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Etienne Blanchard, Ilja Gagic

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ON UNITAL $C(X)$-ALGEBRAS AND $C(X)$-VALUED CONDITIONAL EXPECTATIONS OF FINITE INDEX

ETIENNE BLANCHARD AND ILJA GOGIĆ

Abstract. Let $X$ be a compact Hausdorff space and let $A$ be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of the centre of $A$. We consider the problem of characterizing the existence of a conditional expectation $E : A \to C(X)$ of finite index in terms of the underlying $C^*$-bundle of $A$ over $X$. More precisely, we show that if $A$ admits a $C(X)$-valued conditional expectation of finite index, then $A$ is necessarily a continuous $C(X)$-algebra, and there exists a positive integer $N$ such that every fibre $A_x$ of $A$ is finite-dimensional, with $\dim A_x \leq N$. We also give some sufficient conditions on $A$ that ensure the existence of a $C(X)$-valued conditional expectation of finite index.

1. INTRODUCTION

Let $B \subseteq A$ be two unital $C^*$-algebras with the same unit element. A conditional expectation (abbreviated by C.E.) from $A$ to $B$ is a completely positive contraction $E : A \to B$ such that $E(b) = b$ for all $b \in B$, and which is $B$-bilinear, i.e.

\[ E(b_1 a b_2) = b_1 E(a) b_2 \]

for all $a \in A$ and $b_1, b_2 \in B$. By a result of Y. Tomiyama (see [22, Theorem 1] or [4, Theorem II.6.10.2]), a map $E : A \to B$ is a C.E. if and only if $E$ is a projection of norm one.

If $E(a^* a) = 0$ ($a \in A$) implies $a = 0$, $E$ is said to be faithful. Every faithful conditional expectation $E : A \to B$ introduces a pre-Hilbert $B$-module structure on $A$, whose inner product is defined by

\[ \langle a_1, a_2 \rangle_E := E(a_1^* a_2) \quad (a_1, a_2 \in A) \]

The notion of finite index was introduced by V. F. R. Jones [14] in order to classify the subfactors of a type II$_1$ factor. Soon afterwards H. Kosaki [16] extended the Jones index theory to arbitrary factors. In order to generalize the results of [14, 16], M. Pimsner and S. Popa introduced in [19, 20] a definition for conditional expectations of finite index in the context of $W^*$-algebras: There must exist a constant $K \geq 1$ such that the map $K \cdot E - \text{id}_A$ is positive on $A$. Then, following the idea of M. Baillet, Y. Denizeau and J.-F. Havet (see [3]), the index of $E$ can be defined in the following way: Since the map $K \cdot E - \text{id}_A$ is positive, $E$ defines a (complete) Hilbert $B$-module structure on $A$, with respect to the inner product...
If \( \{ x_i \} \) is a quasi-orthonormal basis in \( A \), the index of \( E \) is the sum \( \sum_{i=1}^{\infty} x_i^* x_i \), with respect to the ultraweak topology.

Y. Watatani also considered C.E. of (algebraically) finite index, when the original \( C^* \)-algebra \( A \) is a finitely generated Hilbert \( C^* \)-module over \( B \) (see [23]).

The results of M. Baillet, Y. Denizeau and J.-F. Havet in [3] also indicated that there might occur some difficulties in order to extend the notion of “finite index” for conditional expectations of \( C^* \)-algebras with arbitrary centres. However, this problem was solved by M. Frank and E. Kirchberg in [11]. The main result of their paper is [11, Theorem 1]:

**Theorem 1.1** (M. Frank and E. Kirchberg). For a C.E. \( E : A \to B \), where \( B \subseteq A \) are unital \( C^* \)-algebras with the same unit element, the following conditions are equivalent:

(i) There exists a constant \( K \geq 1 \) such that the map \( K \cdot E - \text{id}_A \) is positive.

(ii) There exists a constant \( L \geq 1 \) such that the map \( L \cdot E - \text{id}_A \) is completely positive.

(iii) \( A \) becomes a (complete) Hilbert \( B \)-module when equipped with the inner product (1.1).

Moreover, if

\[
K(\varepsilon) := \inf \{ K \geq 1 : K \cdot E - \text{id}_A \text{ is positive} \},
\]

\[
L(\varepsilon) := \inf \{ L \geq 1 : L \cdot E - \text{id}_A \text{ is completely positive} \},
\]

with \( K(\varepsilon) = \infty \) or \( L(\varepsilon) = \infty \) if no such number \( K \) or \( L \) exists, then

\[
K(\varepsilon) \leq L(\varepsilon) \leq \lfloor K(\varepsilon) \rfloor K(\varepsilon),
\]

where \( \lfloor \cdot \rfloor \) denotes the integer part of a real number.

The importance of this result is that it gives the right general definition for conditional expectations on \( C^* \)-algebras to be of finite index:

**Definition 1.2.** If \( B \subseteq A \) are two unital \( C^* \)-algebras with the same unit element, then a C.E. \( E : A \to B \) is said to be of finite index (abbreviated C.E.F.I.) if \( E \) satisfies one of the equivalent conditions of Theorem 1.1.

In this case the index value of \( E \) can be calculated in the enveloping von Neumann algebra \( A^{**} \) (see [11, Definition 3.1]).

For a unital inclusion \( A \subseteq B \) of unital \( C^* \)-algebras we introduce the following constant

\[
K(A, B) := \inf \{ K(\varepsilon) : E : A \to B \text{ is C.E.F.I.} \},
\]

with \( K(A, B) = \infty \), if no such C.E.F.I. exists. This constant will play an important role in this paper.

More recently, A. Pavlov and E. Troitsky considered in [17] the problem of existence of a C.E.F.I. \( E : C(Y) \to C(X) \) for a unital inclusion \( \varphi : C(X) \hookrightarrow C(Y) \) of unital commutative \( C^* \)-algebras. The main result of their paper is [17, Theorem 1.1], which shows that such a C.E.F.I. exists if and only if the transpose map \( \varphi^* : Y \to X \) is a branched covering. This means that \( \varphi^* \) is an open map with uniformly bounded number of pre-images (i.e. \( \sup_{x \in X} |\varphi^*_x(x)| < \infty \)). This result motivated A. Pavlov and E. Troitsky to define the noncommutative branched coverings, as unital inclusion \( B \subseteq A \) of unital \( C^* \)-algebras such that there exists a C.E.F.I. from \( A \) to \( B \) (see [17, Definition 1.2]).
Using the above inclusion $\varphi : C(X) \rightarrow C(Y)$ we may consider $C(Y)$ as a $C(X)$-algebra. Then the map $\varphi$ is open if and only if $C(Y)$ is a continuous $C(X)$-algebra, and $\varphi$, has uniformly bounded number of pre-images if and only if $C(Y)$ is subhomogeneous $C(X)$-algebra. This means that there exists a positive integer $N$ such that every fibre $C(Y)_x$ of $C(Y)$ is finite-dimensional with $\dim C(Y)_x \leq N$ (see Section 2). Therefore, we can restate [17, Theorem 1.1] in terms of $C(X)$-algebras as follows:

**Theorem 1.3** (A. Pavlov and E. Troitsky). *Let $A$ be a unital commutative $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of $A$. Then $A$ admits a $C(X)$-valued C.E.F.I. if and only if $A$ is a continuous subhomogeneous $C(X)$-algebra.*

The purpose of the present paper is to consider a possible extension of Theorem 1.3 to the case when $A$ is an arbitrary (not necessarily commutative) unital $C(X)$-algebra. The necessary condition for the existence of a $C(X)$-valued C.E.F.I. appears to be identical to the one of Theorem 1.3:

**Theorem 1.4.** *Let $A$ be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of the centre of $A$. If $A$ admits a $C(X)$-valued C.E.F.I., then $A$ is a continuous subhomogeneous $C(X)$-algebra. Moreover, in this case the following inequality holds:*

$$K(A, C(X)) \geq r(A),$$

*where $r(A)$ is the rank of $A$, i.e.

$$r(A) = \max \left\{ \sum_{\pi_x \in \tilde{A}_x} \dim \pi_x : x \in X \right\}.$$

We shall prove Theorem 1.4 in Section 3. At the moment we do not know if the converse of Theorem 1.4 also holds. However, if all the fibres of a continuous unital $C(X)$-algebra $A$ are $*$-isomorphic to the same finite-dimensional $C^*$-algebra (i.e. $A$ is a homogeneous $C(X)$-algebra), then there exists a unique C.E. $E : A \rightarrow C(X)$ such that the map $r(A) \cdot E - \text{id}_A$ is positive (Proposition 3.4). In particular, we have the equality $K(A, C(X)) = r(A)$ in this case. Also, a direct consequence of this fact is that any unital $C(X)$-algebra $A$ which can be embedded as a $C(X)$-subalgebra of some continuous homogeneous unital $C(X)$-algebra also admits a $C(X)$-valued C.E.F.I.. However, this embedding condition is not necessary for the existence of such C.E.F.I.. Indeed, there exists a continuous unital $C(X)$-algebra $A$ over a second-countable compact Hausdorff space $X$ with fibres $M_2(\mathbb{C})$ or $\mathbb{C}$ which admits a $C(X)$-valued C.E.F.I., but which cannot be embedded as a $C(X)$-subalgebra into any continuous homogeneous unital $C(X)$-algebra (Example 3.6). At the end of this paper we also show that any continuous unital $C(X)$-algebra $A$ of rank 2 admits a C.E. $E : A \rightarrow C(X)$ such that the map $2 \cdot E - \text{id}_A$ is positive (Proposition 3.7). In particular, the equality $K(A, C(X)) = r(A)$ also holds in this class of $C(X)$-algebras.

2. Notation and preliminaries

Throughout this paper $A$ will be a $C^*$-algebra. We denote by $A_{sa}$ and $A_+$ the self-adjoint and the positive parts of $A$. The centre of $A$ is denoted by $Z(A)$. By
A and \( \text{Prim}(A) \) we respectively denote the spectrum of \( A \) (i.e. the set of all classes of irreducible representations of \( A \)) and the primitive spectrum of \( A \) (i.e. the set of all primitive ideals of \( A \)), equipped with the Jacobson topology. By a dimension of \([\pi] \in \hat{A}\), which is denoted by \( \dim \pi \), we mean the dimension of the underlying Hilbert space of some representative of \([\pi]\).

Let \( X \) be a compact Hausdorff space. For each point \( x \in X \) let

\[ C_x(X) := \{ f \in C(X) : f(x) = 0 \} \]

be the corresponding maximal ideal of \( C(X) \).

**Definition 2.1.** A \( C(X) \)-algebra is a \( C^* \)-algebra \( A \) endowed with a unital \(*\)-homomorphism \( \psi_A \) from \( C(X) \) to the centre of the multiplier algebra of \( A \).

**Remark 2.2.** Given \( f \in C(X) \) and \( a \in A \), we write \( fa \) for the product \( \psi_A(f) \cdot a \) if no confusion is possible.

There is a natural connection between \( C(X) \)-algebras and upper semicontinuous \( C^* \)-bundles over \( X \). We first give a formal definition of such bundles:

**Definition 2.3.** Following [24] by an upper semicontinuous \( C^* \)-bundle we mean a triple \( \mathfrak{A} = (p, A, X) \) where \( A \) is a topological space with a continuous open surjection \( p : A \to X \), together with operations and norms making each fibre \( A_x := p^{-1}(x) \) into a \( C^* \)-algebra, such that the following conditions are satisfied:

1. The maps \( \mathbb{C} \times A \to A, A \times_X A \to A, A \times_X A \to A \) and \( A \to A \) given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous (\( \mathcal{A} \times_X \mathcal{A} \) denotes the Whitney sum over \( X \)).
2. The map \( A \to \mathbb{R} \), defined by norm on each fibre, is upper semicontinuous.
3. If \( x \in X \) and if \( (a_n) \) is a net in \( A \) such that \( ||a_n|| \to 0 \) and \( p(a_n) \to x \) in \( X \), then \( a_n \to 0_x \) in \( A \) (\( 0_x \) denotes the zero-element of \( A_x \)).

If "upper semicontinuous" in (A2) is replaced by "continuous", then we say that \( \mathfrak{A} \) is a continuous \( C^* \)-bundle.

By a section of an upper semicontinuous \( C^* \)-bundle \( \mathfrak{A} \) we mean a map \( s : X \to A \) such that \( p(s(x)) = x \) for all \( x \in X \). We denote by \( \Gamma(\mathfrak{A}) \) the set of all continuous sections of \( \mathfrak{A} \). Then \( \Gamma(\mathfrak{A}) \) becomes a \( C(X) \)-algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, given a \( C(X) \)-algebra \( A \), one can always associate an upper semicontinuous \( C^* \)-bundle \( \mathfrak{A} \) over \( X \) such that \( A \cong \Gamma(\mathfrak{A}) \), as follows. Set \( J_x := C_x(X) \cdot A \) and note that \( J_x \) is a closed two-sided ideal in \( A \) (by Cohen factorization theorem [7], [6, Theorem A.6.2])). The quotient \( A_x := A/J_x \) is called the fibre at the point \( x \), and we denote by \( a_x \) the image in \( A_x \) of an element \( a \in A \). Let

\[ A := \bigsqcup_{x \in X} A_x, \]

and let \( p : A \to X \) be the canonical associated projection. For \( a \in A \) we define the map \( \hat{a} : X \to A \) by \( \hat{a}(x) := a_x \), and let \( \Omega := \{ \hat{a} : a \in A \} \). Since for each \( a \in A \) we have

\[ ||a_x|| \leq \inf\{||1 - f(x) - a|| : f \in C(X)\}, \]

the norm function \( x \mapsto ||a_x|| \) is upper semicontinuous on \( X \). Hence, by Fell’s theorem [24, Theorem C.25] there exists a unique topology on \( A \) for which \( \mathfrak{A} := (p, A, X) \)
becomes an upper semicontinuous $C^*$-bundle such that $\Omega \subseteq \Gamma(\mathfrak{A})$. Moreover, by Lee’s theorem [24, Theorem C.26], $\Omega = \Gamma(\mathfrak{A})$, and the generalized Gelfand transform $\mathcal{G} : a \in A \mapsto \hat{a} \in \Gamma(\mathfrak{A})$, is an isomorphism of $C(X)$-algebras, from $A$ onto $\Gamma(\mathfrak{A})$.

**Definition 2.4.** Let $A$ be a $C(X)$-algebra. If all the norm functions $x \mapsto \|a_x\|$ ($a \in A$) are continuous on $X$, we say that $A$ is a continuous $C(X)$-algebra.

Note that the $C(X)$-algebra $A$ is continuous if and only if $\mathfrak{A}$ is continuous as a $C^*$-bundle.

The $C^*$-algebra $A$ is said to be
- $(n)$-homogeneous ($n \in \mathbb{N}$), if $\dim \pi = n$ for all $[\pi] \in \hat{A}$,
- $(n)$-subhomogeneous ($n \in \mathbb{N}$), if $\sup_{[\pi] \in \hat{A}} \dim \pi = n$.

We shall now define the similar notions for $C(X)$-algebras. To do this, first recall that if $D$ is a finite-dimensional $C^*$-algebra, then there is a finite number of central pairwise orthogonal projections $p_1, \ldots, p_m \in \mathbb{Z}(D)$ with $\sum_{i=1}^m p_i = 1_D$, such that

$$D = p_1D \oplus \cdots \oplus p_mD,$$

and each $p_iD$ is $*$-isomorphic to the matrix algebra $M_{n_i}(\mathbb{C})$ (see e.g. [21, Theorem I.11.9]). We define the rank of $D$ as

$$r(D) := \sum_{i=1}^m p_i = \sum_{[\pi] \in \hat{D}} \dim \pi.$$

**Definition 2.5.** Let $A$ be a $C(X)$-algebra. We say that $A$ is
- homogeneous all the fibres of $A$ are $*$-isomorphic to the same finite-dimensional $C^*$-algebra,
- subhomogeneous if there exists a positive integer $N$ such that every fibre $A_x$ of $A$ is finite-dimensional with $\dim A_x \leq N$.

**Remark 2.6.** Let $A$ be a $C(X)$-algebra.

(i) $A$ is subhomogeneous if and only if

$$r(A) := \sup \{r(A_x) : x \in X\} < \infty$$

As in the finite-dimensional case, we call the number $r(A)$ the rank of $A$.

(ii) If $A$ is continuous and homogeneous, then by [10, Lemma 3.1] the underlying $C^*$-bundle $\mathfrak{A}$ is locally trivial.

### 3. Results

**Remark 3.1.** If $A$ is a unital $C(X)$-algebra, we always assume in this section that the map $\psi_A : C(X) \to Z(A)$ is injective, so that we can identify $C(X)$ with the unital $C^*$-subalgebra $\psi_A(C(X))$ of $Z(A)$.

In order to prove Theorem 1.4 we shall need the following two auxiliary results.

**Lemma 3.2.** Let $D$ be a unital $C^*$-algebra. Then $K(D, \mathbb{C}) := K(D, \mathbb{C}_{1_D}) < \infty$ if and only if $D$ is finite-dimensional. In this case we have:

(i) The constant $K(\omega)$ is finite for every faithful state $\omega$ on $D$, which we identify with the corresponding faithful C.E.

$$d \in D \mapsto \omega(d) \cdot 1_D \in \mathbb{C} \cdot 1_D \quad (d \in D).$$
(ii) $K(D, \mathbb{C}) = r(D)$. Moreover, there exists a unique state $\tau$ on $D$ such that

$$r(D) \cdot \tau(d) 1_D \geq d \quad \text{for all } d \in D_+.$$  

**Proof.** The equivalence $K(D, \mathbb{C}) < \infty \iff \dim D < \infty$ follows from [13, Lemma 4.5]. Hence, suppose that $D$ is finite-dimensional and let $\omega$ be a faithful state on $D$. The proof will now proceed in two steps.

**Step 1.** Assume that $D$ is simple, i.e. $D = M_n(\mathbb{C})$ for some $n$. If $\text{tr}(\cdot)$ is the standard trace of $M_n(\mathbb{C})$, then there exists a strictly positive matrix $a \in M_n(\mathbb{C})$ with $\text{tr}(a) = 1$ such that

$$\omega(d) = \text{tr}(ad) \quad (d \in M_n(\mathbb{C})).$$

Let $a = u^* \cdot \text{diag}(\lambda_1, \ldots, \lambda_n) \cdot u$ be a diagonalisation of $a$, where $u \in M_n(\mathbb{C})$ is a unitary and $\lambda_1, \ldots, \lambda_n > 0$ are the eigenvalues of $a$. Then for all $d \in M_n(\mathbb{C})$ one has

$$\omega(u^*du) = \text{tr}(au^*du) = \text{tr}(uau^*d) = \text{tr}(\text{diag}(\lambda_1, \ldots, \lambda_n)d).$$

The constant $K(\omega)$ is by definition the smallest $K \geq 1$ satisfying

$$K \cdot \omega(d) 1_D \geq d \quad \text{for all } d \in D_+.$$

Thus, (3.2) and (3.3) for rank 1 projections in $D$ imply that

$$K(\omega) = \max\{\lambda_i^{-1} : 1 \leq i \leq n\}.$$  

As $1 = \omega(1) = \sum_{i=1}^n \lambda_i$, one has $K(\omega) \geq n$ for any faithful state $\omega$ on $D$. Also, $K(\omega) = n$ if and only if $\omega = \tau := \frac{1}{n} \text{tr}(\cdot)$. In particular, if $D = M_n(\mathbb{C})$, we have $K(D, \mathbb{C}) = r(D) = n$, and $\tau$ is the unique state on $D$ satisfying (3.1).

**Step 2.** Suppose that $D$ is an arbitrary finite-dimensional $C^*$-algebra. We decompose $D$ as in (2.1). For each $1 \leq i \leq m$

$$\omega_i(p_id) := \frac{1}{\omega(p_i)} \cdot \omega(p_id) \quad (d \in D)$$

defines a faithful state on $p_iD$. By Step 1 we have $n_i \leq K(\omega_i) < \infty$ for all $1 \leq i \leq m$. Put

$$K_\omega := \max \left\{ \frac{K(\omega_i)}{\omega(p_i)} : 1 \leq i \leq m \right\}.$$  

We claim that $K(\omega) = K_\omega$. Indeed, for all $d \in D_+$ we have

$$K_\omega \cdot \omega(d) 1_D = \sum_{i=1}^m K_\omega \cdot \omega(p_i)\omega_i(p_id) 1_D \geq \sum_{i=1}^m K(\omega_i) \cdot \omega_i(p_id)p_i$$

$$\geq \sum_{i=1}^m p_id = d,$$

which shows $K(\omega) \leq K_\omega$. On the other hand, for each $d \in D_+$ we have

$$[\omega(p_i)K(\omega)] \cdot \omega_i(p_id)p_i \geq p_id,$$

so that

$$\omega(p_i)K(\omega) \geq K(\omega_i) \quad (1 \leq i \leq m).$$
This shows $K(\omega) = K_\omega$, as wanted. Also,

$$K(\omega) = \sum_{i=1}^{m} \omega(p_i)K(\omega) \geq \sum_{i=1}^{m} K(\omega) \geq \sum_{i=1}^{m} n_i = r(D),$$

so that $K(D, \mathbb{C}) \geq r(D)$.

It remains to show that there exists a unique state $\tau$ on $D$ satisfying (3.1). To do this, suppose that $r(D) = m$, and for each $1 \leq i \leq m$ let $\tau_i$ be the only faithful tracial state on $p_i D \cong M_{n_i}(\mathbb{C})$. Define the state $\tau$ on $D$ by

$$\tau(d) := \frac{1}{n} \sum_{i=1}^{m} n_i \cdot \tau_i(p_i d) \quad (d \in D).$$

As $\tau(p_i) = \frac{n_i}{n}$ and $K(\tau_i) = n_i$ for all $1 \leq i \leq m$, we have $K(\tau) = K_\tau = n$. In particular, $K(D, \mathbb{C}) = n = r(D)$.

To show the uniqueness of this state $\tau$, suppose that $\omega$ is another state on $D$ with $K(\omega) = n$. Then using (3.4) we have

$$\sum_{i=1}^{m} K(\omega_i) = \sum_{i=1}^{m} \omega(p_i)K(\omega) = K(\omega) = n.$$

But since $K(\omega_i) \geq n_i$, and $\sum_{i=1}^{m} n_i = n$, we must have $K(\omega_i) = n_i$ for all $1 \leq i \leq m$. By the uniqueness part of Step 1 we conclude that

$$(3.6) \quad \omega_i = \tau_i \quad \text{for all} \quad 1 \leq i \leq m.$$

Also, $K_\omega = K(\omega) = n$ and $K(\omega_i) = n_i$ imply $\omega(p_i) \geq \frac{n_i}{n}$ for all $1 \leq i \leq m$. Since $\omega$ is a state on $D$ and $\sum_{i=1}^{m} p_i = 1_D$, we must have

$$(3.7) \quad \omega(p_i) = \frac{n_i}{n} \quad \text{for all} \quad 1 \leq i \leq m.$$

Finally, (3.6) and (3.7) imply that

$$\omega(d) = \sum_{i=1}^{m} \omega(p_i)\omega_i(p_i d) = \frac{1}{n} \sum_{i=1}^{m} n_i \cdot \tau_i(p_i d) = \tau(d),$$

for all $d \in D$, which finishes the proof. \qed

**Proposition 3.3.** Let $A$ be a unital $C(X)$-algebra. If $A$ admits a faithful $C(X)$-valued C.E., then $A$ is a continuous $C(X)$-algebra.

**Proof.** This can be deduced from [5, Section 2]. For completeness, we include a short proof of this fact. It suffices to show that all norm functions $x \mapsto \|a_x\|$ ($a \in A$) are lower semicontinuous on $X$. To prove this, let $E : A \to C(X)$ be a faithful C.E. and let $L^2(A, E)$ be the completion of the pre-Hilbert $C(X)$-module $A$, with respect to the inner product (1.1). For $a \in A$ let $\Phi(a) : L^2(A, E) \to L^2(A, E)$ denote the continuous extension of the left multiplication map $a_1 \mapsto aa_1$ ($a \in A$). Since $E$ is faithful and since

$$\langle \Phi(a)(a_1), a_2 \rangle_E = \langle aa_1, a_2 \rangle_E = E(a^*_1a^*a_2) = \langle a_1, a^*a_2 \rangle_E = \langle a_1, \Phi(a^*)(a_2) \rangle_E,$$

we have
for all $a_1, a_2 \in A$, the map $\Phi$ defines an injective $C(X)$-linear morphism from $A$ to the $C(X)$-algebra $\mathcal{B}_{C(X)}(L^2(A, E))$ of bounded adjointable $C(X)$-linear operators on $L^2(A, E)$. Therefore, for $a \in A$ and $x \in X$ we have
\[
\|a_x\| = \|\Phi(a)x\| = \sup \{|\langle \Phi(a)(a_1), a_2 \rangle_E(x)\} : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\} = \sup \{|E(a_1^* a_2)(x)\} : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\}.
\]
In particular, the function $x \mapsto \|a_x\|$ is a supremum of continuous functions $x \mapsto |E(a_1^* a_2)(x)|$ such that $\|a_1\|_E = \|a_2\|_E = 1$, so it must be lower semicontinuous on $X$. □

**Proof of Theorem 1.4.** Let $E : A \to C(X)$ be a C.E.F.I.. As the conditional expectation $E$ is faithful, Proposition 3.3 implies that the $C(X)$-algebra $A$ is continuous (note that in this case $(A, \langle \cdot, \cdot \rangle_E)$ is already a complete Hilbert $C(X)$-module by Theorem 1.1). It remains to show that each fibre $A_x$ ($x \in X$) is finite-dimensional and satisfies $r(A_x) \leq K(E)$. Indeed, for a fixed point $x \in X$ and $\varepsilon > 0$,
\[
\omega_x : a_x \mapsto E(a)(x)
\]
defines a state on a fibre $A_x$ satisfying
\[
(K(E) + \varepsilon) \cdot \omega_x(a_x) 1_x \geq \omega_x,
\]
for all $a_x \in (A_x)_+$. Lemma 3.2 now yields $r(A_x) \leq K(E)$, as wanted. □

We shall now give some sufficient conditions on a continuous unital subhomogeneous $C(X)$-algebra $A$ to ensure the existence of a $C(X)$-valued C.E.F.I..

**Proposition 3.4.** Every continuous homogeneous unital $C(X)$-algebra $A$ admits a unique C.E. $E : A \to C(X)$ such that the map $r(A) : E - \text{id}_A$ is positive. In particular, $K(A, C(X)) = r(A)$ in this case.

**Proof.** The construction of such a C.E. $E : A \to C(X)$ can be deduced from the proof of [13, Lemma 4.6]. But we include here the main steps of the proof for completeness. By assumption all fibres of $A$ are $\ast$-isomorphic to a fixed finite-dimensional $C^*$-algebra $D$. Suppose that $r(D) = n$, and let $\tau$ be a state on $D$ defined by (3.5). It is easy check that $\tau$ is invariant under the group $\text{Aut}(D)$ of $\ast$-automorphisms of $D$. Since the $C(X)$-algebra $A$ is continuous and homogeneous, its underlying bundle $\mathfrak{A}$ is locally trivial by Remark 2.6. Hence, there exists an open covering $\{U_\alpha\}$ of $X$ such that $\Phi_\alpha : \mathfrak{A}|_{U_\alpha} \cong U_\alpha \times D$, where
- $\Phi_\alpha$ is an isomorphism of $C^*$-bundles, and
- $\mathfrak{A}|_U$ is the restriction bundle over a subset $U \subseteq X$.

Fix an element $a \in A$. For $x \in X$ choose an index $\alpha$ such that $x \in U_\alpha$, and define
\[
E(a)(x) := \tau(\Phi_\alpha(a_x)).
\]

Since $\tau$ is invariant under the group $\text{Aut}(D)$, the value $E(a)(x)$ is well defined, and the local triviality of $\mathfrak{A}$ implies that the function $E(a) : x \mapsto E(a)(x)$ is continuous on $X$. It is now easy to see that the map $E : a \to E(a)$ defines a $C(X)$-valued C.E.F.I. on $A$. Moreover, by (3.1) we have
\[
n \cdot E(a)(x) 1_x \geq a_x, \quad \text{for all } a \in A_+ \text{ and } x \in X.
\]
Thus, the map $n \cdot E - \text{id}_A$ is positive and $E$ is the only C.E. with this property (Lemma 3.2). In particular, $K(A, C(X)) \leq r(A)$, so Theorem 1.4 yields that $K(A, C(X)) = n$. □
Corollary 3.5. If the unital \( C(X) \)-algebra \( A \) admits a \( C(X) \)-linear embedding into some homogeneous continuous unital \( C(X) \)-algebra \( A' \), then \( A \) admits a \( C(X) \)-valued C.E.F.I..

Proof. By Proposition 3.4 there exists a C.E. \( E' : A' \to C(X) \) of finite index. Then the restriction \( E'|_A : A \to C(X) \) defines a convenient C.E.F.I.

Note that the embedding condition of Corollary 3.5 is not necessary for the existence of a \( C(X) \)-valued C.E.F.I.. Indeed, in Example 3.6 we show that there exists a continuous unital \( C(X) \)-algebra \( A \) of rank 2 which does not admit a \( C(X) \)-linear embedding into any continuous homogeneous unital \( C(X) \)-algebra. On the other hand, a direct consequence of Proposition 3.7 is that \( A \) admits a \( C(X) \)-valued C.E.F.I..

To do this, first recall that a \( C^* \)-algebra \( A \) is said to be central if it satisfies the following two conditions:

(i) \( A \) is quasi-central (i.e. no primitive ideal of \( A \) contains \( Z(A) \));
(ii) If \( P, Q \in \text{Prim}(A) \) and \( P \cap Z(A) = Q \cap Z(A) \), then \( P = Q \)

(see [1, 8, 12, 15]). By [8, Proposition 3] a quasi-central \( C^* \)-algebra \( A \) is central if and only if \( \text{Prim}(A) \) is Hausdorff.

Example 3.6. By [18, Example 3.5] there exists a continuous \( M_2(\mathbb{C}) \)-bundle \( \mathfrak{A}_0 \) over the second countable locally compact space \( X_0 := \bigsqcup_{n=1}^\infty \mathbb{C}P^n \), where \( \mathbb{C}P^n \) is the complex projective space of dimension \( n \), which is not of finite type (that is, \( X_0 \) does not admit a finite open cover \( \{U_i\} \) such that each restriction bundle \( \mathfrak{A}_0|_{U_i} \) is trivial, as a \( C^* \)-bundle). Let \( A_0 \) be the \( C^* \)-algebra \( \Gamma_0(\mathfrak{A}_0) \) consisting of all continuous sections of \( \mathfrak{A}_0 \) which vanish at infinity. Then \( A_0 \) is a 2-homogeneous \( C^* \)-algebra with \( \text{Prim}(A_0) = X_0 \). In particular \( A_0 \) is a central \( C^* \)-algebra with centre \( C_0(X_0) \). Let \( X := X_0 \cup \{\infty\} \) be the one-point compactification of \( X_0 \), and let \( A \) be the minimal unitisation of \( A_0 \). By [8, Proposition 3] (or [12, Proposition 3.12]) \( A \) is also a central \( C^* \)-algebra with \( \text{Prim}(A) = X \) and centre \( C(X) \). In particular, by [4, I.6.5.8] all norm functions \( x \mapsto \|a_x\| \) (\( a \in A \)) are continuous on \( X \), so that \( A \) is a continuous unital \( C(X) \)-algebra with fibres \( A_x = M_2(\mathbb{C}) \) (\( x \in X_0 \)) and \( A_\infty = \mathbb{C} \). Suppose that \( A \) is \( C(X) \)-subalgebra of some continuous homogeneous \( C(X) \)-algebra \( A' \). Then the underlying \( C^* \)-bundle \( \mathfrak{A} \) of \( A \) over \( X \) is a \( C^* \)-subbundle of the underlying \( C^* \)-bundle \( \mathfrak{A}' \) of \( A' \) over \( X \). Since \( A' \) is continuous and homogeneous, \( \mathfrak{A}' \) is locally trivial by Remark 2.6. Hence, since \( X \) is compact, \( \mathfrak{A}' \) is of finite type. Using [18, Lemma 2.6] we conclude that \( \mathfrak{A} \) is of finite type as a vector bundle. In particular, \( \mathfrak{A}_0 \) is of finite type as a vector bundle, since \( \mathfrak{A}_0 = \mathfrak{A}|_{X_0} \). As \( \mathfrak{A}_0 \) is a \( M_2(\mathbb{C}) \)-bundle, this implies by [18, Proposition 2.9] that \( \mathfrak{A}_0 \) is also of finite type as a \( C^* \)-bundle; a contradiction.

On the other hand, the \( C(X) \)-algebra \( A \) of Example 3.6 also admits a \( C(X) \)-valued C.E.F.I.. This follows from the following more general fact:

Proposition 3.7. Let \( A \) be a continuous unital \( C(X) \)-algebra. If \( r(A) = 2 \), then there exists a conditional expectation \( E : A \to C(X) \) such that the map \( 2 \cdot E - \text{id}_A \) is positive. In particular, \( K(A, C(X)) = r(A) \) in this case.

In order to prove Proposition 3.7, let us first make the following observation:

Lemma 3.8. Let \( A \) be a unital \( C(X) \)-algebra and let \( a \in A_{sa} \). For each point \( x \in X \) let \( \lambda_{\max}(a) \) and \( \lambda_{\min}(a) \) respectively denote the largest and the smallest numbers in
the spectrum of $a_x$. Then the functions $x \mapsto \lambda_{\max}(a_x)$ and $x \mapsto \lambda_{\min}(a_x)$ are upper semicontinuous on $X$. Furthermore, these functions are continuous on $X$, whenever $A$ is a continuous $C(X)$-algebra.

Proof. This follows directly from the equations

$$\lambda_{\max}(a_x) = \|a\|1_x + a_x\| - \|a\| \quad \text{and} \quad \lambda_{\min}(a_x) = \|a\| - \|a\|1_x - a_x\|.$$

\[\Box\]

Proof of Proposition 3.7. As $r(A) = 2$, any fibre $A_x$ is isomorphic to $\mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}$ or $M_2(\mathbb{C})$. Therefore, for each point $x \in X$ we can choose a unital embedding $\varphi_x: A_x \hookrightarrow M_2(\mathbb{C})$. For $a \in A$ and $x \in X$ we define

$$E(a)(x) := \frac{1}{2}\text{tr}(\varphi_x(a_x)).$$

Obviously $E(a)$ is a $C(X)$-linear map. If $a \in A_{sa}$, note that

\begin{equation}
E(a)(x) = \frac{1}{2}(\lambda_{\min}(a_x) + \lambda_{\max}(a_x))
\end{equation}

for all $x \in X$. By Remark 3.8, $E(a)$ is a continuous function on $X$ for all $a \in A_{sa}$. As $A$ is the linear span of $A_{sa}$, we conclude that $E(a) \in C(X)$ for all $a \in A$. Therefore, $E$ defines a C. E. from $A$ onto $C(X)$. Further, by (3.8) for all $a \in A_+$ and $x \in X$ we have

$$2 \cdot E(a)(x)1_x = (\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \cdot 1_x \geq a_x.$$

This shows that the map $2 \cdot E - \text{id}_A$ is positive, so that $K(A, C(X)) = 2$ by Theorem 1.4. \[\Box\]

Let $A$ be a unital $C^*$-algebra and let $\check{Z}$ be the maximal ideal space of $Z(A)$. We may consider $A$ as a $C(\check{Z})$-algebra, with respect to the action

$$f \cdot a := \mathcal{G}^{-1}(f)a \quad (f \in C(X), \ a \in A),$$

where $\mathcal{G} : Z(A) \to C(\check{Z})$ is the Gelfand transform. We say that $A$ is quasi-standard if $A$ is a continuous $C(\check{Z})$-algebra and each (Glimm) ideal $J_x = C_{\check{Z}}(\check{Z})A$ is primal (see [2]).

Corollary 3.9. For a unital $C^*$-algebra $A$ the following conditions are equivalent:

(i) There exist a C. E. $E : A \to Z(A)$ such that the map $2 \cdot E - \text{id}_A$ is positive.

(ii) $A$ is either commutative or quasi-standard and $2$-subhomogeneous.

Proof. (i) $\Rightarrow$ (ii). Suppose that there exists a C. E. $E : A \to Z(A)$ such that the map $2 \cdot E - \text{id}_A$ is positive. Then by Theorem 1.4 $A$ is a continuous $C(\check{Z})$-algebra and $r(A_x) \leq 2$ for all $x \in \check{Z}$. In particular, $A$ as a $C^*$-algebra is $n$-subhomogeneous, where $n \in \{1, 2\}$. Hence, by [13, Proposition 4.1] every Glimm ideal of $A$ is primal. Also, $n = 1$ if and only if $A$ is commutative.

(ii) $\Rightarrow$ (i). If $A$ is commutative we have nothing to prove, so suppose that $A$ is quasi-standard and $2$-subhomogeneous. Then by [9, Corollary 1, p. 388] for each point $x \in X$ we have

$$r(A_x) = \sum_{[\pi_x] \in A_x} \dim \pi_x \leq 2.$$

It remains to apply Proposition 3.7. \[\Box\]
Remark 3.10. At the end of this paper we note that every separable continuous unit C(X)-algebra A admits a faithful C.E. $E : A \to C(X)$ (see e.g. [5]). In particular, this result applies to continuous subhomogeneous unit C(X)-algebras, when X is second-countable. In this case for each point $x \in X$, the map $E_x : a_x \mapsto E(a)(x)$ defines a faithful state on $A_x$, so Lemma 3.2 implies $K(E_x) < \infty$. However, this does not imply that $E$ is of finite index. That is, it may happen that $\sup_{x \in X} K(E_x) = \infty$. Consider for instance the following example:

- Let $X$ be the closed compact subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $[0,1]$. Let $A$ be the continuous $C(X)$-subalgebra of $C(X) \oplus C(X)$ consisting of all pairs $(f,g) \in C(X) \oplus C(X)$ such that $f(0) = g(0)$.
- Let $E : A \to C(X)$ be a C.E. fixed by the relations

$$E(f \oplus g) \left( \frac{1}{n} \right) = \begin{cases} \frac{n}{n+1} f \left( \frac{1}{n} \right) + \frac{1}{n+1} g \left( \frac{1}{n} \right) & \text{if } n \text{ is odd} \\ \frac{1}{n+1} f \left( \frac{1}{n} \right) + \frac{n}{n+1} g \left( \frac{1}{n} \right) & \text{otherwise} \end{cases}$$

where $(f,g) \in A$.

Then $E$ is a faithful C.E. which is not of finite index. Indeed, one has

$$E(f \oplus 0) \left( \frac{1}{2n} \right) = \frac{1}{2n+1} f \left( \frac{1}{2n} \right)$$

for all $f \in C_0(X \setminus \{0\})$ and all integers $n \in \mathbb{N}$. Consequently, a convenient constant $K$ would satisfy $K \geq 2n + 1$ for all $n \in \mathbb{N}$, which is impossible.

We end this paper with some unresolved problems:

Problem 3.11. Is the converse of Theorem 1.4 also true? Moreover, does every continuous subhomogeneous unit C(X)-algebra A admit a C.E. $E : A \to C(X)$ such that the map $\tau(A) \cdot E - \text{id}_A$ is positive? In particular, do we always have $K(A,C(X)) = \tau(A)$?

References

12. I. Gogić, Derivations which are inner as completely bounded maps, Oper. Matrices, 4 (2010), 193-211.