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ON UNITAL C(X)-ALGEBRAS AND C(X)-VALUED CONDITIONAL EXPECTATIONS OF FINITE INDEX

ETIENNE BLANCHARD AND ILJA GOGIĆ

Abstract. Let \( X \) be a compact Hausdorff space and let \( A \) be a unital \( C(X) \)-algebra, where \( C(X) \) is embedded as a unital \( C^* \)-subalgebra of the centre of \( A \). We consider the problem of characterizing the existence of a conditional expectation \( E : A \to C(X) \) of finite index in terms of the underlying \( C^* \)-bundle of \( A \) over \( X \). More precisely, we show that if \( A \) admits a \( C(X) \)-valued conditional expectation of finite index, then \( A \) is necessarily a continuous \( C(X) \)-algebra, and there exists a positive integer \( N \) such that every fibre \( A_x \) of \( A \) is finite-dimensional, with \( \text{dim} \ A_x \leq N \). We also give some sufficient conditions on \( A \) that ensure the existence of a \( C(X) \)-valued conditional expectation of finite index.

1. Introduction

Let \( B \subseteq A \) be two unital \( C^* \)-algebras with the same unit element. A conditional expectation (abbreviated by C.E.) from \( A \) to \( B \) is a completely positive contraction \( E : A \to B \) such that \( E(b) = b \) for all \( b \in B \), and which is \( B \)-bilinear, i.e.
\[
E(b_1ab_2) = b_1E(a)b_2
\]
for all \( a \in A \) and \( b_1, b_2 \in B \). By a result of Y. Tomiyama (see [22, Theorem 1] or [4, Theorem II.6.10.2]), a map \( E : A \to B \) is a C.E. if and only if \( E \) is a projection of norm one.

If \( E(a^*a) = 0 \) \((a \in A)\) implies \( a = 0 \), \( E \) is said to be faithful. Every faithful conditional expectation \( E : A \to B \) introduces a pre-Hilbert \( B \)-module structure on \( A \), whose inner product is defined by
\[
\langle a_1, a_2 \rangle_E := E(a_1^*a_2) \quad (a_1, a_2 \in A).
\]

The notion of finite index was introduced by V. F. R. Jones [14] in order to classify the subfactors of a type II\(_1\) factor. Soon afterwards H. Kosaki [16] extended the Jones index theory to arbitrary factors. In order to generalize the results of [14, 16], M. Pimsner and S. Popa introduced in [19, 20] a definition for conditional expectations of finite index in the context of \( W^* \)-algebras: There must exist a constant \( K \geq 1 \) such that the map \( K \cdot E - \text{id}_A \) is positive on \( A \). Then, following the idea of M. Baillet, Y. Denizeau and J.-F. Havet (see [3]), the index of \( E \) can be defined in the following way: Since the map \( K \cdot E - \text{id}_A \) is positive, \( E \) defines a (complete) Hilbert \( B \)-module structure on \( A \), with respect to the inner product.

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(1.1). If \( \{x_i\} \) is a quasi-orthonormal basis in \( A \), the index of \( E \) is the sum \( \sum_{i=1}^{\infty} x_i^* x_i \), with respect to the ultraweak topology.

Y. Watatani also considered C.E. of (algebraically) finite index, when the original \( C^* \)-algebra \( A \) is a finitely generated Hilbert \( C^* \)-module over \( B \) (see [23]).

The results of M. Baillet, Y. Denizeau and J.-F. Havet in [3] also indicated that there might occur some difficulties in order to extend the notion of ”finite index” for conditional expectations of \( C^* \)-algebras with arbitrary centres. However, this problem was solved by M. Frank and E. Kirchberg in [11]. The main result of their paper is [11, Theorem 1]:

**Theorem 1.1** (M. Frank and E. Kirchberg). For a C.E. \( E : A \to B \), where \( B \subseteq A \) are unital \( C^* \)-algebras with the same unit element, the following conditions are equivalent:

(i) There exists a constant \( K \geq 1 \) such that the map \( K \cdot E - \text{id}_A \) is positive.

(ii) There exists a constant \( L \geq 1 \) such that the map \( L \cdot E - \text{id}_A \) is completely positive.

(iii) \( A \) becomes a (complete) Hilbert \( B \)-module when equipped with the inner product (1.1).

Moreover, if \( K(E) := \inf \{ K \geq 1 : K \cdot E - \text{id}_A \ \text{is positive} \} \), \( L(E) := \inf \{ L \geq 1 : L \cdot E - \text{id}_A \ \text{is completely positive} \} \), with \( K(E) = \infty \) or \( L(E) = \infty \) if no such number \( K \) or \( L \) exists, then

\[
K(E) \leq L(E) \leq \lfloor K(E) \rfloor K(E),
\]

where \( \lfloor \cdot \rfloor \) denotes the integer part of a real number.

The importance of this result is that it gives the right general definition for conditional expectations on \( C^* \)-algebras to be of finite index:

**Definition 1.2.** If \( B \subseteq A \) are two unital \( C^* \)-algebras with the same unit element, then a C.E. \( E : A \to B \) is said to be of finite index (abbreviated C.E.F.I.) if \( E \) satisfies one of the equivalent conditions of Theorem 1.1.

In this case the index value of \( E \) can be calculated in the enveloping von Neumann algebra \( A^{**} \) (see [11, Definition 3.1]).

For a unital inclusion \( A \subseteq B \) of unital \( C^* \)-algebras we introduce the following constant \( K(A,B) := \inf \{ K(E) : E : A \to B \ \text{is C.E.F.I.} \} \), with \( K(A,B) = \infty \), if no such C.E.F.I. exists. This constant will play an important role in this paper.

More recently, A. Pavlov and E. Troitsky considered in [17] the problem of existence of a C.E.F.I. \( E : C(Y) \to C(X) \) for a unital inclusion \( \varphi : C(X) \hookrightarrow C(Y) \) of unital commutative \( C^* \)-algebras. The main result of their paper is [17, Theorem 1.1], which shows that such a C.E.F.I. exists if and only if the transpose map \( \varphi^* : Y \to X \) is a branched covering. This means that \( \varphi^* \) is an open map with uniformly bounded number of pre-images (i.e. \( \sup_{x \in X} |\varphi^{-1}(x)| < \infty \)). This result motivated A. Pavlov and E. Troitsky to define the noncommutative branched coverings, as unital inclusion \( B \subseteq A \) of unital \( C^* \)-algebras such that there exists a C.E.F.I. from \( A \) to \( B \) (see [17, Definition 1.2]).
Using the above inclusion $\varphi : C(X) \to C(Y)$ we may consider $C(Y)$ as a $C(X)$-algebra. Then the map $\varphi_*$ is open if and only if $C(Y)$ is a continuous $C(X)$-algebra, and $\varphi_*$ has uniformly bounded number of pre-images if and only if $C(Y)$ is subhomogeneous $C(X)$-algebra. This means that there exists a positive integer $N$ such that every fibre $C(Y)_x$ of $C(Y)$ is finite-dimensional with $\dim C(Y)_x \leq N$ (see Section 2). Therefore, we can restate [17, Theorem 1.1] in terms of $C(X)$-algebras as follows:

**Theorem 1.3** (A. Pavlov and E. Troitsky). Let $A$ be a unital commutative $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of $A$. Then $A$ admits a $C(X)$-valued C.E.F.I. if and only if $A$ is a continuous subhomogeneous $C(X)$-algebra.

The purpose of the present paper is to consider a possible extension of Theorem 1.3 to the case when $A$ is an arbitrary (not necessarily commutative) unital $C(X)$-algebra. The necessary condition for the existence of a $C(X)$-valued C.E.F.I. appears to be identical to the one of Theorem 1.3:

**Theorem 1.4.** Let $A$ be a unital $C(X)$-algebra, where $C(X)$ is embedded as a unital $C^*$-subalgebra of the centre of $A$. If $A$ admits a $C(X)$-valued C.E.F.I., then $A$ is a continuous subhomogeneous $C(X)$-algebra. Moreover, in this case the following inequality holds:

$$K(A, C(X)) \geq r(A),$$

where $r(A)$ is the rank of $A$, i.e.

$$r(A) = \max \left\{ \sum_{[\pi_x] \in \hat{A}_x} \dim \pi_x : x \in X \right\}.$$

We shall prove Theorem 1.4 in Section 3. At the moment we do not know if the converse of Theorem 1.4 also holds. However, if all the fibres of a continuous unital $C(X)$-algebra $A$ are $*$-isomorphic to the same finite-dimensional $C^*$-algebra (i.e. $A$ is a homogeneous $C(X)$-algebra), then there exists a unique C.E. $E : A \to C(X)$ such that the map $r(A) \cdot E - \text{id}_A$ is positive (Proposition 3.4). In particular, we have the equality $K(A, C(X)) = r(A)$ in this case. Also, a direct consequence of this fact is that any unital $C(X)$-algebra $A$ which can be embedded as a $C(X)$-subalgebra of some continuous homogeneous unital $C(X)$-algebra also admits a $C(X)$-valued C.E.F.I.. However, this embedding condition is not necessary for the existence of such C.E.F.I.. Indeed, there exists a continuous unital $C(X)$-algebra $A$ over a second-countable compact Hausdorff space $X$ with fibres $M_2(C)$ or $C$ which admits a $C(X)$-valued C.E.F.I., but which cannot be embedded as a $C(X)$-subalgebra into any continuous homogeneous unital $C(X)$-algebra (Example 3.6). At the end of this paper we also show that any continuous unital $C(X)$-algebra $A$ of rank 2 admits a C.E. $E : A \to C(X)$ such that the map $2 \cdot E - \text{id}_A$ is positive (Proposition 3.7). In particular, the equality $K(A, C(X)) = r(A)$ also holds in this class of $C(X)$-algebras.

2. Notation and preliminaries

Throughout this paper $A$ will be a $C^*$-algebra. We denote by $A_{sa}$ and $A_+$ the self-adjoint and the positive parts of $A$. The centre of $A$ is denoted by $Z(A)$. By
\(\hat{\mathcal{A}}\) and \(\operatorname{Prim}(A)\) we respectively denote the \textit{spectrum} of \(A\) (i.e. the set of all classes of irreducible representations of \(A\)) and the \textit{primitive spectrum} of \(A\) (i.e. the set of all primitive ideals of \(A\)), equipped with the Jacobson topology. By a \textit{dimension} of \([\pi]\) \(\in\hat{\mathcal{A}}\), which is denoted by \(\dim \pi\), we mean the dimension of the underlying Hilbert space of some representative of \([\pi]\).

Let \(X\) be a compact Hausdorff space. For each point \(x \in X\) let 
\[
C_x(X) := \{ f \in C(X) : f(x) = 0 \}
\]
be the corresponding maximal ideal of \(C(X)\).

**Definition 2.1.** A \((C(X))\)-\textit{algebra} is a \(C^*\)-algebra \(A\) endowed with a unital \(*\)-homomorphism \(\psi_A\) from \(C(X)\) to the centre of the multiplier algebra of \(A\).

**Remark 2.2.** Given \(f \in C(X)\) and \(a \in A\), we write \(fa\) for the product \(\psi_A(f) \cdot a\) if no confusion is possible.

There is a natural connection between \((C(X))\)-algebras and upper semicontinuous \(C^*\)-bundles over \(X\). We first give a formal definition of such bundles:

**Definition 2.3.** Following [24] by an \textit{upper semicontinuous} \(C^*\)-\textit{bundle} we mean a triple \(\mathfrak{A} = (\mathcal{A}, A, X)\) where \(\mathcal{A}\) is a topological space with a continuous open surjection \(p : A \to X\), together with operations and norms making each fibre \(A_x := p^{-1}(x)\) into a \(C^*\)-algebra, such that the following conditions are satisfied:

(A1) The maps \(\mathcal{C} \times A \to A\), \(\mathcal{A} \times_X A \to A\), \(\mathcal{A} \times_X \mathcal{A} \to \mathcal{A}\) and \(A \to A\) given in each fibre by scalar multiplication, addition, multiplication and involution, respectively, are continuous (\(\mathcal{A} \times_X \mathcal{A}\) denotes the Whitney sum over \(X\)).

(A2) The map \(A \to \mathbb{R}\), defined by norm on each fibre, is upper semicontinuous.

(A3) If \(x \in X\) and if \((a_n)\) is a net in \(A\) such that \(\|a_n\| \to 0\) and \(p(a_n) \to x\) in \(X\), then \(a_n \to 0_x\) in \(A\) (\(0_x\) denotes the zero-element of \(A_x\)).

If "upper semicontinuous" in (A2) is replaced by "continuous", then we say that \(\mathfrak{A}\) is a \textit{continuous} \(C^*\)-bundle.

By a \textit{section} of an upper semicontinuous \(C^*\)-bundle \(\mathfrak{A}\) we mean a map \(s : X \to A\) such that \(p(s(x)) = x\) for all \(x \in X\). We denote by \(\Gamma(\mathfrak{A})\) the set of all continuous sections of \(\mathfrak{A}\). Then \(\Gamma(\mathfrak{A})\) becomes a \((C(X))\)-algebra with respect to the natural pointwise operations and sup-norm.

On the other hand, given a \((C(X))\)-algebra \(A\), one can always associate an upper semicontinuous \(C^*\)-bundle \(\mathfrak{A}\) over \(X\) such that \(A \cong \Gamma(\mathfrak{A})\), as follows. Set \(J_x := C_x(X) \cdot A\) and note that \(J_x\) is a closed two-sided ideal in \(A\) (by Cohen factorization theorem [7], [6, Theorem A.6.2]). The quotient \(A_x := A/J_x\) is called the \textit{fibre} at the point \(x\), and we denote by \(a_x\) the image in \(A_x\) of an element \(a \in A\). Let

\[
\mathcal{A} := \bigcup_{x \in X} A_x,
\]
and let \(p : A \to X\) be the canonical associated projection. For \(a \in A\) we define the map \(\hat{a} : X \to A\) by \(\hat{a}(x) := a_x\), and let \(\Omega := \{ \hat{a} : a \in A \}\). Since for each \(a \in A\) we have

\[
\|a_x\| = \inf\{\|1 - f + f(x)\| : f \in C(X)\},
\]
the norm function \(x \mapsto \|a_x\|\) is upper semicontinuous on \(X\). Hence, by Fell’s theorem [24, Theorem C.25] there exists a unique topology on \(\mathcal{A}\) for which \(\mathfrak{A} := (\mathcal{A}, A, X)\)
becomes an upper semicontinuous $C^*$-bundle such that $\Omega \subseteq \Gamma(\mathfrak{A})$. Moreover, by Lee’s theorem [24, Theorem C.26], $\Omega = \Gamma(\mathfrak{A})$, and the *generalized Gelfand transform* $G : a \in \mathfrak{A} \mapsto \hat{a} \in \Gamma(\mathfrak{A})$, is an isomorphism of $C(X)$-algebras, from $\mathfrak{A}$ onto $\Gamma(\mathfrak{A})$.

**Definition 2.4.** Let $\mathfrak{A}$ be a $C(X)$-algebra. If all the norm functions $x \mapsto \|a_x\|$ ($a \in \mathfrak{A}$) are continuous on $X$, we say that $\mathfrak{A}$ is a continuous $C(X)$-algebra.

Note that the $C(X)$-algebra $\mathfrak{A}$ is continuous if and only if $\mathfrak{A}$ is continuous as a $C^*$-bundle.

The $C^*$-algebra $\mathfrak{A}$ is said to be
- (n-)homogeneous ($n \in \mathbb{N}$), if $\dim \pi = n$ for all $[\pi] \in \hat{\mathfrak{A}$,
- (n-)subhomogeneous ($n \in \mathbb{N}$), if $\sup_{[\pi] \in \hat{\mathfrak{A}}} \dim \pi = n$.

We shall now define the similar notions for $C(X)$-algebras. To do this, first recall that if $D$ is a finite-dimensional $C^*$-algebra, then there is a finite number of central pairwise orthogonal projections $p_1, \ldots, p_m \in \mathcal{Z}(D)$ with $\sum_{i=1}^{m} p_i = 1_D$, such that

$$D = p_1D \oplus \cdots \oplus p_mD,$$

and each $p_iD$ is $*$-isomorphic to the matrix algebra $M_{n_i}(\mathbb{C})$ (see e.g. [21, Theorem I.11.9]). We define the *rank* of $D$ as

$$r(D) := \sum_{i=1}^{m} p_i = \sum_{[\pi] \in \hat{\mathfrak{A}}} \dim \pi.$$

**Definition 2.5.** Let $\mathfrak{A}$ be a $C(X)$-algebra. We say that $\mathfrak{A}$ is
- homogeneous all the fibres of $\mathfrak{A}$ are $*$-isomorphic to the same finite-dimensional $C^*$-algebra.
- subhomogeneous if there exists a positive integer $N$ such that every fibre $A_x$ of $\mathfrak{A}$ is finite-dimensional with $\dim A_x \leq N$.

**Remark 2.6.** Let $\mathfrak{A}$ be a $C(X)$-algebra.

(i) $\mathfrak{A}$ is subhomogeneous if and only if

$$r(\mathfrak{A}) := \sup \{ r(A_x) \; : \; x \in X \} < \infty$$

As in the finite-dimensional case, we call the number $r(\mathfrak{A})$ the *rank* of $\mathfrak{A}$.

(ii) If $\mathfrak{A}$ is continuous and homogeneous, then by [10, Lemma 3.1] the underlying $C^*$-bundle $\mathfrak{A}$ is locally trivial.

3. Results

**Remark 3.1.** If $\mathfrak{A}$ is a unital $C(X)$-algebra, we always assume in this section that the map $\psi_A : C(X) \to Z(\mathfrak{A})$ is injective, so that we can identify $C(X)$ with the unital $C^*$-subalgebra $\psi_A(C(X))$ of $Z(\mathfrak{A})$.

In order to prove Theorem 1.4 we shall need the following two auxiliary results.

**Lemma 3.2.** Let $D$ be a unital $C^*$-algebra. Then $K(D, \mathbb{C}) := K(D, \mathbb{C}1_D) < \infty$ if and only if $D$ is finite-dimensional. In this case we have:

(i) The constant $K(\omega)$ is finite for every faithful state $\omega$ on $D$, which we identify with the corresponding faithful $C.E.$

$$d \in D \mapsto \omega(d) \cdot 1_D \in \mathbb{C} \cdot 1_D \quad (d \in D).$$
(ii) $K(D, \mathbb{C}) = r(D)$. Moreover, there exists a unique state $\tau$ on $D$ such that
\begin{equation}
(3.1) \quad r(D) \cdot \tau(d)1_D \geq d \quad \text{for all } d \in D_+.
\end{equation}

Proof. The equivalence $K(D, \mathbb{C}) < \infty \iff \dim D < \infty$ follows from [13, Lemma 4.5]. Hence, suppose that $D$ is finite-dimensional and let $\omega$ be a faithful state on $D$. The proof will now proceed in two steps.

Step 1. Assume that $D$ is simple, i.e. $D = M_n(\mathbb{C})$ for some $n$. If $\text{tr}(\cdot)$ is the standard trace of $M_n(\mathbb{C})$, then there exists a strictly positive matrix $a \in M_n(\mathbb{C})$ with $\text{tr}(a) = 1$ such that
\[\omega(d) = \text{tr}(ad) \quad (d \in M_n(\mathbb{C})).\]
Let $a = u^* \cdot \text{diag}(\lambda_1, \ldots, \lambda_n) \cdot u$ be a diagonalisation of $a$, where $u \in M_n(\mathbb{C})$ is a unitary and $\lambda_1, \ldots, \lambda_n > 0$ are the eigenvalues of $a$. Then for all $d \in M_n(\mathbb{C})$ one has
\begin{equation}
(3.2) \quad \omega(u^* du) = \text{tr}(au^* du) = \text{tr}(ua u^* d) = \text{tr}(\text{diag}(\lambda_1, \ldots, \lambda_n)d).
\end{equation}
The constant $K(\omega)$ is by definition the smallest $K \geq 1$ satisfying
\begin{equation}
(3.3) \quad K \cdot \omega(d)1_D \geq d \quad \text{for all } d \in D_+.
\end{equation}
Thus, (3.2) and (3.3) for rank 1 projections in $D$ imply that
\[K(\omega) = \max\{\lambda_i^{-1} : 1 \leq i \leq n\}.\]
As $1 = \omega(1) = \sum_{i=1}^n \lambda_i$, one has $K(\omega) \geq n$ for any faithful state $\omega$ on $D$. Also, $K(\omega) = n$ if and only if $\omega = \tau := \frac{1}{n}\text{tr}(\cdot)$. In particular, if $D = M_n(\mathbb{C})$, we have $K(D, \mathbb{C}) = r(D) = n$, and $\tau$ is the unique state on $D$ satisfying (3.1).

Step 2. Suppose that $D$ is an arbitrary finite-dimensional $C^*$-algebra. We decompose $D$ as in (2.1). For each $1 \leq i \leq m$
\[\omega_i(p_i d) := \frac{1}{\omega(p_i)} \cdot \omega(p_i d) \quad (d \in D)\]
defines a faithful state on $p_iD$. By Step 1 we have $n_i \leq K(\omega_i) < \infty$ for all $1 \leq i \leq m$. Put
\[K_\omega := \max \left\{ \frac{K(\omega_i)}{\omega(p_i)} : 1 \leq i \leq m \right\}.\]
We claim that $K(\omega) = K_\omega$. Indeed, for all $d \in D_+$ we have
\[K_\omega \cdot \omega(d)1_D = \sum_{i=1}^m K_\omega \cdot \omega(p_i)\omega_i(p_i d)1_D \geq \sum_{i=1}^m K(\omega_i) \cdot \omega_i(p_i d)p_i \]
\[\geq \sum_{i=1}^m p_i d = d,
\]
which shows $K(\omega) \leq K_\omega$. On the other hand, for each $d \in D_+$ we have
\[\omega(p_i)K(\omega) \cdot \omega_i(p_i d)p_i \geq p_i d,\]
so that
\begin{equation}
(3.4) \quad \omega(p_i)K(\omega) \geq K(\omega_i) \quad (1 \leq i \leq m).
\end{equation}
This shows $K(\omega) = K_\omega$, as wanted. Also,

$$K(\omega) = \sum_{i=1}^{m} \omega(p_i)K(\omega) \geq \sum_{i=1}^{m} K(\omega_i) \geq \sum_{i=1}^{m} n_i = \tau(D),$$

so that $K(D, \mathbb{C}) \geq \tau(D)$.

It remains to show that there exists a unique state $\tau$ on $D$ satisfying (3.1). To do this, suppose that $\tau(D) = n$, and for each $1 \leq i \leq m$ let $\tau_i$ be the only faithful tracial state on $p_i D \cong M_{n_i}(\mathbb{C})$. Define the state $\tau$ on $D$ by

(3.5) $$\tau(d) := \frac{1}{n} \sum_{i=1}^{m} n_i \cdot \tau_i(p_i d) \quad (d \in D).$$

As $\tau(p_i) = \frac{n_i}{n}$ and $K(\tau_i) = n_i$ for all $1 \leq i \leq m$, we have $K(\tau) = K_\tau = n$. In particular, $K(D, \mathbb{C}) = n = \tau(D)$.

To show the uniqueness of this state $\tau$, suppose that $\omega$ is another state on $D$ with $K(\omega) = n$. Then using (3.4) we have

$$\sum_{i=1}^{m} K(\omega_i) \leq \sum_{i=1}^{m} \omega(p_i)K(\omega) = K(\omega) = n.$$

But since $K(\omega_i) \geq n_i$ and $\sum_{i=1}^{m} n_i = n$, we must have $K(\omega_i) = n_i$ for all $1 \leq i \leq m$. By the uniqueness part of Step 1 we conclude that

(3.6) $$\omega_i = \tau_i \quad \text{for all} \quad 1 \leq i \leq m.$$

Also, $K_\omega = K(\omega) = n$ and $K(\omega_i) = n_i$ imply $\omega(p_i) \geq \frac{n_i}{n}$ for all $1 \leq i \leq m$. Since $\omega$ is a state on $D$ and $\sum_{i=1}^{m} p_i = 1_D$, we must have

(3.7) $$\omega(p_i) = \frac{n_i}{n} \quad \text{for all} \quad 1 \leq i \leq m.$$

Finally, (3.6) and (3.7) imply that

$$\omega(d) = \sum_{i=1}^{m} \omega(p_i)\omega_i(p_i d) = \frac{1}{n} \sum_{i=1}^{m} n_i \cdot \tau_i(p_i d) = \tau(d),$$

for all $d \in D$, which finishes the proof. \hfill \Box

**Proposition 3.3.** Let $A$ be a unital $C(X)$-algebra. If $A$ admits a faithful $C(X)$-valued $C.E.$, then $A$ is a continuous $C(X)$-algebra.

**Proof.** This can be deduced from [5, Section 2]. For completeness, we include a short proof of this fact. It suffices to show that all norm functions $x \mapsto \|a_x\|$ ($a \in A$) are lower semicontinuous on $X$. To prove this, let $E : A \rightarrow C(X)$ be a faithful $C.E.$ and let $L^2(A, E)$ be the completion of the pre-Hilbert $C(X)$-module $A$, with respect to the inner product (1.1). For $a \in A$ let $\Phi(a) : L^2(A, E) \rightarrow L^2(A, E)$ denote the continuous extension of the left multiplication map $a_1 \mapsto aa_1$ ($a \in A$). Since $E$ is faithful and since

$$\langle \Phi(a)(a_1), a_2 \rangle_E = \langle a_1, a_2 \rangle_E = E(a_1^* a_2) = \langle a_1, a^* a_2 \rangle_E = \langle a_1, \Phi(a^*)(a_2) \rangle_E,$$

(1)
for all \(a_1, a_2 \in A\), the map \(\Phi\) defines an injective \(C(X)\)-linear morphism from \(A\) to the \(C(X)\)-algebra \(\mathcal{B}_{C(X)}(L^2(A,E))\) of bounded adjointable \(C(X)\)-linear operators on \(L^2(A,E)\). Therefore, for \(a \in A\) and \(x \in X\) we have
\[
\|a_x\| = \|\Phi(a)_x\| = \sup\{|\langle \Phi(a)(a_1), a_2 \rangle_E(x)\} : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\} = \sup\{|E(a^*_1 a^*_2)(x)| : a_1, a_2 \in A, \|a_1\|_E = \|a_2\|_E = 1\}.
\]
In particular, the function \(x \mapsto \|a_x\|\) is a supremum of continuous functions \(x \mapsto |E(a^*_1 a^*_2)(x)| (\|a_1\|_E = \|a_2\|_E = 1)\), so it must be lower semicontinuous on \(X\).

**Proof of Theorem 1.4.** Let \(E : A \to C(X)\) be a C.E.F.I. As the conditional expectation \(E\) is faithful, Proposition 3.3 implies that the \(C(X)\)-algebra \(A\) is continuous (note that in this case \((A, \langle \cdot, \cdot \rangle_E)\) is already a complete Hilbert \(C(X)\)-module by Theorem 1.1). It remains to show that each fibre \(A_x (x \in X)\) is finite-dimensional and satisfies \(r(A_x) \leq K(E)\), so it must be lower semicontinuous on \(X\).

We shall now give some sufficient conditions on a continuous unital subhomogeneous \(C(X)\)-algebra \(A\) to ensure the existence of a \(C(X)\)-valued C.E.F.I.

**Proposition 3.4.** Every continuous homogeneous unital \(C(X)\)-algebra \(A\) admits a unique C.E. \(E : A \to C(X)\) such that the map \(r(A) : E \to \text{id}_A\) is positive. In particular, \(K(A, C(X)) = r(A)\) in this case.

**Proof.** The construction of such a C.E. \(E : A \to C(X)\) can be deduced from the proof of [13, Lemma 4.6]. But we include here the main steps of the proof for completeness. By assumption all fibres of \(A\) are \(*\)-isomorphic to a fixed finite-dimensional \(C^*\)-algebra \(D\). Suppose that \(r(D) = n\), and let \(\tau\) be a state on \(D\) defined by (3.5). It is easy check that \(\tau\) is invariant under the group \(\text{Aut}(D)\) of \(*\)-automorphisms of \(D\). Since the \(C(X)\)-algebra \(A\) is continuous and homogeneous, its underlying bundle \(\mathfrak{A}\) is locally trivial by Remark 2.6. Hence, there exists an open covering \(\{U_\alpha\}\) of \(X\) such that \(\Phi_\alpha : \mathfrak{A}|_{U_\alpha} \cong U_\alpha \times D\), where
- \(\Phi_\alpha\) is an isomorphism of \(C^*\)-bundles, and
- \(\mathfrak{A}|_{U}\) is the restriction bundle over a subset \(U \subseteq X\).

Fix an element \(a \in A\). For \(x \in X\) choose an index \(\alpha\) such that \(x \in U_\alpha\), and define
\[
E(a)(x) := \tau(\Phi_\alpha(a_x)).
\]
Since \(\tau\) is invariant under the group \(\text{Aut}(D)\), the value \(E(a)(x)\) is well defined, and the local triviality of \(\mathfrak{A}\) implies that the function \(E(a) : x \mapsto E(a)(x)\) is continuous on \(X\). It is now easy to see that the map \(E : a \to E(a)\) defines a \(C(X)\)-valued C.E.F.I. on \(A\). Moreover, by (3.1) we have
\[
n \cdot E(a)(x)1_x \geq a_x, \quad \text{for all } a \in A_+ \text{ and } x \in X.
\]
Thus, the map \(n \cdot E - \text{id}_A\) is positive and \(E\) is the only C.E. with this property (Lemma 3.2). In particular, \(K(A, C(X)) \leq r(A)\), so Theorem 1.4 yields that \(K(A, C(X)) = n\). \qed
Corollary 3.5. If the unital $C(X)$-algebra $A$ admits a $C(X)$-linear embedding into some homogeneous continuous unital $C(X)$-algebra $A'$, then $A$ admits a $C(X)$-valued C.E.F.I..

Proof. By Proposition 3.4 there exists a C.E. $E' : A' \to C(X)$ of finite index. Then the restriction $E'|_A : A \to C(X)$ defines a convenient C.E.F.I. \hfill \Box

Note that the embedding condition of Corollary 3.5 is not necessary for the existence of a $C(X)$-valued C.E.F.I.. Indeed, in Example 3.6 we show that there exists a continuous unital $C(X)$-algebra $A$ of rank 2 which does not admit a $C(X)$-linear embedding into any continuous homogeneous unital $C(X)$-algebra. On the other hand, a direct consequence of Proposition 3.7 is that $A$ admits a $C(X)$-valued C.E.F.I..

To do this, first recall that a $C^*$-algebra $A$ is said to be central if it satisfies the following two conditions:

(i) $A$ is quasi-central (i.e. no primitive ideal of $A$ contains $Z(A)$);
(ii) If $P, Q \in \text{Prim}(A)$ and $P \cap Z(A) = Q \cap Z(A)$, then $P = Q$

(see [1, 8, 12, 15]). By [8, Proposition 3] a quasi-central $C^*$-algebra $A$ is central if and only if $\text{Prim}(A)$ is Hausdorff.

Example 3.6. By [18, Example 3.5] there exists a continuous $M_2(\mathbb{C})$-bundle $\mathfrak{A}_0$ over the second countable locally compact space $X_0 := \bigsqcup_{n=1}^{\infty} CP^n$, where $CP^n$ is the complex projective space of dimension $n$, which is not of finite type (that is, $X_0$ does not admit a finite open cover $\{U_i\}$ such that each restriction bundle $\mathfrak{A}_0|_{U_i}$ is trivial, as a $C^*$-bundle). Let $A_0$ be the $C^*$-algebra $\Gamma_0(\mathfrak{A}_0)$ consisting of all continuous sections of $\mathfrak{A}_0$ which vanish at infinity. Then $A_0$ is a 2-homogeneous $C^*$-algebra with $\text{Prim}(A_0) = X_0$. In particular $A_0$ is a central $C^*$-algebra with centre $C_0(X_0)$. Let $X := X_0 \cup \{\infty\}$ be the one-point compactification of $X_0$, and let $A$ be the minimal unitisation of $A_0$. By [8, Proposition 3] (or [12, Proposition 3.12]) $A$ is also a central $C^*$-algebra with $\text{Prim}(A) = X$ and centre $C(X)$. In particular, by [4, II.6.5.8] all norm functions $x \mapsto \|a_x\|$ ($a \in A$) are continuous on $X$, so that $A$ is a continuous unital $C(X)$-algebra with fibres $A_x = M_2(\mathbb{C})$ ($x \in X$) and $A_{\infty} = \mathbb{C}$. Suppose that $A$ is a $C(X)$-subalgebra of some continuous homogeneous $C(X)$-algebra $A'$. Then the underlying $C^*$-bundle $\mathfrak{A}$ of $A$ over $X$ is a $C^*$-subbundle of the underlying $C^*$-bundle $\mathfrak{A}'$ of $A'$ over $X$. Since $A'$ is continuous and homogeneous, $\mathfrak{A}'$ is locally trivial by Remark 2.6. Hence, since $X$ is compact, $\mathfrak{A}'$ is of finite type. Using [18, Lemma 2.6] we conclude that $\mathfrak{A}$ is of finite type as a vector bundle. In particular, $\mathfrak{A}_0$ is of finite type as a vector bundle, since $\mathfrak{A}_0 = \mathfrak{A}|_{X_0}$. As $\mathfrak{A}_0$ is a $M_2(\mathbb{C})$-bundle, this implies by [18, Proposition 2.9] that $\mathfrak{A}_0$ is also of finite type as a $C^*$-bundle; a contradiction.

On the other hand, the $C(X)$-algebra $A$ of Example 3.6 also admits a $C(X)$-valued C.E.F.I. This follows from the following more general fact:

Proposition 3.7. Let $A$ be a continuous unital $C(X)$-algebra. If $r(A) = 2$, then there exists a conditional expectation $E : A \to C(X)$ such that the map $2 \cdot E - \text{id}_A$ is positive. In particular, $K(A, C(X)) = r(A)$ in this case.

In order to prove Proposition 3.7, let us first make the following observation:

Lemma 3.8. Let $A$ be a unital $C(X)$-algebra and let $a \in A_{sa}$. For each point $x \in X$ let $\lambda_{\max}(a)$ and $\lambda_{\min}(a)$ respectively denote the largest and the smallest numbers in
the spectrum of $a_x$. Then the functions $x \mapsto \lambda_{\max}(a_x)$ and $x \mapsto \lambda_{\min}(a_x)$ are upper semicontinuous on $X$. Furthermore, these functions are continuous on $X$, whenever $A$ is a continuous $C(X)$-algebra.

Proof. This follows directly from the equations

$$\lambda_{\max}(a_x) = \|a\| 1_x + a_x\| - \|a\| \quad \text{and} \quad \lambda_{\min}(a_x) = \|a\| - \|a\| 1_x - a_x\|.$$

\hfill \square

Proof of Proposition 3.7. As $r(A) = 2$, any fibre $A_x$ is isomorphic to $\mathbb{C}$, $\mathbb{C} \oplus \mathbb{C}$ or $M_2(\mathbb{C})$. Therefore, for each point $x \in X$ we can choose a unital embedding $\varphi_x : A_x \hookrightarrow M_2(\mathbb{C})$. For $a \in A$ and $x \in X$ we define

$$E(a)(x) := \frac{1}{2} \text{tr}(\varphi_x(a_x)).$$

Obviously $E(a)$ is a $C(X)$-linear map. If $a \in A_{sa}$, note that

$$(3.8) \quad E(a)(x) = \frac{1}{2}(\lambda_{\min}(a_x) + \lambda_{\max}(a_x))$$

for all $x \in X$. By Remark 3.8, $E(a)$ is a continuous function on $X$ for all $a \in A_{sa}$. As $A$ is the linear span of $A_{sa}$, we conclude that $E(a) \in C(X)$ for all $a \in A$. Therefore, $E$ defines a C. E. from $A$ onto $C(X)$. Further, by (3.8) for all $a \in A_+$ and $x \in X$ we have

$$2 \cdot E(a)(x) 1_x = (\lambda_{\min}(a_x) + \lambda_{\max}(a_x)) \cdot 1_x \geq a_x.$$

This shows that the map $2 \cdot E - \text{id}_A$ is positive, so that $K(A, C(X)) = 2$ by Theorem 1.4. \hfill \square

Let $A$ be a unital $C^\ast$-algebra and let $\hat{Z}$ be the maximal ideal space of $Z(A)$. We may consider $A$ as a $C(\hat{Z})$-algebra, with respect to the action

$$f \cdot a := \mathcal{G}^{-1}(f)a \quad (f \in C(X), \ a \in A),$$

where $\mathcal{G} : Z(A) \to C(\hat{Z})$ is the Gelfand transform. We say that $A$ is quasi-standard if $A$ is a continuous $C(\hat{Z})$-algebra and each (Glimm) ideal $J_x = C_x(\hat{Z})A$ is primal (see [2]).

Corollary 3.9. For a unital $C^\ast$-algebra $A$ the following conditions are equivalent:

(i) There exist a C. E. $E : A \to Z(A)$ such that the map $2 \cdot E - \text{id}_A$ is positive.

(ii) $A$ is either commutative or quasi-standard and 2-subhomogeneous.

Proof. (i) $\Rightarrow$ (ii). Suppose that there exists a C. E. $E : A \to Z(A)$ such that the map $2 \cdot E - \text{id}_A$ is positive. Then by Theorem 1.4 $A$ is a continuous $C(\hat{Z})$-algebra and $r(A_x) \leq 2$ for all $x \in \hat{Z}$. In particular, $A$ as a $C^\ast$-algebra is $n$-subhomogeneous, where $n \in \{1, 2\}$. Hence, by [13, Proposition 4.1] every Glimm ideal of $A$ is primal. Also, $n = 1$ if and only if $A$ is commutative.

(ii) $\Rightarrow$ (i). If $A$ is commutative we have nothing to prove, so suppose that $A$ is quasi-standard and 2-subhomogeneous. Then by [9, Corollary 1, p. 388] for each point $x \in X$ we have

$$r(A_x) = \sum_{[\pi_x] \in \mathcal{A}_x} \dim \pi_x \leq 2.$$

It remains to apply Proposition 3.7. \hfill \square
Remark 3.10. At the end of this paper we note that every separable continuous unital $C(X)$-algebra $A$ admits a faithful C.E. $E : A \rightarrow C(X)$ (see e.g. [5]). In particular, this result applies to continuous subhomogeneous unital $C(X)$-algebras, when $X$ is second-countable. In this case for each point $x \in X$, the map $E_x : a_x \mapsto E(a)(x)$ defines a faithful state on $A_x$, so Lemma 3.2 implies $K(E_x) < \infty$. However, this does not imply that $E$ is of finite index. That is, it may happen that $\sup_{x \in X} K(E_x) = \infty$. Consider for instance the following example:

- Let $X$ be the closed compact subset $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ of $[0, 1]$. 
- Let $A$ be the continuous $C(X)$-subalgebra of $C(X) \oplus C(X)$ consisting of all pairs $(f, g) \in C(X) \oplus C(X)$ such that $f(0) = g(0)$. 
- Let $E : A \rightarrow C(X)$ be a C.E. fixed by the relations

$$E(f \oplus g) \left(\frac{1}{n}\right) = \begin{cases} \frac{n}{n+1} f\left(\frac{1}{n}\right) + \frac{1}{n+1} g\left(\frac{1}{n}\right) & \text{if } n \text{ is odd} \\ \frac{n}{n+1} f\left(\frac{1}{n}\right) + \frac{n}{n+1} g\left(\frac{1}{n}\right) & \text{otherwise} \end{cases}$$

where $(f, g) \in A$.

Then $E$ is a faithful C.E. which is not of finite index. Indeed, one has

$$E(f \oplus 0) \left(\frac{1}{2n}\right) = \frac{1}{2n} + f\left(\frac{1}{2n}\right)$$

for all $f \in C_0(X \setminus \{0\})$ and all integers $n \in \mathbb{N}$. Consequently, a convenient constant $K$ would satisfy $K \geq 2n + 1$ for all $n \in \mathbb{N}$, which is impossible.

We end this paper with some unresolved problems:

**Problem 3.11.** Is the converse of Theorem 1.4 also true? Moreover, does every continuous subhomogeneous unital $C(X)$-algebra $A$ admit a C.E. $E : A \rightarrow C(X)$ such that the map $\gamma(A) \cdot E - \text{id}_A$ is positive? In particular, do we always have $K(A, C(X)) = \gamma(A)$?

References