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K\textsubscript{1}-INJECTIVITY FOR PROPERLY INFINITE C\textsuperscript{*}-ALGEBRAS

ÉTIENNE BLANCHARD

Dedicated to Alain Connes on the occasion of his 60th birthday.

1. Introduction

One of the main tools to classify C\textsuperscript{*}-algebras is the study of its projections and its unitaries. It was proved by J. Cuntz in [Cun81] that if \( A \) is a purely infinite simple C\textsuperscript{*}-algebra, then the kernel of the natural map for the unitary group \( \mathcal{U}(A) \) to the K\textsubscript{1}-theory group \( K\textsubscript{1}(A) \) is reduced to the connected component \( \mathcal{U}^0(A) \), i.e. \( A \) is \( K\textsubscript{1} \)-injective (see §3). We study in this note a finitely generated C\textsuperscript{*}-algebra, the \( K\textsubscript{1} \)-injectivity of which would imply the \( K\textsubscript{1} \)-injectivity of all unital properly infinite C\textsuperscript{*}-algebras.

Note that such a question was already considered in [Blac07], [BRR08].

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2. Preliminaries

Let us first review briefly the theory introduced by J. Cuntz ([Cun78]) of comparison of positive elements in a C\textsuperscript{*}-algebra.

Definition 2.1. ([Cun78], [Rør92]) Given two positive elements \( a, b \) in a C\textsuperscript{*}-algebra \( A \), one says that:
- \( a \) is dominated by \( b \) (written \( a \precsim b \)) if and only if there is a sequence \( \{ d_k; k \in \mathbb{N} \} \) in \( A \) such that \( \| d_k^* b d_k - a \| \to 0 \) when \( k \to \infty \),
- \( a \) is properly infinite if \( a \neq 0 \) and \( a \oplus a \precsim a \oplus 0 \) in the C\textsuperscript{*}-algebra \( M_2(A) := M_2(\mathbb{C}) \otimes A \).

This leads to the following notions of infiniteness for C\textsuperscript{*}-algebras.

Definition 2.2. ([Cun78], [Cun81], [KR00]) A unital C\textsuperscript{*}-algebra \( A \) is said to be:
- properly infinite if its unit \( 1_A \) is properly infinite in \( A \),
- purely infinite if all the non zero positive elements in \( A \) are properly infinite in \( A \).

Remark 2.3. E. Kirchberg and M. Rørdam have proved in [KR00] Theorem 4.16 that a C\textsuperscript{*}-algebra \( A \) is properly infinite (in the above sense) if and only if there is no character on the C\textsuperscript{*}-algebra \( A \) and any positive element \( a \) in \( A \) which lies in the closed two-sided ideal generated by another positive element \( b \) in \( A \) satisfies \( a \precsim b \).
The first examples of such C*-algebras were given by J. Cuntz in [Cun81]: For any integer \( n \geq 2 \), the C*-algebra \( \mathcal{T}_n \) is the universal unital C*-algebra generated by \( n \) isometries \( s_1, \ldots, s_n \) satisfying the relation
\[
 s_1 s_1^* + \ldots + s_n s_n^* \leq 1 \tag{2.1}
\]
Then, the closed two sided ideal in \( \mathcal{T}_n \) generated by the minimal projection \( p_{n+1} := 1 - s_1 s_1^* - \ldots - s_n s_n^* \) is isomorphic to the C*-algebra \( K \) of compact operators on an infinite dimension separable Hilbert space and one has an exact sequence
\[
 0 \to K \to \mathcal{T}_n \xrightarrow{\pi} \mathcal{O}_n \to 0, \tag{2.2}
\]
where the quotient \( \mathcal{O}_n \) is a purely infinite simple unital nuclear C*-algebra ([Cun81]).

**Remark 2.4.** A unital C*-algebra \( A \) is properly infinite if and only if there exists a unital *-homomorphism \( \mathcal{T}_2 \to A \).

### 3. \( K_1 \)-injectivity of \( \mathcal{T}_n \)

Given a unital C*-algebra \( A \) with unitary group \( \mathcal{U}(A) \), denote by \( \mathcal{U}^0(A) \) the connected component of \( 1_A \) in \( \mathcal{U}(A) \). For each strictly positive integer \( k \geq 1 \), the upper diagonal embedding \( u \in \mathcal{U}(M_k(A)) \to (u \oplus 1_A) \in \mathcal{U}(M_{k+1}(A)) \) sends the connected component \( \mathcal{U}^0(M_k(A)) \) into \( \mathcal{U}^0(M_{k+1}(A)) \), whence a canonical homomorphism \( \Theta_A \) from \( \mathcal{U}(A)/\mathcal{U}^0(A) \) to \( K_1(A) := \lim_{k \to \infty} \mathcal{U}(M_k(A))/\mathcal{U}^0(M_k(A)) \). As noticed by B. Blackadar in [Blac07], this map is (1) neither injective, (2) nor surjective in general:

1. If \( \Omega_2 \) denotes the compact unitary group of the matrix C*-algebra \( M_2(\mathbb{C}) \), \( A := C(\Omega_2 \times \Omega_2, M_2(\mathbb{C})) \) and \( u, v \in \mathcal{U}(A) \) are the two unitaries \( u(x, y) = x \) and \( v(x, y) = y \), then \( z := uvu^*v^* \) is not unitarily homotopic to \( 1_A \) in \( \mathcal{U}(A) \) but the unitary \( z \oplus 1_A \) belongs to \( \mathcal{U}^0(M_2(A)) \) ([AJT60]).
2. If \( A = C(\mathbb{T}^3) \), then \( \mathcal{U}(A)/\mathcal{U}^0(A) \cong \mathbb{Z}^4 \) but \( K_1(A) \cong \mathbb{Z}^4 \).

**Definition 3.1.** The unital C*-algebra \( A \) is said to be \( K_1 \)-injective if the map \( \Theta_A \) is injective.

J. Cuntz proved in [Cun81] that \( \Theta_A \) is surjective as soon as the C*-algebra \( A \) is properly infinite and that it is also injective if the C*-algebra \( A \) is simple and purely infinite. Now, the \( K \)-theoretical six-term cyclic exact sequence associated to the exact sequence (2.2) implies that \( K_1(\mathcal{T}_n) = 0 \) since \( K_1(K) = K_1(\mathcal{O}_n) = 0 \). Thus, the map \( \Theta_{\mathcal{T}_n} \) is zero.

**Proposition 3.2.** For all \( n \geq 2 \), the C*-algebra \( \mathcal{T}_n \) is \( K_1 \)-injective, i.e. any unitary \( u \in \mathcal{U}(\mathcal{T}_n) \) is unitarily homotopic to \( 1_{\mathcal{T}_n} \) in \( \mathcal{U}(\mathcal{T}_n) \) (written \( u \sim_h 1_{\mathcal{T}_n} \)).

**Proof.** The C*-algebras \( \mathcal{T}_n \) have real rank zero by Proposition 2.3 of [Zha90]. And Lin proved that any unital C*-algebra of real rank zero is \( K_1 \)-injective ([Lin01, Corollary 4.2.10]). □

**Corollary 3.3.** If \( \alpha : \mathcal{T}_3 \to \mathcal{T}_3 \) is a unital *-endomorphism, then its restriction to the unital copy of \( \mathcal{T}_2 \) generated by the two isometries \( s_1, s_2 \) is unitarily homotopic to \( \text{id}_{\mathcal{T}_2} \) among all unital *-homomorphisms \( \mathcal{T}_2 \to \mathcal{T}_3 \).
Proof. The isometry $\sum_{k=1,2} \alpha(s_k) s_k^*$ extends to a unitary $u \in U(T_3)$ such that $\alpha(s_k) = us_k$ for $k = 1, 2$ ([BRR08, Lemma 2.4]). But Proposition 3.2 yields that $U(T_3) = U^0(T_3)$, whence a homotopy $u \sim_h 1$ in $U(T_3)$, and so the desired result holds. □

Remark 3.4. The unital map $\iota : C \to T_2$ induces an isomorphism in $K$-theory. Indeed, $[1_{T_2}] = [s_1 s_1^*] + [s_2 s_2^*] + [p_3] = 2[1_{T_2}] + [p_3]$ and so $[1_{T_2}] = -[p_3]$ is invertible in $K_0(T_2)$.

4. $K_1$-injectivity of properly infinite $C^*$-algebras

Denote by $T_2 \ast_C T_2$ the universal unital free product with amalgamation over $C$ (in the sequel called full unital free product) of two copies of $T_2$ amalgamated over $C$ and let $j_0, j_1$ be the two canonical unital inclusions of $T_2$ in $T_2 \ast_C T_2$. We show in this section that the $K_1$-injectivity of $T_2 \ast_C T_2$ is equivalent to the $K_1$-injectivity of all the unital properly infinite $C^*$-algebras. The proof is similar to that of the universality of the full unital free product $O_\infty \ast_C O_\infty$ (see Theorem 5.5 of [BRR08]).

Definition 4.1. ([Blan09], [BRR08, §2]) If $X$ is a compact Hausdorff space, a unital $C(X)$-algebra is a unital $C^*$-algebra $A$ endowed with a unital $*$-homomorphism from the $C^*$-algebra $C(X)$ of continuous functions on $X$ to the centre of $A$.

For any non-empty closed subset $Y$ of $X$, we denote by $\pi_Y^A$ (or simply by $\pi_Y$ if no confusion is possible) the quotient map from $A$ to the quotient $A_Y$ of $A$ by the (closed) ideal $C_0(X \setminus Y) \cdot A$. For any point $x \in X$, we also denote by $A_x$ the quotient $A_{\{x\}}$ and by $\pi_x$ the quotient map $\pi_{\{x\}}$.

Proposition 4.2. The following assertions are equivalent.

(i) $T_2 \ast_C T_2$ is $K_1$-injective.

(ii) $D := \{f \in C([0, 1], T_2 \ast_C T_2); f(0) \in j_0(T_2) \text{ and } f(1) \in j_1(T_2)\}$ is properly infinite.

(iii) There exists a unital $*$-homomorphism $\theta : T_2 \to D$.

(iv) There exists a projection $q \in D$ with $\pi_0(q) = j_0(s_1 s_1^*)$ and $\pi_1(q) = j_1(s_1 s_1^*)$.

(v) Any unital properly infinite $C^*$-algebra $A$ is $K_1$-injective.

Proof. (i)$\Rightarrow$(ii) We have a pull-back diagram

$$
\begin{array}{ccc}
D & \xrightarrow{\pi_2} & D_{[\frac{1}{2}, 1]} \\
\downarrow & & \downarrow \\
T_2 \ast_C T_2 & \xrightarrow{\pi_2} & T_2 \ast_C T_2
\end{array}
$$

and the two $C^*$-algebras $D_{[0, \frac{1}{2}]}$ and $D_{[\frac{1}{2}, 1]}$ are properly infinite (Remark 2.4). Hence, the implication follows from [BRR08, Proposition 2.7].

(ii)$\Rightarrow$(iii) is Remark 2.4 applied to the $C^*$-algebra $D$.

(iii)$\Rightarrow$(iv) The two full, properly infinite projections $j_0(s_1 s_1^*)$ and $\pi_0 \circ \theta(s_1 s_1^*)$ are unitarily equivalent in $j_0(T_2)$ by [LLR00, Lemma 2.2.2] and [BRR08, Proposition 2.3].
Thus, they are homotopic among the projections in the C∗-algebra \( J_0(T_2) \) (written \( j_0(s_1s_1^*) \sim \pi_0 \circ \theta(s_1s_1^*) \)) by Proposition 3.2. Similarly, \( \pi_1 \circ \theta(s_1s_1^*) \sim \pi_1 \circ \theta(s_1s_1^*) \) in \( J_1(T_2) \).

Further, \( \pi_0 \circ \theta(s_1s_1^*) \sim \pi_0 \circ \theta(s_1s_1^*) \) in \( T_2 \ast_C T_2 \) by hypothesis, whence the result by composition.

(iv) \( \Rightarrow (v) \) By [BRR08] Proposition 5.1, it is equivalent to prove that if \( p \) and \( p' \) are two properly infinite full projections in \( A \), then there exist full properly infinite projections \( p_0 \) and \( p'_0 \) in \( A \) such that \( p_0 \leq p \), \( p'_0 \leq p' \) and \( p_0 \sim_h p'_0 \).

Fix two such projections \( p \) and \( p' \) in \( A \). Then, there exist unital ∗-homomorphisms \( \sigma : T_2 \to pAp \), \( \sigma' : T_2 \to p'Ap' \) and isometries \( t, t' \in A \) such that \( 1_A = t^*pt = (t')^*p't' \). Now, one thoroughly defines unital ∗-homomorphisms \( \alpha_0 : T_2 \to A \) and \( \alpha_1 : T_2 \to A \) by

\[
\alpha_0(s_k) := \sigma(s_k) \cdot t \quad \text{and} \quad \alpha_1(s_k) := \sigma'(s_k) \cdot t' \quad \text{for} \quad k = 1, 2,
\]

whence a unital ∗-homomorphism \( \alpha := \alpha_0 \ast \alpha_1 : T_2 \ast_C T_2 \to A \) such that \( \alpha \circ j_0 = \alpha_0 \) and \( \alpha \circ j_1 = \alpha_1 \).

The two full properly infinite projections \( p_0 = \alpha_0(s_1s_1^*) \) and \( p'_0 = \alpha_1(s_1s_1^*) \) satisfy \( p_0 \leq p \) and \( p'_0 \leq p' \). Further, the projection \( (id \ast \alpha)(q) \) gives a continuous path of projections in \( A \) from \( p_0 \) to \( p'_0 \).

**Remark 4.3.** The C∗-algebra \( M_2(D) \) is properly infinite by [BRR08] Proposition 2.7.

**Lemma 4.4.** \( K_0(T_2 \ast_C T_2) = \mathbb{Z} \) and \( K_1(T_2 \ast_C T_2) = 0 \)

**Proof.** The commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\iota_1} & T_2 \\
\downarrow{\iota_0} & & \downarrow{\iota_1} \\
T_2 & \xrightarrow{j_0} & T_2 \ast_C T_2
\end{array}
\]

yields by [Ger97] Theorem 2.2 a six-term cyclic exact sequence

\[
K_0(\mathbb{C}) = \mathbb{Z} \xrightarrow{(i_0 \oplus i_1)_*} K_0(T_2 \oplus T_2) = \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(j_0)_* - (j_1)_*} K_0(T_2 \ast_C T_2) \\
K_1(T_2 \ast_C T_2) \xleftarrow{} K_1(T_2 \oplus T_2) = 0 \oplus 0 \xleftarrow{} K_1(\mathbb{C}) = 0
\]

Now, Remark 3.4 implies that the map \((i_0 \oplus i_1)_*\) is injective, whence the equalities.

**Remark 4.5.** G. Skandalis noticed that the C∗-algebra \( T_2 \) is KK-equivalent to \( \mathbb{C} \) and so \( T_2 \ast_C T_2 \) is KK-equivalent to \( \mathbb{C} \ast_C \mathbb{C} = \mathbb{C} \).

This Lemma entails that the \( K_1 \)-injectivity question for unital properly infinite C∗-algebras boils down to knowing whether \( U(T_2 \ast_C T_2) = U(T_2 \ast_C T_2) \). Note that Proposition 3.2 already yields that \( U(T_2) \ast_T U(T_2) \subset U(T_2 \ast_C T_2) \).

But the following holds.

**Proposition 4.6.** Set \( p_3 = 1 - s_1s_1^* - s_2s_2^* \) in the Cuntz algebra \( T_2 \) and let \( u \) be the canonical unitary generating \( C^*(\mathbb{Z}) \).

(i) The relations \( j_0(s_k) \to s_k \) and \( j_1(s_k) \to us_k \) (\( k = 1, 2 \)) uniquely define a unital ∗-homomorphism \( T_2 \ast_C T_2 \to T_2 \ast_C C^*(\mathbb{Z}) \) which is injective but not \( K_1 \)-surjective.
(ii) The two projections $j_0(p_3)$ and $j_1(p_3)$ satisfy $j_1(p_3) \neq j_0(p_3)$ in $T_2 \ast_C T_2$.

(iii) There is no $v \in \mathcal{U}(T_2 \ast_C T_2)$ such that $j_1(s_1s_1^* + s_2s_2^*) = v j_0(s_1s_1^* + s_2s_2^*) v^*$.

(iv) There is a unitary $v \in \mathcal{U}(T_2 \ast_C T_2)$ such that $j_1(s_1s_1^*) = v j_0(s_1s_1^*) v^*$.

**Proof.** (i) The unital $C^*$-subalgebra of $O_3$ generated by the two isometries $s_1$ and $s_2$ is isomorphic to $T_2$, whence a unital $C^*$-embedding $T_2 \ast_C T_2 \subset O_3 \ast_C O_3$ ([ADEL04]). Let $\Phi$ be the $*$-homomorphism from $O_3 \ast_C O_3$ to the free product $O_3 \ast_C C^*(\mathbb{Z}) = C^*(s_1, s_2, s_3, u)$ fixed by the relations

$$\Phi(j_0(s_k)) = s_k \quad \text{and} \quad \Phi(j_1(s_k)) = us_k \quad \text{for} \quad k = 1, 2, 3$$

and let $\Psi : O_3 \ast_C C^*(\mathbb{Z}) \rightarrow O_3 \ast_C O_3$ be the only $*$-homomorphism such that

$$\Psi(u) = \sum_{i=1}^3 j_1(s_i)j_0(s_i)^* \quad \text{and} \quad \Psi(s_k) = j_0(s_k) \quad \text{for} \quad k = 1, 2, 3.$$ 

For all $k = 1, 2, 3$, we have the equalities:

$$- \Psi \circ \Phi(j_0(s_k)) = \Psi(s_k) = j_0(s_k),$$

$$- \Psi \circ \Phi(j_1(s_k)) = \Psi(us_k) = j_1(s_k),$$

$$- \Phi \circ \Psi(s_k) = \Phi(j_0(s_k)) = s_k.$$ 

Also, $\Psi(u)^* \Psi(u) = \sum_{i,i'} j_0(s_{i'})(s_i')^* j_1(s_i)j_0(s_i)^* = 1_{O_3 \ast_C O_3} = \Psi(u)^* \Psi(u)$, i.e. $\Psi(u)$ is a unital $*$-homomorphism which satisfies:

$$- \Phi \circ \Psi(u) = \sum_{i=1,2,3} \Phi(j_1(s_i)) \Phi(j_0(s_i)^*) = u.$$ 

Thus, $\Phi$ is an invertible unital $*$-homomorphism with inverse $\Psi$ ([Blac07]), and the restriction of $\Phi$ to the $C^*$-subalgebra $T_2 \ast_C T_2$ takes values in $T_2 \ast_C C^*(\mathbb{Z}) \subset O_3 \ast_C C^*(\mathbb{Z})$.

Now, there is (see [Ger97]) a six-term cyclic exact sequence

$$K_0(\mathbb{C}) = \mathbb{Z} \quad \hookrightarrow \quad K_0(T_2 \oplus C^*(\mathbb{Z})) = \mathbb{Z} \oplus \mathbb{Z} \quad \twoheadrightarrow \quad K_0(T_2 \ast_C C^*(\mathbb{Z}))$$

$$K_1(T_2 \ast_C C^*(\mathbb{Z})) \quad \hookleftarrow \quad K_1(T_2 \oplus C^*(\mathbb{Z})) = 0 \oplus \mathbb{Z} \quad \twoheadleftarrow \quad K_1(\mathbb{C}) = 0$$

and so, $K_1(T_2 \ast_C C^*(\mathbb{Z})) = \mathbb{Z}$, whereas $K_1(T_2 \ast_C T_2) = 0$ by Lemma 4.4

(ii) Let $\pi_0 : T_2 \rightarrow L(H)$ be a unital $*$-representation on a separable Hilbert space $H$ such that $\pi_0(p_3)$ is a rank one projection, let $\pi_1 : T_2 \rightarrow L(H)$ be a unital $*$-representation such that $\pi_1(p_3)$ is a rank two projection and consider the induced unital $*$-representation $\pi = \pi_0 \ast \pi_1$ of the unital free product $T_2 \ast_C T_2$.

Then the two projections $\pi[j_0(p_3)] = \pi_0(p_3)$ and $\pi[j_1(p_3)] = \pi_1(p_3)$ have distinct ranks and so cannot be equivalent in $L(H)$. Hence, $j_0(p_3) \nleftrightarrow j_1(p_3)$ in $T_2 \ast_C T_2$.

(iii) This is just a rewriting of the previous assertion since $s_1s_1^* + s_2s_2^* = 1 - p_3$. Indeed, the partial isometry $b = j_1(s_1)j_0(s_1)^* + j_1(s_2)j_0(s_2)^*$ defines a Murray-von Neumann equivalence in $T_2 \ast_C T_2$ between the projections $j_0(s_1s_1^* + s_2s_2^*) = 1 - j_0(p_3)$ and $j_1(s_1s_1^* + s_2s_2^*) = 1 - j_1(p_3)$. Thus, they are unitarily equivalent in $T_2 \ast_C T_2$ if and only if $j_0(p_3) \sim j_1(p_3)$ in $T_2 \ast_C T_2$ ([LLR00 Proposition 2.2.2]).
(iv) There exists a unitary $v \in \mathcal{U}(T_2 \ast C T_2)$ (which is necessarily $K_1$-trivial by Lemma 4.4) such that $j_1(s_1 s_1^*) = v j_0(s_1 s_1^*) v^*$. Indeed, we have the inequalities

$$1 > s_2 s_2^* + p_3 > s_2 s_2^* > s_2 (s_2 s_2^* + p_3) s_2^* + s_2 s_2 (p_3) s_2 s_2^* \quad \text{in } T_2.$$  

Thus, if we set $w := j_1(s_1) j_0(s_1)^*$, then $1 - w^* w = j_0(s_2 s_2^* + p_3)$ and $1 - w w^* = j_1(s_2 s_2^* + p_3)$ are two properly infinite and full $K_0$-equivalent projections in $T_2 \ast C T_2$. Thus, there is a partial isometry $a \in T_2 \ast C T_2$ with $a^* a = 1 - w^* w$ and $a a^* = 1 - w w^*$ (Cuntz81). The sum $v = a + w$ has the required properties (BRR08 Lemma 2.4). 

**Remarks 4.7.** (i) The equivalence (iv)$\iff$(v) in Proposition 4.2 implies that all unital properly infinite $C^*$-algebras are $K_1$-injective if and only if the unitary $v \in \mathcal{U}(T_2 \ast C T_2)$ constructed in Proposition 4.6(iv) belongs to the connected component $\mathcal{U}^0(T_2 \ast C T_2)$.

Note that $v \oplus 1 \sim_h 1 \oplus 1$ in $\mathcal{U}(M_2(T_2 \ast C T_2))$ by [LLR00, Exercise 8.11].

(ii) Let $\sigma \in \mathcal{U}(T_2)$ be the symmetry $\sigma = s_1 s_1^* + s_2 s_2^* + p_3$, let $v \in \mathcal{U}(T_2 \ast C T_2)$ be a unitary such that $j_1(s_1 s_1^*) = v j_0(s_1 s_1^*) v^*$ (Proposition 4.6(iv)) and set $z := v^* j_1(\sigma) v j_0(\sigma)$. Then, $q_1 = j_0(s_1 s_1^*)$, $q_2 = j_0(s_2 s_2^*)$ and $q_3 = z j_0(s_2 s_2^*) z^*$ are three properly infinite full projections in $T_2 \ast C T_2$ which satisfy:

- $q_1 q_3 = j_0(s_1 s_1^*) v^* j_1(s_2 s_2^*) v = v^* j_1(s_1 s_1^*) j_1(s_2 s_2^*) v = 0 = q_1 q_2$,

- $q_2 \sim_h q_1 \sim_h q_3$ in $T_2 \ast C T_2$ since $\sigma \in \mathcal{U}^0(T_2)$ and so $z \sim_h v^* v = 1$ in $\mathcal{U}(T_2 \ast C T_2)$,

- $q_1 + q_3 = v^* j_1(s_1 s_1^* + s_2 s_2^*) v \not\sim_h j_0(s_1 s_1^* + s_2 s_2^*) = q_1 + q_2$ in $T_2 \ast C T_2$ by Proposition 4.6(iii).

**Addendum**

(iii) Let $\alpha = \alpha_0 \ast \alpha_1$ be the unital $\ast$-endomorphism of the free product $T_2 \ast C T_2$ defined by $\alpha_0(s_k) = j_0(s_k)$ and $\alpha_1(s_k) = v^* j_1(s_k)$ for $k = 1, 2$. Then $\alpha_0(s_2 s_2^* + p_3) = 1 - \alpha_0(s_1 s_1^*) = 1 - \alpha_1(s_1 s_1^*) = \alpha_1(s_2 s_2^* + p_3)$ and $\alpha_0(s_2 s_2^*) \sim_h \alpha_0(s_1 s_1^*) = \alpha_1(s_2 s_2^*) \sim_h \alpha_1(s_2 s_2^*)$ among the projections in $\alpha(T_2 \ast C T_2)$. But $\alpha_0(p_3) \not\sim \alpha_1(p_3)$ in $\alpha(T_2 \ast C T_2)$.

**References**


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