cdot)$ and $x_2(\cdot)$ reach z with different directions.

Corollary 2: Assume that the hypotheses of Proposition 3 and Condition G hold true. Let Γ be a cycle on S and let S_1 and S_2 denote the two connected components of $S \setminus \Gamma$. If $z(\cdot)$ is an extremal solution on S such that $z(0) \in S_1$ then $z(t) \in S_1 \cup \Gamma$ for all $t \geq 0$. Moreover if $z(t_*) \in \Gamma$ for some t_* then $z(t) \in \Gamma$ for $t \geq t_*$.

V. Strict convexity of Barabanov balls

In this section, we focus on the strict convexity of Barabanov balls. In the following, given two points $x_0, x_1 \in \mathbb{R}^n$ we will indicate as (x_0, x_1) the open segment connecting x_0 with x_1. Similarly, we will use the bracket symbols "[" and "]" to denote left and right closed segments.

Proposition 4: Assume that $\mathcal{M}$ is a convex compact irreducible subset of $\mathbb{R}^{n\times n}$, not containing singular matrices and $\rho(\mathcal{M}) = 0$. Then, the intersection of the Barabanov unit sphere S with any hyperplane P has empty relative interior in P.

The proof is based on the following intermediate results.

Lemma 1: Let $x_0, x_1 \in S$ such that $x_0 \neq x_1$ and $R(\cdot)$ an evolution operator associated with some switching law $A(\cdot)$, i.e., $\dot{R}(t) = A(t)R(t)$ and $R(0) = Id$. Suppose that the open segment $(R(t)x_0, R(t)x_1)$ intersects S for some $t \geq 0$. Then the whole segment $(R(t)x_0, R(t)x_1)$ belongs to S.

Proof: Let $z \in S \cap (R(t)x_0, R(t)x_1)$, i.e. $z = \alpha R(t)x_0 + (1-\alpha)R(t)x_1$ for some $\alpha \in (0,1)$. Let $\hat{z} = \beta R(t)x_0 + (1-\beta)R(t)x_1$ where $\beta \in (0,1)$. We first have $v(\hat{z}) \leq 1$. Indeed, $v(\hat{z}) \leq \beta v(R(t)x_0) + (1-\beta)v(R(t)x_1) \leq 1$. Let us prove that $v(\hat{z}) \geq 1$. Since $\hat{z} \in \big(R(t)x_0, R(t)x_1\big)$, then either $z \in [\hat{z}, R(t)x_1)$ or $(R(t)x_0, \hat{z}]$. If $z \in [\hat{z}, R(t)x_1)$, then $1 = v(z) \leq \alpha v(\hat{z}) + (1-\alpha)v(R(t)x_1) \leq \alpha v(\hat{z}) + 1 - \alpha$, implying that $v(\hat{z}) \geq 1$. The second case can be treated with the same arguments. ■

Lemma 2: Let $x_0, x_1 \in S$ such that $x_0 \neq x_1$ and define $x_\lambda := \lambda x_1 + (1-\lambda)x_0 \ \forall \lambda \in [0,1]$. Assume $x_{\bar\lambda} \in S$ for some $\bar\lambda \in (0,1)$ and let $\gamma_{\bar\lambda}(t) := R(t)x_{\bar\lambda}$ be an extremal solution of System (1) with $\dot{R}(t) = A(t)R(t)$ and $R(0) = Id$ for some $A(\cdot)$. Then, $R(t)x_\mu \in S \ \forall \mu \in [0,1]$ and $\forall t \geq 0$.

Proof: One has that $\gamma_{\bar\lambda}(t) = \bar\lambda R(t)x_1 + (1-\bar\lambda)R(t)x_0$ for every $t \geq 0$. Therefore for every $t \geq 0$, we have

$$1 = v(\gamma_{\bar\lambda}(t)) \leq \bar\lambda v(R(t)x_1) + (1-\bar\lambda)v(R(t)x_0) \leq \bar\lambda v(x_1) + (1-\bar\lambda)v(x_0) = 1.$$

Hence $v(R(t)x_1) = 1$ and $v(R(t)x_0) = 1$ for every $t \geq 0$. According to Lemma 1, we get the conclusion. ■

Proof of Proposition 4: We are now ready to end the proof of Proposition 4. Clearly the conclusion is true if $0 \in P$. So assume that the conclusion is false for some P with $0 \notin P$. Let x_0 be a point in the interior of $P \cap S$ admitting an extremal trajectory $R(t)x_0$ differentiable at x_0 and with $\dot{R}(0) = A \in \mathcal{M}$. By Lemma 1 for any $x \in S \cap P$ there exists a segment in $S \cap P$ connecting x and x_0 and containing x_0 in its interior. Then Lemma 2 says that $R(t)x \in S \cap P$ for small t and for all x in the interior of $S \cap P$, which implies that Ax is tangent to P, that is $l^T Ax = 0$ where l is orthogonal to P. But then this is also true on a cone of $\mathbb{R}^n$ with non-empty interior and thus on the whole $\mathbb{R}^n$, which is impossible since A is non-singular. This proves Proposition 4. ■

The following corollary of Proposition 4 provides an answer to the **Open Problem 3** when $n = 2$.

Corollary 3: If $n = 2$ and the hypotheses of Proposition 4 hold true, then the Barabanov balls are strictly convex.

The analysis of strict convexity of Barabanov balls appears to be a very complicated task in the general case. The following result shows that the Barabanov unit ball is strictly convex under some regularity condition on $\mathcal{M}$.

Theorem 5: If $\mathcal{M}$ is a $\mathcal{C}^1$ domain of $\mathbb{R}^{n\times n}$, irreducible and with $\rho(\mathcal{M}) = 0$, then Barabanov balls are strictly convex.

Sketch of the proof: By contradiction assume that there exist $x_0, x_1 \in S$ and $\lambda \in (0,1)$ such that $x_\lambda := \lambda x_1 + (1-\lambda)x_0 \in S$. For every $(x, l) \in S \times (\mathbb{R}^n \setminus \{0\})$, we define the linear functional $\phi_{(x,l)}(A) = l^T Ax$ on $\mathbb{R}^{n\times n}$. Notice that $\forall A \in \mathcal{M}$ there exists at most one supporting hyperplane of $\mathcal{M}$ at A of the form $\big\{B \in \mathbb{R}^{n\times n} : \phi_{(\bar{x}(A), \bar{l}(A))}(B) = 0\big\}$ for some $(\bar{x}(A), \bar{l}(A)) \in S \times (\mathbb{R}^n \setminus \{0\})$. One has also that $(\bar{x}(A), \bar{l}(A))$ is uniquely defined if it exists. Let $x_\lambda(t) = R(t)x_\lambda$ be an extremal solution. Then $x_\mu(t) := R(t)x_\mu \in S \ \forall \mu \in [0,1]$ and for all $t \geq 0$ where x_μ is defined in Lemma 2. Therefore one gets $x_\mu(t) = \pm\bar{x}(A(t))$ for a.e. t and $\forall \mu$ which implies that $x_0(t) = \pm x_1(t)$ for t sufficiently small. Since $[x_0, x_1] \subset S$, then $x_0 = x_1$. Hence the contradiction proves Theorem 5. ■

VI. A Poincaré-Bendixson Theorem for extremal solutions

In this section we first show a characterization of the extremal flows in the framework of linear differential inclusions. We then prove a Poincaré-Bendixson theorem for extremal solutions of System (1). A similar result for a very particular case has been given in [4].

Definition 4: A multifunction $F(\cdot) : p \mapsto F(p)$ is a mapping defined on a subset D of $\mathbb{R}^m$ for some positive integer m such that $\emptyset \neq F(p) \subset \mathbb{R}^m$ is closed $\forall p \in D$.

We define $\alpha(A, B) = \sup_{a\in A} d(a, B)$ for every closed sets A and B of $\mathbb{R}^m$ where $d(a, B)$ is the Euclidean distance between the point $a \in A$ and the set B.

We say that $F(\cdot)$ is upper semicontinuous at $p \in D$ if $\lim_{q\to p} \alpha(F(q), F(p)) = 0$. If $F(\cdot)$ is upper semicontinuous $\forall p \in D$, we say that $F(\cdot)$ is upper semicontinuous on D.

Lemma 3: For every $x \in \mathbb{R}^n$, define $F(x) := \{Ax : A \in \mathcal{M}\}$. Then $F(\cdot)$ is upper semicontinuous on $\mathbb{R}^n$ and $F(x)$ is non-empty, closed, bounded and convex for every $x \in \mathbb{R}^n$.

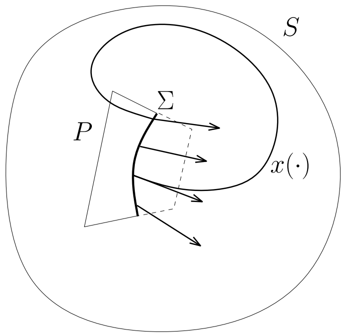

Fig. 2. Transverse section and proof of Poincaré-Bendixson theorem.

Consider now the linear differential inclusion

$$\dot{x} \in \hat{F}(x) := \{Ax : A \in \mathcal{M}, A \text{ verifies } \mathcal{P}(x)\}, \tag{13}$$

where we say that A verifies $\mathcal{P}(x)$ if there exists $l \in \partial v(x)$ such that $l^T Ax = \max_{B \in \mathcal{M}} l^T Bx = 0$.

Definition 5: A solution $x(\cdot)$ of System (1) is said to be solution of System (13) if $\dot{x}(t) \in \hat{F}(x(t))$ for a.e. t in the domain of $x(\cdot)$.

Based on Theorem 4 it is not difficult to show the following result, which provides a necessary and sufficient condition for a solution of System (1) to be extremal.

Proposition 5: The solutions of System (13) coincide with the extremal solutions of System (1).

In the last part of this section, we focus on the asymptotic behaviour of the extremal solutions of System (1), and in particular we state a Poincaré-Bendixson theorem for extremal trajectories. For this purpose from now on we will assume $n = 3$, so that extremal trajectories live on a two-dimensional (Lipschitz) surface.

Remark 5: Let us notice that classical Poincaré-Bendixson results for planar differential inclusions (see e.g. [8, Theorem 3, page 137]) do not apply in our case, since, beside the fact that our system is defined on a non-smooth manifold instead of $\mathbb{R}^2$, we cannot ensure that some classical requirements such as the convexity of $\hat{F}(x)$ or the upper semicontinuity of $\hat{F}(\cdot)$ are satisfied.

Definition 6: A transverse section Σ of S for System (13) is a connected subset of the intersection of S with a plane P such that for every $x \in \Sigma$ and $u \in \hat{F}(x)$ we have $w^T u > 0$ where w is an orthogonal vector to P (see Figure 2).

Definition 7: We say that $b \in S$ is a stationary point of System (1) if $0 \in F(b)$. Otherwise, we say that b is a nonstationary point.

The following lemma allows one to follow a strategy which is similar to the classical one in order to prove a Poincaré-Bendixson result.

Lemma 4: For every nonstationary point $b \in S$, there exists a local transverse section Σ of S containing b.

We now state our Poincaré-Bendixson result for extremal solutions of System (1).

Theorem 6: Assume that $n = 3$, Condition G holds true and every matrix of $\mathcal{M}$ is non-singular. Then every extremal solution of System (1) tends to a periodic solution of (1).

Sketch of the proof: Let $x_0 \in S$ and $x(\cdot)$ be an extremal solution of (1) starting at x_0. Notice first that one can assume without loss of generality that $x(t_1) \neq x(t_2)$ for every $t_1 \neq t_2$, otherwise, by Corollary 2, $x(\cdot)$ is periodic on $[T, +\infty)$ for some $T \geq 0$ and the theorem is proved. By using a classical procedure to prove Poincaré-Bendixson theorem based on Jordan separation theorem (see also Figure 2), one can show that the ω-limit set $\omega(x(\cdot))$ of the solution $x(\cdot)$ is the union of periodic extremal trajectories $x_b(\cdot)$ on S passing through $b \in \omega(x(\cdot))$. We then define $\Gamma_b := \{x_b(t) : t \geq 0\}$, $\Gamma := \{x(t) : t \geq 0\}$ and, by Corollary 2, we assume without loss of generality that $\Gamma \cap \Gamma_b = \emptyset$ for every b. It is possible to show that there is at most a finite number of distinct Γ_b. Hence by connectedness of $\omega(x(\cdot))$ and the fact that such sets are pairwise disjoint and closed, one concludes easily. ■

VII. Conclusion

In this paper several questions concerning Barabanov norms and extremal trajectories have been addressed and some partial answers have been provided. In particular, uniqueness of Barabanov norms up to homogeneity has been studied as well as the problem of finding sufficient conditions for the strict convexity of Barabanov balls. In addition, a characterization of extremal trajectories by means of Pontryagin maximum principle has been obtained and we provided a Poincaré-Bendixson theorem for extremal solutions under an additional regularity condition. Several questions remain substantially open and will be addressed in future works.

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