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May 19, 2014

Abstract

The Grundy number of a graph G, denoted by $\Gamma(G)$, is the largest k such that there exists a partition of $V(G)$, into k independent sets $V_1, \ldots, V_k$ and every vertex of V_i is adjacent to at least one vertex in V_j, for every $j < i$. The objects which are studied in this article are families of r-regular graphs such that $\Gamma(G) = r + 1$. Using the notion of independent module, a characterization of this family is given for $r = 3$. Moreover, we determine classes of graphs in this family, in particular the class of r-regular graphs without induced C_4, for $r \leq 4$. Furthermore, our propositions imply results on partial Grundy number.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph $G = (V, E)$, a *proper k-coloring* of G is a surjective mapping $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E$; the *color class* V_i is the set $\{u \in V | c(u) = i\}$ and a vertex v has color i if $v \in V_i$. A vertex v of color i is a *Grundy vertex* if v is adjacent to at least one vertex colored j, for every $j < i$. A *Grundy k-coloring* is a proper k-coloring such that every vertex is a Grundy vertex. A *partial Grundy k-coloring* is a proper k-coloring such that every color class contains a Grundy vertex. The *Grundy number* (*partial Grundy number*, respectively) of G denoted by $\Gamma(G)$ ($\partial\Gamma(G)$, respectively) is the largest k such that G admits a Grundy k-coloring (partial Grundy k-coloring, respectively).

Let $N(v) = \{u \in V(G) | uv \in E(G)\}$ be the neighborhood of v. A set X of vertices is an *independent module* if X is an independent set and all vertices

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in X have the same neighborhood. The vertices in an independent module of size 2 are called *false twins*. Let P_n, C_n, K_n and I_n be respectively, the path, cycle complete and empty graph of order n. The concepts of Grundy k-coloring and domination are connected. In a Grundy coloring, V_1 is a dominating set. Given a graph G and an ordering ϕ on $V(G)$ with $\phi = v_1, \ldots, v_n$, the greedy algorithm assigns to v_i the minimum color that was not assigned in the set $\{v_1, \ldots, v_{i-1}\} \cap N(v_i)$. Let $\Gamma_\phi(G)$ be the number of colors used by the greedy algorithm with the ordering ϕ on G. We obtain the following result [7]: $\Gamma(G) = \max\limits_{\phi \in S_n}(\Gamma_\phi(G))$.

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a given graph, even for complements of bipartite graphs, is an NP-complete problem. However, for a fixed t, determining if a given graph has Grundy number at least t is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called t-atoms, such that any graph with Grundy number at least t contains a t-atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for the Grundy number [8, 15], that there exist upper bounds for d-degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between the products of graphs and the Grundy number [6, 1, 4]. Recently, Havet and Sampaio [9] have proven that the problem of deciding if for a given graph G we have $\Gamma(G) = \Delta(G)+1$, even if G is bipartite, is NP-complete. Moreover, they have proven that the dual of Grundy k-coloring problem is in FPT by finding an algorithm in $O(2k^{2k}.|E| + 2^{2k}k^{3k+5/2})$ time.
Note that a Grundy k-coloring is a partial Grundy k-coloring, hence $\Gamma(G) \leq \partial\Gamma(G)$. Given a graph G and a positive integer k, the problem of determining if a partial Grundy k-coloring exists, even for chordal graphs, is NP-complete but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the *b-chromatic number*, denoted by $\varphi(G)$, which is the largest k such that there exists a proper k-coloring and for every color class V_i, there exists a vertex adjacent to at least one vertex colored j, for every j, with $j \neq i$. Note that a b-coloring is a partial Grundy k-coloring, hence $\varphi(G) \leq \partial\Gamma(G)$. The b-chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: A characterization of the Grundy number of every cubic graph and the following theorem: For $r \leq 4$, every r-regular graphs without induced C_4 has Grundy number $r + 1$. We conjecture that this assertion is also true for $r > 4$.

Conjecture 1. *For any integer $r \geq 1$, every r-regular graph without induced C_4 has Grundy number $r + 1$.*

Section 2 gives characterizations of some classes of graphs with Grundy number at most k, $2 \leq k \leq \Delta(G)$, using the notion of independent module. Section 3 contains the first main theorem: A description of the cubic graphs with Grundy number at most 3 that also allows us to prove that every cubic graph except

$K_{3,3}$ has partial Grundy number 4. This theorem implies the existence of a linear algorithm to determine the Grundy number of cubic graphs. In Section 4, we present examples of infinite families of regular graphs with Grundy number exactly or at most k, $3 \leq k \leq r$. To determine these families we use recursive definitions. The last section contains the second main theorem of this article: 4-regular graphs without induced C_4 have Grundy number 5.

2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy k-coloring.

Definition 2.1. *Let G be a graph. A Grundy partial k-coloring is a Grundy k-coloring of a subset S of $V(G)$.*

Observation 2.2 ([1],[6])**.** *If G admits a Grundy partial k-coloring, then $\Gamma(G) \geq k$.*

This property has an important consequence: For a graph G, with $\Gamma(G) \geq t$ and any Grundy partial t-coloring, there exist smallest subgraphs H of G such that $\Gamma(H) = t$. The family of t-atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of t-atoms is finite and the presence of a t-atom can be determined in polynomial time for a fixed t. The following definition is slightly different from Zaker's one, insisting more on the construction of every t-atom.

Definition 2.3 ([15])**.** *For any integer t, we define the family of t-atoms, denoted by $\mathscr{A}_t$, $t = 1, \ldots$ by induction. Let the family $\mathscr{A}_1$ contain only K_1. A graph G is in $\mathscr{A}_{t+1}$ if there exists a graph G' in $\mathscr{A}_t$ and an integer m, $m \leq |V(G')|$, such that G is composed of G' and an independent set I_m of order m, adding edges between G' and I_m such that every vertex in G' is connected to at least one vertex in I_m. Moreover a t-atom A is minimal, if there is no t-atom included in A other than itself.*

Theorem 1 ([15])**.** *For a given graph G, $\Gamma(G) \geq t$ if and only if G contains an induced minimal t-atom.*

We now present conditions related to the presence of modules that allows us to upper-bound the Grundy number.

Proposition 2.4 ([1])**.** *Let G be a graph and X be an independent module. In every Grundy coloring of G, the vertices in X must have the same color.*

Definition 2.5. *Let G be an r-regular graph. A vertex v is a $(0, \ell)$-twin-vertex if there exists an independent module of cardinality $r + 2 - \ell$ that contains v.*

Proposition 2.6. *Let G be an r-regular graph. The color of an $(0, \ell)$-twin-vertex is at most ℓ in every Grundy coloring of G.*

Proof. Let v be a $(0, \ell)$-twin-vertex colored $\ell + 1$ in G. By Definition, v is in an independent module X of cardinality $r + 2 - \ell$ and by Proposition 2.4, every other vertex of X should be colored $\ell + 1$. Let u be a neighbor of v. There are at most $\ell - 2$ neighbors of u in $V(G - X)$. Therefore, u cannot be colored ℓ. □

Definition 2.7. *A vertex v of a graph G is a $(1, \ell)$-twin-vertex if $N(v)$ can be partitioned into at least $\ell - 1$ independent modules.*

Proposition 2.8. *Let G be a graph. The color of an $(1, \ell)$-twin-vertex is at most ℓ in every Grundy coloring of G.*

Proof. By Proposition 2.4, vertices of the neighborhood of v can only have $\ell - 1$ different colors. Therefore, the color of v is at most ℓ. □

Definition 2.9. *A vertex v of a graph G is a $(2, \ell)$-twin-vertex if $N(v)$ is independent and every vertex in $N(v)$ is a $(1, \ell)$-twin-vertex.*

Proposition 2.10. *Let G be a graph. The color of an $(2, \ell)$-twin-vertex is at most ℓ in every Grundy coloring of G.*

Proof. Let v be a $(2, \ell)$-twin-vertex in G. Every vertex in $N(v)$ is a $(1, \ell)$-twin-vertex. If a vertex in $N(v)$ is colored ℓ, then v could only have a color at most $\ell - 1$. If the vertices in the neighborhood of v have colors at most $\ell - 1$, then in every Grundy coloring of G, v has a color at most ℓ. □

Corollary 2.11. *Let G be a graph. If every vertex is a $(1, \ell)$-twin-vertex or a $(2, \ell)$-twin-vertex, then $\Gamma(G) \leq \ell$.*

Corollary 2.12. *Let G be a regular graph. If every vertex is an (i, ℓ)-twin-vertex, for some i, $0 \leq i \leq 2$, then $\Gamma(G) \leq \ell$.*

Proposition 2.13 ([1],[15])**.** *Let G be a graph. We have $\Gamma(G) \leq 2$ if and only if $G = K_{n,m}$ for some integers $n > 0$ and $m > 0$.*

3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial k-colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures cannot be adjacent as it would contradict the hypothesis.

Proposition 3.1 ([6])**.** *Let G be a connected 2-regular graph. $\partial\Gamma(G) = \Gamma(G) = 2$ if and only if $G = C_4$.*

The following definition gives a construction of the cubic graphs in which every vertex is an $(i, 3)$-twin-vertex, for some i, $0 \leq i \leq 2$. Figure 2 gives the list of every graph of order at most 16 in this family.

Definition 3.2. *Let $K_{2,3}$ and $K_{3,3}^*$ be the graphs from Figure 1. We define recursively the family of graphs $\mathscr{F}_3^*$ as follows:*

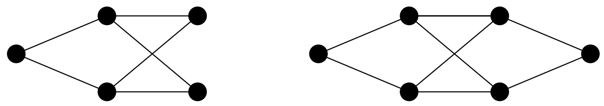

Figure 1: The graphs $K_{2,3}$ (on the left) and $K^*_{3,3}$ (on the right).

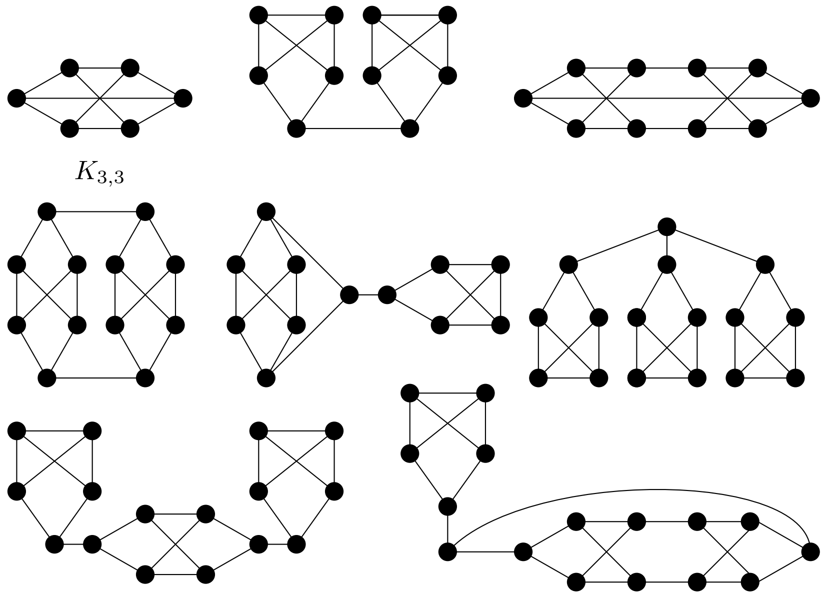

Figure 2: The cubic graphs G such that $|V(G)| < 18$ and $\Gamma(G) < 4$.

1. *$K_{2,3} \in \mathscr{F}^*_3$ and $K^*_{3,3} \in \mathscr{F}^*_3$;*
2. *the disjoint union of two elements of $\mathscr{F}^*_3$ is in $\mathscr{F}^*_3$;*
3. *if G is a graph in $\mathscr{F}^*_3$, then the graph H obtained from G by adding an edge between two vertices of degree at most 2 is also in $\mathscr{F}^*_3$;*
4. *if G is a graph in $\mathscr{F}^*_3$, then the graph H obtained from G by adding a new vertex adjacent to three vertices of degree at most 2 is in $\mathscr{F}^*_3$.*

The family $\mathscr{F}_3$ is the subfamily of cubic graphs in $\mathscr{F}^*_3$.

Proposition 3.3. *Let G be a cubic graph. Every vertex of $V(G)$ is an $(i,3)$-twin vertex, for some i, $0 \leq i \leq 2$, if and only if $G \in \mathscr{F}_3$.*

Proof. Every graph G in $\mathscr{F}_3$ has three kind of vertices: $(0,3)$-twin-vertices (called also false twins), vertices where an edge is added by Point 3 and vertices added by Point 4. Vertices where an edge is added by Point 3 are $(1, 3)$-twin-vertex and vice versa. Vertices added by Point 4 are $(2, 3)$-twin-vertices and vice versa. $\square$

Theorem 2. *Let G be a cubic graph. $\Gamma(G) \leq 3$ if and only if every vertex is an $(i, 3)$-twin-vertex, for some i, $0 \leq i \leq 2$.*

Proof. By Corollary 2.12, the "if" part is proven. Assume that G contains a vertex v which is not an $(i, 3)$-twin-vertex, for some i, $0 \leq i \leq 2$ and $\Gamma(G) < 4$. In every configuration we want to either find a Grundy partial 4-coloring, contradicting $\Gamma(G) < 4$ or proving that v is an $(i, 3)$-twin-vertex, for some i, with $0 \leq i \leq 2$. We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases: v or a neighbor of v is in a C_3, v is in an induced C_4 and v or a neighbor of v are not in a C_3 and v is not in an induced C_4. Let C be an induced cycle of order 3 or 4 which contains v or a neighbor of v and let $D_1 = \{x \in V(G) | d(x, C) = 1\}$, where $d(x, C)$ is the distance from x to C in the graph G. To simplify notation, D_1 will also denote the subgraph of G induced by D_1.

Case 1: Assume that v or a neighbor of v is in C and $C = C_3$. If $|D_1| = 1$, then $G = K_4$ and $\Gamma(K_4) = 4$. If $|D_1| = 2$ and $D_1 = P_2$, then v is a $(0, 3)$-twin-vertex or a $(1, 3)$-twin-vertex. If $D_1 = I_2$ then Figure 3.1.a yields a Grundy partial 4-coloring of G. If $|D_1| = 3$, then we have four subcases: D_1 is C_3 or P_3 (Figure 3.1.b), $P_2 \cup I_1$ (Figure 3.1.c) or I_3 (Figure 3.1.d). In every case G admits a Grundy partial 4-coloring.

Case 2: Assume that v is in C and $C = C_4$. Note that for two non adjacent vertices of C who have a common neighbor in D_1, the vertex v is a $(0, 3)$-twin-vertex or a $(1, 3)$-twin-vertex. Hence, we will not consider these cases. If $|D_1| = 2$, then $D_1 = P_2$ or $D_1 = I_2$ (Figure 3.2.a) and in both cases, G admits a Grundy partial 4-coloring. If $|D_1| = 3$, Figure 3.2.b yields a Grundy partial 4-coloring of G. In the case $|D_1| = 4$, we first assume that two adjacent vertices of C have their neighbors in D_1 adjacent (Figure 3.2.c). Afterwards, we suppose that the previous case does not happen and that two non adjacent vertices of C have their neighbors in D_1 adjacent (Figure 3.2.d). In the case $D_1 = I_4$, we first suppose that two vertices of D_1 which have two adjacent vertices of C as neighbor, are not adjacent to two common vertices (Figure 3.2.e) and after consider they are (Figure 3.2.f).

Case 3: Assume that v or a neighbor of v is not in a C_3 and v is not in an induced C_4. Firstly, suppose that a neighbor u of v is in an induced C_4. Using the coloring from the previous case, G admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of v in the C_4 have a common neighbor outside the C_4. However, this case cannot happen for every neighbor of v, otherwise v would be a $(2, 3)$-twin-vertex. Assume that u is the neighbor of v not in the previous configuration. If u is in an induced C_4, then using the coloring from the previous case, G

admits a Grundy partial 4-coloring. If u is not in an induced C_4, then Figure 3.3.a yields a Grundy partial 4-coloring of G. In this figure, the color 2 is given to a neighbor of u not adjacent to both f_1 and f_2. Secondly, suppose that v is in an induced C_5. Figure 3.3.b yields a Grundy partial 4-coloring of G. Thirdly, if v is not in an induced C_5, then Figure 3.3.c yields a Grundy partial 4-coloring of G.

Therefore, if $\Gamma(G) \leq 3$, then every vertex is an $(i,3)$-twin-vertex, for some i, $0 \leq i \leq 2$. □

Observe that if an edge is added between the two vertices of degree 2 in $K^*_{3,3}$, then we obtain $K_{3,3}$ which has Grundy number 2. By Proposition 3.3, in all the remaining cases, the cubic graphs which have Grundy number at most 3 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

Corollary 3.4. *A cubic graph G does not contain any induced minimal subcubic 4-atom if and only if every vertex is an $(i,3)$-twin-vertex, for some i, $0 \leq i \leq 2$.*

Corollary 3.5. *Let G be a cubic graph. If G is without induced C_4, then $\Gamma(G) = 4$.*

Proof. As every graph G with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K^*_{3,3}$, the graph G always contains a square if $\Gamma(G) < 4$. □

For a fixed integer t, the largest $(t+1)$-atom has order 2^t. Thus, for a graph G of maximum degree t, there exists an $O(n^{2^t})$-time algorithm to determine if $\Gamma(G) < t+1$ (which verifies if the graph contains an induced $(t+1)$-atom). For a cubic graph, we obtain an $O(n^8)$-time algorithm, whereas our characterization yields a linear-time algorithm.

Observation 3.6. *Let G be a cubic graph of order n. There exists an $O(n)$-time algorithm*[1] *to determine the Grundy number of G.*

Proof. Suppose we have a cubic graph G with its adjacency list. Verifying if G is $K_{3,3}$ can be done in constant time. We suppose now that G is not $K_{3,3}$. For each vertex v, the algorithm verifies that v is an $(i,3)$-twin-vertex, for some i, $0 \leq i \leq 2$. If the condition is true for all vertices, then $\Gamma(G) = 3$, else $\Gamma(G) = 4$. To determine if a vertex v is a $(0,3)$-twin-vertex, it suffices to verify that there is a common vertex other than v in the adjacency lists of the neighbors of v. To determine if a vertex v is a $(1,3)$-twin-vertex, it suffices to verify that there are two neighbors of v which have the same adjacency list. To determine if a vertex v is a $(2,3)$-twin-vertex, it suffices to verify that the neighborhood of v is independent and that every neighbor is a $(1,3)$-twin-vertex. Hence, checking if a vertex is an $(i,3)$-twin vertex can be done in constant time, so the algorithms runs in linear time. □

[1] Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.

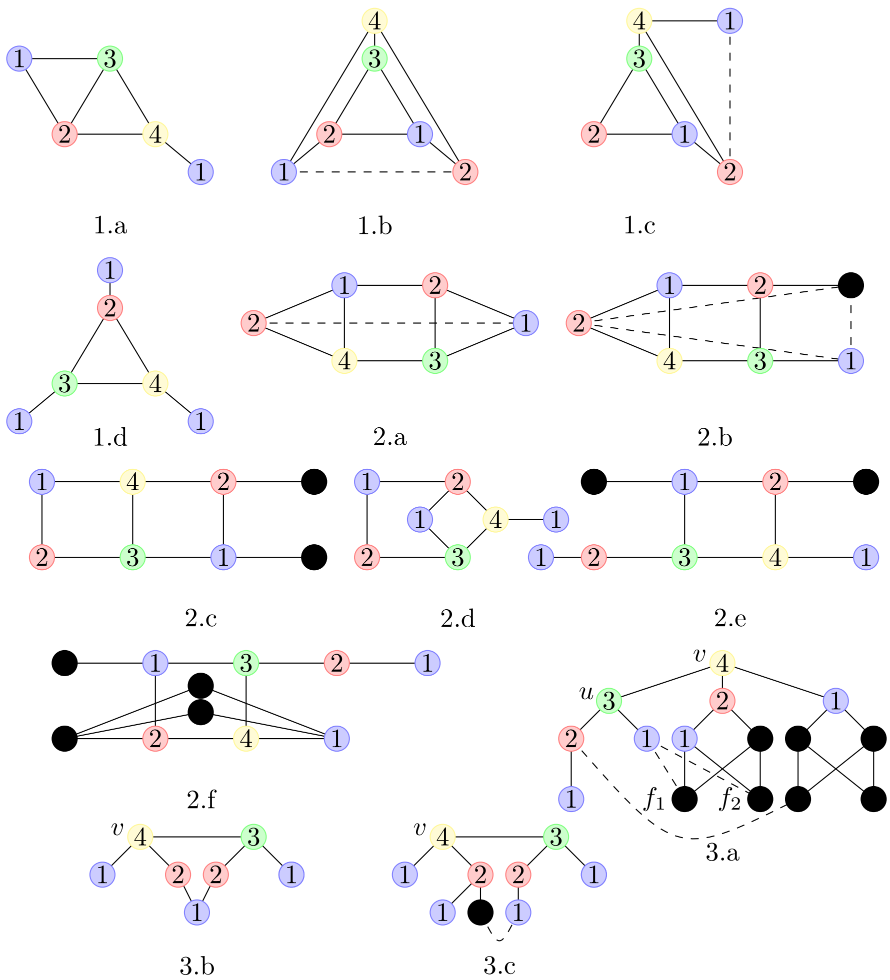

Figure 3: Possible configurations in a cubic graph (bold vertices: Uncolored vertices, vertices with number i: Vertices of color i).

Proposition 3.7. *If G is a connected cubic graph and $G \neq K_{3,3}$, then $\partial\Gamma(G) = 4$.*

Proof. Let G be a cubic connected graph. Note that if $\Gamma(G) = 4$ then $\partial\Gamma(G) = 4$. Every graph G with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K^*_{3,3}$. If G contains more than two copies (so it is different from $K_{3,3}$), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial\Gamma(G) = 4$. □

Only $K_{3,3}$ and three other cubic graphs have b-chromatic number at most 3 [10]. Thus, our result is coherent with the results on the b-chromatic number. Shi et al. [13] proved that there exists a smallest integer N_r such that every r-regular graph G with more than N_r vertices has $\partial\Gamma(G) = r+1$. Observe that we have $N_2 = 4$ and $N_3 = 6$. It is an open question to determine N_r for $r \geq 4$. However, using results on b-chromatic number [3], we have $N_r \leq 2r^3 - r^2 + r$.

4 Properties on the Grundy number of r-regular graphs

Definition 4.1. *Let $r \geq 2$ be an integer. We define recursively the family of graphs $\mathscr{G}^*_r$ as follows:*

1. *$K_{r-k,k+2} \in \mathscr{G}^*_r$, for any k, $0 \leq k \leq (r-2)/2$;*
2. *the disjoint union of two elements of $\mathscr{G}^*_r$ is in $\mathscr{G}^*_r$;*
3. *if G is a graph in $\mathscr{G}^*_r$, then the graph H obtained from G by adding an edge between two vertices of degree at most $r-1$ is also in $\mathscr{G}^*_r$;*
4. *if G is a graph in $\mathscr{G}^*_r$, then the graph H obtained from G by adding a new vertex adjacent to r vertices of degree at most $r-1$ is in $\mathscr{G}^*_r$.*

The family $\mathscr{G}_r$ is the subfamily of r-regular graphs in $\mathscr{G}^*_r$.

Proposition 4.2. *Let G be an r-regular graph. If $G \in \mathscr{G}_r$, then $\Gamma(G) < r+1$.*

Proof. By I_{r-k} and I_{k+2}, with $|I_{r-k}| = r-k$ and $|I_{k+2}| = k+2$, we denote the two sets of vertices in the bipartition of an induced subgraph $K_{r-k,k+2}$ in G. Firstly, suppose there exists a vertex u in an induced subgraph $K_{r-k,k+2}$ colored $r+1$. Without loss of generality, suppose u is in I_{r-k}. The r neighbors of u should have colors from 1 to r. Among the neighbors of u, $k+2$ neighbors are in I_{k+2}. Let v be the neighbor of u in I_{k+2} with the largest color in I_{k+2}. The vertex v has color at least $k+2$. Hence, there exists an integer $s \geq 0$ such that the color of v is $k+2+s$. Note that there are s vertices in $N(u) \setminus I_{k+2}$ which have colors at most $k+2+s$. The colors of the s vertices are the only one possible remaining colors at most $k+2+s$ in I_{r-k}. Hence, as there are k vertices in $N(v) \setminus I_{r-k}$, the neighbors of v can only have at most $k+s$ different colors at most $k+2+s$. Therefore, we have a contradiction and u cannot have

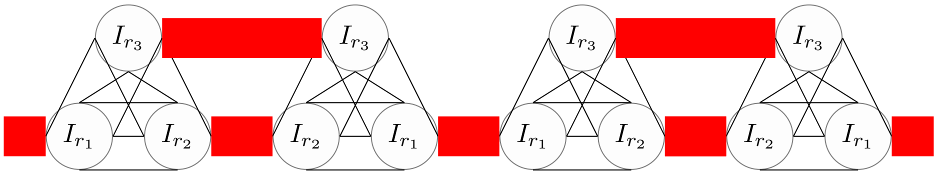

Figure 4: The Graph $G_{r,4,i}$, $i \geq 2$, $r = r_1 + r_2 + r_3$.

color $r+1$. Secondly, suppose there exists a vertex u added by Point 4 which has color $r+1$. As a neighbor of u in an induced $K_{r-k,k+2}$ should be colored r, the argument is completely similar to the previous one. $\square$

Corollary 4.3. *Let G be a 4-regular graph. If $G \in \mathscr{G}_4$, then $\Gamma(G) < 5$.*

The reader can believe that the family of 4-regular graphs with $\Gamma(G) < 5$ contains only the family $\mathscr{G}_4$. However, there exist graphs with Grundy number r which are not inside this family. For example, the power graph (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle C_7^2 satisfies $\Gamma(C_7^2) < 5$ and is not in $\mathscr{G}_4$.

The next proposition shows that unlike the b-chromatic number, r-regular graphs of order arbitrarily large with Grundy number k can be constructed for any r and any k, $3 \leq k \leq r+1$.

Proposition 4.4. *Let $r \geq 4$ and $3 \leq k \leq r+1$ be integers. There exists an infinite family $\mathscr{H}$ of connected r-regular graphs such that for all G in $\mathscr{H}$, $\Gamma(G) = k$.*

Proof. Let $i \geq 2$ be a positive integer and $r_1, \ldots, r_{k-1}$ be a sequence of positive integers such that $r = r_1 + \ldots + r_{k-1}$. We construct a graph $G_{r,k,i}$ as follows: Take $2i$ copies of $K_{r_1,\ldots,r_{k-1}}$. Let H_{j-1} be the copy number j of $K_{r_1,\ldots,r_{k-1}}$ and H_{j,r_l} be the independent r_l-set in H_j. If $j \equiv 1 \pmod 2$, do the graph join of $H_{j \pmod{2i}, r_1}$ and $H_{j-1 \pmod{2i}, r_1}$ and for an integer l, $1 < l < k$, do the graph join of $H_{j \pmod{2i}, r_l}$ and $H_{j+1 \pmod{2i}, r_l}$. The r-regular graph obtained is the graph $G_{r,k,i}$. Figure 4 gives $G_{r,k,i}$, for $k = 4$ and $i \geq 2$. Note that H_{j,r_i} is an independent module. Thus, every vertex is a $(0,k)$-twin-vertex. By Proposition 2.6, $\Gamma(G_{r,k,i}) \leq k$.
For an integer l, $1 < l < k$, color one vertex $l-1$ in H_{1,r_l} and H_{2,r_l}. Afterwards, color one vertex $k-1$ in H_{1,r_1} and one vertex k in H_{2,r_1}. The given coloring is a Grundy partial k-coloring of $G_{r,k,i}$ for $i \geq 2$. Therefore, $\Gamma(G_{r,k,i}) = k$, for $i \geq 2$. $\square$

5 Grundy number of 4-regular graphs without induced C_4

The following lemmas will be useful to prove the second main theorem of this paper: The family of 4-regular graphs without induced C_4 contains only graphs with Grundy number 5.

Lemma 5.1. *Let G be a 4-regular graph without induced C_4. If G contains (an induced) K_4 then $\Gamma(G) = 5$.*

Proof. Note that if $G = K_5$, we have $\Gamma(G) = 5$. If G is not K_5 then every pair of neighbors of vertices of K_4 cannot be adjacent (G would contain a C_4). Giving the color 1 to each neighbor of the vertices of K_4 and colors 2, 3, 4, 5 to the vertices of K_4, we obtain a Grundy partial 5-coloring of G. □

Lemma 5.2. *Let G be a 4-regular graph without induced C_4 and let W be the graph from Figure 5. If G contains an induced W then $\Gamma(G) = 5$.*

Proof. The names of the vertices of W come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of G will be given using their references on Figure 5. Let D_1 be the set of vertices at distance 1 from vertices of W in $G - W$. Suppose that two vertices of W have a common neighbor in D_1. This two vertices could only be u_4 and u_5 or u_3 and u_5 (or u_1 and u_4, by symmetry). In the case that u_4 and u_5 have a common neighbor in D_1, colors will be given to neighbors of u_3 in D_1, depending if they are adjacent (Figure 5.1.a) or not (Figure 5.1.b). In the case that u_3 and u_5 have a common neighbor w in D_1, w can be adjacent with a neighbor of u_3 in D_1 (Figure 5.2.a) or not (Figure 5.2.b). Suppose now that no vertices in W have a common neighbor in D_1. Let w_1 and w_2 be the neighbors of u_3 in D_1. We first consider that w_1 and w_2 are adjacent (Figure 5.3.a). Secondly, we consider that w_1 and w_2 are not adjacent and that u_5, u_3 and w_1 are in an induced C_5 (Figure 5.3.b). Finally, we consider that the previous configurations are impossible (Figure 5.3.c). □

Proposition 5.3. *Let G be a 4-regular graph without induced C_4. If G contains C_3 then $\Gamma(G) = 5$.*

Proof. Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of G in Figure 6 will be given. Let M_i, $i = 2$ or 3, be the graph of order $2 + i$ containing two adjacent vertices u_1 and u_2 which have exactly i common neighbors, $\{v_1, \ldots, v_i\}$, that form an independent set. Let D_1 be the set of vertices at distance 1 from an induced M_i in $G - M_i$, for $2 \le i \le 3$.

Case 1: Firstly, assume that G contains an induced M_3 and a vertex of M_3 has its two neighbors in D_1 adjacent (Figure 6.1.a). Secondly, assume that G contains an induced M_2 and a vertex of M_2 has its two neighbors in D_1

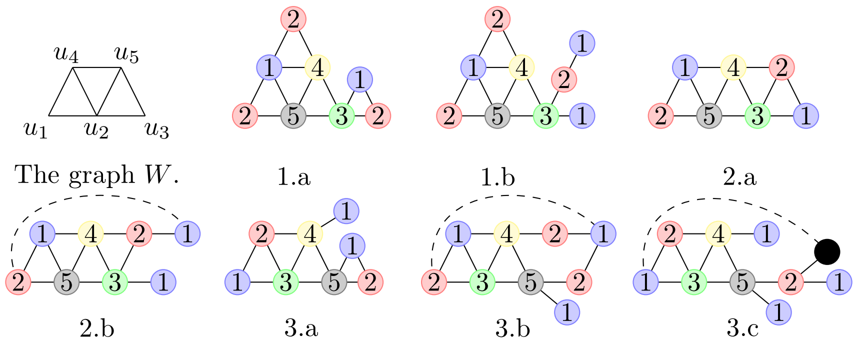

Figure 5: Possible configurations when G contains an induced W.

adjacent (Figure 6.1.b). Note that these Grundy partial 5-colorings use the fact that G cannot contain a K_4 by Lemma 5.1.

Case 2: Assume that G contains an induced M_3 excluding the previous configuration. There are three cases: u_1, v_2 and v_3 are in an induced C_5 (Figure 6.2.a), u_1, v_2 and v_3 are in an induced C_6 and not in an induced C_5 (Figure 6.2.b) and u_1, v_2 and v_3 are neither in an induced C_5 nor C_6 (Figure 6.2.c).

Case 3: Suppose that G contains an induced M_2 excluding the previous configurations. Firstly, we suppose that u_1, v_1 and v_2 are in an induced C_5 (Figure 6.3.a). Secondly, we suppose that u_1, v_1 are in an induced C_5 excluding the previous case (Figure 6.3.b). Thirdly, we suppose that u_1, v_1 and v_2 are in an induced C_6 and not in an induced C_5 (Figure 6.3.c) and finally neither in an induced C_5 nor C_6 (Figure 6.3.d).

Suppose that G contains a 3-cycle C and no induced M_2. Let u_1, u_2 and u_3 be the vertices of C. Let w_1 and w_2 be the neighbors of u_1 outside C, let w_1' and w_2' be the neighbors of u_2 outside C and let w_1'' and w_2'' be the neighbors of u_3 outside C.

Case 4: Firstly, suppose that u_1, u_2, w_1 and w_1' are in a 5-cycle and a neighbor of u_1, say w_1, has a common neighbor with w_1' (Figure 6.4.a). Secondly, excluding the previous configuration, suppose that u_1, u_2, w_1 and w_1' are in a 5-cycle; w_1'', v_1, u_1 and w_1 are in another 5-cycle and w_1 is in a triangle (Figure 6.4.b). We suppose that w_1 is not in a triangle (Figure 6.4.c). Thirdly, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of C are in a 5-cycle (Figure 6.4.d). Fourthly, we suppose that two vertices of C cannot be in a 5-cycle (Figure 6.4.e).

□

In the following two lemmas, we consider a graph G of girth $g = 5$ and possibly containing an induced Petersen graph. Let u_1, u_2, u_3, u_4 and u_5 be

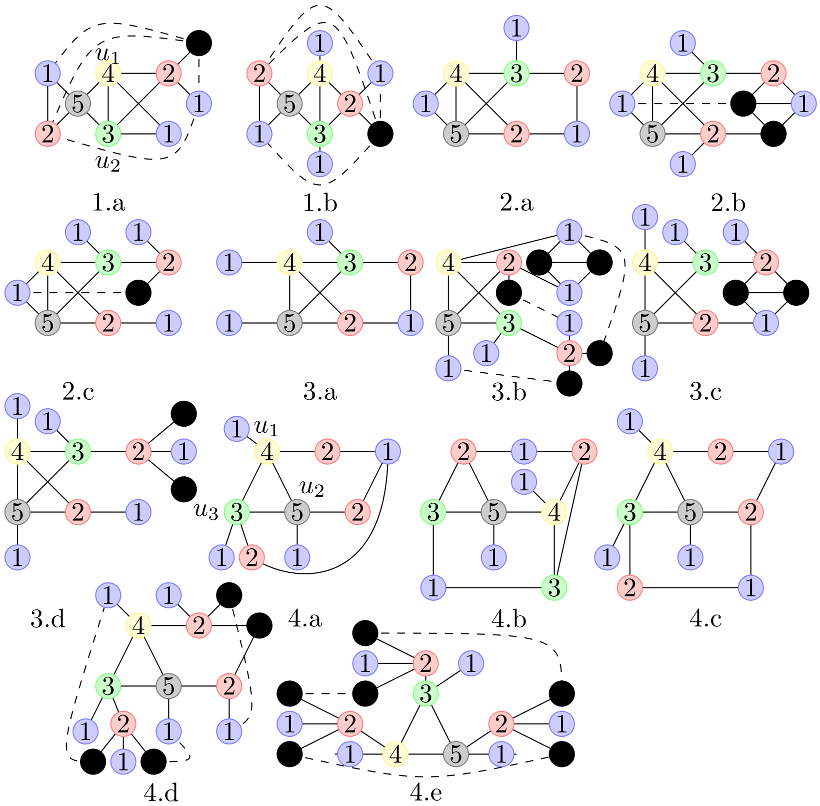

Figure 6: Possible configurations when G an induced C_3.

the vertices in an induced C_5 (or in the the outer cycle of a Petersen graph, if any). Let v_1, v_1', v_2, v_2', v_3, v_3', v_4, v_4', v_5 and v_5' be the remaining neighbors of respectively u_1, u_2, u_3, u_4 and u_5 (all different as $g = 5$).

Lemma 5.4. *Let G be a 4-regular graph with girth $g = 5$. If G contains the Petersen graph as induced subgraph then $\Gamma(G) = 5$.*

Proof. Suppose that v_1, v_2, v_3, v_4 and v_5 form an induced C_5 (the inner cycle of the Petersen graph). Let u_2' and u_5' be the remaining neighbors of respectively v_2 and v_5. Observe that v_1' can be adjacent with no more than three vertices among v_3', v_4', u_2' and u_5'. Firstly, suppose that v_1' is not adjacent with v_3' (or v_4', without loss of generality since the configuration is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph G. Secondly, assume that v_1' is not adjacent with u_5' (or u_2', without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph G. □

In a graph G, let a *neighbor-connected* C_n be an n-cycle C such that the set

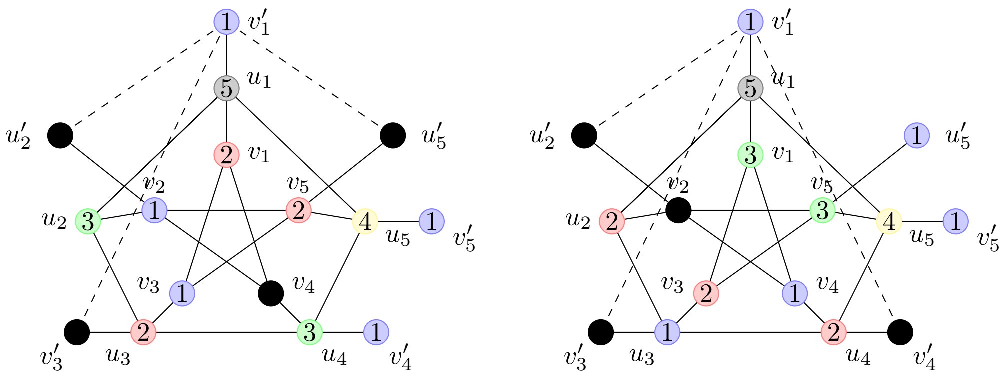

Figure 7: Two Grundy partial 5-colorings of a subgraph containing an induced Petersen graph.

of vertices of G at distance 1 from C is not independent.

Lemma 5.5. *Let G be a 4-regular graph with girth $g = 5$. If G contains a neighbor-connected C_5 as induced subgraph, then $\Gamma(G) = 5$.*

Proof. Let C be a neighbor-connected C_5 in G. By Lemma 5.4 we can suppose that the neighbors of the vertices of C do not form an induced C_5 (otherwise a Petersen would be an induced subgraph). Hence, we can assume that the neighbors of the vertices of C form a subgraph of a C_{10}. If there are two edges between the neighbors of the vertices of C, then Figure 8 illustrates Grundy partial 5-colorings of the graph G. Suppose that two neighbors are adjacent, say v_1 and v_3' and the graph G does not contain the previous configuration. Note that v_3' can be adjacent with v_1' and v_5'. Let w_1, w_2 and w_3 be the three neighbors of v_2 different from u_2. We suppose that w_1 can be possibly adjacent with v_3' and w_2 can be possibly adjacent with v_1'. Figure 9 illustrates a Grundy partial 5-coloring of G in this case. In this figure, the vertex w_3 can be possibly adjacent with v_5' or v_4, but in this case we can switch the color 1 from v_5' to v_5 or from v_4' to v_4. □

Proposition 5.6. *If G is a 4-regular graph with girth $g = 5$, then $\Gamma(G) = 5$.*

Proof. Let C be a 5-cycle in G. Assume that two neighbors of consecutive vertices of C, for example v_1 and v_5, have a common neighbor w_1. The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph G. In this figure the vertex w_1 can be possibly adjacent with v_2', v_3' or v_4, but in this case we can switch the color 1 from v_2' to v_2, from v_3' to v_3 or from v_4 to v_4'. Hence, we can suppose that no neighbors of consecutive vertices of C are adjacent. Among the neighbors of v_1, there exists one vertex w_1 not adjacent with both v_4 and v_4' (otherwise G would contain a C_4). Among the neighbor of v_5', there exists one

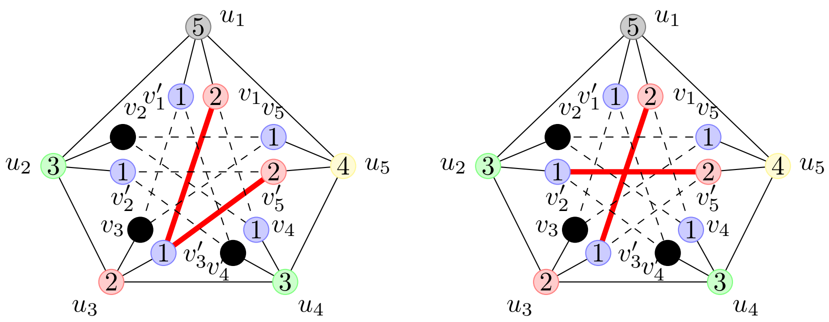

Figure 8: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected C_5.

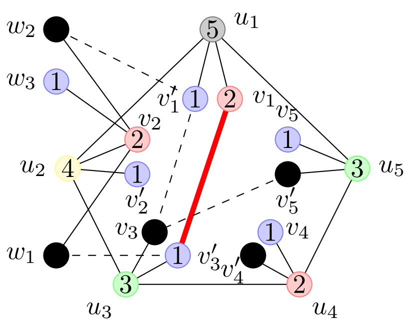

Figure 9: A Grundy partial 5-coloring of a subgraph containing an induced neighbor-connected C_5.

vertex, say w_2, not adjacent with w_1. The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph G. In this figure the vertex w_1 can be possibly adjacent with v_4 and the vertex w_2 can be possibly adjacent with v_2' or v_4, but in these cases we can switch the color 1 from v_2' to v_2 or from v_4 to v_4'. □

In the following lemma and proposition, we consider a graph G of girth $g = 6$. Let u_1, u_2, u_3, u_4, u_5 and u_6 be the vertices in an induced C_6. Let v_1, v_1', v_2, v_2', v_3, v_3', v_4, v_4', v_5, v_5', v_6 and v_6' be the remaining neighbors of respectively u_1, u_2, u_3, u_4, u_5 and u_6 (all different as $g = 6$).

Lemma 5.7. *If G is a 4-regular graph with girth $g = 6$ which contains a neighbor-connected C_6 as induced subgraph, then $\Gamma(G) = 5$.*

Proof. Firstly, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let w_1 be a neighbor of v_1' not adjacent with v_4. The graph G admits a Grundy partial 5-coloring as the

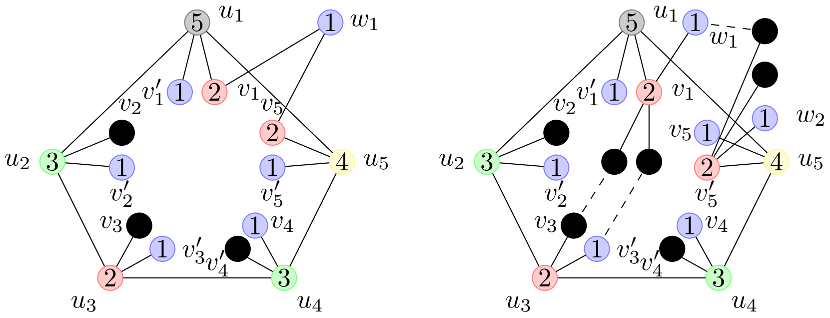

Figure 10: Two Grundy partial 5-colorings of a subgraph containing an induced C_5.

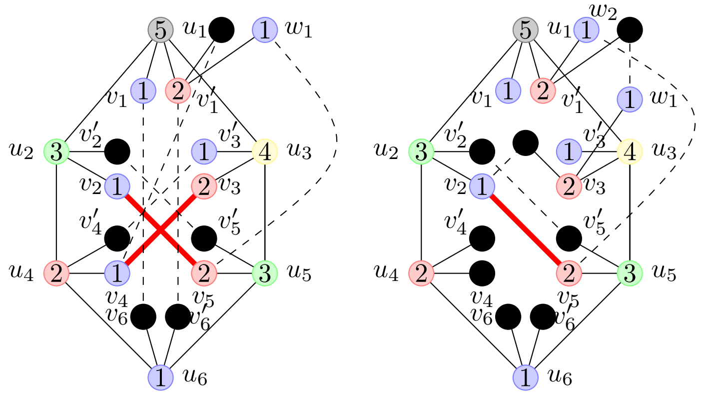

Figure 11: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected C_6.

left part of Figure 11 illustrates it. Secondly, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let w_1 be a neighbor of v_3 not adjacent with v_2 and let w_2 be a neighbor of v_1' not adjacent with w_1. The graph G admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it. $\square$

Proposition 5.8. *If G is a 4-regular graph with girth $g = 6$, then $\Gamma(G) = 5$.*

Proof. By Lemma 5.7, assume that no neighbors of the vertices of the induced C_6 are adjacent. Firstly, suppose that there are two neighbors at distance 4 along the cycle C_6, for example v_1' and v_5, which have a common neighbor w_1. Let w_2 be a neighbor of v_3 not adjacent with w_1. G admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Secondly, suppose that there

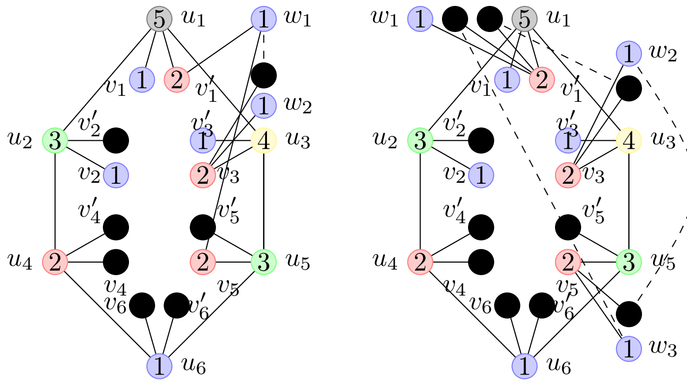

Figure 12: Two Grundy partial 5-colorings of a subgraph containing an induced C_6.

are no two neighbors at distance 4 along the cycle C_6 which have a common neighbor. Let w_1 be a neighbor of v_1' not adjacent with a neighbor of v_5 or a neighbor of v_3, let w_2 be a neighbor of v_3 not adjacent with a neighbor of v_5, and let w_3 be a neighbor of v_5. The graph G admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it. □

Proposition 5.9. *If G is a 4-regular graph with girth $g \geq 7$, then $\Gamma(G) = 5$.*

Proof. Suppose that G contains a 7-cycle. We denote the 5-atom which is a tree by T_5 (the binomial tree with maximum degree 4). It can be easily verified that G contains T_5 where two leaves are merged (which is a 5-atom). Moreover, if G does not contain a 7-cycle, then it contains T_5 as induced subgraph. □

Theorem 3. *Let G be a 4-regular graph. If G does not contain an induced C_4, then $\Gamma(G) = 5$.*

Proof. Suppose that G does not contain an induced C_4. Using Proposition 5.9 for the case $g \geq 7$, Propositions 5.6 and 5.8 for the case $g = 5, 6$, and Proposition 5.3 when G contains a C_3 yields the desired result. □

By Proposition 3.1, Corollary 3.5 and Theorem 3, any r-regular graph with $r \leq 4$ and without induced C_4 has Grundy number $r+1$. Therefore, it is natural to propose Conjecture 1.

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