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Etienne Blanchard, Ilja Gogic

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ON UNITAL $C(X)$-ALGEBRAS AND $C(X)$-VALUED CONDITIONAL EXPECTATIONS OF FINITE INDEX

ETIENNE BLANCHARD AND ILJA GOGIĆ

Abstract. Let $X$ be a compact Hausdorff space and let $A$ be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of the centre of $A$. We consider the problem of characterizing the existence of a conditional expectation $E : A \to C(X)$ of finite index in terms of the underlying $C^*$-bundle of $A$ over $X$. More precisely, we show that if $A$ admits a $C(X)$-valued conditional expectation of finite index, then $A$ is necessarily a continuous $C(X)$-algebra, and there exists a positive integer $N$ such that every fibre $A_x$ of $A$ is finite-dimensional, with $\dim A_x \leq N$. We also give some sufficient conditions on $A$ that ensure the existence of a $C(X)$-valued conditional expectation of finite index.

1. Introduction

Let $B \subseteq A$ be two unital $C^*$-algebras with the same unit element. A conditional expectation (abbreviated by C.E.) from $A$ to $B$ is a completely positive contraction $E : A \to B$ such that $E(b) = b$ for all $b \in B$, and which is $B$-bilinear, i.e.

$$E(b_1ab_2) = b_1E(a)b_2$$

for all $a \in A$ and $b_1, b_2 \in B$. By a result of Y. Tomiyama (see [?, Theorem 1] or [?, Theorem II.6.10.2]), a map $E : A \to B$ is a C.E. if and only if $E$ is a projection of norm one.

If $E(a^*a) = 0$ ($a \in A$) implies $a = 0$, $E$ is said to be faithful. Every faithful conditional expectation $E : A \to B$ introduces a pre-Hilbert $B$-module structure on $A$, whose inner product is defined by

$$(1.1) \quad \langle a_1, a_2 \rangle_E := E(a_1^*a_2) \quad (a_1, a_2 \in A).$$

The notion of finite index was introduced by V. F. R. Jones [?] in order to classify the subfactors of a type II$_1$ factor. Soon afterwards H. Kosaki [?] extended the Jones index theory to arbitrary factors. In order to generalize the results of [?, ?], M. Pimsner and S. Popa introduced in [?, ?] a definition for conditional expectations of finite index in the context of $W^*$-algebras: There must exist a constant $K \geq 1$ such that the map $K \cdot E - \text{id}_A$ is positive on $A$. Then, following the idea of M. Baillet, Y. Denizeau and J.-F. Havet (see [?]), the index of $E$ can be defined in the following way: Since the map $K \cdot E - \text{id}_A$ is positive, $E$ defines a (complete) Hilbert $B$-module structure on $A$, with respect to the inner product $(??)$. If $\{x_i\}$ is
a quasi-orthonormal basis in $A$, the index of $E$ is the sum $\sum_{i=1}^{\infty} x_i^* x_i$, with respect to the ultraweak topology.

Y. Watatani also considered C.E. of (algebraically) finite index, when the original $C^*$-algebra $A$ is a finitely generated Hilbert $C^*$-module over $B$ (see [?]).

The results of M. Baillet, Y. Denizeau and J.-F. Havet in [?] also indicated that there might occur some difficulties in order to extend the notion of "finite index" for conditional expectations of $C^*$-algebras with arbitrary centres. However, this problem was solved by M. Frank and E. Kirchberg in [?]. The main result of their paper is [?, Theorem 1]:

**Theorem 1.1** (M. Frank and E. Kirchberg). For a C.E. $E : A \to B$, where $B \subseteq A$ are unital $C^*$-algebras with the same unit element, the following conditions are equivalent:

(i) There exists a constant $K \geq 1$ such that the map $K \cdot E - \text{id}_A$ is positive.

(ii) There exists a constant $L \geq 1$ such that the map $L \cdot E - \text{id}_A$ is completely positive.

(iii) $A$ becomes a (complete) Hilbert $B$-module when equipped with the inner product $\langle \cdot, \cdot \rangle$.

Moreover, if

$$K(E) := \inf\{ K \geq 1 : K \cdot E - \text{id}_A \text{ is positive} \},$$

$$L(E) := \inf\{ L \geq 1 : L \cdot E - \text{id}_A \text{ is completely positive} \},$$

with $K(E) = \infty$ or $L(E) = \infty$ if no such number $K$ or $L$ exists, then

$$K(E) \leq L(E) \leq \lfloor K(E) \rfloor K(E),$$

where $\lfloor \cdot \rfloor$ denotes the integer part of a real number.

The importance of this result is that it gives the right general definition for conditional expectations on $C^*$-algebras to be of finite index:

**Definition 1.2.** If $B \subseteq A$ are two unital $C^*$-algebras with the same unit element, then a C.E. $E : A \to B$ is said to be of finite index (abbreviated C.E.F.I.) if $E$ satisfies one of the equivalent conditions of Theorem ??

In this case the index value of $E$ can be calculated in the enveloping von Neumann algebra $A^{**}$ (see [?, Definition 3.1]).

For a unital inclusion $A \subseteq B$ of unital $C^*$-algebras we introduce the following constant

$$K(A, B) := \inf\{ K(E) : E : A \to B \text{ is C.E.F.I.} \},$$

with $K(A, B) = \infty$, if no such C.E.F.I. exists. This constant will play an important role in this paper.

More recently, A. Pavlov and E. Troitsky considered in [?] the problem of existence of a C.E.F.I. $E : C(Y) \to C(X)$ for a unital inclusion $\varphi : C(X) \hookrightarrow C(Y)$ of unital commutative $C^*$-algebras. The main result of their paper is [?, Theorem 1.1], which shows that such a C.E.F.I. exists if and only if the transpose map $\varphi^* : Y \to X$ is a branched covering. This means that $\varphi^*$ is an open map with uniformly bounded number of pre-images (i.e. $\sup_{x \in X} |\varphi^*^{-1}(x)| < \infty$). This result motivated A. Pavlov and E. Troitsky to define the noncommutative branched coverings, as unital inclusion $B \subseteq A$ of unital $C^*$-algebras such that there exists a C.E.F.I. from $A$ to $B$ (see [?, Definition 1.2]).
Using the above inclusion \( \varphi : C(X) \to C(Y) \) we may consider \( C(Y) \) as a \( C(X) \)-algebra. Then the map \( \varphi_* \) is open if and only if \( C(Y) \) is a continuous \( C(X) \)-algebra, and \( \varphi_* \) has uniformly bounded number of pre-images if and only if \( C(Y) \) is subhomogeneous \( C(X) \)-algebra. This means that there exists a positive integer \( N \) such that every fibre \( C(Y)_x \) of \( C(Y) \) is finite-dimensional with \( \dim C(Y)_x \leq N \) (see Section 2). Therefore, we can restate [?, Theorem 1.1] in terms of \( C(X) \)-algebras as follows:

**Theorem 1.3** (A. Pavlov and E. Troitsky). Let \( A \) be a unital commutative \( C(X) \)-algebra, where \( C(X) \) is embedded as a unital \( C^* \)-subalgebra of \( A \). Then \( A \) admits a \( C(X) \)-valued C.E.F.I. if and only if \( A \) is a continuous subhomogeneous \( C(X) \)-algebra.

The purpose of the present paper is to consider a possible extension of Theorem 2 to the case when \( A \) is an arbitrary (not necessarily commutative) unital \( C(X) \)-algebra. The necessary condition for the existence of a \( C(X) \)-valued C.E.F.I. appears to be identical to the one of Theorem 2:

**Theorem 1.4.** Let \( A \) be a unital \( C(X) \)-algebra, where \( C(X) \) is embedded as a unital \( C^* \)-subalgebra of the centre of \( A \). If \( A \) admits a \( C(X) \)-valued C.E.F.I., then \( A \) is a continuous subhomogeneous \( C(X) \)-algebra. Moreover, in this case the following inequality holds:

\[
K(A,C(X)) \geq r(A),
\]

where \( r(A) \) is the rank of \( A \), i.e.

\[
r(A) = \max \left\{ \sum_{[\pi_x] \in \widehat{A}_x} \dim \pi_x : x \in X \right\}.
\]

We shall prove Theorem 2 in Section 2. At the moment we do not know if the converse of Theorem 2 also holds. However, if all the fibres of a continuous unital \( C(X) \)-algebra \( A \) are \( * \)-isomorphic to the same finite-dimensional \( C^* \)-algebra (i.e. \( A \) is a homogeneous \( C(X) \)-algebra), then there exists a unique C.E. \( E : A \to C(X) \) such that the map \( r(A) \cdot E - \text{id}_A \) is positive (Proposition 2). In particular, we have the equality \( K(A,C(X)) = r(A) \) in this case. Also, a direct consequence of this fact is that any unital \( C(X) \)-algebra \( A \) which can be embedded as a \( C(X) \)-subalgebra of some continuous homogeneous unital \( C(X) \)-algebra also admits a \( C(X) \)-valued C.E.F.I. However, this embedding condition is not necessary for the existence of such C.E.F.I.. Indeed, there exists a continuous unital \( C(X) \)-algebra \( A \) over a second-countable compact Hausdorff space \( X \) with fibres \( M_2(\mathbb{C}) \) or \( \mathbb{C} \) which admits a \( C(X) \)-valued C.E.F.I., but which cannot be embedded as a \( C(X) \)-subalgebra into any continuous homogeneous unital \( C(X) \)-algebra (Example 2). At the end of this paper we also show that any continuous unital \( C(X) \)-algebra \( A \) of rank 2 admits a C.E. \( E : A \to C(X) \) such that the map \( 2 \cdot E - \text{id}_A \) is positive (Proposition 2). In particular, the equality \( K(A,C(X)) = r(A) \) also holds in this class of \( C(X) \)-algebras.

2. Notation and preliminaries

Throughout this paper \( A \) will be a \( C^* \)-algebra. We denote by \( A_{sa} \) and \( A_+ \) the self-adjoint and the positive parts of \( A \). The centre of \( A \) is denoted by \( Z(A) \). By
$\hat{A}$ and $\text{Prim}(A)$ we respectively denote the spectrum of $A$ (i.e. the set of all classes of irreducible representations of $A$) and the primitive spectrum of $A$ (i.e. the set of all primitive ideals of $A$), equipped with the Jacobson topology. By a dimension of $[\pi] \in \hat{A}$, which is denoted by $\dim \pi$, we mean the dimension of the underlying Hilbert space of some representative of $[\pi]$.

Let $X$ be a compact Hausdorff space. For each point $x \in X$ let

$$C_x(X) := \{f \in C(X) : f(x) = 0\}$$

be the corresponding maximal ideal of $C(X)$.

**Definition 2.1.** A $C(X)$-algebra is a $C^*$-algebra $A$ endowed with a unital *-homomorphism $\psi_A$ from $C(X)$ to the centre of the multiplier algebra of $A$.

**Remark 2.2.** Given $f \in C(X)$ and $a \in A$, we write $fa$ for the product $\psi_A(f) \cdot a$ if no confusion is possible.

There is a natural connection between $C(X)$-algebras and upper semicontinuous $C^*$-bundles over $X$. We first give a formal definition of such bundles:

**Definition 2.3.** Following [?] by an upper semicontinuous $C^*$-bundle we mean a triple $\mathfrak{A} = (p, \mathcal{A}, X)$ where $\mathcal{A}$ is a topological space with a continuous open surjection $p : \mathcal{A} \to X$, together with operations and norms making each fibre $\mathcal{A}_x := p^{-1}(x)$ into a $C^*$-algebra, such that the following conditions are satisfied:

(A1) The maps $\mathbb{C} \times \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$, $\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{A}$ given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous ($\mathcal{A} \times_X \mathcal{A}$ denotes the Whitney sum over $X$).

(A2) The map $\mathcal{A} \to \mathbb{R}$, defined by norm on each fibre, is upper semicontinuous.

(A3) If $x \in X$ and if $(a_n)$ is a net in $\mathcal{A}$ such that $\|a_n\| \to 0$ and $p(a_n) \to x$ in $X$, then $a_n \to 0_x$ in $\mathcal{A}$ ($0_x$ denotes the zero-element of $\mathcal{A}_x$).

If "upper semicontinuous" in (A2) is replaced by "continuous", then we say that $\mathfrak{A}$ is a continuous $C^*$-bundle.

By a *section* of an upper semicontinuous $C^*$-bundle $\mathfrak{A}$ we mean a map $s : X \to \mathcal{A}$ such that $p(s(x)) = x$ for all $x \in X$. We denote by $\Gamma(\mathfrak{A})$ the set of all continuous sections of $\mathfrak{A}$. Then $\Gamma(\mathfrak{A})$ becomes a $C(X)$-algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, given a $C(X)$-algebra $A$, one can always associate an upper semicontinuous $C^*$-bundle $\mathfrak{A}$ over $X$ such that $A \cong \Gamma(\mathfrak{A})$, as follows. Set $J_x := C_x(X) \cdot A$ and note that $J_x$ is a closed two-sided ideal in $A$ (by Cohen factorization theorem [?], [?, Theorem A.6.2])). The quotient $A_x := A/J_x$ is called the fibre at the point $x$, and we denote by $a_x$ the image in $A_x$ of an element $a \in A$. Let

$$\mathcal{A} := \bigsqcup_{x \in X} A_x,$$

and let $p : \mathcal{A} \to X$ be the canonical associated projection. For $a \in A$ we define the map $\hat{a} : X \to \mathcal{A}$ by $\hat{a}(x) := a_x$, and let $\Omega := \{\hat{a} : a \in A\}$. Since for each $a \in A$ we have

$$\|a_x\| = \inf \{\| [1 - f + f(x)] \cdot a \| : f \in C(X) \},$$

the norm function $x \mapsto \|a_x\|$ is upper semicontinuous on $X$. Hence, by Fell’s theorem [?, Theorem C.25] there exists a unique topology on $\mathcal{A}$ for which $\mathfrak{A} := (p, \mathcal{A}, X)$
becomes an upper semicontinuous $C^*$-bundle such that $\mathcal{O} \subseteq \Gamma(\mathfrak{A})$. Moreover, by Lee’s theorem [3, Theorem C.26], $\Omega = \Gamma(\mathfrak{A})$, and the generalized Gelfand transform $\mathcal{G} : a \in A \mapsto \hat{a} \in \Gamma(\mathfrak{A})$, is an isomorphism of $C(X)$-algebras, from $A$ onto $\Gamma(\mathfrak{A})$.

**Definition 2.4.** Let $A$ be a $C(X)$-algebra. If all the norm functions $x \mapsto \|a_x\|$ ($a \in A$) are continuous on $X$, we say that $A$ is a continuous $C(X)$-algebra.

Note that the $C(X)$-algebra $A$ is continuous if and only if $\mathfrak{A}$ is continuous as a $C^*$-bundle.

The $C^*$-algebra $A$ is said to be
- (n-)homogeneous ($n \in \mathbb{N}$), if $\dim \pi = n$ for all $[\pi] \in \hat{A}$,
- (n-)subhomogeneous ($n \in \mathbb{N}$), if $\sup_{[\pi] \in \hat{A}} \dim \pi = n$.

We shall now define the similar notions for $C(X)$-algebras. To do this, first recall that if $D$ is a finite-dimensional $C^*$-algebra, then there is a finite number of central pairwise orthogonal projections $p_1, \ldots, p_m \in \mathcal{Z}(D)$ with $\sum_{i=1}^{m} p_i = 1_D$, such that

$$D = p_1 D \oplus \cdots \oplus p_m D,$$

and each $p_iD$ is $*$-isomorphic to the matrix algebra $M_{n_i}(\mathbb{C})$ (see e.g. [3, Theorem I.11.9]). We define the rank of $D$ as

$$r(D) := \sum_{i=1}^{m} n_i = \sum_{[\pi] \in \hat{D}} \dim \pi.$$

**Definition 2.5.** Let $A$ be a $C(X)$-algebra. We say that $A$ is
- homogeneous all the fibres of $A$ are $*$-isomorphic to the same finite-dimensional $C^*$-algebra.
- subhomogeneous if there exists a positive integer $N$ such that every fibre $A_x$ of $A$ is finite-dimensional with $\dim A_x \leq N$.

**Remark 2.6.** Let $A$ be a $C(X)$-algebra.

(i) $A$ is subhomogeneous if and only if

$$r(A) := \sup \{ r(A_x) : x \in X \} < \infty$$

As in the finite-dimensional case, we call the number $r(A)$ the rank of $A$.

(ii) If $A$ is continuous and homogeneous, then by [3, Lemma 3.1] the underlying $C^*$-bundle $\mathfrak{A}$ is locally trivial.

3. Results

**Remark 3.1.** If $A$ is a unital $C(X)$-algebra, we always assume in this section that the map $\psi_A : C(X) \rightarrow Z(A)$ is injective, so that we can identify $C(X)$ with the unital $C^*$-subalgebra $\psi_A(C(X))$ of $Z(A)$.

In order to prove Theorem ?? we shall need the following two auxiliary results.

**Lemma 3.2.** Let $D$ be a unital $C^*$-algebra. Then $K(D, \mathbb{C}) := K(D, \mathbb{C}1_D) < \infty$ if and only if $D$ is finite-dimensional. In this case we have:

(i) The constant $K(\omega)$ is finite for every faithful state $\omega$ on $D$, which we identify with the corresponding faithful C.E.

$$d \in D \mapsto \omega(d) \cdot 1_D \in \mathbb{C} \cdot 1_D \quad (d \in D).$$
(ii) $K(D, \mathbb{C}) = r(D)$. Moreover, there exists a unique state $\tau$ on $D$ such that
\begin{equation}
(3.1) \quad r(D) \cdot \tau(d)1_D \geq d \quad \text{for all } d \in D_+.
\end{equation}

**Proof.** The equivalence $K(D, \mathbb{C}) < \infty \iff \dim D < \infty$ follows from [?, Lemma 4.5]. Hence, suppose that $D$ is finite-dimensional and let $\omega$ be a faithful state on $D$. The proof will now proceed in two steps.

**Step 1.** Assume that $D$ is simple, i.e. $D = M_n(\mathbb{C})$ for some $n$. If $\text{tr}(\cdot)$ is the standard trace of $M_n(\mathbb{C})$, then there exists a strictly positive matrix $a \in M_n(\mathbb{C})$ with $\text{tr}(a) = 1$ such that
\[ \omega(d) = \text{tr}(ad) \quad (d \in M_n(\mathbb{C})). \]
Let $a = u^* \cdot \text{diag}(\lambda_1, \ldots, \lambda_n) \cdot u$ be a diagonalisation of $a$, where $u \in M_n(\mathbb{C})$ is a unitary and $\lambda_1, \ldots, \lambda_n > 0$ are the eigenvalues of $a$. Then for all $d \in M_n(\mathbb{C})$ one has
\begin{equation}
(3.2) \quad \omega(u^* du) = \text{tr}(au^* du) = \text{tr}(uau^* d) = \text{tr}(\text{diag}(\lambda_1, \ldots, \lambda_n)d).
\end{equation}
The constant $K(\omega)$ is by definition the smallest $K \geq 1$ satisfying
\begin{equation}
(3.3) \quad K \cdot \omega(d)1_D \geq d \quad \text{for all } d \in D_+.
\end{equation}
Thus, (??) and (??) for rank 1 projections in $D$ imply that
\[ K(\omega) = \max \{ \lambda_i^{-1} : 1 \leq i \leq n \}. \]
As $1 = \omega(1) = \sum_{i=1}^n \lambda_i$, one has $K(\omega) \geq n$ for any faithful state $\omega$ on $D$. Also, $K(\omega) = n$ if and only if $\omega = \tau := \frac{1}{n} \text{tr}(\cdot)$. In particular, if $D = M_n(\mathbb{C})$, we have $K(D, \mathbb{C}) = r(D) = n$, and $\tau$ is the unique state on $D$ satisfying (??).

**Step 2.** Suppose that $D$ is an arbitrary finite-dimensional $C^*$-algebra. We decompose $D$ as in (??). For each $1 \leq i \leq m$
\[ \omega_i(p_i d) := \frac{1}{\omega(p_i)} \cdot \omega(p_i d) \quad (d \in D) \]
defines a faithful state on $p_i D$. By Step 1 we have $n_i \leq K(\omega_i) < \infty$ for all $1 \leq i \leq m$. Put
\[ K_\omega := \max \left\{ \frac{K(\omega_i)}{\omega(p_i)} : 1 \leq i \leq m \right\}. \]
We claim that $K(\omega) = K_\omega$. Indeed, for all $d \in D_+$ we have
\begin{align*}
K_\omega \cdot \omega(d)1_D &= \sum_{i=1}^m K_\omega \cdot \omega(p_i)\omega_i(p_i d)1_D \geq \sum_{i=1}^m K(\omega_i) \cdot \omega_i(p_i d)p_i \\
&\geq \sum_{i=1}^m p_i d = d,
\end{align*}
which shows $K(\omega) \leq K_\omega$. On the other hand, for each $d \in D_+$ we have
\[ [\omega(p_i)K(\omega)] \cdot \omega_i(p_i d)p_i \geq p_i d, \]
so that
\begin{equation}
(3.4) \quad \omega(p_i)K(\omega) \geq K(\omega_i) \quad (1 \leq i \leq m).
\end{equation}
This shows $K(\omega) = K_\omega$, as wanted. Also,

$$K(\omega) = \sum_{i=1}^m \omega(p_i)K(\omega) \geq \sum_{i=1}^m K(\omega_i) \geq \sum_{i=1}^m n_i = r(D),$$

so that $K(D, \mathbb{C}) \geq r(D)$.

It remains to show that there exists a unique state $\tau$ on $D$ satisfying (3.5). To do this, suppose that $r(D) = n$, and for each $1 \leq i \leq m$ let $\tau_i$ be the only faithful tracial state on $p_iD \cong M_{n_i}(\mathbb{C})$. Define the state $\tau$ on $D$ by

$$\tau(d) := \frac{1}{n} \sum_{i=1}^m n_i \cdot \tau_i(p_id) \quad (d \in D).$$

As $\tau(p_i) = \frac{n_i}{n}$ and $K(\tau_i) = n_i$ for all $1 \leq i \leq m$, we have $K(\tau) = K_\tau = n$. In particular, $K(D, \mathbb{C}) = n = r(D)$.

To show the uniqueness of this state $\tau$, suppose that $\omega$ is another state on $D$ with $K(\omega) = n$. Then using (3.5) we have

$$\sum_{i=1}^m K(\omega_i) \leq \sum_{i=1}^m \omega(p_i)K(\omega) = K(\omega) = n.$$

But since $K(\omega_i) \geq n_i$ and $\sum_{i=1}^m n_i = n$, we must have $K(\omega_i) = n_i$ for all $1 \leq i \leq m$. By the uniqueness part of Step 1 we conclude that

$$\omega_i = \tau_i \quad \text{for all } 1 \leq i \leq m.$$

Also, $K_\omega = K(\omega) = n$ and $K(\omega_i) = n_i$ imply $\omega(p_i) \geq \frac{n_i}{n}$ for all $1 \leq i \leq m$. Since $\omega$ is a state on $D$ and $\sum_{i=1}^m p_i = 1_D$, we must have

$$\omega(p_i) = \frac{n_i}{n} \quad \text{for all } 1 \leq i \leq m.$$

Finally, (3.5) and (3.6) imply that

$$\omega(d) = \sum_{i=1}^m \omega(p_i)\omega_i(p_id) = \frac{1}{n} \sum_{i=1}^m n_i \cdot \tau_i(p_id) = \tau(d),$$

for all $d \in D$, which finishes the proof.

\[\square\]

**Proposition 3.3.** Let $A$ be a unital $C(X)$-algebra. If $A$ admits a faithful $C(X)$-valued C.E., then $A$ is a continuous $C(X)$-algebra.

**Proof.** This can be deduced from [?, Section 2]. For completeness, we include a short proof of this fact. It suffices to show that all norm functions $x \mapsto \|a_x\|$ $(a \in A)$ are lower semicontinuous on $X$. To prove this, let $E : A \rightarrow C(X)$ be a faithful C.E. and let $L^2(A, E)$ be the completion of the pre-Hilbert $C(X)$-module $A$, with respect to the inner product (3.7). For $a \in A$ let $\Phi(a) : L^2(A, E) \rightarrow L^2(A, E)$ denote the continuous extension of the left multiplication map $a_1 \mapsto aa_1$ $(a \in A)$. Since $E$ is faithful and since

$$(\Phi(a)(a_1), a_2)_E = \langle aa_1, a_2 \rangle_E = E(a^*_1a_2) = \langle a_1, a^*a_2 \rangle_E = \langle a_1, \Phi(a^*)(a_2) \rangle_E,$$

(3.7) holds.
for all \( a_1, a_2 \in A \), the map \( \Phi \) defines an injective \( C(X) \)-linear morphism from \( A \) to the \( C(X) \)-algebra \( \mathcal{B}_{C(X)}(L^2(A, E)) \) of bounded adjointable \( C(X) \)-linear operators on \( L^2(A, E) \). Therefore, for \( a \in A \) and \( x \in X \) we have

\[
\|a_x\| = \|\Phi(a)x\| = \sup\{\|\Phi(a)(a_1), a_2\|_E(x) : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\} = \sup\{\|E(a_1^*a_2^*)(x)\| : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\}.
\]

In particular, the function \( x \mapsto \|a_x\| \) is a supremum of continuous functions \( x \mapsto \|E(a_1^*a_2^*)(x)\| \), so it must be lower semicontinuous on \( X \).

\textbf{Proof of Theorem 3.4.} Let \( E : A \to C(X) \) be a C.E.F.I. As the conditional expectation \( E \) is faithful, Proposition 3.4 implies that the \( C(X) \)-algebra \( A \) is continuous (note that in this case \((A, \langle \cdot, \cdot \rangle_E)\) is already a complete Hilbert \( C(X) \)-module by Theorem 3.4). It remains to show that each fibre \( A_x \) \((x \in X)\) is finite-dimensional and satisfies \( r(A_x) \leq K(E) \). Indeed, for a fixed point \( x \in X \) and \( \varepsilon > 0 \),

\[
\omega_x : a_x \mapsto E(a)(x)
\]

defines a state on a fibre \( A_x \) satisfying

\[
(K(E) + \varepsilon) \cdot \omega_x(a_x)1_x \geq a_x
\]

for all \( a_x \in (A_x)_+ \). Lemma 3.4 now yields \( r(A_x) \leq K(E) \), as wanted. \( \square \)

We shall now give some sufficient conditions on a continuous unital subhomogeneous \( C(X) \)-algebra \( A \) to ensure the existence of a \( C(X) \)-valued C.E.F.I.

\textbf{Proposition 3.4.} Every continuous homogeneous unital \( C(X) \)-algebra \( A \) admits a unique C.E. \( E : A \to C(X) \) such that the map \( r(A) \cdot E - \text{id}_A \) is positive. In particular, \( K(A, C(X)) = r(A) \) in this case.

\textbf{Proof.} The construction of such a C.E. \( E : A \to C(X) \) can be deduced from the proof of [?], Lemma 4.6. But we include here the main steps of the proof for completeness. By assumption all fibres of \( A \) are \(*\)-isomorphic to a fixed finite-dimensional \( C^* \)-algebra \( D \). Suppose that \( r(D) = n \), and let \( \tau \) be a state on \( D \) defined by (3.4). It is easy check that \( \tau \) is invariant under the group \( \text{Aut}(D) \) of \(*\)-automorphisms of \( D \). Since the \( C(X) \)-algebra \( A \) is continuous and homogenous, its underlying bundle \( \mathfrak{A} \) is locally trivial by Remark 3.4. Hence, there exists an open covering \( \{U_\alpha\} \) of \( X \) such that \( \Phi_\alpha : \mathfrak{A}|_{U_\alpha} \cong U_\alpha \times D \), where

- \( \Phi_\alpha \) is an isomorphism of \( C^* \)-bundles, and
- \( \mathfrak{A}|_U \) is the restriction bundle over a subset \( U \subseteq X \).

Fix an element \( a \in A \). For \( x \in X \) choose an index \( \alpha \) such that \( x \in U_\alpha \), and define

\[
E(a)(x) := \tau(\Phi_\alpha(a_x)).
\]

Since \( \tau \) is invariant under the group \( \text{Aut}(D) \), the value \( E(a)(x) \) is well defined, and the local triviality of \( \mathfrak{A} \) implies that the function \( E(a) : x \mapsto E(a)(x) \) is continuous on \( X \). It is now easy to see that the map \( E : a \mapsto E(a) \) defines a \( C(X) \)-valued C.E.F.I. on \( A \). Moreover, by (3.4) we have

\[
n \cdot E(a)(x)1_x \geq a_x, \text{ for all } a \in A_+ \text{ and } x \in X.
\]

Thus, the map \( n \cdot E - \text{id}_A \) is positive and \( E \) is the only C.E. with this property (Lemma 3.4). In particular, \( K(A, C(X)) \leq r(A) \), so Theorem 3.4 yields that \( K(A, C(X)) = n \). \( \square \)
Corollary 3.5. If the unital \( C(X) \)-algebra \( A \) admits a \( C(X) \)-linear embedding into some homogeneous continuous unital \( C(X) \)-algebra \( A' \), then \( A \) admits a \( C(X) \)-valued C.E.F.I.

**Proof.** By Proposition ?? there exists a C.E. \( E' : A' \to C(X) \) of finite index. Then the restriction \( E'|_A : A \to C(X) \) defines a convenient C.E.F.I. \( \square \)

Note that the embedding condition of Corollary ?? is not necessary for the existence of a \( C(X) \)-valued C.E.F.I.. Indeed, in Example ?? we show that there exists a continuous unital \( C(X) \)-algebra \( A \) of rank 2 which does not admit a \( C(X) \)-linear embedding into any continuous homogeneous unital \( C(X) \)-algebra. On the other hand, a direct consequence of Proposition ?? is that \( A \) admits a \( C(X) \)-valued C.E.F.I..

To do this, first recall that a \( C^* \)-algebra \( A \) is said to be **central** if it satisfies the following two conditions:

(i) \( A \) is quasi-central (i.e. no primitive ideal of \( A \) contains \( Z(A) \));

(ii) If \( P, Q \in \text{Prim}(A) \) and \( P \cap Z(A) = Q \cap Z(A) \), then \( P = Q \)

(see [?, ?, ?, ?]). By [?, Proposition 3] a quasi-central \( C^* \)-algebra \( A \) is central if and only if \( \text{Prim}(A) \) is Hausdorff.

**Example 3.6.** By [?, Example 3.5] there exists a continuous \( M_2(\mathbb{C}) \)-bundle \( \mathfrak{A}_0 \) over the second countable locally compact space \( X_0 := \bigsqcup_{n=1}^{\infty} \mathbb{C}P^n \), where \( \mathbb{C}P^n \) is the complex projective space of dimension \( n \), which is not of finite type (that is, \( X_0 \) does not admit a finite open cover \( \{U_i\} \) such that each restriction bundle \( \mathfrak{A}_0|_{U_i} \) is trivial, as a \( C^* \)-bundle). Let \( A_0 \) be the \( C^* \)-algebra \( \Gamma_0(\mathfrak{A}_0) \) consisting of all continuous sections of \( \mathfrak{A}_0 \) which vanish at infinity. Then \( A_0 \) is a \( 2 \)-homogeneous \( C^* \)-algebra with \( \text{Prim}(A_0) = X_0 \). In particular, \( A_0 \) is a central \( C^* \)-algebra with centre \( C_0(X_0) \). Let \( X := X_0 \sqcup \{\infty\} \) be the one-point compactification of \( X_0 \), and let \( A \) be the minimal unitisation of \( A_0 \). By [?, Proposition 3] (or [?, Proposition 3.12]) \( A \) is also a central \( C^* \)-algebra with \( \text{Prim}(A) = X \) and centre \( C(X) \).

In particular, by [?, II.6.5.8] all norm functions \( x \mapsto \|a_x\| \) \((a \in A)\) are continuous on \( X \), so that \( A \) is a continuous unital \( C(X) \)-algebra with fibres \( A_x = M_2(\mathbb{C}) \) \((x \in X_0)\) and \( A_\infty = \mathbb{C} \). Suppose that \( A \) is \( C(X) \)-subalgebra of some continuous homogeneous \( C(X) \)-algebra \( A' \). Then the underlying \( C^* \)-bundle \( \mathfrak{A} \) of \( A \) over \( X \) is a \( C^* \)-subbundle of the underlying \( C^* \)-bundle \( \mathfrak{A}' \) of \( A' \) over \( X \). Since \( A' \) is continuous and homogeneous, \( \mathfrak{A}' \) is locally trivial by Remark ??.

On the other hand, the \( C(X) \)-algebra \( A \) of Example ?? also admits a \( C(X) \)-valued C.E.F.I.. This follows from the following more general fact:

**Proposition 3.7.** Let \( A \) be a continuous unital \( C(X) \)-algebra. If \( r(A) = 2 \), then there exists a conditional expectation \( E : A \to C(X) \) such that the map \( 2 \cdot E - \text{id}_A \) is positive. In particular, \( K(A, C(X)) = r(A) \) in this case.

In order to prove Proposition ??, let us first make the following observation:

**Lemma 3.8.** Let \( A \) be a unital \( C(X) \)-algebra and let \( a \in A_{sa} \). For each point \( x \in X \) let \( \lambda_{\max}(a) \) and \( \lambda_{\min}(a) \) respectively denote the largest and the smallest numbers in
the spectrum of \( a_x \). Then the functions \( x \mapsto \lambda_{\max}(a_x) \) and \( x \mapsto \lambda_{\min}(a_x) \) are upper semicontinuous on \( X \). Furthermore, these functions are continuous on \( X \), whenever \( A \) is a continuous \( C(X) \)-algebra.

**Proof.** This follows directly from the equations

\[
\lambda_{\max}(a_x) = \| a \| 1_x + a_x \| - \| a \| \quad \text{and} \quad \lambda_{\min}(a_x) = \| a \| - \| a \| 1_x - a_x \|.
\]

\( \square \)

**Proof of Proposition 3.8.** As \( r(A) = 2 \), any fibre \( A_x \) is isomorphic to \( \mathbb{C}, \mathbb{C} \oplus \mathbb{C} \) or \( M_2(\mathbb{C}) \). Therefore, for each point \( x \in X \) we can choose a unital embedding \( \varphi_x : A_x \hookrightarrow M_2(\mathbb{C}) \). For \( a \in A \) and \( x \in X \) we define

\[
E(a)(x) := \frac{1}{2} \text{tr}(\varphi_x(a_x)).
\]

Obviously \( E(a) \) is a \( C(X) \)-linear map. If \( a \in A_{sa} \), note that

\[
E(a)(x) = \frac{1}{2}(\lambda_{\min}(a_x) + \lambda_{\max}(a_x))
\]

for all \( x \in X \). By Remark 3.9, \( E(a) \) is a continuous function on \( X \) for all \( a \in A_{sa} \). As \( A \) is the linear span of \( A_{sa} \), we conclude that \( E(a) \in C(X) \) for all \( a \in A \). Therefore, \( E \) defines a C. E. from \( A \) onto \( C(X) \). Further, by (3.8) for all \( a \in A_+ \) and \( x \in X \) we have

\[
2 \cdot E(a)(x)1_x = (\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \cdot 1_x \geq a_x.
\]

This shows that the map \( 2 \cdot E \circ \text{id}_A \) is positive, so that \( K(A, C(X)) = 2 \) by Theorem 3.9. \( \square \)

Let \( A \) be a unital \( C^* \)-algebra and let \( \hat{Z} \) be the maximal ideal space of \( Z(A) \). We may consider \( A \) as a \( C(\hat{Z}) \)-algebra, with respect to the action

\[
f \cdot a := G^{-1}(f)a \quad (f \in C(X), \ a \in A),
\]

where \( G : Z(A) \to C(\hat{Z}) \) is the Gelfand transform. We say that \( A \) is quasi-standard if \( A \) is a continuous \( C(\hat{Z}) \)-algebra and each (Glimm) ideal \( J_x = C_x(\hat{Z})A \) is primal (see [?]).

**Corollary 3.9.** For a unital \( C^* \)-algebra \( A \) the following conditions are equivalent:

(i) There exist a C. E. \( E : A \to Z(A) \) such that the map \( 2 \cdot E \circ \text{id}_A \) is positive.

(ii) \( A \) is either commutative or quasi-standard and \( 2 \)-subhomogeneous.

**Proof.** (i) \( \Rightarrow \) (ii). Suppose that there exists a C. E. \( E : A \to Z(A) \) such that the map \( 2 \cdot E \circ \text{id}_A \) is positive. Then by Theorem 3.8 \( A \) is a continuous \( C(\hat{Z}) \)-algebra and \( r(A_x) \leq 2 \) for all \( x \in \hat{Z} \). In particular, \( A \) as a \( C^* \)-algebra is \( n \)-subhomogeneous, where \( n \in \{1, 2\} \). Hence, by [3, Proposition 4.1] every Glimm ideal of \( A \) is primal. Also, \( n = 1 \) if and only if \( A \) is commutative.

(ii) \( \Rightarrow \) (i). If \( A \) is commutative we have nothing to prove, so suppose that \( A \) is quasi-standard and \( 2 \)-subhomogeneous. Then by [3, Corollary 1, p. 388] for each point \( x \in X \) we have

\[
r(A_x) = \sum_{[\pi_x] \in A_x} \dim \pi_x \leq 2.
\]

It remains to apply Proposition 3.8. \( \square \)
Remark 3.10. At the end of this paper we note that every separable continuous unital C(X)-algebra A admits a faithful C.E. \( E : A \to C(X) \) (see e.g. [?]). In particular, this result applies to continuous subhomogeneous unital C(X)-algebras, when \( X \) is second-countable. In this case for each point \( x \in X \), the map \( E_x : a_x \mapsto E(a)(x) \) defines a faithful state on \( A_x \), so Lemma ?? implies \( K(E_x) < \infty \). However, this does not imply that \( E \) is of finite index. That is, it may happen that \( \sup_{x \in X} K(E_x) = \infty \). Consider for instance the following example:

- Let \( X \) be the closed compact subset \( \{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\} \) of \([0, 1]\).
- Let \( A \) be the continuous \( C(X)\)-subalgebra of \( C(X) \oplus C(X) \) consisting of all pairs \((f, g) \in C(X) \oplus C(X)\) such that \( f(0) = g(0) \).
- Let \( E : A \to C(X) \) be a C.E. fixed by the relations

\[
E(f \oplus g) \left( \frac{1}{n} \right) = \begin{cases} 
\frac{n}{n+1} f\left( \frac{1}{n} \right) + \frac{1}{n+1} g\left( \frac{1}{n} \right) & \text{if } n \text{ is odd} \\
\frac{1}{n+1} f\left( \frac{1}{n} \right) + \frac{n}{n+1} g\left( \frac{1}{n} \right) & \text{otherwise}
\end{cases}
\]

where \((f, g) \in A\).

Then \( E \) is a faithful C.E. which is not of finite index. Indeed, one has

\[
E(f \oplus 0) \left( \frac{1}{2n} \right) = \frac{1}{2n+1} f\left( \frac{1}{2n} \right)
\]

for all \( f \in C_0(X \setminus \{0\}) \) and all integers \( n \in \mathbb{N} \). Consequently, a convenient constant \( K \) would satisfy \( K \geq 2n + 1 \) for all \( n \in \mathbb{N} \), which is impossible.

We end this paper with some unresolved problems:

Problem 3.11. Is the converse of Theorem ?? also true? Moreover, does every continuous subhomogeneous unital C(X)-algebra \( A \) admit a C.E. \( E : A \to C(X) \) such that the map \( r(A) \cdot E - \text{id}_A \) is positive? In particular, do we always have \( K(A, C(X)) = r(A) \)?

References

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Institut de Mathématiques de Jussieu, Bâtiment Sophie Germain, Case 7012, F-75205 Paris cedex 13
E-mail address: Etienne.Blanchard@math.jussieu.fr

Department of Mathematics, University of Zagreb, Bijenicka 30, 10000 Zagreb, Croatia, and Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21000 Novi Sad, Serbia
E-mail address: ilja@math.hr