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HAL Id: hal-00923208
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Submitted on 2 Jan 2014

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On the continuity of the eigenvalues of a sublaplacian

Amine Aribi, Sorin Dragomir, Ahmad El Soufi

Abstract. We study the behavior of the eigenvalues of a sublaplacian $\Delta_b$ on a compact strictly pseudoconvex CR manifold $M$, as functions on the set $P_+$ of positively oriented contact forms on $M$ by endowing $P_+$ with a natural metric topology.

1. Introduction

Let $M$ be a compact strictly pseudoconvex CR manifold, of CR dimension $n$, without boundary. Let $P$ be the set of all $C^\infty$ pseudohermitian structures on $M$. Every $\theta \in P$ is a contact form on $M$ i.e. $\theta \wedge (d\theta)^n$ is a volume form. Let $P_+$ be the sets of $\theta \in P$ such that the Levi form $G_\theta$ is positive definite (respectively negative definite). For $\theta \in P_+$ let $\Delta_b$ be the sublaplacian

$$\Delta_b u = -\text{div}(\nabla^H u)$$

of $(M, \theta)$ acting on smooth real valued functions $u \in C^\infty(M, \mathbb{R})$. As $\Delta_b$ is a subelliptic operator (of order $1/2$) it has a discrete spectrum

$$0 = \lambda_0(\theta) < \lambda_1(\theta) \leq \lambda_2(\theta) \leq \cdots \uparrow +\infty$$

(the eigenvalues of $\Delta_b$ are counted with their multiplicities). Each eigenvalue $\lambda_\nu(\theta)$, $\nu = 0, 1, 2, \cdots$, is thought of as a function of $\theta \in P_+$. We shall deal mainly with the following problem: Is there a natural topology on $P_+$ such that each eigenvalue function $\lambda_\nu : P_+ \to \mathbb{R}$ is continuous? The analogous problem for the spectrum of the Laplace-Beltrami operator on a compact Riemannian manifold was solved by S. Bando & H. Urakawa, [2], and our main result is imitative of their Theorem 2.2 (cf. op. cit., p. 155).

We shall establish

Corollary 1. For every compact strictly pseudoconvex CR manifold $M$ the space of positively oriented contact forms $P_+$ admits a natural complete
distance function $d : \mathcal{P}_+ \times \mathcal{P}_+ \to [0, +\infty)$ such that each eigenvalue function $\lambda_k : \mathcal{P}_+ \to \mathbb{R}$ is continuous relative to the $d$-topology.

By a result of J.M. Lee, [8], for every $\theta \in \mathcal{P}_+$ there is a Lorentzian metric $F_\theta \in \text{Lor}(C(M))$ (the Fefferman metric) on the total space $C(M)$ of the canonical circle bundle $\pi : S^1 \to C(M) \to M$. Also if $\square$ is the Laplace-Beltrami operator of $F_\theta$ (the wave operator) then $\sigma(\Delta_0) \subset \sigma(\square)$. Therefore the eigenvalues $\lambda_k$ may be thought of as functions $\lambda_k^1 : C \to \mathbb{R}$ on the set $C = \{F_\theta \in \text{Lor}(C(M)) : \theta \in \mathcal{P}_+\}$ of all Fefferman metrics on $C(M)$. On the other hand $\text{Lor}(C(M))$ may be endowed with the distance function $d_{\text{e}}^\omega$ considered by P. Mounoud, [10] (associated to a fixed Riemannian metric $g$ on $C(M)$) and hence $(C, d_{\text{e}}^\omega)$ is itself a metric space. It is then a natural question whether $\lambda_k^1$ are continuous functions relative to the $d_{\text{e}}^\omega$-topology.

The paper is organized as follows. In § 2 we recall the needed material on CR and pseudohermitian geometry. The distance function $d$ (in Corollary 1) is built in § 3. In § 4 we establish a Max-Mini principle (cf. Proposition 2) for the eigenvalues of a sublaplacian. Then Corollary 1 follows from Theorem 1 in § 5. In § 6 we prove the continuity of the eigenvalues with respect to the Fefferman metric (cf. Corollary 2) though only as functions on $C_+ = \{e^{i\theta}F_0 : u \in C^\omega(M, \mathbb{R}), \ u > 0\}$.

2. Review of CR and pseudohermitian geometry

Let $(M, T_{1,0}(M))$ be a CR manifold, of CR dimension $n$, where $T_{1,0}(M) \subset T(M) \oplus \mathbb{C}$ is its CR structure. Cf. e.g. [5], p. 3-4. The Levi distribution is $H(M) = \text{Re}(T_{1,0}(M) \oplus \overline{T_{1,0}(M)})$. The Levi distribution carries the complex structure $J : H(M) \to H(M)$ given by $J(Z - \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(M)$ (here $i = \sqrt{-1}$). A pseudohermitian structure is a globally defined nowhere zero section $\theta \in C^\omega(H(M)^\perp)$ in the conormal bundle $H(M)^\perp \subset T^*(M)$. Pseudohermitian structures do exist by the mere assumption that $M$ be orientable. Let $\mathcal{P}$ be the set of all pseudohermitian structures on $M$. As $H(M)^\perp \to M$ is a real line bundle for any $\theta, \theta_0 \in \mathcal{P}$ there is a $C^\omega$ function $\lambda : M \to \mathbb{R} \setminus \{0\}$ such that $\theta = \lambda \theta_0$. Given $\theta \in \mathcal{P}$ the Levi form is $G_\theta(X, Y) = (d\theta)(X, JY)$ for every $X, Y \in \chi(M)$. Then $G_{\lambda \theta_0} = \lambda G_{\theta_0}$. The CR manifold $M$ is strictly pseudoconvex if $G_\theta$ is positive definite (write $G_\theta > 0$) for some $\theta \in \mathcal{P}$. If $M$ is strictly pseudoconvex then each $\theta \in \mathcal{P}$ is a contact form i.e. $\Psi_\theta = \theta \wedge (d\theta)^n$ is a volume form on $M$. Clearly, if $G_\theta$ is positive definite then $G_{-\theta}$ is negative definite. Hence $\mathcal{P}$ admits a natural orientation $\mathcal{P}_+$ ($G_\theta > 0$ for each $\theta \in \mathcal{P}_+$). Let $M$ be a strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$. The Reeb vector field is the globally defined, nowhere zero, tangent vector field $T \in \chi(M)$, transverse to $H(M)$, determined by $\theta(T) = 1$ and $(d\theta)(T, X) = 0$ for any $X \in \chi(M)$ (cf. Proposition 1.2 in [5], p. 8).
Webster metric is the Riemannian metric $g_\theta$ on $M$ given by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for every $X, Y \in H(M)$. Let $S^1 \to C(M) \to M$ be the canonical circle bundle (cf. Definition 2.9 in [5], p. 119). For every $\theta \in \mathcal{P}_+$ there is a Lorentzian metric $F_\theta$ on $C(M)$ (the Fefferman metric, cf. Definition 2.15 in [5], p. 128) such that the set $C = \{F_\theta : \theta \in \mathcal{P}_+\}$ of all Fefferman metrics is given by $C = \{e^{i\omega}F_\theta : u \in C^\omega(M, \mathbb{R})\}$ for each fixed contact form $\theta \in \mathcal{P}_+$ (by a result of J.M. Lee, [8], or Theorem 2.3 in [5], p. 128). $C$ is also referred to as the restricted conformal class of $F_\theta$ and it is a CR invariant.

If $u \in C^\omega(M, \mathbb{R})$ then the horizontal gradient $\nabla^H u \in C^\omega(H(M))$ is given by $\nabla^H u = \Pi_H \nabla u$. Here $\Pi_H : T(M) \to H(M)$ is the projection relative to the decomposition $T(M) = H(M) \oplus \mathbb{R}T$ and $\nabla u$ is the gradient of $u$ with respect to the Webster metric i.e. $g_\theta(\nabla u, X) = X(u)$ for any $X \in \mathfrak{X}(M)$. The divergence operator $\text{div} : \mathfrak{X}(M) \to C^\omega(M, \mathbb{R})$ is meant with respect to the volume form $\Psi_\theta$ i.e. $L_X \Psi_\theta = \text{div}(X) \Psi_\theta$ for any $X \in \mathfrak{X}(M)$. The sublaplacian $\Delta_b$ of $(M, \theta)$ is then the formally self-adjoint, second order, degenerate elliptic (in the sense of J.M. Bony, [4]) operator given by $\Delta_b u = -\text{div}(\nabla^H u)$ for any $u \in C^\omega(M, \mathbb{R})$. A systematic application of functional analysis methods to the study of sublaplacians (on domains in strictly pseudoconvex CR manifolds) was started in [3]. By a result following essentially from work in [9] (cf. also [12]) if $M$ is compact then $\Delta_b$ has a discrete spectrum $\sigma(\Delta_b) = \{\lambda_\nu : \nu \geq 0\}$ such that $\lambda_0 = 0$ and $\lambda_\nu \uparrow +\infty$ as $\nu \to \infty$.

3. A topology on the space of oriented contact forms

Let $(U_i)_{i \in A}$ be a finite open covering of $M$ such that the closure of each $U_i$ is contained in a larger open set $V_i$ which is both the domain of a local frame $\{X_\alpha : 1 \leq \alpha \leq 2n\} \subset C^\omega(V_i, H(M))$ with $X_{\alpha+n} = JX_\alpha$ for any $1 \leq \alpha \leq n$, and a coordinate neighborhood with the local coordinates $(x^1, \ldots, x^{2n+1})$. For each point $x \in M$ let $P_x$ (respectively $S_x$) be the set of all symmetric positive definite (respectively merely symmetric) bilinear forms on $T_x(M)$. Let us consider the anti-reflexive partial order relation on $S_x$ defined by

$$\varphi < \psi \iff \psi - \varphi \in P_x, \quad \varphi, \psi \in S_x.$$ 

Next let $\rho_x'' : P_x \times P_x \to [0, +\infty)$ be the distance function given by

$$\rho_x''(\varphi, \psi) = \inf \{\delta > 0 : \exp(-\delta) \varphi < \psi < \exp(\delta) \varphi\}$$

for any $\varphi, \psi \in P_x$. Then $(P_x, \rho_x'')$ is a complete metric space (by (iii) of Lemma 1.1 in [2], p. 158).

Let $\mathcal{M}$ be the set of all Riemannian metrics on $M$, so that $g_\theta \in \mathcal{M}$ for every $\theta \in \mathcal{P}_+$. Following [2] one may endow $\mathcal{M}$ with a complete distance
function $\rho$. Indeed as $M$ is compact one may set
\[ \rho''(g_1, g_2) = \sup_{x \in M} \rho''_x(g_{1,x}, g_{2,x}), \quad g_1, g_2 \in M. \]
Also let $S(M)$ be the space of all $C^\infty$ symmetric $(0, 2)$-tensor fields on $M$, organized as a Fréchet space by the family of seminorms $\{|\cdot|_k : k \in \mathbb{N} \cup \{0\}\}$ where
\[ |g|_k = \sum_{x \in X} |g|_{x,k}, \quad |g|_{i,k} = \sup_{x \in U_x} \sum_{|i| \leq k} |D^i g_{ij}(x)|, \]
where
\[ D^i = \partial_i x^1 \partial(x^1)^{\alpha_1} \cdots \partial(x^{2n+1})^{\alpha_{2n+1}}, \quad g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j) \in C^\infty(V_t, \mathbb{R}), \]
for any $g \in S(M)$. The topology of $S(M)$ as a locally convex space is compatible to the distance function
\[ \rho'(g_1, g_2) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|g_1 - g_2|_k}{1 + |g_1 - g_2|_k}, \quad g_1, g_2 \in S(M). \]
In particular $(S(M), \rho')$ is a complete metric space. If
\[ \rho(g_1, g_2) = \rho'(g_1, g_2) + \rho''(g_1, g_2) \]
then $(M, \rho)$ is a complete metric space (cf. Proposition 2 in [2], p. 158). Each metric $g \in M$ determines a Laplace-Beltrami operator $\Delta_g$ hence the eigenvalues of $\Delta_g$ may be thought of as functions of $g$ and as such the eigenvalues are (by Theorem 2.2 in [2], p. 161) continuous functions on $(M, \rho)$. To deal with the similar problem for the spectrum of a sublaplacian, we start by observing that the natural counterpart of $M$ in the category of strictly pseudoconvex CR manifolds is the set $M_H$ of all sub-Riemannian metrics on $(M, H(M))$. Nevertheless only a particular sort of sub-Riemannian metric gives rise to a sublaplacian i.e. $\Delta_g$ is associated to $G_0 \in M_H$ for some positively oriented contact form $\theta_0 \in \mathcal{P}_+$. Of course $\mathcal{P}_+ \subset \Omega^1(M)$ and one may endow $\Omega^1(M)$ with the $C^\infty$ topology. One may then attempt to repeat the arguments in [2] (by replacing $S(M)$ with $\Omega^1(M)$). The situation at hand is however much simpler since, once a contact form $\theta_0 \in \mathcal{P}_+$ is fixed, all others are parametrized by $C^\infty(M, \mathbb{R})$ i.e. for any $\theta \in \mathcal{P}_+$ there is a unique $u \in C^\infty(M, \mathbb{R})$ such that $\theta = e^u \theta_0$. We may then use the canonical Fréchet space structure (and corresponding complete distance function) of $C^\infty(M, \mathbb{R})$. Precisely, for every $u \in C^\infty(M, \mathbb{R})$, $\lambda \in \Lambda$ and $k \in \mathbb{N} \cup \{0\}$ we set
\[ p_{\lambda,k}(u) = \sup_{x \in U_x} \sum_{|\alpha| \leq k} |D^\alpha u(x)|, \]
\[ p_k(u) = \sum_{\lambda \in \Lambda} p_{\lambda,k}(u), \quad |u|_{C^\infty} = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{p_k(u)}{1 + p_k(u)}. \]
If \( \theta_0 \in \mathcal{P}_+ \) is a fixed contact form then we set
\[
d'(\theta_1, \theta_2) = |u_1 - u_2|_{C^\infty}, \quad \theta_1, \theta_2 \in \mathcal{P}_+,
\]
where \( u_i \in C^\infty(M, \mathbb{R}) \) are given by \( \theta_i = e^{\alpha} \theta_0 \) for any \( i \in \{1, 2\} \). The definition of \( d' \) doesn’t depend upon the choice of \( \theta_0 \in \mathcal{P}_+ \).

**Lemma 1.** \((\mathcal{P}_+, d')\) is a complete metric space.

**Proof.** Let \( \{\theta_v\}_{v \geq 1} \) be a Cauchy sequence in \((\mathcal{P}_+, d')\). If \( u_v \in C^\infty(M, \mathbb{R}) \) is the function determined by \( \theta_v = e^{\alpha} \theta_0 \) then (by the very definition of \( d' \)) \( \{u_v\}_{v \geq 1} \) is a Cauchy sequence in \( C^\infty(M, \mathbb{R}) \). Here \( C^\infty(M, \mathbb{R}) \) is organized as a Fréchet space by the (countable, separating) family of seminorms \( \{p_k : k \in \mathbb{N} \cup \{0\}\} \). Hence there is \( u \in C^\infty(M, \mathbb{R}) \) such that \( |u_v - u|_{C^\infty} \to 0 \) as \( v \to \infty \). Finally if \( \theta = e^{\alpha} \theta_0 \in \mathcal{P}_+ \) then \( d'(\theta_v, \theta) \to 0 \) as \( v \to \infty \). Q.e.d.

Let \( S(H) \subset H(M)^* \otimes H(M)^* \) be the subbundle of all bilinear symmetric forms on \( H(M) \). For every \( G \in C^\infty(S(H)), k \in \mathbb{Z}, k \geq 0, \) and \( \lambda \in \Lambda \) we set
\[
|G|_{\Lambda k} = \sup_{x \in U_\lambda} \sum_{|\lambda| \leq k, a, b = 1} 2^n |D^a G_{ab}(x)|, \\
|G|_k = \sum_{\lambda \in \Lambda} |G|_{\Lambda k}, \quad |G|_{C^\infty} = \sum_{k=0}^{\infty} \frac{1}{2^k 1 + |G|_k},
\]
where \( G_{ab} = G(X_a, X_b) \in C^\infty(V_\lambda, \mathbb{R}) \). Moreover we set
\[
\rho'_H(G_1, G_2) = |G_1 - G_2|_{C^\infty}, \quad G_1, G_2 \in C^\infty(S(H)).
\]

**Lemma 2.** \( \{\cdot|_k : k \in \mathbb{N} \cup \{0\}\} \) is a countable separating family of seminorms organizing \( \mathfrak{X} = C^\infty(S(H)) \) as a Fréchet space. In particular \( (\mathfrak{X}, \rho'_H) \) is a complete metric space.

**Proof.** For each \( k \in \mathbb{N} \cup \{0\} \) and \( N \in \mathbb{N} \) we set
\[
(3) \quad V(k, N) = \{G \in \mathfrak{X} : |G|_k < 1/N\}.
\]
Let \( \mathcal{B} \) be the collection of all finite intersections of sets (3). Then \( \mathcal{B} \) is (cf. e.g. Theorem 1.37 in [11], p. 27) a convex balanced local base for a topology \( \tau \) on \( \mathfrak{X} \) which makes \( \mathfrak{X} \) into a locally convex space such that every seminorm \( |\cdot|_k \) is continuous and a set \( E \subset \mathfrak{X} \) is bounded if and only if every \( |\cdot|_k \) is bounded on \( E \). \( \tau \) is compatible with the distance function \( \rho'_H \). Let \( \{G_m\}_{m \geq 1} \in \mathfrak{X} \) be a Cauchy sequence relative to \( \rho'_H \). Thus for every fixed \( k \in \mathbb{N} \cup \{0\} \) and \( N \in \mathbb{N} \) one has \( G_m - G_p \in V(k, N) \) for \( m, p \) sufficiently large. Consequently
\[
|D^a(G_m)_{ab}(x) - D^a(G_p)_{ab}(x)| < 1/N, \\
x \in U_\lambda, \quad \lambda \in \Lambda, \quad |\lambda| \leq k, \quad 1 \leq a, b \leq 2n.
\]
It follows that each sequence $\{D^m(G_m)_{ab}\}_{m \geq 1}$ converges uniformly on $\overline{U}_J$ to a function $G^a_{ab}$. In particular for $\alpha = 0$ one has $(G_m)_{ab}(x) \to G^0_{ab}(x)$ as $m \to \infty$, uniformly in $x \in \overline{U}_J$. If $\Lambda, \Lambda' \in \Lambda$ are such that $U_J \cap U_{J'} \neq \emptyset$ and

$$X'_b = A'_b X_a, \quad A \equiv [A'_b] : U_J \cap U_{J'} \to \text{GL}(2n, \mathbb{R}),$$

is a local transformation of the frame in $H(M)$ then

$$(G_m)_{ab} = A'_b A'_a (G_m)_{cd} \text{ on } U_J \cap U_{J'}$$

so that (for $m \to \infty$) $G^0_{ab} = A'_b A'_a G^0_{cd}$ on $U_J \cap U_{J'}$. Thus $G^0_{ab} \in C^\infty(U_J)$ glue up to a (globally defined) bilinear symmetric form $G^0$ on $H(M)$ and $G_m \to G^0$ in $\mathcal{X}$ as $m \to \infty$. Q.e.d.

For each point $x \in M$ let $P(H)_x$ be the set of all symmetric positive definite bilinear forms on $H(M)_x$. We endow $S(H)_x$ with the anti-reflexive partial order relation

$$\varphi < \psi \iff \varphi, \psi \in P(H)_x, \quad \varphi, \psi \in S(H)_x.$$

Next let $\rho''_x : P(H)_x \times P(H)_x \to [0, +\infty)$ be given by

$$\rho''_x(\varphi, \psi) = \inf \{\delta > 0 : \exp(-\delta) \varphi < \psi < \exp(\delta) \psi\}$$

for any $\varphi, \psi \in P(H)_x$.

**Lemma 3.** $\rho''_x$ is a distance function on $P(H)_x$.

*Proof.* As $e^{-\delta} \varphi < \psi < e^\delta \varphi$ is equivalent to $e^{-\delta} \psi < \varphi < e^\delta \psi$, it follows that $\rho''_x$ is symmetric. To prove the triangle inequality we assume that $\rho''_x(\varphi, \psi) > \rho''_x(\varphi, \chi) + \rho''_x(\chi, \psi)$ for some $\varphi, \psi, \chi \in P(H)_x$. Then

$$\rho''_x(\varphi, \psi) - \rho''_x(\varphi, \chi) > \inf \{\delta > 0 : \exp(-\delta) \chi < \psi < \exp(\delta) \chi\}$$

hence there is $\delta_2 > 0$ such that $e^{-\delta_2} \chi < \psi < e^{\delta_2} \chi$ and $\rho''_x(\varphi, \psi) - \rho''_x(\varphi, \chi) > \delta_2$. Similarly

$$\rho''_x(\varphi, \psi) - \delta_2 > \inf \{\delta > 0 : \exp(-\delta) \varphi < \chi < \exp(\delta) \varphi\}$$

yields the existence of a number $\delta_1 > 0$ such that $e^{-\delta_1} \varphi < \chi < e^{\delta_1} \varphi$ and $\rho''_x(\varphi, \psi) - \delta_2 > \delta_1$. Let us set $\delta \equiv \delta_1 + \delta_2$. The inequalities written so far show that $e^{-\delta} \varphi < \psi < e^{\delta} \varphi$ and $\rho''_x(\varphi, \psi) > \delta$, a contradiction. Finally, let us assume that $\rho''_x(\varphi, \psi) = 0$ so that for any $k \in \mathbb{N}$

$$\inf \{\delta > 0 : \exp(-\delta) \varphi < \psi < \exp(\delta) \varphi\} < 1/k$$

i.e. there is $\delta_k > 0$ such that $e^{-\delta_k} \varphi < \psi < e^{\delta_k} \varphi$ and $\delta_k < 1/k$. Thus $\lim_{k \to \infty} \delta_k = 0$ and $\psi - e^{-\delta_k} \varphi \in P(H)_x$ shows (by passing to the limit with $k \to \infty$ in $\psi(v, v) - e^{-\delta_k} \varphi(v, v) > 0$, $v \in H(M)_x \setminus \{0\}$) that $\varphi < \psi$. Similarly $e^{\delta_k} \varphi - \psi \in P(H)_x$ yields in the limit $\psi < \varphi$, and we may conclude that $\varphi = \psi$. Viceversa, if $\varphi \in P(H)_x$ then

$$\{\delta > 0 : (1 - e^{-\delta}) \varphi < (e^\delta - 1) \varphi \in P(H)_x\} = (0, +\infty)$$
Lemma 4. i) \((P(H)_x, \rho''_x)\) is a complete metric space.

ii) Let \(\{\varphi_j\}_{j \in \mathbb{N}} \subset P(H)_x\) such that \(\lim_{j \to \infty} \varphi_j = \varphi \in P(H)_x\) in the \(\rho''_x\)-topology. Then \(\lim_{j \to \infty} \varphi_j(v, w) = \varphi(v, w)\) for any \(v, w \in H(M)_x\).

Proof. i) Let \(\{\varphi_j\}_{j \in \mathbb{N}} \subset P(H)_x\) be a Cauchy sequence in the \(\rho''_x\)-topology i.e. for any \(\epsilon > 0\) there is \(j_\epsilon \in \mathbb{N}\) such that \(\rho''_x(\varphi_j, \varphi_j) > \epsilon\) for any \(j \geq j_\epsilon\) and any \(p = 1, 2, \cdots\). Hence there is \(\delta_\epsilon > 0\) such that \(e^{-\delta_\epsilon} \varphi_j < \varphi_{j+p} < e^{\delta_\epsilon} \varphi_j\) and \(\delta_\epsilon < \epsilon\). Consequently

\[
|\log \varphi_{j+p}(v, v) - \log \varphi_j(v, v)| < \delta_\epsilon < \epsilon
\]

for any \(v \in H(M)_x\setminus\{0\}\). Therefore if

\[
\{\xi_j\}_{j \in \mathbb{N}} = (\log \varphi_j(v, v), \cdots, \log \varphi_j(v, v)) \in \mathbb{R}^{2n}
\]

then \(\{\xi_j\}_{j \in \mathbb{N}}\) is a Cauchy sequence in \(\mathbb{R}^{2n}\). Let then \(\xi = \lim_{j \to \infty} \xi_j\) and let \(\varphi : H(M)_x \times H(M)_x \to \mathbb{R}\) be the bilinear form given by \(\varphi(v, v) = \exp(\xi''_x)\) for any \(v \in H(M)_x\setminus\{0\}\) followed by polarization. Here \(\xi''_x = (\xi''_1, \cdots, \xi''_{2n})\). Then \(\varphi \in P(H)_x\) and \(\lim_{j \to \infty} \varphi_j = \varphi\) in the \(\rho''_x\)-topology.

ii) If \(\varphi_j \to \varphi\) as \(j \to \infty\) then \(\log \varphi_j(v, v) \to \log \varphi(v, v)\) as \(j \to \infty\), for any \(v \in H(M)_x\setminus\{0\}\). Then \(\lim_{j \to \infty} \varphi_j(v, v) = \varphi(v, v)\) uniformly in \(v\) and statement (ii) follows by polarization. Q.e.d.

As \(M\) is compact we may set

\[
\rho''_H(G_1, G_2) = \sup_{x \in M} \rho''_x(G_{1,x}, G_{2,x}),
\]

\[
\rho_H(G_1, G_2) = \rho'_H(G_1, G_2) + \rho''_H(G_1, G_2), \quad G_1, G_2 \in M_H.
\]

Also let \(d\) be the distance function on \(P_+\) given by

\[
d(\theta_1, \theta_2) = d'(\theta_1, \theta_2) + \rho''_H(G_{\theta_1}, G_{\theta_2}), \quad \theta_1, \theta_2 \in P_+.
\]

Proposition 1. i) \((M_H, \rho_H)\) is a complete metric space.

ii) The map \(\theta \in P_+ \mapsto G_\theta \in M_H\) of \((P_+, d)\) into \((M_H, \rho_H)\) is continuous.

iii) \((P_+, d)\) is a complete metric space.

Proof. i) Let \(\{G_j\}_{j \geq 1}\) be a Cauchy sequence in \((M_H, \rho_H)\). Then \(\{G_j\}_{j \geq 1}\) is a Cauchy sequence in both \((\mathfrak{X}, \rho'_H)\) and \((M_H, \rho''_H)\). Yet \((\mathfrak{X}, \rho'_H)\) is complete (by Lemma 2). Thus \(\rho'_H(G_j, G) \to 0\) as \(j \to \infty\) for some \(G \in \mathfrak{X}\). In particular

\[
\lim_{j \to \infty} G_{j,x}(v, w) = G_x(v, w)
\]

for every \(x \in M\) and \(v, w \in H(M)_x\). On the other hand, as \(\{G_j\}_{j \geq 1}\) is Cauchy in \((M_H, \rho''_H)\), for every \(\epsilon > 0\) there is \(N_\epsilon \geq 1\) such that

\[
\rho''_x(G_{j,x}, G_{j,x}) \leq \rho''_H(G_j, G_j) < \epsilon
\]
for every \( i, j \geq N_c \) and \( x \in M \). Thus \( \{ G_{j,k} \}_{j \geq 1} \) is Cauchy in the complete (by Lemma 4) metric space \( (P(H), \rho_u) \) so that \( \rho_u''(G_{j,k}, \varphi) \to 0 \) as \( j \to \infty \) for some \( \varphi \in P(H) \). Then (by (iii) in Lemma 4) \( \lim_{j \to \infty} G_{j,k}(v, w) = \varphi(v, w) \) for every \( v, w \in H(M) \), hence \( G_{j,k} = \varphi \) yielding \( G \in M_H \).

ii) Let \( \{ \theta_v \}_{v \geq 1} \subset \mathcal{P}_+ \) such that \( d(\theta_v, \theta) \to 0 \) for \( \nu \to \infty \) for some \( \theta \in \mathcal{P}_+ \). If \( \theta_v = e^{\theta_0} \theta_0 \) and \( \theta = e^{\theta_0} \theta_0 \) then \( |u_{\theta_v} - u|_{C^\infty} \to 0 \) as \( \nu \to \infty \). Then \( G_{\theta_v} = e^{\theta_0} G_{\theta_0} \) and \( G_\theta = e^{\theta_0} G_{\theta_0} \). Since \( D^2 u_{\nu} \to D^2 u \) as \( \nu \to \infty \), uniformly on \( \overline{U}_\lambda \), for any \( \lambda \in \Lambda, |a| \leq k \) and \( k \in \mathbb{N} \cup \{0\} \), it follows that \( D^a(G_{\theta_v})_{ab} \to D^a(G_\theta)_{ab} \) as \( \nu \to \infty \) uniformly on \( \overline{U}_\lambda \) for any \( 1 \leq a, b \leq 2n \). Hence \( G_{\theta_v} \to G_\theta \) in \( X \) so that (by the very definition of \( d \) and \( \rho_H \)) \( \rho_H(G_{\theta_v}, G_\theta) \to 0 \). Q.e.d.

iii) If \( \{ \theta_v \}_{v \geq 1} \) is a Cauchy sequence in \( (\mathcal{P}_+, d) \) then \( \{ u_{\theta_v} \}_{v \geq 1} \) is Cauchy in \( (\mathcal{P}_+, d') \) as well. Yet (by Lemma 1) \( (\mathcal{P}_+, d') \) is complete hence \( d'(\theta_v, \theta) \to 0 \) for some \( \theta \in \mathcal{P}_+ \). Then, as a byproduct of the proof of statement (ii), one has \( G_{\theta_v} \to G_\theta \) in \( X \). Finally, the verbatim repetition of the arguments in the proof of statement (i) yields \( \rho_H''(G_{\theta_v}, G_\theta) \to 0 \) so that \( d(\theta_v, \theta) \to 0 \). Q.e.d.

4. A MAX-MIN PRINCIPLE

For each \( k \in \mathbb{N} \cup \{0\} \) we consider a \((k+1)\)-dimensional real subspace \( L_{k+1} \subset C^\infty(M, \mathbb{R}) \) and set

\[
\Lambda_\theta(L_{k+1}) = \sup \left\{ \frac{||\nabla^H f||^2_{L^2}}{||f||^2_{L^2}} : f \in L_{k+1} \setminus \{0\} \right\}.
\]

Here

\[
||f||_{L^2} = \left( \int_M f^2 \Psi_\theta \right)^{1/2}, \quad ||X||_{L^2} = \left( \int_M g_{\theta}(X, X) \Psi_\theta \right)^{1/2},
\]

for any \( f \in C^\infty(M, \mathbb{R}) \) and any \( X \in \mathfrak{X}(M) \). Let \( \{ u_v \}_{v \geq 0} \subset C^\infty(M, \mathbb{R}) \) be a complete orthonormal system relative to the \( L^2 \) inner product \( (f, g)_{L^2} = \int_M fg \Psi_\theta \) such that \( u_v \in \text{Eigen}(\Delta_\theta ; \lambda_v(\theta)) \) for every \( \nu \geq 0 \). If \( f \in C^\infty(M, \mathbb{R}) \) then \( f = \sum_{v=0}^\infty a_v(f) u_v \) \((L^2 \text{ convergence})\) for some \( a_v(f) \in \mathbb{R} \). Let \( L^0_{k+1} \) be the subspace of \( C^\infty(M, \mathbb{R}) \) spanned by \( \{ u_v : 0 \leq v \leq k \} \). Let \((\nabla^H)^*\) be the formal adjoint of \( \nabla^H \) i.e.

\[
(\nabla^H f, X)_{L^2} = (f, (\nabla^H)^* X)_{L^2}
\]

for any \( f \in C^\infty(M, \mathbb{R}) \) and \( X \in C^\infty(H(M)) \). Mere integration by parts shows that

\[
(\nabla^H)^* X = -\text{div}(X) \quad X \in C^\infty(H(M)),
\]

implying (by (1)) the useful identity

\[
||\nabla^H f||^2_{L^2} = (f, \Delta_\theta f)_{L^2}, \quad f \in C^\infty(M, \mathbb{R}).
\]
Let $f \in L^0_{k+1} \setminus \{0\}$ so that $f = \sum_{v=0}^{k} a_v u_v$ for some $a_v \in \mathbb{R}$. Then (by (6))

$$\|\nabla^H f\|^2_{L^2} = \sum_{v=0}^{k} a_v^2 \lambda_v(\theta) \leq \lambda_k(\theta) \sum_{v=0}^{k} a_v^2 = \lambda_k(\theta) \|f\|^2_{L^2}$$

hence

(7) \[ \Lambda_0(L^0_{k+1}) \leq \lambda_k(\theta). \]

Our purpose in this section is to establish

**Proposition 2.** Let $M$ be a compact strictly pseudoconvex CR manifold and $\theta \in \mathcal{P}_+$ a positively oriented contact form. Then

(8) \[ \lambda_k(\theta) = \inf_{L_{k+1}} \Lambda_0(L_{k+1}) \]

where the g.l.b. is taken over all subspaces $L_{k+1} \subset C^\infty(M, \mathbb{R})$ with $\dim_{\mathbb{R}} L_{k+1} = k + 1$.

So far (by (7)) $\lambda_k(\theta) \geq \Lambda_0(L^0_{k+1}) \geq \inf_{L_{k+1}} \Lambda_0(L_{k+1})$. The proof of Proposition 2 is by contradiction. We assume that $\lambda_k(\theta) > \inf_{L_{k+1}} \Lambda_0(L_{k+1})$ i.e. there is a $(k+1)$-dimensional subspace $L_{k+1} \subset C^\infty(M, \mathbb{R})$ such that $\Lambda_0(L_{k+1}) < \lambda_k(\theta)$. Then $\Lambda_0(L_{k+1})$ is finite and

$$\|f\|^2_{L^2} \Lambda_0(L_{k+1}) \geq \|\nabla^H f\|^2_{L^2}, \quad f \in L_{k+1}.$$

Then (by (6))

$$\sum_{v=0}^{\infty} a_v(f)^2 \Lambda_0(L_{k+1}) \geq \sum_{v=0}^{\infty} a_v(\theta) a_v(f)^2$$

so that

(9) \[ \sum_{\Lambda_0(L_{k+1}) \geq \lambda_v(\theta)} a_v(f)^2 [\Lambda_0(L_{k+1}) - \lambda_v(\theta)] \geq \sum_{\Lambda_0(L_{k+1}) < \lambda_v(\theta)} a_v(f)^2 [\lambda_v(\theta) - \Lambda_0(L_{k+1})]. \]

Let $\Phi : L_{k+1} \to C^\infty(M, \mathbb{R})$ be the linear map given by

$$\Phi(f) = \sum_{v=0}^{m} a_v(f) u_v, \quad f \in L_{k+1},$$

where $m = \max\{v \geq 0 : \lambda_v(\theta) \leq \Lambda_0(L_{k+1})\}$. Note that $0 \leq m \leq k - 1$ (by the contradiction assumption). We claim that

(10) \[ \ker(\Phi) \neq \{0\}. \]

Of course (10) is only true within the contradiction loop. The statement follows from $\dim_{\mathbb{R}} \Phi(L_{k+1}) \leq m + 1 \leq k < k + 1$ (hence $\Phi$ cannot be injective). Let (by (10)) $f_0 \in L_{k+1}$ such that $\Phi(f_0) = 0$ and $f_0 \neq 0$. Then $a_v(f_0) = 0$ for any $0 \leq v \leq m$ i.e. whenever $\Lambda_0(L_{k+1}) \geq \lambda_v(\theta)$. Applying
(9) to \( f = f_0 \) yields \( a_r(f_0) = 0 \) whenever \( \Lambda_\theta(L_{k+1}) < \lambda_r(\theta) \). Thus \( f_0 = 0 \), a contradiction.

5. Continuity of eigenvalues

The scope of § 5 is to establish

**Theorem 1.** Let \( M \) be a compact strictly pseudoconvex CR manifold. If \( \delta > 0 \) and \( \theta, \hat{\theta} \in \mathcal{P}_+ \) are two contact forms on \( M \) such that \( d(\theta, \hat{\theta}) < \delta \) then \( e^{-\delta} \lambda_k(\theta) \leq \lambda_k(\hat{\theta}) \leq e^\delta \lambda_k(\theta) \) for any \( k \geq 0 \).

**Proof.** For any \( x \in M \)
\[
\delta > \inf \left\{ \epsilon > 0 : e^{-\epsilon} G_{\theta,x} < G_{\hat{\theta},x} < e^\epsilon G_{\theta,x} \right\}
\]
i.e. there is \( 0 < \epsilon < \delta \) such that \( G_{\hat{\theta},x} - e^{-\epsilon} G_{\theta,x} \in P(H)_x \) and \( e^\epsilon G_{\theta,x} - G_{\hat{\theta},x} \in P(H)_x \). There is a unique \( u \in C^\infty(M, \mathbb{R}) \) such that \( \hat{\theta} = e^u \theta \). Consequently
\[
(11) \quad \hat{\theta} \wedge (d\hat{\theta})^n = e^{(n+1)u} \theta \wedge (d\theta)^n.
\]
On the other hand \( e^{-\delta} G_{\theta,x}(v, v) < G_{\hat{\theta},x}(v, v) < e^\delta G_{\theta,x}(v, v) \) for any \( v \in H(M)_x \setminus \{0\} \) implies \( |u| < \delta \). Then for every \( f \in C^\infty(M) \) (by (11))
\[
(12) \quad e^{-(n+1)\delta} \int_M f^2 \Psi_\theta \leq \int_M f^2 \Psi_{\hat{\theta}} \leq e^{(n+1)\delta} \int_M f^2 \Psi_\theta.
\]
Moreover
\[
(13) \quad \hat{\nabla}^H f = e^{-u} \nabla^H f
\]
where \( \hat{\nabla}^H f \) is the horizontal gradient of \( f \) with respect to \( \hat{\theta} \). Thus (by (13))
\[
||\hat{\nabla}^H f||^2_{\hat{\theta}} = e^{-u}||\nabla^H f||^2_{\theta} < e^\delta ||\nabla^H f||^2_{\theta}
\]
so that (by (11))
\[
(14) \quad e^{-(n+2)\delta} \int_M ||\nabla^H f||^2_{\theta} \Psi_\theta \leq \int_M ||\hat{\nabla}^H f||^2_{\hat{\theta}} \Psi_{\hat{\theta}} \leq e^{(n+2)\delta} \int_M ||\nabla^H f||^2_{\theta} \Psi_\theta.
\]
Finally (by (12)-(13))
\[
\frac{e^{-\delta} \frac{||\nabla^H f||^2_{L^2}}{||f||^2_{L^2}}}{\frac{||\hat{\nabla}^H f||^2_{L^2}}{||f||^2_{L^2}}} \leq \frac{\int_M ||\hat{\nabla}^H f||^2_{\hat{\theta}} \Psi_{\hat{\theta}}}{\int_M f^2 \Psi_{\hat{\theta}}} \leq e^\delta \frac{||\nabla^H f||^2_{L^2}}{||f||^2_{L^2}},
\]
so that (by the Max-Mini principle)
\[
(15) \quad e^{-\delta} \lambda_k(\theta) \leq \lambda_k(\hat{\theta}) \leq e^\delta \lambda_k(\theta).
\]
Theorem 1 is proved. Corollary 1 follows from (15).
6. Spectra of $\Delta_b$ and $\Box$

Let $F_\theta$ be the Fefferman metric of $(M, \theta)$ and $\Box$ the corresponding wave operator (the Laplace-Beltrami operator of $(C(M), F_\theta)$). We set $\mathcal{M} = C(M)$ for simplicity. Let $g$ be a fixed Riemannian metric on $\mathcal{M}$. The space $S(\mathcal{M})$ of all symmetric tensor fields may be identified with the space of all fields of endomorphisms of $T(\mathcal{M})$ which are symmetric with respect to $g$ i.e. for each $h \in S(\mathcal{M})$ let $\tilde{h} \in C^\infty(\text{End}(T(\mathcal{M})))$ be given by

$$g(\tilde{h}X, Y) = h(X, Y), \quad X, Y \in T(\mathcal{M}).$$

From now on we assume that $M$ is compact. Then $\mathcal{M}$ is compact as well (as $\mathcal{M}$ is the total space of a principal bundle with compact base and compact fibres) and we endow $S(\mathcal{M})$ with the distance function

$$d_k^\infty(h_1, h_2) = \sup_{\varphi \in \mathcal{M}} \left[\text{trace}\left(\varphi^2\right)\right]^{1/2}, \quad h_1, h_2 \in S(\mathcal{M}),$$

where $\varphi = \tilde{h}_1 - \tilde{h}_2$ and $\varphi^2 = \varphi \circ \varphi$. The set $\text{Lor}(\mathcal{M})$ of all Lorentz metrics on $\mathcal{M}$ is an open set of $(S(\mathcal{M}), d_k^\infty)$ and for any pair $g_1, g_2$ of Riemannian metrics on $\mathcal{M}$ the distance functions $d_{g_1}$ and $d_{g_2}$ are uniformly equivalent (cf. e.g. [10], p. 49). We shall use the topology induced by $d_k^\infty$ on $\text{Lor}(\mathcal{M})$ (and therefore on $C \subset \text{Lor}(\mathcal{M})$). By a result of J.M. Lee, [8], the sublaplacian $\Delta_b$ of $(M, \theta)$ is the pushforward of the wave operator i.e. $\pi_* \Box = \Delta_b$. In particular $\sigma(\Delta_b) \subset \sigma(\Box)$. Thus each $\lambda_k : \mathcal{P}_+ \to \mathbb{R}$ may be thought of as a function $\lambda_k^1 : C \to \mathbb{R}$ such that $\lambda_k^1 \circ F = \lambda_k$ for every $k \geq 0$, where $F : \mathcal{P}_+ \to C$ is the map given by $F(\theta) = F_\theta$ for every $\theta \in \mathcal{P}_+$. As another consequence of Theorem 1 we establish

**Corollary 2.** Let $M$ be a compact strictly pseudoconvex CR manifold and let $g$ be an arbitrary Riemannian metric on $\mathcal{M} = C(M)$. Let $\theta_0 \in \mathcal{P}_+$ be a fixed contact form and $\mathcal{P}_+^\ast = \{e^u \theta_0 : u \in C^\infty(M, \mathbb{R}), \quad u > 0\}$. If $C_+ = \{F_\theta : \theta \in \mathcal{P}_+\}$ then for every $k \in \mathbb{N} \cup \{0\}$ the function $\lambda_k^1 : C_+ \to \mathbb{R}$ is continuous relative to the $d_k^\infty$-topology.

**Proof.** Let $\theta_i \in \mathcal{P}_+, i \in \{1, 2\}$, and let us set $\varphi = \tilde{F}_{\theta_1} - \tilde{F}_{\theta_2}$. Let $\{E_p : 1 \leq p \leq 2n + 2\}$ be a local $g$-orthonormal frame on $T(\mathcal{M})$, defined on the open set $\mathcal{U} \subset \mathcal{M}$. Then

$$\text{trace}\left(\varphi^2\right) = \sum_{p=1}^{2n+2} g(\varphi^2 E_p, E_p) = \sum_p \left\{F_{\theta_1}(\varphi E_p, E_p) - F_{\theta_2}(\varphi E_p, E_p)\right\}$$

on $\mathcal{U}$. On the other hand if $\varphi E_p = \varphi_p^q E_q$ then $\varphi_q^q = F(\theta_1)(E_p, E_q) - F(\theta_2)(E_p, E_q)$ hence

$$\text{trace}\left(\varphi^2\right) = (e^{\theta_1}_*\varphi - e^{\theta_2}_*\varphi)^2 \|F_{\theta_0}\|^2_g$$

(16)
where \( u_\ell \in C^\infty(M, \mathbb{R}) \) is given by \( \theta_\ell = e^{u_\ell} \theta_0 \) and \( \| F_\theta_0 \|_g \) is the norm of \( F_\theta_0 \) as a \((0, 2)\)-tensor field on \( \mathcal{M} \) with respect to \( g \). Then (by (16))

\[
(17) \quad d_g^\infty(F_{\theta_1}, F_{\theta_2}) = \sup_{\gamma} |e^{u_{1, \gamma}} - e^{u_{2, \gamma}}| \| F_{\theta_0} \|_g.
\]

As \( \mathcal{M} \) is compact \( a = \inf_{\gamma \in \mathcal{M}} \| F_{\theta_0} \|_{g, z_0} > 0 \). Indeed (by compactness) \( a = \| F_{\theta_0} \|_{g, z_0} \) for some \( z_0 \in \mathcal{M} \). If \( a = 0 \) then \( F_{\theta_0}, z_0 = 0 \), a contradiction (as \( F_{\theta_0} \) is Lorentzian, and hence nondegenerate). Let \( \epsilon > 0 \) such that \( d_g^\infty(F_{\theta_1}, F_{\theta_2}) < \epsilon \). Then \( |e^{u_1} - e^{u_2}| < \epsilon / a \) everywhere on \( M \). As both \( u_1 > 0 \) and \( u_2 > 0 \) it follows that \( |u_1 - u_2| < \log(1 + \epsilon / a) \). Indeed \( e^{u_1} - e^{u_2} < \epsilon / a \) is equivalent to \( e^{u_1 - u_2} < 1 + (\epsilon / a) e^{u_2} \) hence (as \( u_2 > 0 \))

\[
|u_1 - u_2| < \log(1 + (\epsilon / a) e^{u_2}) < \log(1 + \epsilon / a).
\]

Therefore

\[
(1 + \epsilon / a)^{-1} G_{\theta_1, x}(v, v) < G_{\theta_2, x}(v, v) < (1 + \epsilon / a) G_{\theta_1, x}(v, v)
\]

for any \( v \in H(M)_x \setminus \{0\} \) and any \( x \in M \). Consequently \( \rho''(G_{\theta_1} \setminus \{0\}) < \log(1 + \epsilon / a) \). The arguments in \$5 \$ then yield

\[
(1 + \epsilon / a)^{-1} J_k^1(F_{\theta_1}) \leq J_k^1(F_{\theta_2}) \leq (1 + \epsilon / a) J_k^1(F_{\theta_1})
\]

and Corollary 2 follows. The problem of the behavior of \( J_k^1 : C \to \mathbb{R} \) is open. So does the more general problem of the behavior of the spectrum of the wave operator on \( \mathcal{M} \) with respect to a change of \( F \in \text{Lor}(\mathcal{M}) \). Further work (cf. \[1\]) on the behavior of \( \sigma(\Delta_a) \) under analytic 1-parameter deformations \( \{ \theta(t) \}_{t \in \mathbb{R}} \) of a given contact form \( \theta_0 \in \mathcal{P}_+ \) builds on the Riemannian counterpart in \[6\] and the functional analysis results in \[7\].

References


