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AMALGAMATED PRODUCTS OF C*-BUNDLES

ÉTIENNE BLANCHARD

Abstract. We describe which classical amalgamated products of continuous C*-bundles are continuous C*-bundles and we analyse the involved extension problems for continuous C*-bundles.

Introduction

Different (fibrewise) amalgamated products of continuous C*-bundles have been studied over the last years ([1], [8], [6], [4]), one of the main questions being to know when these amalgamated products are still continuous C*-bundles.

In order to gather these different approaches in a joint survey, we first recall a few definitions from the theory of deformations of C*-algebras and we fix several notations which will be used in the sequel.

Then we present a few possible extension properties for continuous C*-bundles. More precisely, given a compact Hausdorff space X which is perfect, i.e. without any isolated point, we first recall in §2 that there is no general C(X)-linear version of the Hahn-Banach extension theorem for continuous C(X)-algebra. But we describe in §3 a Tietze extension property for continuous C(X)-algebras which will enable us to characterize in the following sections:
- when the canonical fiberwise amalgamated tensor products of a given continuous C(X)-algebra A with any other continuous C(X)-algebra B is a continuous C(X)-algebra ([6, Theorem 1.1 and Theorem 1.2]),
- when the canonical fiberwise amalgamated free products of a given continuous C(X)-algebra A with any other continuous C(X)-algebra B is a continuous C(X)-algebra ([4, Theorem 3.7 and Corollary 4.8]).

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1. C(X)-algebras

We recall first a few definitions from deformation theory for C*-algebras and we fix the notations which will be used in the sequel.

Let X be a compact Hausdorff space and C(X) the C*-algebra of continuous functions on X with values in the complex field ℂ.

Definition 1.1. A C(X)-algebra is a C*-algebra A endowed with a unital *-homomorphism from C(X) to the centre of the multiplier C*-algebra M(A) of A.

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Given a closed subset $Y \subset X$, we denote by $C_0(X \setminus Y)$ the closed ideal of continuous functions on $X$ that vanish of $Y$. If $A$ is a $C(X)$-algebra, then the subset $C_0(X \setminus Y).A$ is a closed ideal in $A$ (by Cohen factorisation Theorem) and we denote by $\pi_0^\vee$ the quotient map $A \to A/C_0(X \setminus Y).A$.

If the closed subset $Y$ is reduced to a point $x$ and the element $a$ belongs to the $C(X)$-algebra $A$, we usually write $\pi_x$, $A_x$ and $a_x$ for $\pi_x^X$, $\pi_x^X(A)$ and $\pi_x^X(a)$.

Note that the function
\begin{equation}
    x \mapsto \|a_x\| = \inf \{ \| [1 - f + f(x)]a \| ; f \in C(X) \}
\end{equation}
is always upper semi-continuous by construction. And the $C(X)$-algebra $A$ is said to be continuous (or to be a continuous $C^*$-bundle over $X$) if the function $x \mapsto \|a_x\|$ is actually continuous for all $a$ in $A$.

**Definition 1.2.** A continuous field of states on a unital $C(X)$-algebra $A$ is a unital positive $C(X)$-linear map $\varphi : A \to C(X)$.

**Remark 1.3.** A (unital) separable $C(X)$-algebra $A$ is continuous if and only if (iff) there exists a continuous field of states $\varphi : A \to C(X)$ such that for all $x \in X$, the induced state $\varphi_x : a_x \in A_x \mapsto \varphi(a)(x)$ is faithful on $A_x$ (\[\text{(2)}\]).

### 2. Hahn-Banach Extension Properties

Given a compact Hausdorff space $X$, a continuous unital $C(X)$-algebra $A$, a unital $C(X)$-subalgebra $B \subset A$ and a continuous field of states $\phi : B \to C(X)$, there does not exist in general a $C(X)$-linear positive unital map $\varphi : A \to C(X)$ extending $\phi$, i.e. a continuous field of states $\varphi$ on $A$ making the following diagram commutative:

\[
\begin{array}{ccc}
B & \xrightarrow{\phi} & C(X) \\
\cap & \| & \\
A & \xrightarrow{\varphi} & C(X) \\
\end{array}
\]

The problem happens as soon as the interior of $X$ is non empty. Indeed, consider:

- the compact space $X := \{0\} \cup \{ \frac{1}{n} : n \in \mathbb{N}^* \}$,
- the unital continuous $C(X)$-algebra $A := C(X) \oplus C(X)$ and
- the $C(X)$-subalgebra $B := C(X).1_A + \left(C_0(X \setminus \{0\}) \oplus C_0(X \setminus \{0\}) \right) \subset A$

And let $\phi : B \to C(X)$ be the continuous field of states on $B$ fixed by the formulae

\[
\phi\left( (b_1, b_2) \right)\left( \frac{1}{n} \right) = \begin{cases} 
    b_1\left( \frac{1}{n} \right) & \text{if } n \text{ is odd} \\
    b_2\left( \frac{1}{n} \right) & \text{otherwise}
\end{cases} \quad \text{for } (b_1, b_2) \in C_0(X \setminus \{0\}) \oplus C_0(Y \setminus \{0\})
\]

Then, there cannot be any continuous field of states $\varphi : A \to C(X)$ such that $\varphi(b) = \phi(b)$ for all $b \in B$. Indeed, if $a = 1 \oplus 0 \in A$, one has that:
Proposition 3.1. And let \( \bar{\phi} \) order to study extensions in the general case, let us define in

\[ \pi^Z_X(A) \xrightarrow{\phi} C(X) \]

\[ A \xrightarrow{\varphi} C(Z) \]

\[ A \xrightarrow{\varphi} C(Z) \]

3. Tietze Extension Properties

Given a second countable compact Hausdorff space \( X \) and a closed non empty sub-

space \( Y \subset X \), we describe in this section when a continuous \( C(Y) \)-algebra \( A \) can be extended to \( X \), i.e. when there exists a continuous \( C(X) \)-algebra \( D \) with a \( C(Y) \)-
algebra \( \pi^X_Y(D) \cong A \).

If the \( C^* \)-algebra \( A \) is an exact separable \( C^* \)-algebra, then there exists a unital embedding of the \( C(Y) \)-algebra \( A \) into the trivial \( C(Y) \)-algebra \( C(Y; \mathcal{O}_2) \cong C(Y) \otimes \mathcal{O}_2 \),

where \( \mathcal{O}_2 \) is the unital Cuntz \( C^* \)-algebra generated by two isometries \( s_1, s_2 \) satisfying the relation \( 1_{\mathcal{O}_2} = s_1(s_1)^* + s_2(s_2)^* \) (3). Hence, the continuous \( C(X) \)-algebra \( D := \{ f \in C(X, \mathcal{O}_2) ; \pi^X_Y(f) \in A \} \) answers the question in that case.

But there are continuous \( C(Y) \)-algebras which are not exact \( C^* \)-algebras. Thus, in

order to study extensions in the general case, let us define in \( X \times Y \times [0, 1] \):

- the open subspace \( U = \{(x, y, t) \in X \times Y \times [0, 1] ; 0 < t \} \) and

- the closed subspace \( Z = \{(x, y, t) \in X \times Y \times [0, 1] ; 0 \leq t, d(x, Y) \leq 2d(x, Y) - d(x, y) \} \).

And let \( \bar{d} \) be the metric on \( Z \) given by \( \bar{d}((x, y, t), (x', y', t')) = d(x, x') + d(y, y') + |t - t'| \).

Proposition 3.1. (6) The coordinate map \( p_1 : (x, y, t) \mapsto x \) gives a structure of

\( C(X) \)-algebra on \( C(Z) \) and the ideal \( C_0(U \cap Z) \) is a continuous \( C(X) \)-algebra such that

\( C_0(U \cap Z) \mid_Y \cong C_0(Y \times (0, 1)) \), i.e. the map \( (x, y, t) \in U \cap Z \mapsto x \in X \) is open.

Proof. Given a function \( f \) in \( C_0(U \cap Z) \), let us prove the continuity of the function

\[ x \in X \mapsto \| \pi^X_x(f) \| = \sup \{|f(z)| ; z \in p_1^{-1}\{x\}\} \]

This map is already upper semi-continuous (u. s. c.) by construction. Hence, it

only remains to show that for any point \( x_0 \in X \) and any constant \( \varepsilon > 0 \), one has

\[ \| \pi^X_x(f) \| > \| \pi^X_{x_0}(f) \| - \varepsilon \]

for all points \( x \) in a neighbourhood of \( x_0 \) in \( X \).

The uniform continuity of the function \( f \) implies that there exists \( \delta > 0 \) such that

\[ |f(z) - f(z')| < \varepsilon \]

for all \( z, z' \) in \( Z \) with \( \bar{d}(z, z') < \delta \). Now three cases can appear:
1) If $x_0 \in Y$ and $x \in Y$ satisfies $d(x_0, x) < \delta/2$, then $|f(x, x, t) - f(x_0, x_0, t)| < \varepsilon$ for all $t \in [0, 1]$. And so $\|\pi_x^X(f)\| > \|\pi_{x_0}^X(f)\| - \varepsilon$.

2) If $x_0 \in Y$ and $x \in X \setminus Y$ satisfies $d(x_0, x) < \delta/4$, then for all $y \in Y$, the relation $d(x, y) \leq 2d(x, Y)$ implies that $d(y, x_0) \leq d(y, x) + d(x, x_0) \leq 2d(x, Y) + d(x, x_0) \leq \frac{3}{4}\delta$ and so $|f(x, y, t) - f(x_0, x_0, t)| < \varepsilon$ for all $t \in [0, 2 - \frac{d(x, y)}{d(x, Y)}]$. Whence the inequality $\|\pi_x^X(f)\| > 2 - \frac{d(x, y)}{d(x, Y)}$.

3) If $x_0 \not\in Y$ and the triple $(x_0, y_0, t_0) \in U \cap Z$ satisfies $|f(x_0, y_0, t_0)| = \|\pi_x^X(f)\| \neq 0$, then $d(x_0, y_0) < 2d(x_0, Y)$. Thus, there exists by continuity a constant $\alpha(x_0) \in \{0, \delta/2\}$ such that all $x \in X$ in the ball of radius $\alpha(x_0)$ around $x_0$ satisfy:

\begin{itemize}
  \item[a)] $d(x, Y) > 0$, \quad \item[b)] $d(x, y) < 2d(x, Y)$, \quad \item[c)] $t_0 < 2 - \frac{d(x_0, y_0)}{d(x_0, Y)} + \delta/2$.
\end{itemize}

And so $\|\pi_x^X(f)\| \geq \|f(x, y_0, \inf\{t_0, 2 - \frac{d(x_0, y_0)}{d(x_0, Y)}\})\| > \|\pi_{x_0}^X(f)\| - \varepsilon$. \hfill \qed

Remark 3.2. S. Wassermann pointed out that if $Y = \{0, 1\} \subset X = [0, 1]$, then $Z = \{(x, 0, t) \in [0, 1] \times \{0\} \times [0, 1] ; t \leq \frac{2 - 3t}{2} \} \cup \{(x, 1, t) \in [0, 1] \times \{1\} \times [0, 1] ; t \leq \frac{3t - 1}{x}\}$. Hence, the $C(X)$-algebra $C(Z)$ is not continuous at $x = \frac{1}{3}$ and $x = \frac{2}{3}$.

But the essential ideal $C_0\{(0, 1) \times \{0, 1\} \times (0, 1)\}$ is a continuous $C(X)$-algebra.

The following Corollary will be essential in the proof of Proposition 4.2.

Corollary 3.3. Let $X$ be a second countable compact space, $Y \subset X$ a non zero closed subset and $A$ a continuous $C(Y)$-algebra.

\begin{itemize}
  \item[a)] $B := C(X) \otimes A \otimes C([0, 1])$ is a continuous $C(X \times Y \times [0, 1])$-algebra.
  \item[b)] $D := [C_0(U).B]|_Z = C_0(U).B/C_0(U\setminus U \cap Z).B$ is a continuous $C(X)$-algebra.
  \item[c)] There is an isomorphism of $C(Y)$-algebras $D|_Y \cong A \otimes C_0((0, 1])$.
\end{itemize}

Proof. a) holds because the $C^*$-algebras $C(X)$ and $C_0((0, 1])$ are nuclear.

b) Let $b \in D$. Then for all $x \in X$, we have

\begin{equation}
\|\pi_x^X(b)\| = \|b + C_0(X \setminus \{x\})D\| = \sup\{\|\pi_z^Z(b)\| ; z \in p_1^{-1}(\{x\})\},
\end{equation}

whence the continuity of the map $x \mapsto \|\pi_x^X(b)\|$ by a) and Proposition 3.1

c) One has $D|_Y \cong [C_0(U).B]|_Y \cong A \otimes C_0((0, 1])$ by Proposition 3.1. \hfill \qed

4. **Amalgamated tensor products of continuous $C(X)$-algebras**

Given a fixed compact Hausdorff space $X$, we study in this section the continuity properties of the different tensor products amalgamated over $C(X)$ of two given continuous $C(X)$-algebras $A$ and $B$.

More precisely, let $A \otimes B$ denote the algebraic tensor product (over $\mathbb{C}$) of $A$ and $B$, let $\mathcal{I}_X(A, B)$ be the ideal in $A \otimes B$ generated by the differences $af \otimes b - a \otimes fb$ ($a \in A$, $b \in B$, $f \in C(X)$) and let $A \otimes B$ denote the quotient of $A \otimes B$ by $\mathcal{I}_X(A, B)$.

If $C_\Delta(X \times X) \subset C(X \times X)$ is the ideal of continuous function of $X \times X$ which are
zero on the diagonal and $A \otimes m B$ (resp. $A \otimes M B$) is the minimal (resp. maximal) tensor product over $\mathbb{C}$ of the two continuous $C(X)$-algebras $A$ and $B$, then the quotient

\[ A \otimes_c m B := A \otimes m B / C\Delta(X \times X)A \otimes m B \quad (\text{resp. } A \otimes_c M B := A \otimes M B / C\Delta(X \times X)A \otimes M B) \]

is the minimal (resp. maximal) completion of the algebraic amalgamated tensor product $A \otimes_c B$. Further, the $*$-algebra $A \otimes_c B$ embeds in the $C(X)$-algebra $A \otimes c M B$ (\[ \Pi \]) and we have

\[ (4.1) \quad \forall x \in X, \quad (A \otimes_c m B)_x \cong A_x \otimes m B_x \quad \text{and} \quad (A \otimes_c M B)_x \cong A_x \otimes M B_x. \]

Let us also recall a characterisation of exactness given by Kirchberg and Wassermann.

**Proposition 4.1.** ([8, Theorem 4.5]) Let $Y = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$ and let $D$ be a $C^*$-algebra. Then the following assertions are equivalent.

i) The $C^*$-algebra $A$ is exact.

ii) For all continuous $C(Y)$-algebra $B$, the minimal tensor product $A \otimes m B$ is a continuous $C(Y)$-algebra with fibres $A_x \otimes m B_y (y \in Y)$.

It induces the following results for fibrewise tensor products of continuous $C(X)$-algebras.

**Proposition 4.2.** ([6, \[ \Pi \]) Let $X$ be a second countable compact Hausdorff space and $A$ a separable unital continuous $C(X)$-algebra.

If the topological space $X$ is perfect (i.e. without isolated point), then the following assertions $\alpha_e$ and $\beta_e$ (resp. $\alpha_n$ and $\beta_n$) are equivalent.

$\alpha_e$) The $C^*$-algebra $A$ is exact.

$\beta_e$) For all continuous $C(X)$-algebra $B$, the amalgamated tensor product $A \otimes_c m B$ is a continuous $C(X)$-algebra with fibres $A_x \otimes m B_x (x \in X)$.

$\alpha_n$) The $C^*$-algebra $A$ is nuclear.

$\beta_n$) For all continuous $C(X)$-algebra $B$, the amalgamated tensor product $A \otimes_c M B$ is a continuous $C(X)$-algebra with fibres $A_x \otimes M B_x (x \in X)$.

**Proof.** $\alpha_e \Rightarrow \beta_e$) If the $C^*$-algebra $A$ is exact, then the spatial tensor product $A \otimes D$ is a continuous $C(X \times X)$-algebra with fibres $A_x \otimes m D_x' (x, x' \in X)$ ([8]). Hence, its restriction to the diagonal is as desired.

$\beta_e \Rightarrow \alpha_e$) Suppose conversely that the $C(X)$-algebra $A$ satisfies $\beta_e$. And let us prove step by step that the $C^*$-algebra $A$ is exact.

**Step a)** All the fibres $A_x$ are exact ($x \in X$). Indeed, given a point $x$ in $X$, take a sequence of points $x_n$ in $X$ converging to $x$ such that there is a topological isomorphism
Let $X:=\{x_n; n \in \mathbb{N}\} \cup \{x\} \cong \mathbb{N} \cup \{\infty\}$. Then, for any separable continuous $C(Y)$-algebra $B$, there is a continuous $C(X)$-algebra $\mathcal{B}$ such that $\mathcal{B}|_Y = B \otimes C_0((0, 1])$ (Corollary 3.3).

Now, the continuity of the $C(X)$-algebra $\mathcal{B} \otimes \bigotimes m_{C(X)} A$ given by $\beta_c$) implies that of its restriction $\bigotimes m_{C(X)} A|_Y \cong (C_0((0, 1]) \otimes B) \otimes m_{C(Y)} A_Y$, whence that of the $C(Y)$-algebra $B \otimes \bigotimes m_{C(Y)} A$ since there is an isometric $C(Y)$-linear embedding $B \hookrightarrow \mathcal{B}|_Y$. And this implies the exactness of the $C^*$-algebra $A_x$ by Proposition 4.1.

Step b) If $B$ is a $C^*$-algebra and $\mathcal{B}$ is the constant $C(X)$-algebra $C(X; B)$, then for all $x \in X$, we have the exact sequence

$$0 \to C_x(X) A \otimes B \to (A \otimes \bigotimes m_{C(X)} B)_x = A_x \otimes B \to A_x \otimes \bigotimes m_{C(Y)} B \to 0.$$  

Step c) If $B$ is a $C(X)$-algebra, then for all $x \in X$, we have the sequence of epimorphisms $(A \otimes \bigotimes m_{C(X)} B)_x \twoheadrightarrow (A_x \otimes \bigotimes m_{C(Y)} B)_x \twoheadrightarrow A_x \otimes \bigotimes m_{C(Y)} B_x$

Step d) Now, let $B$ be a $C^*$-algebra, $K \triangleleft B$ a closed two sided ideal in $B$ and take an element $d \in \ker\{A \otimes B \to A \otimes B/K\}$. Then for all $x \in X$, we have

$$d_x \in \ker\{(A \otimes B)_x \to (A \otimes B/K)_x\}$$

$$= \ker\{A_x \otimes B \to A_x \otimes B/K\} \quad \text{by } b)$$

$$= A_x \otimes K \quad \text{by } a)$$

Thus, $d \in A \otimes K$. And so, the $C^*$-algebra $A$ is exact.

$a_n) \Rightarrow \beta_n$) has a similar proof to that of $\alpha_n) \Rightarrow \beta_c$).

$\beta_n) \Rightarrow a_n$) If a $C^*$-algebra $A$ satisfies $\beta_n$), then all the fibres $A_x (x \in X)$ are nuclear by [3, Theorem 3.2] and so the $C^*$-algebra $A$ itself is nuclear (see e.g. [2, Proposition 3.23]).

Remark 4.3. These characterisations do not hold anymore if the compact space $X$ is not perfect. Indeed, if the space $X$ is reduced to a point, then both the amalgamated tensor products $A \otimes \bigotimes m_{C(X)} B$ and $A \otimes \bigotimes m_{C(X)} B$ are constant, hence continuous.

Proposition 4.2 implies the following characterisation of exact continuous $C(X)$-algebras in the framework of $C(X)$-algebras.

Corollary 4.4. Let $X$ be a perfect compact metric space and $A$ be a separable continuous $C(X)$-algebra. Then the following are equivalent

1. The $C^*$-algebra $A$ is exact.
2. For all exact sequence of continuous $C(X)$-algebras $0 \to J \to B \to D \to 0$, the sequence $0 \to A \otimes \bigotimes m_{C(X)} J \to A \otimes \bigotimes m_{C(X)} B \to A \otimes \bigotimes m_{C(X)} D \to 0$ is exact.
Proof. (2) ⇒ (1) If the unital continuous $C(X)$-algebra $A$ satisfies (2) and the sequence $0 \to J_0 \to B_0 \to D_0 \to 0$ is an exact sequence of $C^*$-algebras, then the sequence $0 \to J := C(X) \otimes J_0 \to B := C(X) \otimes B_0 \to D := C(X) \otimes D_0 \to 0$ is an exact sequence of $C(X)$-algebras. And the condition (2) implies the exactness of the sequence

$$0 \to A \overset{m}{\otimes} J = A \otimes J_0 \to A \overset{m}{\otimes} B = A \otimes B_0 \to A \overset{m}{\otimes} D = A \otimes D_0 \to 0.$$ 

whence the exactness of $A$.

(1) ⇒ (2) If the $C(X)$-algebra $A$ is an exact $C^*$-algebra and $0 \to J \to B \to D \to 0$ is an exact sequence of $C(X)$-algebras, then the two first lines of the following diagram are exact by assumption (1)

$C \Delta (X \times X) A \overset{m}{\otimes} J \to C \Delta (X \times X) A \overset{m}{\otimes} B \to C \Delta (X \times X) A \overset{m}{\otimes} D$

$\downarrow$ 

$A \overset{m}{\otimes} J \to A \overset{m}{\otimes} B \to A \overset{m}{\otimes} D$

$\downarrow$ 

$A \overset{m}{\otimes} J \to A \overset{m}{\otimes} B \to A \overset{m}{\otimes} D$

Besides, all the columns of the diagram are exact by definition, whence the exactness of the last line by a diagram chasing. □

5. AMALGAMATED FREE PRODUCTS OF CONTINUOUS $C(X)$-ALGEBRAS

In this section, we describe the continuity properties of different free products amalgamated over $C(X)$ of two given unital continuous $C(X)$-algebras $A$ and $B$.

Proposition 5.1. ([4]) Let $X$ be a second countable perfect compact Hausdorff space and $A$ a separable unital continuous $C(X)$-algebra.

Then the following assertions are equivalent.

α) The $C^*$-algebra $A$ is exact.

γ) For all separable unital continuous $C(X)$-algebra $B$ and all continuous fields of faithful states $\phi : A \to C(X), \psi : B \to C(X)$, the reduced amalgamated free product $(C, \phi \ast \psi) = (A, \phi) \ast (B, \psi)$ is a continuous $C(X)$-algebra with fibres $(C_x, \phi_x \ast \psi_x) = (A_x, \phi_x) \ast (B_x, \psi_x)$.

Proof. γ) ⇒ α) Let $B$ be a unital separable continuous $C(X)$-algebra and let $\psi$ be a continuous field of faithful states $\psi$ on $B$. Set $D = A \overset{m}{\otimes} B$ and let $E$ be the Hilbert $D$-bimodule $E = L^2(D, \phi \otimes \psi) \otimes_{C(X)} D$.

Then, the following assertions are equivalent ([4 Lemma 4.5]).

a) $D$ is a continuous $C(X)$-algebra with fibres $D_x \cong A_x \overset{m}{\otimes} B_x$ ($x \in X$).

b) The Pimsner $C^*$-algebra $T_D(E \oplus D)$ of the full Hilbert $D$-bimodule $E \oplus D$ is a continuous $C(X)$-algebra with fibres $T_{D_x}(E_x \oplus D_x)$ ($x \in X$).
But there is a $C(X)$-linear isomorphism $T_D(E \oplus D) \cong C \times \mathbb{N}$. And so, these two assertions are equivalent to the continuity of the reduced amalgamated free product $(C, \phi \ast \psi) = (A, \phi) \ast_{C(X)} (B, \psi)$ since the group $\mathbb{Z}$ is amenable ([2 Corollaire 5.10]).

Conversely, if $A$ is an exact $C^*$-algebra and $B$ is a unital separable continuous $C(X)$, the amalgamated tensor product $D = A \underset{C(X)}{\otimes} B$ is a continuous $C(X)$-algebra with fibres $D_x \cong A_x \otimes B_x$ for $x \in X$ ([3]). Hence, the reduced amalgamated free product $(C, \phi \ast \psi) = (A, \phi) \ast_{C(X)} (B, \psi)$ is a continuous $C(X)$-algebra ([3 Theorem 4.1]).

**Remark 5.2.** There is no similar result for full amalgamated free product. Indeed, the full amalgamated free product $A \underset{C(X)}{\star} B$ of two unital continuous $C(X)$-algebras $A$ and $B$ is always a continuous $C(X)$-algebra with fibres $A_x \star B_x$ ($x \in X$) ([3 Theorem 3.7]).

**Sketch of proof.** The algebraic amalgamated free product $A \underset{C(X)}{\otimes} B$ is a dense $C(X)$-submodule of the amalgamated Haagerup tensor product $A \underset{C(X)}{h \otimes} B$, which itself is contained in the full amalgamated free product $A \underset{C(X)}{\star} B$ ([9]). And for all $d \in A \underset{C(X)}{\otimes} B$, one has

$$\|d_x\|_{A_x \otimes B_x} = \|d_x\|_{A_x \otimes B_x} = \inf \left\{ \|\sum_i a_i a_i^*\|^{\frac{1}{2}}, \|\sum_i b_i^* b_i\|^{\frac{1}{2}} ; d_x = \sum_i a_i \otimes b_i \right\}$$

$$= \sup \left\{ \left\langle \xi, \sum_i \pi(a_i) \sigma(b_i) \eta \right\rangle ; \xi, \eta \text{ unit vectors in the Hilbert space } \ell^2(\mathbb{N}) \right\}$$

Hence, the map $x \mapsto \|d_x\|$ is both upper and lower semi-continuous if $d \in A \underset{C(X)}{\otimes} B$.

The proof for the continuity of the map $x \mapsto \|d_x\|$ for elements $d$ in the algebraic amalgamated free product $A \underset{C(X)}{\otimes} B$ is similar ([4]).

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