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ON UNITAL $C(X)$-ALGEBRAS AND $C(X)$-VALUED CONDITIONAL EXPECTATIONS OF FINITE INDEX

ETIENNE BLANCHARD AND ILJA GOGIĆ

Abstract. Let $X$ be a compact Hausdorff space and let $A$ be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of the centre of $A$. We consider the problem of characterizing the existence of a conditional expectation $E : A \to C(X)$ of finite index in terms of the underlying $C^*$-bundle of $A$ over $X$. More precisely, we show that if $A$ admits a $C(X)$-valued conditional expectation of finite index, then $A$ is necessarily a continuous $C(X)$-algebra, and there exists a positive integer $N$ such that every fibre $A_x$ of $A$ is finite-dimensional, with $\dim A_x \leq N$. We also give some sufficient conditions on $A$ that ensure the existence of a $C(X)$-valued conditional expectation of finite index.

1. Introduction

Let $B \subseteq A$ be two unital $C^*$-algebras with the same unit element. A conditional expectation (abbreviated by C.E.) from $A$ to $B$ is a completely positive contraction $E : A \to B$ such that $E(b) = b$ for all $b \in B$, and which is $B$-bilinear, i.e.

$$E(b_1ab_2) = b_1E(a)b_2$$

for all $a \in A$ and $b_1, b_2 \in B$. By a result of Y. Tomiyama (see [22, Theorem 1] or [4, Theorem II.6.10.2]), a map $E : A \to B$ is a C.E. if and only if $E$ is a projection of norm one.

If $E(\ast a) = 0$ ($a \in A$) implies $a = 0$, $E$ is said to be faithful. Every faithful conditional expectation $E : A \to B$ introduces a pre-Hilbert $B$-module structure on $A$, whose inner product is defined by

$$\langle a_1, a_2 \rangle_E := E(a_1^*a_2) \quad (a_1, a_2 \in A).$$

The notion of finite index was introduced by V. F. R. Jones [14] in order to classify the subfactors of a type II$_1$ factor. Soon afterwards H. Kosaki [16] extended the Jones index theory to arbitrary factors. In order to generalize the results of [14, 16], M. Pimsner and S. Popa introduced in [19, 20] a definition for conditional expectations of finite index in the context of $W^*$-algebras: There must exist a constant $K \geq 1$ such that the map $K \cdot E - \text{id}_A$ is positive on $A$. Then, following the idea of M. Baillet, Y. Denizeau and J.-F. Havet (see [3]), the index of $E$ can be defined in the following way: Since the map $K \cdot E - \text{id}_A$ is positive, $E$ defines a (complete) Hilbert $B$-module structure on $A$, with respect to the inner product

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(1.1). If \( \{ x_i \} \) is a quasi-orthonormal basis in \( A \), the index of \( E \) is the sum \( \sum_{i=1}^{\infty} x_i^* x_i \), with respect to the ultraweak topology.

Y. Watatani also considered C.E. of (algebraically) finite index, when the original C\(^*\)-algebra \( A \) is a finitely generated Hilbert C\(^*\)-module over \( B \) (see [23]).

The results of M. Baillet, Y. Denizeau and J.-F. Havet in [3] also indicated that there might occur some difficulties in order to extend the notion of "finite index" for conditional expectations of C\(^*\)-algebras with arbitrary centres. However, this problem was solved by M. Frank and E. Kirchberg in [11]. The main result of their paper is [11, Theorem 1]:

**Theorem 1.1** (M. Frank and E. Kirchberg). For a C.E. \( E : A \to B \), where \( B \subseteq A \) are unital C\(^*\)-algebras with the same unit element, the following conditions are equivalent:

(i) There exists a constant \( K \geq 1 \) such that the map \( K \cdot E - \text{id}_A \) is positive.

(ii) There exists a constant \( L \geq 1 \) such that the map \( L \cdot E - \text{id}_A \) is completely positive.

(iii) \( A \) becomes a (complete) Hilbert \( B \)-module when equipped with the inner product (1.1).

Moreover, if 
\[
K(E) := \inf \{ K \geq 1 : K \cdot E - \text{id}_A \text{ is positive} \},
\]
\[
L(E) := \inf \{ L \geq 1 : L \cdot E - \text{id}_A \text{ is completely positive} \},
\]
with \( K(E) = \infty \) or \( L(E) = \infty \) if no such number \( K \) or \( L \) exists, then
\[
K(E) \leq L(E) \leq [K(E)] K(E),
\]
where \([\cdot]\) denotes the integer part of a real number.

The importance of this result is that it gives the right general definition for conditional expectations on C\(^*\)-algebras to be of finite index:

**Definition 1.2.** If \( B \subseteq A \) are two unital C\(^*\)-algebras with the same unit element, then a C.E. \( E : A \to B \) is said to be of finite index (abbreviated C.E.F.I.) if \( E \) satisfies one of the equivalent conditions of Theorem 1.1.

In this case the index value of \( E \) can be calculated in the enveloping von Neumann algebra \( A^{**} \) (see [11, Definition 3.1]).

For a unital inclusion \( A \subseteq B \) of unital C\(^*\)-algebras we introduce the following constant
\[
K(A, B) := \inf \{ K(E) : E : A \to B \text{ is C.E.F.I.} \},
\]
with \( K(A, B) = \infty \), if no such C.E.F.I. exists. This constant will play an important role in this paper.

More recently, A. Pavlov and E. Troitsky considered in [17] the problem of existence of a C.E.F.I. \( E : C(Y) \to C(X) \) for a unital inclusion \( \varphi : C(X) \hookrightarrow C(Y) \) of unital commutative C\(^*\)-algebras. The main result of their paper is [17, Theorem 1.1], which shows that such a C.E.F.I. exists if and only if the transpose map \( \varphi^* : Y \to X \) is a branched covering. This means that \( \varphi^* \) is an open map with uniformly bounded number of pre-images (i.e. \( \sup_{x \in X} |\varphi^{-1}_x(x)| < \infty \)). This result motivated A. Pavlov and E. Troitsky to define the noncommutative branched coverings, as unital inclusion \( B \subseteq A \) of unital C\(^*\)-algebras such that there exists a C.E.F.I. from \( A \) to \( B \) (see [17, Definition 1.2]).
Using the above inclusion $\varphi : C(X) \to C(Y)$ we may consider $C(Y)$ as a $C(X)$-algebra. Then the map $\varphi_*$ is open if and only if $C(Y)$ is a continuous $C(X)$-algebra, and $\varphi_*$ has uniformly bounded number of pre-images if and only if $C(Y)$ is subhomogeneous $C(X)$-algebra. This means that there exists a positive integer $N$ such that every fibre $C(Y)_x$ of $C(Y)$ is finite-dimensional with $\dim C(Y)_x \leq N$ (see Section 2). Therefore, we can restate [17, Theorem 1.1] in terms of $C(X)$-algebras as follows:

**Theorem 1.3** (A. Pavlov and E. Troitsky). Let $A$ be a unital commutative $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of $A$. Then $A$ admits a $C(X)$-valued C.E.F.I. if and only if $A$ is a continuous subhomogeneous $C(X)$-algebra.

The purpose of the present paper is to consider a possible extension of Theorem 1.3 to the case when $A$ is an arbitrary (not necessarily commutative) unital $C(X)$-algebra. The necessary condition for the existence of a $C(X)$-valued C.E.F.I. appears to be identical to the one of Theorem 1.3:

**Theorem 1.4.** Let $A$ be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of the centre of $A$. If $A$ admits a $C(X)$-valued C.E.F.I., then $A$ is a continuous subhomogeneous $C(X)$-algebra. Moreover, in this case the following inequality holds:

$$K(A, C(X)) \geq r(A),$$

where $r(A)$ is the rank of $A$, i.e.

$$r(A) = \max \left\{ \sum_{\pi_x \in \hat{A}_x} \dim \pi_x : x \in X \right\}.$$

We shall prove Theorem 1.4 in Section 3. At the moment we do not know if the converse of Theorem 1.4 also holds. However, if all the fibres of a continuous unital $C(X)$-algebra $A$ are $*$-isomorphic to the same finite-dimensional $C^*$-algebra (i.e. $A$ is a homogeneous $C(X)$-algebra), then there exists a unique C.E. $E : A \to C(X)$ such that the map $r(A) \cdot E - \text{id}_A$ is positive (Proposition 3.4). In particular, we have the equality $K(A, C(X)) = r(A)$ in this case. Also, a direct consequence of this fact is that any unital $C(X)$-algebra $A$ which can be embedded as a $C(X)$-subalgebra of some continuous homogeneous unital $C(X)$-algebra also admits a $C(X)$-valued C.E.F.I.. However, this embedding condition is not necessary for the existence of such C.E.F.I.. Indeed, there exists a continuous unital $C(X)$-algebra $A$ over a second-countable compact Hausdorff space $X$ with fibres $M_2(\mathbb{C})$ or $\mathbb{C}$ which admits a $C(X)$-valued C.E.F.I., but which cannot be embedded as a $C(X)$-subalgebra into any continuous homogeneous unital $C(X)$-algebra (Example 3.6).

At the end of this paper we also show that any continuous unital $C(X)$-algebra $A$ of rank 2 admits a C.E. $E : A \to C(X)$ such that the map $2 \cdot E - \text{id}_A$ is positive (Proposition 3.7). In particular, the equality $K(A, C(X)) = r(A)$ also holds in this class of $C(X)$-algebras.

2. **Notation and preliminaries**

Throughout this paper $A$ will be a $C^*$-algebra. We denote by $A_{sa}$ and $A_+$ the self-adjoint and the positive parts of $A$. The centre of $A$ is denoted by $Z(A)$. By
Ã and Prin(Ã) we respectively denote the spectrum of A (i.e. the set of all classes of irreducible representations of A) and the primitive spectrum of A (i.e. the set of all primitive ideals of A), equipped with the Jacobson topology. By a dimension of [π] ∈ Ā, which is denoted by dim π, we mean the dimension of the underlying Hilbert space of some representative of [π].

Let X be a compact Hausdorff space. For each point \( x \in X \) let
\[
C_*(x) := \{ f \in C(X) : f(x) = 0 \}
\]
be the corresponding maximal ideal of \( C(X) \).

**Definition 2.1.** A \( C(X) \)-algebra is a \( C^* \)-algebra \( A \) endowed with a unital \( * \)-homomorphism \( \psi_A \) from \( C(X) \) to the centre of the multiplier algebra of \( A \).

**Remark 2.2.** Given \( f \in C(X) \) and \( a \in A \), we write \( fa \) for the product \( \psi_A(f) \cdot a \) if no confusion is possible.

There is a natural connection between \( C(X) \)-algebras and upper semicontinuous \( C^* \)-bundles over \( X \). We first give a formal definition of such bundles:

**Definition 2.3.** Following [24] by an upper semicontinuous \( C^* \)-bundle we mean a triple \( \mathfrak{A} = (p, \mathcal{A}, X) \) where \( \mathcal{A} \) is a topological space with a continuous open surjection \( p : \mathcal{A} \rightarrow X \), together with operations and norms making each fibre \( \mathcal{A}_x := p^{-1}(x) \) into a \( C^* \)-algebra, such that the following conditions are satisfied:

A1 The maps \( C \times \mathcal{A} \rightarrow \mathcal{A}, \mathcal{A} \times_X \mathcal{A} \rightarrow \mathcal{A}, \mathcal{A} \rightarrow \mathcal{A} \) and \( \mathcal{A} \rightarrow \mathcal{A} \) given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous (\( \mathcal{A} \times_X \mathcal{A} \) denotes the Whitney sum over \( X \)).

A2 The map \( \mathcal{A} \rightarrow \mathbb{R} \), defined by norm on each fibre, is upper semicontinuous.

A3 If \( x \in X \) and if \( (a_n) \) is a net in \( \mathcal{A} \) such that \( \|a_n\| \rightarrow 0 \) and \( p(a_n) \rightarrow x \) in \( X \), then \( a_n \rightarrow 0_x \) in \( \mathcal{A} \) (\( 0_x \) denotes the zero-element of \( \mathcal{A}_x \)).

If ”upper semicontinuous” in (A2) is replaced by ”continuous”, then we say that \( \mathfrak{A} \) is a continuous \( C^* \)-bundle.

By a section of an upper semicontinuous \( C^* \)-bundle \( \mathfrak{A} \) we mean a map \( s : X \rightarrow \mathcal{A} \) such that \( p(s(x)) = x \) for all \( x \in X \). We denote by \( \Gamma(\mathfrak{A}) \) the set of all continuous sections of \( \mathfrak{A} \). Then \( \Gamma(\mathfrak{A}) \) becomes a \( C(X) \)-algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, given a \( C(X) \)-algebra \( A \), one can always associate an upper semicontinuous \( C^* \)-bundle \( \mathfrak{A} \) over \( X \) such that \( A \cong \Gamma(\mathfrak{A}) \), as follows. Set \( J_x := C_*(x) \cdot A \) and note that \( J_x \) is a closed two-sided ideal in \( A \) (by Cohen factorization theorem [7], [6, Theorem A.6.2])). The quotient \( A_x := A/J_x \) is called the fibre at the point \( x \), and we denote by \( a_x \) the image in \( A_x \) of an element \( a \in A \). Let
\[
\mathcal{A} := \bigsqcup_{x \in X} A_x,
\]
and let \( p : \mathcal{A} \rightarrow X \) be the canonical associated projection. For \( a \in A \) we define the map \( \tilde{a} : X \rightarrow \mathcal{A} \) by \( \tilde{a}(x) := a_x \), and let \( \Omega := \{ \tilde{a} : a \in A \} \). Since for each \( a \in A \) we have
\[
\|a_x\| = \inf\{\|1 - f + f(x)\| : f \in C(X)\},
\]
the norm function \( x \mapsto \|a_x\| \) is upper semicontinuous on \( X \). Hence, by Fell’s theorem [24, Theorem C.25] there exists a unique topology on \( \mathcal{A} \) for which \( \mathfrak{A} := (p, \mathcal{A}, X) \)
becomes an upper semicontinuous $C^*$-bundle such that $\Omega \subseteq \Gamma(\mathfrak{A})$. Moreover, by Lee’s theorem [24, Theorem C.26], $\Omega = \Gamma(\mathfrak{A})$, and the generalized Gelfand transform $\mathcal{G} : a \in A \mapsto \hat{a} \in \Gamma(\mathfrak{A})$, is an isomorphism of $C(X)$-algebras, from $A$ onto $\Gamma(\mathfrak{A})$.

**Definition 2.4.** Let $A$ be a $C(X)$-algebra. If all the norm functions $x \mapsto \|a_x\|$ ($a \in A$) are continuous on $X$, we say that $A$ is a *continuous* $C(X)$-algebra.

Note that the $C(X)$-algebra $A$ is continuous if and only if $\mathfrak{A}$ is continuous as a $C^*$-bundle. The $C^*$-algebra $A$ is said to be

- *(n)*-homogeneous ($n \in \mathbb{N}$), if $\dim \pi = n$ for all $[\pi] \in \hat{A}$,
- *(n)*-subhomogeneous ($n \in \mathbb{N}$), if $\sup_{[\pi] \in \hat{A}} \dim \pi = n$.

We shall now define the similar notions for $C(X)$-algebras. To do this, first recall that if $D$ is a finite-dimensional $C^*$-algebra, then there is a finite number of central pairwise orthogonal projections $p_1, \ldots, p_m \in Z(D)$ with $\sum_{i=1}^m p_i = 1_D$, such that

$$D = p_1D \oplus \cdots \oplus p_mD,$$

and each $p_iD$ is $*$-isomorphic to the matrix algebra $M_{n_i}(\mathbb{C})$ (see e.g. [21, Theorem I.11.9]). We define the **rank** of $D$ as

$$r(D) := \sum_{i=1}^m p_i = \sum_{[\pi] \in \hat{D}} \dim \pi.$$

**Definition 2.5.** Let $A$ be a $C(X)$-algebra. We say that $A$ is

- **homogeneous** all the fibres of $A$ are $*$-isomorphic to the same finite-dimensional $C^*$-algebra,
- **subhomogeneous** if there exists a positive integer $N$ such that every fibre $A_x$ of $A$ is finite-dimensional with $\dim A_x \leq N$.

**Remark 2.6.** Let $A$ be a $C(X)$-algebra.

(i) $A$ is subhomogeneous if and only if

$$r(A) := \sup \{r(A_x) : x \in X\} < \infty$$

As in the finite-dimensional case, we call the number $r(A)$ the **rank** of $A$.

(ii) If $A$ is continuous and homogeneous, then by [10, Lemma 3.1] the underlying $C^*$-bundle $\mathfrak{A}$ is locally trivial.

### 3. Results

**Remark 3.1.** If $A$ is a unital $C(X)$-algebra, we always assume in this section that the map $\psi_A : C(X) \to Z(A)$ is injective, so that we can identify $C(X)$ with the unital $C^*$-subalgebra $\psi_A(C(X))$ of $Z(A)$.

In order to prove Theorem 1.4 we shall need the following two auxiliary results.

**Lemma 3.2.** Let $D$ be a unital $C^*$-algebra. Then $K(D, \mathbb{C}) := K(D, \mathbb{C}1_D) < \infty$ if and only if $D$ is finite-dimensional. In this case we have:

(i) The constant $K(\omega)$ is finite for every faithful state $\omega$ on $D$, which we identify with the corresponding faithful C.E.

$$d \in D \mapsto \omega(d) \cdot 1_D \in \mathbb{C} \cdot 1_D \quad (d \in D).$$
(ii) \( K(D, \mathbb{C}) = r(D) \). Moreover, there exists a unique state \( \tau \) on \( D \) such that
\[
K(D, \mathbb{C}) = r(D) \cdot \tau(d)1_D \geq d \quad \text{for all } d \in D_+.
\]

Proof. The equivalence \( K(D, \mathbb{C}) < \infty \iff \dim D < \infty \) follows from [13, Lemma 4.5]. Hence, suppose that \( D \) is finite-dimensional and let \( \omega \) be a faithful state on \( D \). The proof will now proceed in two steps.

**Step 1.** Assume that \( D \) is simple, i.e. \( D = M_n(\mathbb{C}) \) for some \( n \). If \( \text{tr}(\cdot) \) is the standard trace of \( M_n(\mathbb{C}) \), then there exists a strictly positive matrix \( a \in M_n(\mathbb{C}) \) with \( \text{tr}(a) = 1 \) such that
\[
\omega(d) = \text{tr}(ad) \quad (d \in M_n(\mathbb{C})).
\]
Let \( a = u^* \cdot \text{diag}(\lambda_1, \ldots, \lambda_n) \cdot u \) be a diagonalisation of \( a \), where \( u \in M_n(\mathbb{C}) \) is a unitary and \( \lambda_1, \ldots, \lambda_n > 0 \) are the eigenvalues of \( a \). Then for all \( d \in M_n(\mathbb{C}) \) one has
\[
\omega(u^*du) = \text{tr}(au^*du) = \text{tr}(uau^*d) = \text{tr}(\text{diag}(\lambda_1, \ldots, \lambda_n)d).
\]
The constant \( K(\omega) \) is by definition the smallest \( K \geq 1 \) satisfying
\[
K \cdot \omega(d)1_D \geq d \quad \text{for all } d \in D_+.
\]
Thus, (3.2) and (3.3) for rank 1 projections in \( D \) imply that
\[
K(\omega) = \max\{\lambda_i^{-1} : 1 \leq i \leq n\}.
\]
As \( 1 = \omega(1) = \sum_{i=1}^n \lambda_i \), one has \( K(\omega) \geq n \) for any faithful state \( \omega \) on \( D \). Also, \( K(\omega) = n \) if and only if \( \omega = \tau := \frac{1}{n} \text{tr}(\cdot) \). In particular, if \( D = M_n(\mathbb{C}) \), we have \( K(D, \mathbb{C}) = r(D) = n \), and \( \tau \) is the unique state on \( D \) satisfying (3.1).

**Step 2.** Suppose that \( D \) is an arbitrary finite-dimensional \( C^* \)-algebra. We decompose \( D \) as in (2.1). For each \( 1 \leq i \leq m \)
\[
\omega_i(p_i d) := \frac{1}{\omega(p_i)} \cdot \omega(p_i d) \quad (d \in D)
\]
defines a faithful state on \( p_i D \). By Step 1 we have \( n_i \leq K(\omega_i) < \infty \) for all \( 1 \leq i \leq m \). Put
\[
K_\omega := \max \left\{ \frac{K(\omega_i)}{\omega(p_i)} : 1 \leq i \leq m \right\}.
\]
We claim that \( K(\omega) = K_\omega \). Indeed, for all \( d \in D_+ \) we have
\[
K_\omega \cdot \omega(d)1_D = \sum_{i=1}^m K_\omega \cdot \omega(p_i)\omega_i(p_i d)1_D \geq \sum_{i=1}^m K(\omega_i) \cdot \omega_i(p_i d)p_i
\]
\[
\geq \sum_{i=1}^m p_i d = d,
\]
which shows \( K(\omega) \leq K_\omega \). On the other hand, for each \( d \in D_+ \) we have
\[
[\omega(p_i) K(\omega)] \cdot \omega_i(p_i d)p_i \geq p_i d,
\]
so that
\[
(3.4) \quad \omega(p_i) K(\omega) \geq K(\omega_i) \quad (1 \leq i \leq m).
\]
This shows \( K(\omega) = K_\omega \), as wanted. Also,

\[
K(\omega) = \sum_{i=1}^{m} \omega(p_i)K(\omega) \geq \sum_{i=1}^{m} K(\omega_i) \geq \sum_{i=1}^{m} n_i = r(D),
\]

so that \( K(D, \mathbb{C}) \geq r(D) \).

It remains to show that there exists a unique state \( \tau \) on \( D \) satisfying (3.1). To do this, suppose that \( r(D) = n \), and for each \( 1 \leq i \leq m \) let \( \tau_i \) be the only faithful tracial state on \( p_i D \cong M_{n_i} (\mathbb{C}) \). Define the state \( \tau \) on \( D \) by

\[
(3.5) \quad \tau(d) := \frac{1}{n} \sum_{i=1}^{m} n_i \cdot \tau_i(p_id) \quad (d \in D).
\]

As \( \tau(p_i) = \frac{n_i}{n} \) and \( K(\tau_i) = n_i \) for all \( 1 \leq i \leq m \), we have \( K(\tau) = K_\tau = n \). In particular, \( K(D, \mathbb{C}) = n = r(D) \).

To show the uniqueness of this state \( \tau \), suppose that \( \omega \) is another state on \( D \) with \( K(\omega) = n \). Then using (3.4) we have

\[
\sum_{i=1}^{m} K(\omega_i) \leq \sum_{i=1}^{m} \omega(p_i)K(\omega) = K(\omega) = n.
\]

But since \( K(\omega_i) \geq n_i \) and \( \sum_{i=1}^{m} n_i = n \), we must have \( K(\omega_i) = n_i \) for all \( 1 \leq i \leq m \). By the uniqueness part of Step 1 we conclude that

\[
(3.6) \quad \omega_i = \tau_i \quad \text{for all} \quad 1 \leq i \leq m.
\]

Also, \( K_\omega = K(\omega) = n \) and \( K(\omega_i) = n_i \) imply \( \omega(p_i) \geq \frac{n_i}{n} \) for all \( 1 \leq i \leq m \). Since \( \omega \) is a state on \( D \) and \( \sum_{i=1}^{m} p_i = 1_D \), we must have

\[
(3.7) \quad \omega(p_i) = \frac{n_i}{n} \quad \text{for all} \quad 1 \leq i \leq m.
\]

Finally, (3.6) and (3.7) imply that

\[
\omega(d) = \sum_{i=1}^{m} \omega(p_i)\omega_i(p_id) = \frac{1}{n} \sum_{i=1}^{m} n_i \cdot \tau_i(p_id) = \tau(d),
\]

for all \( d \in D \), which finishes the proof. \( \square \)

**Proposition 3.3.** Let \( A \) be a unital \( C(X) \)-algebra. If \( A \) admits a faithful \( C(X) \)-valued \( C.E. \), then \( A \) is a continuous \( C(X) \)-algebra.

**Proof.** This can be deduced from [5, Section 2]. For completeness, we include a short proof of this fact. It suffices to show that all norm functions \( x \mapsto \|a_x\| \) \((a \in A)\) are lower semicontinuous on \( X \). To prove this, let \( E : A \to C(X) \) be a faithful C.E. and let \( L^2(A, E) \) be the completion of the pre-Hilbert \( C(X) \)-module \( A \), with respect to the inner product (1.1). For \( a \in A \) let \( \Phi(a) : L^2(A, E) \to L^2(A, E) \) denote the continuous extension of the left multiplication map \( a_1 \mapsto aa_1 \) \((a \in A)\).

Since \( E \) is faithful and since

\[
(\Phi(a)(a_1), a_2)_E = (a_1, a_2)_E = E(a_1^*a_2) = (a_1, a^*a_2)_E \quad = (a_1, \Phi(a^*)(a_2))_E,
\]

we conclude that \( \Phi(a) \) is a continuous map \((a \in A)\). To complete the proof, we need to show that \( \Phi \) is a homomorphism. This follows from the ultraweak continuity of \( \Phi \) and the fact that \( \Phi(a)x = ax \) for all \( x \in X \). \( \square \)
for all $a_1, a_2 \in A$, the map $\Phi$ defines an injective $C(X)$-linear morphism from $A$ to the $C(X)$-algebra $\mathcal{B}_{C(X)}(L^2(A,E))$ of bounded adjointable $C(X)$-linear operators on $L^2(A,E)$. Therefore, for $a \in A$ and $x \in X$ we have

$$\|a_x\| = \|\Phi(a)_x\| = \sup\{|\langle \Phi(a)(a_1), a_2 \rangle_E(x) \rangle : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\} = \sup\{|E(a_1a^*a_2)(x)) : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\}.$$ 

In particular, the function $x \mapsto \|a_x\|$ is a supremum of continuous functions $x \mapsto |E(a_1a^*a_2)(x)|$ with $\|a_1\|_E = \|a_2\|_E = 1$, so it must be lower semicontinuous on $X$. □

**Proof of Theorem 1.4.** Let $E : A \to C(X)$ be a C.E.F.I. As the conditional expectation $E$ is faithful, Proposition 3.3 implies that the $C(X)$-algebra $A$ is continuous (note that in this case $(A, \langle \cdot, \cdot \rangle_E)$ is already a complete Hilbert $C(X)$-module by Theorem 1.1). It remains to show that each fibre $A_x$ ($x \in X$) is finite-dimensional and satisfies $r(A_x) \leq K(E)$. Indeed, for a fixed point $x \in X$ and $\varepsilon > 0$,

$$\omega_x : a_x \mapsto E(a)(x)$$

defines a state on a fibre $A_x$ satisfying

$$(K(E) + \varepsilon) \cdot \omega_x(a_x)1_x \geq a_x$$

for all $a_x \in (A_x)_+$. Lemma 3.2 now yields $r(A_x) \leq K(E)$, as wanted. □

We shall now give some sufficient conditions on a continuous unital subhomogeneous $C(X)$-algebra $A$ to ensure the existence of a $C(X)$-valued C.E.F.I.

**Proposition 3.4.** Every continuous homogeneous unital $C(X)$-algebra $A$ admits a unique C.E. $E : A \to C(X)$ such that the map $r(A) : E - \text{id}_A$ is positive. In particular, $K(A,C(X)) = r(A)$ in this case.

**Proof.** The construction of such a C.E. $E : A \to C(X)$ can be deduced from the proof of [13, Lemma 4.6]. But we include here the main steps of the proof for completeness. By assumption all fibres of $A$ are $*$-isomorphic to a fixed finite-dimensional $C^*$-algebra $D$. Suppose that $r(D) = n$, and let $\tau$ be a state on $D$ defined by (3.5). It is easy check that $\tau$ is invariant under the group Aut($D$) of $*$-automorphisms of $D$. Since the $C(X)$-algebra $A$ is continuous and homogeneous, its underlying bundle $\mathfrak{A}$ is locally trivial by Remark 2.6. Hence, there exists an open covering $\{U_\alpha\}$ of $X$ such that $\Phi_\alpha : \mathfrak{A}|U_\alpha \cong U_\alpha \times D$, where

- $\Phi_\alpha$ is an isomorphism of $C^*$-bundles, and
- $\mathfrak{A}|U$ is the restriction bundle over a subset $U \subseteq X$.

Fix an element $a \in A$. For $x \in X$ choose an index $\alpha$ such that $x \in U_\alpha$, and define

$$E(a)(x) := \tau(\Phi(a)(a_x)).$$

Since $\tau$ is invariant under the group Aut($D$), the value $E(a)(x)$ is well defined, and the local triviality of $\mathfrak{A}$ implies that the function $E(a) : x \mapsto E(a)(x)$ is continuous on $X$. It is now easy to see that the map $E : a \to E(a)$ defines a $C(X)$-valued C.E.F.I. on $A$. Moreover, by (3.1) we have

$$n \cdot E(a)(x)1_x \geq a_x,$$

for all $a \in A_+$ and $x \in X$.

Thus, the map $n \cdot E - \text{id}_A$ is positive and $E$ is the only C.E. with this property (Lemma 3.2). In particular, $K(A,C(X)) \leq r(A)$, so Theorem 1.4 yields that $K(A,C(X)) = n$. □
Corollary 3.5. If the unital $C(X)$-algebra $A$ admits a $C(X)$-linear embedding into some homogeneous continuous unital $C(X)$-algebra $A'$, then $A$ admits a $C(X)$-valued C.E.F.I..

Proof. By Proposition 3.4 there exists a C.E. $E' : A' \to C(X)$ of finite index. Then the restriction $E'|_A : A \to C(X)$ defines a convenient C.E.F.I. 

Note that the embedding condition of Corollary 3.5 is not necessary for the existence of a $C(X)$-valued C.E.F.I.. Indeed, in Example 3.6 we show that there exists a continuous unital $C(X)$-algebra $A$ of rank 2 which does not admit a $C(X)$-linear embedding into any continuous homogeneous unital $C(X)$-algebra. On the other hand, a direct consequence of Proposition 3.7 is that $A$ admits a $C(X)$-valued C.E.F.I..

To do this, first recall that a $C^*$-algebra $A$ is said to be central if it satisfies the following two conditions:

(i) $A$ is quasi-central (i.e. no primitive ideal of $A$ contains $Z(A)$);

(ii) If $P, Q \in \text{Prim}(A)$ and $P \cap Z(A) = Q \cap Z(A)$, then $P = Q$ (see [1, 8, 12, 15]). By [8, Proposition 3] a quasi-central $C^*$-algebra $A$ is central if and only if $\text{Prim}(A)$ is Hausdorff.

Example 3.6. By [18, Example 3.5] there exists a continuous $M_2(\mathbb{C})$-bundle $\mathfrak{A}_0$ over the second countable locally compact space $X_0 := \bigsqcup_{n=1}^{\infty} \mathbb{C}P^n$, where $\mathbb{C}P^n$ is the complex projective space of dimension $n$, which is not of finite type (that is, $X_0$ does not admit a finite open cover $\{U_i\}$ such that each restriction bundle $\mathfrak{A}_0|_{U_i}$ is trivial, as a $C^*$-bundle). Let $A_0$ be the $C^*$-algebra $\Gamma_0(\mathfrak{A}_0)$ consisting of all continuous sections of all continuous sections of $\mathfrak{A}_0$ which vanish at infinity. Then $A_0$ is a 2-homogeneous $C^*$-algebra with $\text{Prim}(A_0) = X_0$. In particular $A_0$ is a central $C^*$-algebra with centre $C_0(X_0)$. Let $X := X_0 \cup \{\infty\}$ be the one-point compactification of $X_0$, and let $A$ be the minimal unitisation of $A_0$. By [8, Proposition 3] (or [12, Proposition 3.12]) $A$ is also a central $C^*$-algebra with $\text{Prim}(A) = X$ and centre $C(X)$. In particular, by [4, II.6.5.8] all norm functions $x \mapsto \|a_x\| (a \in A)$ are continuous on $X$, so that $A$ is a continuous unital $C(X)$-algebra with fibres $A_x = M_2(\mathbb{C}) (x \in X_0)$ and $A_\infty = \mathbb{C}$. Suppose that $A$ is $C(X)$-subalgebra of some continuous homogeneous $C(X)$-algebra $A'$. Then the underlying $C^*$-bundle $\mathfrak{A}$ of $A$ over $X$ is a $C^*$-subbundle of the underlying $C^*$-bundle $\mathfrak{A}'$ of $A'$ over $X$. Since $A'$ is continuous and homogeneous, $\mathfrak{A}'$ is locally trivial by Remark 2.6. Hence, since $X$ is compact, $\mathfrak{A}'$ is of finite type. Using [18, Lemma 2.6] we conclude that $\mathfrak{A}$ is of finite type as a vector bundle. In particular, $\mathfrak{A}_0$ is of finite type as a vector bundle, since $\mathfrak{A}_0 = \mathfrak{A}|_{X_0}$. As $\mathfrak{A}_0$ is a $M_2(\mathbb{C})$-bundle, this implies by [18, Proposition 2.9] that $\mathfrak{A}_0$ is also of finite type as a $C^*$-bundle; a contradiction.

On the other hand, the $C(X)$-algebra $A$ of Example 3.6 also admits a $C(X)$-valued C.E.F.I. This follows from the following more general fact:

Proposition 3.7. Let $A$ be a continuous unital $C(X)$-algebra. If $r(A) = 2$, then there exists a conditional expectation $E : A \to C(X)$ such that the map $2 \cdot E - \text{id}_A$ is positive. In particular, $K(A,C(X)) = r(A)$ in this case.

In order to prove Proposition 3.7, let us first make the following observation:

Lemma 3.8. Let $A$ be a unital $C(X)$-algebra and let $a \in A_{sa}$. For each point $x \in X$ let $\lambda_{\max}(a)$ and $\lambda_{\min}(a)$ respectively denote the largest and the smallest numbers in
the spectrum of \(a_x\). Then the functions \(x \mapsto \lambda_{\max}(a_x)\) and \(x \mapsto \lambda_{\min}(a_x)\) are upper semicontinuous on \(X\). Furthermore, these functions are continuous on \(X\), whenever \(A\) is a continuous \(C(X)\)-algebra.

**Proof.** This follows directly from the equations

\[
\lambda_{\max}(a_x) = \|a\|1_x + a_x\| - \|a\| \quad \text{and} \quad \lambda_{\min}(a_x) = \|a\| - \|a\|1_x - a_x\|
\]

\(\square\)

**Proof of Proposition 3.7.** As \(r(A) = 2\), any fibre \(A_x\) is isomorphic to \(C, C \oplus C\) or \(M_2(C)\). Therefore, for each point \(x \in X\) we can choose a unital embedding \(\varphi_x : A_x \hookrightarrow M_2(C)\). For \(a \in A\) and \(x \in X\) we define

\[
E(a)(x) := \frac{1}{2}\text{tr}(\varphi_x(a_x)).
\]

Obviously \(E(a)\) is a \(C(X)\)-linear map. If \(a \in A_{sa}\), note that

(3.8)

\[
E(a)(x) = \frac{1}{2}(\lambda_{\min}(a_x) + \lambda_{\max}(a_x))
\]

for all \(x \in X\). By Remark 3.8, \(E(a)\) is a continuous function on \(X\) for all \(a \in A_{sa}\). As \(A\) is the linear span of \(A_{sa}\), we conclude that \(E(a) \in C(X)\) for all \(a \in A\). Therefore, \(E\) defines a C. E. from \(A\) onto \(C(X)\). Further, by (3.8) for all \(a \in A_x\) and \(x \in X\) we have

\[
2 \cdot E(a)(x)1_x = (\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \cdot 1_x \geq a_x.
\]

This shows that the map \(2 \cdot E - \text{id}_A\) is positive, so that \(K(A, C(X)) = 2\) by Theorem 1.4.

Let \(A\) be a unital \(C^*\)-algebra and let \(Z\) be the maximal ideal space of \(Z(A)\). We may consider \(A\) as a \((\hat{Z})\)-algebra, with respect to the action

\[
f \cdot a := G^{-1}(f)a \quad (f \in C(X), \ a \in A),
\]

where \(G : Z(A) \to \hat{Z}\) is the Gelfand transform. We say that \(A\) is quasi-standard if \(A\) is a continuous \((\hat{Z})\)-algebra and each (Glimm) ideal \(J_x = C_x(\hat{Z})A\) is primal (see [2]).

**Corollary 3.9.** For a unital \(C^*\)-algebra \(A\) the following conditions are equivalent:

(i) There exist a C. E. \(E : A \to Z(A)\) such that the map \(2 \cdot E - \text{id}_A\) is positive.

(ii) \(A\) is either commutative or quasi-standard and \(2\)-subhomogeneous.

**Proof.** (i) \(\Rightarrow\) (ii). Suppose that there exists a C. E. \(E : A \to Z(A)\) such that the map \(2 \cdot E - \text{id}_A\) is positive. Then by Theorem 1.4 \(A\) is a continuous \((\hat{Z})\)-algebra and \(r(A_x) \leq 2\) for all \(x \in \hat{Z}\). In particular, \(A\) as a \(C^*\)-algebra is \(n\)-subhomogeneous, where \(n \in \{1, 2\}\). Hence, by [13, Proposition 4.1] every Glimm ideal of \(A\) is primal. Also, \(n = 1\) if and only if \(A\) is commutative.

(ii) \(\Rightarrow\) (i). If \(A\) is commutative we have nothing to prove, so suppose that \(A\) is quasi-standard and \(2\)-subhomogeneous. Then by [9, Corollary 1, p. 388] for each point \(x \in X\) we have

\[
r(A_x) = \sum_{[\pi_x] \in A_x} \dim \pi_x \leq 2.
\]

It remains to apply Proposition 3.7. \(\square\)
Remark 3.10. At the end of this paper we note that every separable continuous unital C(X)-algebra \( A \) admits a faithful C.E. \( E : A \to C(X) \) (see e.g. [5]). In particular, this result applies to continuous subhomogeneous unital C(X)-algebras, when \( X \) is second-countable. In this case for each point \( x \in X \), the map \( E_x : a_x \mapsto E(a)(x) \) defines a faithful state on \( A_x \), so Lemma 3.2 implies \( K(E_x) < \infty \). However, this does not imply that \( E \) is of finite index. That is, it may happen that \( \sup_{x \in X} K(E_x) = \infty \). Consider for instance the following example:

- Let \( X \) be the closed compact subset \( \{0\} \cup \{ \frac{1}{n} : n \in \mathbb{N} \} \) of \([0, 1]\).
- Let \( A \) be the continuous \( C(X) \)-subalgebra of \( C(X) \oplus C(X) \) consisting of all pairs \((f, g) \in C(X) \oplus C(X)\) such that \( f(0) = g(0) \).
- Let \( E : A \to C(X) \) be a C.E. fixed by the relations

\[
E(f \oplus g) \left( \frac{1}{n} \right) = \begin{cases} \frac{n}{n+1} f \left( \frac{1}{n} \right) + \frac{1}{n+1} g \left( \frac{1}{n} \right) & \text{if } n \text{ is odd} \\
\frac{1}{n+1} f \left( \frac{1}{n} \right) + \frac{n}{n+1} g \left( \frac{1}{n} \right) & \text{otherwise} 
\end{cases}
\]

where \((f, g) \in A\).

Then \( E \) is a faithful C.E. which is not of finite index. Indeed, one has

\[
E(f \oplus 0) \left( \frac{1}{2n} \right) = \frac{1}{2n+1} f \left( \frac{1}{2n} \right)
\]

for all \( f \in C_0(X \setminus \{0\}) \) and all integers \( n \in \mathbb{N} \). Consequently, a convenient constant \( K \) would satisfy \( K \geq 2n+1 \) for all \( n \in \mathbb{N} \), which is impossible.

We end this paper with some unresolved problems:

Problem 3.11. Is the converse of Theorem 1.4 also true? Moreover, does every continuous subhomogeneous unital C(X)-algebra \( A \) admit a C.E. \( E : A \to C(X) \) such that the map \( r(A) \cdot E - \text{id}_A \) is positive? In particular, do we always have \( K(A, C(X)) = r(A) \)?

References

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