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Abstract

The Grundy number of a graph \( G \), denoted by \( \Gamma(G) \), is the largest \( k \) such that there exists a partition of \( V(G) \), into \( k \) independent sets \( V_1, \ldots, V_k \) and every vertex of \( V_i \) is adjacent to at least one vertex in \( V_j \), for every \( j < i \). The objects which are studied in this article are families of \( r \)-regular graphs such that \( \Gamma(G) = r + 1 \). Using the notion of independent module, a characterization of this family is given for \( r = 3 \). Moreover, we determine classes of graphs in this family, in particular the class of \( r \)-regular graphs without induced \( C_4 \), for \( r \leq 4 \). Furthermore, our propositions imply results on partial Grundy number.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph \( G = (V, E) \), a proper \( k \)-coloring of \( G \) is a surjective mapping \( c : V \to \{1, \ldots, k\} \) such that \( c(u) \neq c(v) \) for any \( uv \in E \); the color class \( V_i \) is the set \( \{u \in V | c(u) = i\} \) and a vertex \( v \) has color \( i \) if \( v \in V_i \). A vertex \( v \) of color \( i \) is a Grundy vertex if \( v \) is adjacent to at least one vertex colored \( j \), for every \( j < i \). A Grundy \( k \)-coloring is a proper \( k \)-coloring such that every vertex is a Grundy vertex. A partial Grundy \( k \)-coloring is a proper \( k \)-coloring such that every color class contains a Grundy vertex. The Grundy number (partial Grundy number, respectively) of \( G \) denoted by \( \Gamma(G) \) (\( \partial \Gamma(G) \), respectively) is the largest \( k \) such that \( G \) admits a Grundy \( k \)-coloring (partial Grundy \( k \)-coloring, respectively).

Let \( N(v) = \{u \in V(G)|uv \in E(G)\} \) be the neighborhood of \( v \). A set \( X \) of vertices is an independent module if \( X \) is an independent set and all vertices

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in $X$ have the same neighborhood. The vertices in an independent module of size 2 are called false twins. Let $P_n$, $C_n$, $K_n$ and $I_n$ be respectively, the path, cycle complete and empty graph of order $n$. The concepts of Grundy $k$-coloring and domination are connected. In a Grundy coloring, $V_1$ is a dominating set. Given a graph $G$ and an ordering $\phi$ on $V(G)$ with $\phi = v_1, \ldots, v_n$, the greedy algorithm assigns to $v_i$ the minimum color that was not assigned in the set $\{v_1, \ldots, v_{i-1}\} \cap N(v_i)$. Let $\Gamma_\phi(G)$ be the number of colors used by the greedy algorithm with the ordering $\phi$ on $G$. We obtain the following result [7]:

$$\Gamma(G) = \max_{\phi \in S_n} \Gamma_\phi(G).$$

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a given graph, even for complements of bipartite graphs, is an NP-complete problem. However, for a fixed $t$, determining if a given graph has Grundy number at least $t$ is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called $t$-atoms, such that any graph with Grundy number at least $t$ contains a $t$-atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for the Grundy number [8, 15], that there exist upper bounds for $d$-degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between the products of graphs and the Grundy number [6, 1, 4].

Recently, Havet and Sampaio [9] have proven that the dual of Grundy $k$-coloring problem is in FPT by finding an algorithm in $O(2k^{2k} |E| + 2^{2k}k^{3k+5/2})$ time. Note that a Grundy $k$-coloring is a partial Grundy $k$-coloring, hence $\Gamma(G) \leq \partial \Gamma(G)$. Given a graph $G$ and a positive integer $k$, the problem of determining if a partial Grundy $k$-coloring exists, even for chordal graphs, is NP-complete but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the $b$-chromatic number, denoted by $\varphi(G)$, which is the largest $k$ such that there exists a proper $k$-coloring and for every color class $V_i$, there exists a vertex adjacent to at least one vertex colored $j$, for every $j$, with $j \neq i$. Note that a $b$-coloring is a partial Grundy $k$-coloring, hence $\varphi(G) \leq \partial \Gamma(G)$. The $b$-chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: A characterization of the Grundy number of every cubic graph and the following theorem: For $r \leq 4$, every $r$-regular graphs without induced $C_4$ has Grundy number $r + 1$. We conjecture that this assertion is also true for $r > 4$.

**Conjecture 1.** For any integer $r \geq 1$, every $r$-regular graph without induced $C_4$ has Grundy number $r + 1$.

Section 2 gives characterizations of some classes of graphs with Grundy number at most $k$, $2 \leq k \leq \Delta(G)$, using the notion of independent module. Section 3 contains the first main theorem: A description of the cubic graphs with Grundy number at most 3 that also allows us to prove that every cubic graph except
$K_{3,3}$ has partial Grundy number 4. This theorem implies the existence of a linear algorithm to determine the Grundy number of cubic graphs. In Section 4, we present examples of infinite families of regular graphs with Grundy number exactly or at most $k$, $3 \leq k \leq r$. To determine these families we use recursive definitions. The last section contains the second main theorem of this article: 4-regular graphs without induced $C_4$ have Grundy number 5.

2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy $k$-coloring.

**Definition 2.1.** Let $G$ be a graph. A Grundy partial $k$-coloring is a Grundy $k$-coloring of a subset $S$ of $V(G)$.

**Observation 2.2 ([1],[6]).** If $G$ admits a Grundy partial $k$-coloring, then $\Gamma(G) \geq k$.

This property has an important consequence: For a graph $G$, with $\Gamma(G) \geq t$ and any Grundy partial $t$-coloring, there exist smallest subgraphs $H$ of $G$ such that $\Gamma(H) = t$. The family of $t$-atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from Zaker’s one, insisting more on the construction of every $t$-atom.

**Definition 2.3 ([15]).** For any integer $t$, we define the family of $t$-atoms, denoted by $\mathcal{A}_t$, $t = 1, \ldots$ by induction. Let the family $\mathcal{A}_t$ contain only $K_1$. A graph $G$ is in $\mathcal{A}_{t+1}$ if there exists a graph $G'$ in $\mathcal{A}_t$ and an integer $m$, $m \leq |V(G')|$, such that $G$ is composed of $G'$ and an independent set $I_m$ of order $m$, adding edges between $G'$ and $I_m$ such that every vertex in $G'$ is connected to at least one vertex in $I_m$. Moreover a $t$-atom $A$ is minimal, if there is no $t$-atom included in $A$ other than itself.

**Theorem 1 ([15]).** For a given graph $G$, $\Gamma(G) \geq t$ if and only if $G$ contains an induced minimal $t$-atom.

We now present conditions related to the presence of modules that allows us to upper-bound the Grundy number.

**Proposition 2.4 ([1]).** Let $G$ be a graph and $X$ be an independent module. In every Grundy coloring of $G$, the vertices in $X$ must have the same color.

**Definition 2.5.** Let $G$ be an $r$-regular graph. A vertex $v$ is a $(0, \ell)$-twin-vertex if there exists an independent module of cardinality $r + 2 - \ell$ that contains $v$.

**Proposition 2.6.** Let $G$ be an $r$-regular graph. The color of an $(0, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$. 

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Proof. Let \( v \) be a \((0, \ell)\)-twin-vertex colored \( \ell + 1 \) in \( G \). By Definition, \( v \) is in an independent module \( X \) of cardinality \( r + 2 - \ell \) and by Proposition 2.4, every other vertex of \( X \) should be colored \( \ell + 1 \). Let \( u \) be a neighbor of \( v \). There are at most \( \ell - 2 \) neighbors of \( u \) in \( V(G - X) \). Therefore, \( u \) cannot be colored \( \ell \). \( \square 

**Definition 2.7.** A vertex \( v \) of a graph \( G \) is a \((1, \ell)\)-twin-vertex if \( N(v) \) can be partitioned into at least \( \ell - 1 \) independent modules.

**Proposition 2.8.** Let \( G \) be a graph. The color of an \((1, \ell)\)-twin-vertex is at most \( \ell \) in every Grundy coloring of \( G \).

*Proof.* By Proposition 2.4, vertices of the neighborhood of \( v \) can only have \( \ell - 1 \) different colors. Therefore, the color of \( v \) is at most \( \ell \). \( \square 

**Definition 2.9.** A vertex \( v \) of a graph \( G \) is a \((2, \ell)\)-twin-vertex if every vertex in \( N(v) \) is independent and every vertex in \( N(v) \) is a \((1, \ell)\)-twin-vertex.

**Proposition 2.10.** Let \( G \) be a graph. The color of an \((2, \ell)\)-twin-vertex is at most \( \ell \) in every Grundy coloring of \( G \).

*Proof.* Let \( v \) be a \((2, \ell)\)-twin-vertex in \( G \). Every vertex in \( N(v) \) is a \((1, \ell)\)-twin-vertex. If a vertex in \( N(v) \) is colored \( \ell \), then \( v \) could only have a color at most \( \ell - 1 \). If the vertices in the neighborhood of \( v \) have colors at most \( \ell - 1 \), then in every Grundy coloring of \( G \), \( v \) has a color at most \( \ell \). \( \square 

**Corollary 2.11.** Let \( G \) be a graph. If every vertex is a \((1, \ell)\)-twin-vertex or a \((2, \ell)\)-twin-vertex, then \( \Gamma(G) \leq \ell \).

**Corollary 2.12.** Let \( G \) be a regular graph. If every vertex is an \((i, \ell)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \), then \( \Gamma(G) \leq \ell \).

**Proposition 2.13** ([1],[15]). Let \( G \) be a graph. We have \( \Gamma(G) \leq 2 \) if and only if \( G = K_{n,m} \) for some integers \( n > 0 \) and \( m > 0 \).

### 3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial \( k \)-colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures cannot be adjacent as it would contradict the hypothesis.

**Proposition 3.1** ([6]). Let \( G \) be a connected 2-regular graph. \( \partial \Gamma(G) = \Gamma(G) = 2 \) if and only if \( G = C_4 \).

The following definition gives a construction of the cubic graphs in which every vertex is an \((i, 3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \). Figure 2 gives the list of every graph of order at most 16 in this family.

**Definition 3.2.** Let \( K_{2,3} \) and \( K_{3,3} \) be the graphs from Figure 1. We define recursively the family of graphs \( \mathcal{F}_3 \) as follows:
The family $F_3$ is the subfamily of cubic graphs in $\mathcal{F}_3^*$.

**Proposition 3.3.** Let $G$ be a cubic graph. Every vertex of $V(G)$ is an $(i, 3)$-twin vertex, for some $i$, $0 \leq i \leq 2$, if and only if $G \in \mathcal{F}_3$.

**Proof.** Every graph $G$ in $\mathcal{F}_3$ has three kind of vertices: $(0, 3)$-twin-vertices (called also false twins), vertices where an edge is added by Point 3 and vertices
added by Point 4. Vertices where an edge is added by Point 3 are (1,3)-twin-vertex and vice versa. Vertices added by Point 4 are (2,3)-twin-vertices and vice versa.

Case 1: Assume that \( v \) is in \( C \) and \( C = C_3 \). If \(|D_1| = 1\), then \( G = K_4 \) and \( \Gamma(K_4) = 4 \). If \(|D_1| = 2 \), then \( G \) admits a Grundy partial 4-coloring. If \(|D_1| = 3\), then we have four subcases:

- \( D_1 = C_4 \) or \( P_3 \) (Figure 3.1.b), \( P_2 \cup I_1 \) (Figure 3.1.c) or \( I_3 \) (Figure 3.1.d).

Case 2: Assume that \( v \) is in \( C \) and \( C = C_4 \). Note that for two non adjacent vertices of \( C \) who have a common neighbor in \( D_1 \), the vertex \( v \) is a (0,3)-twin-vertex or a (1,3)-twin-vertex. Hence, we will not consider these cases. If \(|D_1| = 2\), then \( D_1 = P_2 \) or \( D_1 = I_2 \) (Figure 3.2.a) and in both cases, \( G \) admits a Grundy partial 4-coloring. If \(|D_1| = 3\), Figure 3.2.b yields a Grundy partial 4-coloring of \( G \). In the case \(|D_1| = 4\), we first assume that two adjacent vertices of \( C \) have their neighbors in \( D_1 \) adjacent (Figure 3.2.c). Afterwards, we suppose that the previous case does not happen and that two non adjacent vertices of \( C \) have their neighbors in \( D_1 \) adjacent (Figure 3.2.d). In the case \( D_1 = I_4 \), we first suppose that two vertices of \( D_1 \) which have two adjacent vertices of \( C \) as neighbor, are not adjacent to two common vertices (Figure 3.2.e) and after consider they are (Figure 3.2.f).

Case 3: Assume that \( v \) or a neighbor of \( v \) is not in a \( C_4 \) and \( v \) is not in an induced \( C_4 \). Firstly, suppose that a neighbor \( u \) of \( v \) is in an induced \( C_4 \). Using the coloring from the previous case, \( G \) admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of \( v \) in the \( C_4 \) have a common neighbor outside the \( C_4 \). However, this case cannot happen for every neighbor of \( v \), otherwise \( v \) would be a (2,3)-twin-vertex. Assume that \( u \) is the neighbor of \( v \) not in the previous configuration. If \( u \) is in an induced \( C_4 \), then using the coloring from the previous case, \( G \)

Theorem 2. Let \( G \) be a cubic graph. \( \Gamma(G) \leq 3 \) if and only if every vertex is an \((i,3)\)-twin-vertex, for some \( i, 0 \leq i \leq 2 \).

Proof. By Corollary 2.12, the "if" part is proven. Assume that \( G \) contains a vertex \( v \) which is not an \((i,3)\)-twin-vertex, for some \( i, 0 \leq i \leq 2 \) and \( \Gamma(G) < 4 \). In every configuration we want to either find a Grundy partial 4-coloring, contradicting \( \Gamma(G) < 4 \) or proving that \( v \) is an \((i,3)\)-twin-vertex, for some \( i \), with \( 0 \leq i \leq 2 \). We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases: \( v \) or a neighbor of \( v \) is in a \( C_3 \), \( v \) is in an induced \( C_4 \) and \( v \) or a neighbor of \( v \) are not in a \( C_3 \) and \( v \) is not in an induced \( C_4 \). Let \( C \) be an induced cycle of order 3 or 4 which contains \( v \) or a neighbor of \( v \) and let \( D_1 = \{x \in V(G) \mid d(x, C) = 1\} \), where \( d(x, C) \) is the distance from \( x \) to \( C \) in the graph \( G \). To simplify notation, \( D_1 \) will also denote the subgraph of \( G \) induced by \( D_1 \).
admits a Grundy partial 4-coloring. If \( u \) is not in an induced \( C_4 \), then Figure 3.3.a yields a Grundy partial 4-coloring of \( G \). In this figure, the color 2 is given to a neighbor of \( u \) not adjacent to both \( f_1 \) and \( f_2 \). Secondly, suppose that \( v \) is in an induced \( C_5 \). Figure 3.3.b yields a Grundy partial 4-coloring of \( G \). Thirdly, if \( v \) is not in an induced \( C_5 \), then Figure 3.3.c yields a Grundy partial 4-coloring of \( G \).

Therefore, if \( \Gamma(G) \leq 3 \), then every vertex is an \((i,3)\)-twin-vertex, for some \( i \).

Observe that if an edge is added between the two vertices of degree 2 in \( K_3^* \), then we obtain \( K_3^* \) which has Grundy number 2. By Proposition 3.3, in all the remaining cases, the cubic graphs which have Grundy number at most 3 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

**Corollary 3.4.** A cubic graph \( G \) does not contain any induced minimal subcubic 4-atom if and only if every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

**Corollary 3.5.** Let \( G \) be a cubic graph. If \( G \) is without induced \( C_4 \), then \( \Gamma(G) = 4 \).

**Proof.** As every graph \( G \) with \( \Gamma(G) < 4 \) is composed of copies of \( K_{2,3} \) or \( K_3^* \), the graph \( G \) always contains a square if \( \Gamma(G) < 4 \).

For a fixed integer \( t \), the largest \((t+1)\)-atom has order \( 2^t \). Thus, for a graph \( G \) of maximum degree \( t \), there exists an \( O(n^t) \)-time algorithm to determine if \( \Gamma(G) < t+1 \) (which verifies if the graph contains an induced \((t+1)\)-atom). For a cubic graph, we obtain an \( O(n^8) \)-time algorithm, whereas our characterization yields a linear-time algorithm.

**Observation 3.6.** Let \( G \) be a cubic graph of order \( n \). There exists an \( O(n) \)-time algorithm\(^1\) to determine the Grundy number of \( G \).

**Proof.** Suppose we have a cubic graph \( G \) with its adjacency list. Verifying if \( G \) is \( K_3^* \) can be done in constant time. We suppose now that \( G \) is not \( K_3^* \). For each vertex \( v \), the algorithm verifies that \( v \) is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \). If the condition is true for all vertices, then \( \Gamma(G) = 3 \), else \( \Gamma(G) = 4 \).

To determine if a vertex \( v \) is a \((0,3)\)-twin-vertex, it suffices to verify that there is a common vertex other than \( v \) in the adjacency lists of the neighbors of \( v \). To determine if a vertex \( v \) is a \((1,3)\)-twin-vertex, it suffices to verify that there are two neighbors of \( v \) which have the same adjacency list. To determine if a vertex \( v \) is a \((2,3)\)-twin-vertex, it suffices to verify that the neighborhood of \( v \) is independent and that every neighbor is a \((1,3)\)-twin-vertex. Hence, checking if a vertex is an \((i,3)\)-twin vertex can be done in constant time, so the algorithms runs in linear time.\(^1\)

\(^1\)Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.
Figure 3: Possible configurations in a cubic graph (bold vertices: Uncolored vertices, vertices with number $i$: Vertices of color $i$).
Shi et al. [13] proved that there exists a smallest integer \( \mathcal{N} \). Thus, our result is coherent with the results on the \( r \)-Properties on the Grundy number of Proposition 3.7.

Let \( \Gamma(G) \) be a cubic connected graph. Note that if \( \Gamma(G) = 4 \) then \( \partial \Gamma(G) = 4 \). Every graph \( G \) with \( \Gamma(G) < 4 \) is composed of copies of \( K_{2,3} \) or \( K_{3,3}^* \). If \( G \) contains more than two copies (so it is different from \( K_{3,3} \)), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, \( \partial \Gamma(G) = 4 \).

Only \( K_{3,3} \) and three other cubic graphs have \( b \)-chromatic number at most 3 [10]. Thus, our result is coherent with the results on the \( b \)-chromatic number.

Proof.\( \Box \)

Proposition 3.7. If \( G \) is a connected cubic graph and \( G \neq K_{3,3} \), then \( \partial \Gamma(G) = 4 \).

Proof. Let \( G \) be a cubic connected graph. Note that if \( \Gamma(G) = 4 \) then \( \partial \Gamma(G) = 4 \). Therefore, we have \( N \). Observe that we have \( N_2 = 4 \) and \( N_3 = 6 \). It is an open question to determine \( N_r \) for \( r \geq 4 \). However, using results on \( b \)-chromatic number [3], we have \( N_r \leq 2r^3 - r^2 + r \).

4 Properties on the Grundy number of \( r \)-regular graphs

Definition 4.1. Let \( r \geq 2 \) be an integer. We define recursively the family of graphs \( \mathcal{G}_r^* \) as follows:

1. \( K_{r-k,k+2} \in \mathcal{G}_r^* \), for any \( k \), \( 0 \leq k \leq (r-2)/2 \);
2. the disjoint union of two elements of \( \mathcal{G}_r^* \) is in \( \mathcal{G}_r^* \);
3. if \( G \) is a graph in \( \mathcal{G}_r^* \), then the graph \( H \) obtained from \( G \) by adding an edge between two vertices of degree at most \( r - 1 \) is also in \( \mathcal{G}_r^* \);
4. if \( G \) is a graph in \( \mathcal{G}_r^* \), then the graph \( H \) obtained from \( G \) by adding a new vertex adjacent to \( r \) vertices of degree at most \( r - 1 \) is in \( \mathcal{G}_r^* \).

The family \( \mathcal{G}_r \) is the subfamily of \( r \)-regular graphs in \( \mathcal{G}_r^* \).

Proposition 4.2. Let \( G \) be an \( r \)-regular graph. If \( G \in \mathcal{G}_r \), then \( \Gamma(G) < r + 1 \).

Proof. By \( I_{r-k} \) and \( I_{k+2} \), with \( |I_{r-k}| = r - k \) and \( |I_{k+2}| = k + 2 \), we denote the two sets of vertices in the bipartition of an induced subgraph \( K_{r-k,k+2} \) in \( G \). Firstly, suppose there exists a vertex \( u \) in an induced subgraph \( K_{r-k,k+2} \) colored \( r + 1 \). Without loss of generality, suppose \( u \) is in \( I_{r-k} \). The \( r \) neighbors of \( u \) should have colors from 1 to \( r \). Among the neighbors of \( u \), \( k + 2 \) neighbors are in \( I_{k+2} \). Let \( v \) be the neighbor of \( u \) in \( I_{k+2} \) with the largest color in \( I_{k+2} \). The vertex \( v \) has color at least \( k + 2 \). Hence, there exists an integer \( s \geq 0 \) such that the color of \( v \) is \( k + 2 + s \). Note that there are \( s \) vertices in \( N(u) \setminus I_{k+2} \) which have colors at most \( k + 2 + s \). The colors of the \( s \) vertices are the only one possible remaining colors at most \( k + 2 + s \) in \( I_{r-k} \). Hence, as there are \( k \) vertices in \( N(v) \setminus I_{r-k} \), the neighbors of \( v \) can only have at most \( k + s \) different colors at most \( k + 2 + s \). Therefore, we have a contradiction and \( u \) cannot have
Figure 4: The Graph $G_{r,4,i}$, $i \geq 2$, $r = r_1 + r_2 + r_3$.

color $r + 1$. Secondly, suppose there exists a vertex $u$ added by Point 4 which has color $r + 1$. As a neighbor of $u$ in an induced $K_{r-k,k+2}$ should be colored $r$, the argument is completely similar to the previous one. □

**Corollary 4.3.** Let $G$ be a 4-regular graph. If $G \in \mathcal{G}_4$, then $\Gamma(G) < 5$.

The reader can believe that the family of 4-regular graphs with $\Gamma(G) < 5$ contains only the family $\mathcal{G}_4$. However, there exist graphs with Grundy number $r$ which are not inside this family. For example, the power graph (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle $C_7^2$ satisfies $\Gamma(C_7^2) < 5$ and is not in $\mathcal{G}_4$.

The next proposition shows that unlike the $b$-chromatic number, $r$-regular graphs of order arbitrarily large with Grundy number $k$ can be constructed for any $r$ and any $k$, $3 \leq k \leq r + 1$.

**Proposition 4.4.** Let $r \geq 4$ and $3 \leq k \leq r + 1$ be integers. There exists an infinite family $\mathcal{H}$ of connected $r$-regular graphs such that for all $G$ in $\mathcal{H}$, $\Gamma(G) = k$.

**Proof.** Let $i \geq 2$ be a positive integer and $r_1, \ldots, r_{k-1}$ be a sequence of positive integers such that $r = r_1 + \ldots + r_{k-1}$. We construct a graph $G_{r,k,i}$ as follows: Take $2i$ copies of $K_{r_1,\ldots,r_{k-1}}$. Let $H_{j-1}$ be the copy number $j$ of $K_{r_1,\ldots,r_{k-1}}$ and $H_{j,r_1}$ be the independent $r_1$-set in $H_j$. If $j \equiv 1 \pmod{2}$, do the graph join of $H_{j-1 \pmod{2},r_1}$ and $H_{j-1 \pmod{2},r_1}$ and for an integer $l$, $1 < l < k$, do the graph join of $H_{j-1 \pmod{2},r_1}$ and $H_{j+1 \pmod{2},r_1}$. The $r$-regular graph obtained is the graph $G_{r,k,i}$. Figure 4 gives $G_{r,k,i}$ for $k = 4$ and $i \geq 2$. Note that $H_{j,r_1}$ is an independent module. Thus, every vertex is a $(0,k)$-twin-vertex. By Proposition 2.6, $\Gamma(G_{r,k,i}) \leq k$.

For an integer $l$, $1 < l < k$, color one vertex $l-1$ in $H_{l,r_1}$ and $H_{l,r_1}$. Afterwards, color one vertex $k - 1$ in $H_{l,r_1}$ and one vertex $k$ in $H_{l,r_1}$. The given coloring is a Grundy partial $k$-coloring of $G_{r,k,i}$ for $i \geq 2$. Therefore, $\Gamma(G_{r,k,i}) = k$, for $i \geq 2$. □
5 Grundy number of 4-regular graphs without induced $C_4$

The following lemmas will be useful to prove the second main theorem of this paper: The family of 4-regular graphs without induced $C_4$ contains only graphs with Grundy number 5.

**Lemma 5.1.** Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains (an induced) $K_4$ then $\Gamma(G) = 5$.

**Proof.** Note that if $G = K_5$, we have $\Gamma(G) = 5$. If $G$ is not $K_5$ then every pair of neighbors of vertices of $K_4$ cannot be adjacent ($G$ would contain a $C_4$). Giving the color 1 to each neighbor of the vertices of $K_4$ and colors 2, 3, 4, 5 to the vertices of $K_4$, we obtain a Grundy partial 5-coloring of $G$. □

**Lemma 5.2.** Let $G$ be a 4-regular graph without induced $C_4$ and let $W$ be the graph from Figure 5. If $G$ contains an induced $W$ then $\Gamma(G) = 5$.

**Proof.** The names of the vertices of $W$ come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of $G$ will be given using their references on Figure 5. Let $D_1$ be the set of vertices at distance 1 from vertices of $W$ in $G - W$. Suppose that two vertices of $W$ have a common neighbor in $D_1$. This two vertices could only be $u_4$ and $u_5$ or $u_3$ and $u_5$ (or $u_1$ and $u_4$, by symmetry). In the case that $u_4$ and $u_5$ have a common neighbor in $D_1$, colors will be given to neighbors of $u_3$ in $D_1$, depending if they are adjacent (Figure 5.1.a) or not (Figure 5.1.b). In the case that $u_3$ and $u_5$ have a common neighbor $w$ in $D_1$, $w$ can be adjacent with a neighbor of $u_3$ in $D_1$ (Figure 5.2.a) or not (Figure 5.2.b). Suppose now that no vertices in $W$ have a common neighbor in $D_1$. Let $w_1$ and $w_2$ be the neighbors of $u_3$ in $D_1$. We first consider that $w_1$ and $w_2$ are adjacent (Figure 5.3.a). Secondly, we consider that $w_1$ and $w_2$ are not adjacent and that $u_5$, $u_3$ and $w_1$ are in an induced $C_5$ (Figure 5.3.b). Finally, we consider that the previous configurations are impossible (Figure 5.3.c). □

**Proposition 5.3.** Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains $C_3$ then $\Gamma(G) = 5$.

**Proof.** Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of $G$ in Figure 6 will be given. Let $M_i$, $i = 2$ or 3, be the graph of order $2 + i$ containing two adjacent vertices $u_1$ and $u_2$ which have exactly $i$ common neighbors, $\{v_1, \ldots, v_i\}$, that form an independent set. Let $D_1$ be the set of vertices at distance 1 from an induced $M_i$ in $G - M_i$, for $2 \leq i \leq 3$.

**Case 1:** Firstly, assume that $G$ contains an induced $M_3$ and a vertex of $M_3$ has its two neighbors in $D_1$ adjacent (Figure 6.1.a). Secondly, assume that $G$ contains an induced $M_2$ and a vertex of $M_2$ has its two neighbors in $D_1$.
adjacent (Figure 6.1.b). Note that these Grundy partial 5-colorings use the fact that $G$ cannot contain a $K_4$ by Lemma 5.1.

**Case 2:** Assume that $G$ contains an induced $M_3$ excluding the previous configuration. There are three cases: $u_1$, $v_2$ and $v_3$ are in an induced $C_5$ (Figure 6.2.a), $u_1$, $v_2$ and $v_3$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.2.b) and $u_1$, $v_2$ and $v_3$ are neither in an induced $C_5$ nor $C_6$ (Figure 6.2.c).

**Case 3:** Suppose that $G$ contains an induced $M_2$ excluding the previous configurations. Firstly, we suppose that $u_1$, $v_1$ and $v_2$ are in an induced $C_5$ (Figure 6.3.a). Secondly, we suppose that $u_1$, $v_1$ are in an induced $C_5$ excluding the previous case (Figure 6.3.b). Thirdly, we suppose that $u_1$, $v_1$ and $v_2$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.3.c) and finally neither in an induced $C_5$ nor $C_6$ (Figure 6.3.d).

Suppose that $G$ contains a 3-cycle $C$ and no induced $M_2$. Let $u_1$, $u_2$ and $u_3$ be the vertices of $C$. Let $w_1$ and $w_2$ be the neighbors of $u_1$ outside $C$, let $w'_1$ and $w'_2$ be the neighbors of $u_2$ outside $C$ and let $w''_1$ and $w''_2$ be the neighbors of $u_3$ outside $C$.

**Case 4:** Firstly, suppose that $u_1$, $u_2$, $w_1$ and $w'_1$ are in a 5-cycle and a neighbor of $u_1$, say $w_1$, has a common neighbor with $w'_1$ (Figure 6.4.a). Secondly, excluding the previous configuration, suppose that $u_1$, $u_2$, $w_1$ and $w'_1$ are in a 5-cycle; $w''_1$, $v_1$, $u_1$ and $w_1$ are in another 5-cycle and $w_1$ is in a triangle (Figure 6.4.b). We suppose that $w_1$ is not in a triangle (Figure 6.4.c). Thirdly, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of $C$ are in a 5-cycle (Figure 6.4.d). Fourthly, we suppose that two vertices of $C$ cannot be in a 5-cycle (Figure 6.4.e).

In the following two lemmas, we consider a graph $G$ of girth $g = 5$ and possibly containing an induced Petersen graph. Let $u_1$, $u_2$, $u_3$, $u_4$ and $u_5$ be

Figure 5: Possible configurations when $G$ contains an induced $W$.
the vertices in an induced $C_5$ (or in the the outer cycle of a Petersen graph, if any). Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5$ and $v'_5$ be the remaining neighbors of respectively $u_1, u_2, u_3, u_4$ and $u_5$ (all different as $g = 5$).

**Lemma 5.4.** Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains the Petersen graph as induced subgraph then $\Gamma(G) = 5$.

**Proof.** Suppose that $v_1, v_2, v_3, v_4$ and $v_5$ form an induced $C_5$ (the inner cycle of the Petersen graph). Let $u'_2$ and $u'_5$ be the remaining neighbors of respectively $v_2$ and $v_5$. Observe that $v'_1$ can be adjacent with no more than three vertices among $v'_3, v'_4, u'_2$ and $u'_5$. Firstly, suppose that $v'_1$ is not adjacent with $v'_3$ (or $v'_4$, without loss of generality since the configuration is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$. Secondly, assume that $v'_1$ is not adjacent with $u'_5$ (or $u'_2$, without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$.

In a graph $G$, let a *neighbor-connected* $C_n$ be an $n$-cycle $C$ such that the set
Figure 7: Two Grundy partial 5-colorings of a subgraph containing an induced Petersen graph.

of vertices of $G$ at distance 1 from $C$ is not independent.

Lemma 5.5. Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains a neighbor-connected $C_5$ as induced subgraph, then $\Gamma(G) = 5$.

Proof. Let $C$ be a neighbor-connected $C_5$ in $G$. By Lemma 5.4 we can suppose that the neighbors of the vertices of $C$ do not form an induced $C_5$ (otherwise a Petersen would be an induced subgraph). Hence, we can assume that the neighbors of the vertices of $C$ form a subgraph of a $C_{10}$. If there are two edges between the neighbors of the vertices of $C$, then Figure 8 illustrates Grundy partial 5-colorings of the graph $G$. Suppose that two neighbors are adjacent, say $v_1$ and $v'_3$ and the graph $G$ does not contain the previous configuration. Note that $v'_2$ can be adjacent with $v'_1$ and $v'_5$. Let $w_1, w_2$ and $w_3$ be the three neighbors of $v_2$ different from $u_2$. We suppose that $w_1$ can be possibly adjacent with $v'_1$ and $w_2$ can be possibly adjacent with $v'_4$. Figure 9 illustrates a Grundy partial 5-coloring of $G$ in this case. In this figure, the vertex $w_3$ can be possibly adjacent with $v'_5$ or $v_4$, but in this case we can switch the color 1 from $v'_5$ to $v_5$ or from $v'_4$ to $v_4$.

Proposition 5.6. If $G$ is a 4-regular graph with girth $g = 5$, then $\Gamma(G) = 5$.

Proof. Let $C$ be a 5-cycle in $G$. Assume that two neighbors of consecutive vertices of $C$, for example $v_1$ and $v_5$, have a common neighbor $w_1$. The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph $G$. In this figure the vertex $w_1$ can be possibly adjacent with $v'_2$, $v'_3$ or $v_4$, but in this case we can switch the color 1 from $v'_2$ to $v_2$, from $v'_3$ to $v_3$ or from $v_4$ to $v'_4$. Hence, we can suppose that no neighbors of consecutive vertices of $C$ are adjacent. Among the neighbors of $v_1$, there exists one vertex $w_1$ not adjacent with both $v_4$ and $v'_4$ (otherwise $G$ would contain a $C_4$). Among the neighbor of $v'_5$, there exists one
Figure 8: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected $C_5$.

Figure 9: A Grundy partial 5-coloring of a subgraph containing an induced neighbor-connected $C_5$.

vertex, say $w_2$, not adjacent with $w_1$. The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph $G$. In this figure the vertex $w_1$ can be possibly adjacent with $v_4$ and the vertex $w_2$ can be possibly adjacent with $v'_2$ or $v_4$, but in these cases we can switch the color 1 from $v'_2$ to $v_2$ or from $v_4$ to $v'_4$.

In the following lemma and proposition, we consider a graph $G$ of girth $g = 6$. Let $u_1, u_2, u_3, u_4, u_5$ and $u_6$ be the vertices in an induced $C_6$. Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5, v'_5, v_6$ and $v'_6$ be the remaining neighbors of respectively $u_1, u_2, u_3, u_4, u_5$ and $u_6$ (all different as $g = 6$).

**Lemma 5.7.** If $G$ is a 4-regular graph with girth $g = 6$ which contains a neighbor-connected $C_6$ as induced subgraph, then $\Gamma(G) = 5$.

**Proof.** Firstly, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let $w_1$ be a neighbor of $v'_1$ not adjacent with $v_4$. The graph $G$ admits a Grundy partial 5-coloring as the
Figure 10: Two Grundy partial 5-colorings of a subgraph containing an induced $C_5$.

Figure 11: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected $C_6$.

left part of Figure 11 illustrates it. Secondly, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let $w_1$ be a neighbor of $v_3$ not adjacent with $v_2$ and let $w_2$ be a neighbor of $v'_1$ not adjacent with $w_1$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it.

Proposition 5.8. If $G$ is a 4-regular graph with girth $g = 6$, then $\Gamma(G) = 5$.

Proof. By Lemma 5.7, assume that no neighbors of the vertices of the induced $C_6$ are adjacent. Firstly, suppose that there are two neighbors at distance 4 along the cycle $C_6$, for example $v'_1$ and $v_5$, which have a common neighbor $w_1$. Let $w_2$ be a neighbor of $v_3$ not adjacent with $w_1$. $G$ admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Secondly, suppose that there
are no two neighbors at distance 4 along the cycle $C_6$ which have a common neighbor. Let $w_1$ be a neighbor of $v'_1$ not adjacent with a neighbor of $v_5$ or a neighbor of $v_3$, let $w_2$ be a neighbor of $v_3$ not adjacent with a neighbor of $v_5$, and let $w_3$ be a neighbor of $v_5$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it.

**Proposition 5.9.** If $G$ is a 4-regular graph with girth $g \geq 7$, then $\Gamma(G) = 5$.

**Proof.** Suppose that $G$ contains a 7-cycle. We denote the 5-atom which is a tree by $T_5$ (the binomial tree with maximum degree 4). It can be easily verified that $G$ contains $T_5$ where two leaves are merged (which is a 5-atom). Moreover, if $G$ does not contain a 7-cycle, then it contains $T_5$ as induced subgraph.

**Theorem 3.** Let $G$ be a 4-regular graph. If $G$ does not contain an induced $C_4$, then $\Gamma(G) = 5$.

**Proof.** Suppose that $G$ does not contain an induced $C_4$. Using Proposition 5.9 for the case $g \geq 7$, Propositions 5.6 and 5.8 for the case $g = 5, 6$, and Proposition 5.3 when $G$ contains a $C_3$ yields the desired result.

By Proposition 3.1, Corollary 3.5 and Theorem 3, any $r$-regular graph with $r \leq 4$ and without induced $C_4$ has Grundy number $r + 1$. Therefore, it is natural to propose Conjecture 1.

**References**


