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On the family of $r$-regular graphs with Grundy number $r + 1$

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Abstract

The Grundy number of a graph $G$, denoted by $\Gamma(G)$, is the largest $k$ such that there exists a partition of $V(G)$, into $k$ independent sets $V_1, \ldots, V_k$ and every vertex of $V_i$ is adjacent to at least one vertex in $V_j$, for every $j < i$. The objects which are studied in this article are families of $r$-regular graphs such that $\Gamma(G) = r + 1$. Using the notion of independent module, a characterization of this family is given for $r = 3$. Moreover, we determine classes of graphs in this family, in particular the class of $r$-regular graphs without induced $C_4$, for $r \leq 4$. Furthermore, our propositions imply results on partial Grundy number.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph $G = (V, E)$, a proper $k$-coloring of $G$ is a surjective mapping $c : V \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E$; the color class $V_i$ is the set $\{u \in V | c(u) = i\}$ and a vertex $v$ has color $i$ if $v \in V_i$. A vertex $v$ of color $i$ is a Grundy vertex if $v$ is adjacent to at least one vertex colored $j$, for every $j < i$. A Grundy $k$-coloring is a proper $k$-coloring such that every vertex is a Grundy vertex. A partial Grundy $k$-coloring is a proper $k$-coloring such that every color class contains a Grundy vertex. The Grundy number (partial Grundy number, respectively) of $G$ denoted by $\Gamma(G)$ ($\partial \Gamma(G)$, respectively) is the largest $k$ such that $G$ admits a Grundy $k$-coloring (partial Grundy $k$-coloring, respectively).

Let $N(v) = \{u \in V(G)|uv \in E(G)\}$ be the neighborhood of $v$. A set $X$ of vertices is an independent module if $X$ is an independent set and all vertices

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in $X$ have the same neighborhood. The vertices in an independent module of size 2 are called false twins. Let $P_n$, $C_n$, $K_n$ and $I_n$ be respectively, the path, cycle complete and empty graph of order $n$. The concepts of Grundy $k$-coloring and domination are connected. In a Grundy coloring, $V_1$ is a dominating set. Given a graph $G$ and an ordering $\phi$ on $V(G)$ with $\phi = v_1, \ldots, v_n$, the greedy algorithm assigns to $v_i$ the minimum color that was not assigned in the set $\{v_1, \ldots, v_{i-1}\} \cap N(v_i)$. Let $\Gamma_\phi(G)$ be the number of colors used by the greedy algorithm with the ordering $\phi$ on $G$. We obtain the following result [7]:

$$\Gamma(G) = \max_{\phi \in S_n}(\Gamma_\phi(G)).$$

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a given graph, even for complements of bipartite graphs, is an NP-complete problem. However, for a fixed $t$, determining if a given graph has Grundy number at least $t$ is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called $t$-atoms, such that any graph with Grundy number at least $t$ contains a $t$-atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for the Grundy number [8, 15], that there exist upper bounds for $d$-degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between the products of graphs and the Grundy number [6, 1, 4].

Recently, Havet and Sampaio [9] have proven that the problem of deciding if for a given graph $G$ we have $\Gamma(G) = \Delta(G)+1$, even if $G$ is bipartite, is NP-complete. Moreover, they have proven that the dual of Grundy $k$-coloring problem is in FPT by finding an algorithm in $O(2^{2k2^{kE}} + 2^{2k2^{kE}}) = 5/2$ time.

Another coloring parameter with domination constraints on the colors is the $b$-chromatic number, denoted by $\varphi(G)$, which is the largest $k$ such that there exists a proper $k$-coloring and for every color class $V_i$, there exists a vertex adjacent to at least one vertex colored $j$, for every $j$, with $j \neq i$. Note that a $b$-coloring is a partial Grundy $k$-coloring, hence $\varphi(G) \leq \partial \Gamma(G)$. Given a graph $G$ and a positive integer $k$, the problem of determining if a partial Grundy $k$-coloring exists, even for chordal graphs, is NP-complete but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the $b$-chromatic number, denoted by $\varphi(G)$, which is the largest $k$ such that there exists a proper $k$-coloring and for every color class $V_i$, there exists a vertex adjacent to at least one vertex colored $j$, for every $j$, with $j \neq i$. Note that a $b$-coloring is a partial Grundy $k$-coloring, hence $\varphi(G) \leq \partial \Gamma(G)$. The $b$-chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: A characterization of the Grundy number of every cubic graph and the following theorem: For $r \leq 4$, every $r$-regular graphs without induced $C_4$ has Grundy number $r+1$. We conjecture that this assertion is also true for $r > 4$.

**Conjecture 1.** For any integer $r \geq 1$, every $r$-regular graph without induced $C_4$ has Grundy number $r+1$.

Section 2 gives characterizations of some classes of graphs with Grundy number at most $k$, $2 \leq k \leq \Delta(G)$, using the notion of independent module. Section 3 contains the first main theorem: A description of the cubic graphs with Grundy number at most 3 that also allows us to prove that every cubic graph except
$K_{3,3}$ has partial Grundy number 4. This theorem implies the existence of a linear algorithm to determine the Grundy number of cubic graphs. In Section 4, we present examples of infinite families of regular graphs with Grundy number exactly or at most $k$, $3 \leq k \leq r$. To determine these families we use recursive definitions. The last section contains the second main theorem of this article: 4-regular graphs without induced $C_4$ have Grundy number 5.

2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy $k$-coloring.

**Definition 2.1.** Let $G$ be a graph. A Grundy partial $k$-coloring is a Grundy $k$-coloring of a subset $S$ of $V(G)$.

**Observation 2.2** ([1],[6]). If $G$ admits a Grundy partial $k$-coloring, then $\Gamma(G) \geq k$.

This property has an important consequence: For a graph $G$, with $\Gamma(G) \geq t$ and any Grundy partial $t$-coloring, there exist smallest subgraphs $H$ of $G$ such that $\Gamma(H) = t$. The family of $t$-atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from Zaker’s one, insisting more on the construction of every $t$-atom.

**Definition 2.3** ([15]). For any integer $t$, we define the family of $t$-atoms, denoted by $\mathcal{A}_t$, $t = 1, \ldots$ by induction. Let the family $\mathcal{A}_1$ contain only $K_1$. A graph $G$ is in $\mathcal{A}_{t+1}$ if there exists a graph $G'$ in $\mathcal{A}_t$ and an integer $m$, $m \leq |V(G')|$, such that $G$ is composed of $G'$ and an independent set $I_m$ of order $m$, adding edges between $G'$ and $I_m$ such that every vertex in $G'$ is connected to at least one vertex in $I_m$. Moreover a $t$-atom $A$ is minimal, if there is no $t$-atom included in $A$ other than itself.

**Theorem 1** ([15]). For a given graph $G$, $\Gamma(G) \geq t$ if and only if $G$ contains an induced minimal $t$-atom.

We now present conditions related to the presence of modules that allows us to upper-bound the Grundy number.

**Proposition 2.4** ([1]). Let $G$ be a graph and $X$ be an independent module. In every Grundy coloring of $G$, the vertices in $X$ must have the same color.

**Definition 2.5.** Let $G$ be an $r$-regular graph. A vertex $v$ is a $(0, \ell)$-twin-vertex if there exists an independent module of cardinality $r + 2 - \ell$ that contains $v$.

**Proposition 2.6.** Let $G$ be an $r$-regular graph. The color of an $(0, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$. 
Proof. Let v be a (0, {\ell})-twin-vertex colored {\ell} + 1 in G. By Definition, v is in an independent module X of cardinality {r + 2 - \ell} and by Proposition 2.4, every other vertex of X should be colored {\ell} + 1. Let u be a neighbor of v. There are at most {\ell} - 2 neighbors of u in V(G - X). Therefore, u cannot be colored {\ell}.

Definition 2.7. A vertex v of a graph G is a (1, {\ell})-twin-vertex if N(v) can be partitioned into at least {\ell} - 1 independent modules.

Proposition 2.8. Let G be a graph. The color of an (1, {\ell})-twin-vertex is at most {\ell} in every Grundy coloring of G.

Proof. By Proposition 2.4, vertices of the neighborhood of v can only have {\ell} - 1 different colors. Therefore, the color of v is at most {\ell}.

Definition 2.9. A vertex v of a graph G is a (2, {\ell})-twin-vertex if N(v) is independent and every vertex in N(v) is a (1, {\ell})-twin-vertex.

Proposition 2.10. Let G be a graph. The color of an (2, {\ell})-twin-vertex is at most {\ell} in every Grundy coloring of G.

Proof. Let v be a (2, {\ell})-twin-vertex in G. Every vertex in N(v) is a (1, {\ell})-twin-vertex. If a vertex in N(v) is colored {\ell}, then v could only have a color at most {\ell} - 1. If the vertices in the neighborhood of v have colors at most {\ell} - 1, then in every Grundy coloring of G, v has a color at most {\ell}.

Corollary 2.11. Let G be a graph. If every vertex is a (1, {\ell})-twin-vertex or a (2, {\ell})-twin-vertex, then {\Gamma(G)} \leq {\ell}.

Corollary 2.12. Let G be a regular graph. If every vertex is an (i, {\ell})-twin-vertex, for some i, 0 \leq i \leq 2, then {\Gamma(G)} \leq {\ell}.

Proposition 2.13 ([1], [15]). Let G be a graph. We have {\Gamma(G)} \leq 2 if and only if G = K_{n,m} for some integers n > 0 and m > 0.

3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial k-colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures cannot be adjacent as it would contradict the hypothesis.

Proposition 3.1 ([6]). Let G be a connected 2-regular graph. \partial \Gamma(G) = \Gamma(G) = 2 if and only if G = C_4.

The following definition gives a construction of the cubic graphs in which every vertex is an (i, 3)-twin-vertex, for some i, 0 \leq i \leq 2. Figure 2 gives the list of every graph of order at most 16 in this family.

Definition 3.2. Let K_{2,3} and K_{3,3} be the graphs from Figure 1. We define recursively the family of graphs \mathcal{F}_3 as follows:
Figure 1: The graphs $K_{2,3}$ (on the left) and $K_{3,3}^*$ (on the right).

Figure 2: The cubic graphs $G$ such that $|V(G)| < 18$ and $\Gamma(G) < 4$.

1. $K_{2,3} \in \mathcal{F}_3$ and $K_{3,3}^* \in \mathcal{F}_3^*$;
2. the disjoint union of two elements of $\mathcal{F}_3^*$ is in $\mathcal{F}_3^*$;
3. if $G$ is a graph in $\mathcal{F}_3^*$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most 2 is also in $\mathcal{F}_3^*$;
4. if $G$ is a graph in $\mathcal{F}_3$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to three vertices of degree at most 2 is in $\mathcal{F}_3^*$.

The family $\mathcal{F}_3$ is the subfamily of cubic graphs in $\mathcal{F}_3^*$.

**Proposition 3.3.** Let $G$ be a cubic graph. Every vertex of $V(G)$ is an $(i,3)$-twin vertex, for some $i$, $0 \leq i \leq 2$, if and only if $G \in \mathcal{F}_3$.

**Proof.** Every graph $G$ in $\mathcal{F}_3$ has three kind of vertices: $(0,3)$-twin-vertices (called also false twins), vertices where an edge is added by Point 3 and vertices...
Assume that to $x$ and vice versa. Vertices added by Point 4 are (1,3)-twin-vertex and vice versa. Vertices added by Point 4 are (2,3)-twin-vertices and vice versa.

**Theorem 2.** Let $G$ be a cubic graph. $\Gamma(G) \leq 3$ if and only if every vertex is an $(i,3)$-twin-vertex, for some $i$, $0 \leq i \leq 2$.

**Proof.** By Corollary 2.12, the "if" part is proven. Assume that $G$ contains a vertex $v$ which is not an $(i,3)$-twin-vertex, for some $i$, $0 \leq i \leq 2$ and $\Gamma(G) < 4$. In every configuration we want to either find a Grundy partial 4-coloring, contradicting $\Gamma(G) < 4$ or proving that $v$ is an $(i,3)$-twin-vertex, for some $i$, with $0 \leq i \leq 2$. We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases: $v$ or a neighbor of $v$ is in a $C_3$, $v$ is in an induced $C_4$ and $v$ or a neighbor of $v$ are not in a $C_3$ and $v$ is not in an induced $C_4$. Let $C$ be an induced cycle of order 3 or 4 which contains $v$ or a neighbor of $v$ and let $D_1 = \{x \in V(G) | d(x, C) = 1\}$, where $d(x, C)$ is the distance from $x$ to $C$ in the graph $G$. To simplify notation, $D_1$ will also denote the subgraph of $G$ induced by $D_1$.

**Case 1:** Assume that $v$ or a neighbor of $v$ is in $C$ and $C = C_3$. If $|D_1| = 1$, then $G = K_4$ and $\Gamma(K_4) = 4$. If $|D_1| = 2$ and $D_1 = P_2$, then $v$ is a (0,3)-twin-vertex or a (1,3)-twin-vertex. If $D_1 = I_2$ then Figure 3.1.a yields a Grundy partial 4-coloring of $G$. If $|D_1| = 3$, then we have four subcases: $D_1$ is $C_3$ or $P_3$ (Figure 3.1.b), $P_2 \cup I_1$ (Figure 3.1.c) or $I_3$ (Figure 3.1.d). In every case $G$ admits a Grundy partial 4-coloring.

**Case 2:** Assume that $v$ is in $C$ and $C = C_4$. Note that for two non adjacent vertices of $C$ who have a common neighbor in $D_1$, the vertex $v$ is a (0,3)-twin-vertex or a (1,3)-twin-vertex. Hence, we will not consider these cases. If $|D_1| = 2$, then $D_1 = P_2$ or $D_1 = I_2$ (Figure 3.2.a) and in both cases, $G$ admits a Grundy partial 4-coloring. If $|D_1| = 3$, Figure 3.2.b yields a Grundy partial 4-coloring of $G$. In the case $|D_1| = 4$, we first assume that two adjacent vertices of $C$ have their neighbors in $D_1$ adjacent (Figure 3.2.c). Afterwards, we suppose that the previous case does not happen and that two non adjacent vertices of $C$ have their neighbors in $D_1$ adjacent (Figure 3.2.d). In the case $D_1 = I_4$, we first suppose that two vertices of $D_1$ which have two adjacent vertices of $C$ as neighbor, are not adjacent to two common vertices (Figure 3.2.e) and after consider they are (Figure 3.2.f).

**Case 3:** Assume that $v$ or a neighbor of $v$ is not in a $C_3$ and $v$ is not in an induced $C_4$. Firstly, suppose that a neighbor $u$ of $v$ is in an induced $C_4$. Using the coloring from the previous case, $G$ admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of $v$ in the $C_4$ have a common neighbor outside the $C_4$. However, this case cannot happen for every neighbor of $v$, otherwise $v$ would be a (2,3)-twin-vertex. Assume that $u$ is the neighbor of $v$ not in the previous configuration. If $u$ is in an induced $C_4$, then using the coloring from the previous case, $G$
admits a Grundy partial 4-coloring. If \( u \) is not in an induced \( C_4 \), then Figure 3.3.a yields a Grundy partial 4-coloring of \( G \). In this figure, the color 2 is given to a neighbor of \( u \) not adjacent to both \( f_1 \) and \( f_2 \). Secondly, suppose that \( v \) is in an induced \( C_5 \). Figure 3.3.b yields a Grundy partial 4-coloring of \( G \). Thirdly, if \( v \) is not in an induced \( C_5 \), then Figure 3.3.c yields a Grundy partial 4-coloring of \( G \).

Therefore, if \( \Gamma(G) \leq 3 \), then every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

Observe that if an edge is added between the two vertices of degree 2 in \( K^*_3,3 \), then we obtain \( K_{3,3} \) which has Grundy number 2. By Proposition 3.3, in all the remaining cases, the cubic graphs which have Grundy number at most 3 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

**Corollary 3.4.** A cubic graph \( G \) does not contain any induced minimal subcubic 4-atom if and only if every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

**Corollary 3.5.** Let \( G \) be a cubic graph. If \( G \) is without induced \( C_4 \), then \( \Gamma(G) = 4 \).

**Proof.** As every graph \( G \) with \( \Gamma(G) < 4 \) is composed of copies of \( K_{2,3} \) or \( K^*_3,3 \), the graph \( G \) always contains a square if \( \Gamma(G) < 4 \).

For a fixed integer \( t \), the largest \((t+1)\)-atom has order \( 2^t \). Thus, for a graph \( G \) of maximum degree \( t \), there exists an \( O(n^2) \)-time algorithm to determine if \( \Gamma(G) < t + 1 \) (which verifies if the graph contains an induced \((t+1)\)-atom). For a cubic graph, we obtain an \( O(n^8) \)-time algorithm, whereas our characterization yields a linear-time algorithm.

**Observation 3.6.** Let \( G \) be a cubic graph of order \( n \). There exists an \( O(n) \)-time algorithm\(^1\) to determine the Grundy number of \( G \).

**Proof.** Suppose we have a cubic graph \( G \) with its adjacency list. Verifying if \( G \) is \( K_{3,3} \) can be done in constant time. We suppose now that \( G \) is not \( K_{3,3} \). For each vertex \( v \), the algorithm verifies that \( v \) is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \). If the condition is true for all vertices, then \( \Gamma(G) = 3 \), else \( \Gamma(G) = 4 \). To determine if a vertex \( v \) is a \((0,3)\)-twin-vertex, it suffices to verify that there is a common vertex other than \( v \) in the adjacency lists of the neighbors of \( v \). To determine if a vertex \( v \) is a \((1,3)\)-twin-vertex, it suffices to verify that there are two neighbors of \( v \) which have the same adjacency list. To determine if a vertex \( v \) is a \((2,3)\)-twin-vertex, it suffices to verify that the neighborhood of \( v \) is independent and that every neighbor is a \((1,3)\)-twin-vertex. Hence, checking if a vertex is an \((i,3)\)-twin vertex can be done in constant time, so the algorithm runs in linear time.  

\(^1\)Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.
Figure 3: Possible configurations in a cubic graph (bold vertices: Uncolored vertices, vertices with number $i$: Vertices of color $i$).
Proposition 3.7. If $G$ is a connected cubic graph and $G \neq K_{3,3}$, then $\partial \Gamma(G) = 4$.

Proof. Let $G$ be a cubic connected graph. Note that if $\Gamma(G) = 4$ then $\partial \Gamma(G) = 4$. Every graph $G$ with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K_{3,3}$. If $G$ contains more than two copies (so it is different from $K_{3,3}$), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial \Gamma(G) = 4$.

Only $K_{3,3}$ and three other cubic graphs have $b$-chromatic number at most 3 [10]. Thus, our result is coherent with the results on the $b$-chromatic number. Shi et al. [13] proved that there exists a smallest integer $N$ such that every $r$-regular graph $G$ with more than $N_r$ vertices has $\partial \Gamma(G) = r + 1$. Observe that we have $N_2 = 4$ and $N_3 = 6$. It is an open question to determine $N_r$ for $r \geq 4$. However, using results on $b$-chromatic number [3], we have $N_r \leq 2r^3 - r^2 + r$.

4 Properties on the Grundy number of $r$-regular graphs

Definition 4.1. Let $r \geq 2$ be an integer. We define recursively the family of graphs $\mathcal{G}_r^*$ as follows:

1. $K_{r-k,k+2} \in \mathcal{G}_r^*$, for any $k$, $0 \leq k \leq (r-2)/2$;
2. the disjoint union of two elements of $\mathcal{G}_r^*$ is in $\mathcal{G}_r^*$;
3. if $G$ is a graph in $\mathcal{G}_r^*$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most $r-1$ is also in $\mathcal{G}_r^*$;
4. if $G$ is a graph in $\mathcal{G}_r^*$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to $r$ vertices of degree at most $r-1$ is in $\mathcal{G}_r^*$.

The family $\mathcal{G}_r$ is the subfamily of $r$-regular graphs in $\mathcal{G}_r^*$.

Proposition 4.2. Let $G$ be an $r$-regular graph. If $G \in \mathcal{G}_r$, then $\Gamma(G) < r + 1$.

Proof. By $I_{r-k}$ and $I_{k+2}$, with $|I_{r-k}| = r - k$ and $|I_{k+2}| = k + 2$, we denote the two sets of vertices in the bipartition of an induced subgraph $K_{r-k,k+2}$ in $G$. Firstly, suppose there exists a vertex $u$ in an induced subgraph $K_{r-k,k+2}$ colored $r + 1$. Without loss of generality, suppose $u$ is in $I_{r-k}$. The $r$ neighbors of $u$ should have colors from 1 to $r$. Among the neighbors of $u$, $k + 2$ neighbors are in $I_{k+2}$. Let $v$ be the neighbor of $u$ in $I_{k+2}$ with the largest color in $I_{k+2}$. The vertex $v$ has color at least $k + 2$. Hence, there exists an integer $s \geq 0$ such that the color of $v$ is $k + 2 + s$. Note that there are $s$ vertices in $N(u) \setminus I_{k+2}$ which have colors at most $k + 2 + s$. The colors of the $s$ vertices are the only one possible remaining colors at most $k + 2 + s$ in $I_{r-k}$. Hence, as there are $k$ vertices in $N(v) \setminus I_{r-k}$, the neighbors of $v$ can only have at most $k + s$ different colors at most $k + 2 + s$. Therefore, we have a contradiction and $u$ cannot have
color $r + 1$. Secondly, suppose there exists a vertex $u$ added by Point 4 which has color $r + 1$. As a neighbor of $u$ in an induced $K_{r-k,k+2}$ should be colored $r$, the argument is completely similar to the previous one.

**Corollary 4.3.** Let $G$ be a 4-regular graph. If $G \in \mathcal{G}_4$, then $\Gamma(G) < 5$.

The reader can believe that the family of 4-regular graphs with $\Gamma(G) < 5$ contains only the family $\mathcal{G}_4$. However, there exist graphs with Grundy number $r$ which are not inside this family. For example, the power graph (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle $C_7^2$ satisfies $\Gamma(C_7^2) < 5$ and is not in $\mathcal{G}_4$.

The next proposition shows that unlike the $b$-chromatic number, $r$-regular graphs of order arbitrarily large with Grundy number $k$ can be constructed for any $r$ and any $k$, $3 \leq k \leq r + 1$.

**Proposition 4.4.** Let $r \geq 4$ and $3 \leq k \leq r + 1$ be integers. There exists an infinite family $\mathcal{H}$ of connected $r$-regular graphs such that for all $G$ in $\mathcal{H}$, $\Gamma(G) = k$.

**Proof.** Let $i \geq 2$ be a positive integer and $r_1, \ldots, r_{k-1}$ be a sequence of positive integers such that $r = r_1 + \ldots + r_{k-1}$. We construct a graph $G_{r,k,i}$ as follows: Take $2i$ copies of $K_{r_1,\ldots,r_{k-1}}$. Let $H_j$ be the copy number $j$ of $K_{r_1,\ldots,r_{k-1}}$ and $H_j$ be the independent $r_i$-set in $H_j$. If $j \equiv 1 \pmod{2i}$, do the graph join of $H_j$ and $H_{j+1}$ for an integer $l$, $1 < l < k$, do the graph join of $H_j$ and $H_{j+1}$ for an integer $l$, $1 < l < k$, do the graph join of $H_j$ and $H_{j+1}$ for an integer $l$, $1 < l < k$. The $r$-regular graph obtained is the graph $G_{r,k,i}$. Figure 4 gives $G_{r,k,i}$ for $r = 4$ and $i \geq 2$. Note that $H_{j+r_i}$ is an independent module. Thus, every vertex is a $(0,k)$-twin-vertex. By Proposition 2.6, $\Gamma(G_{r,k,i}) \leq k$.

For an integer $l$, $1 < l < k$, color one vertex $l - 1$ in $H_{l+r_i}$ and $H_{l+r_i}$. Afterwards, color one vertex $k - 1$ in $H_{l+r_i}$ and one vertex $k$ in $H_{l+r_i}$. The given coloring is a Grundy partial $k$-coloring of $G_{r,k,i}$ for $i \geq 2$. Therefore, $\Gamma(G_{r,k,i}) = k$, for $i \geq 2$. \qed
5 Grundy number of 4-regular graphs without induced $C_4$

The following lemmas will be useful to prove the second main theorem of this paper: The family of 4-regular graphs without induced $C_4$ contains only graphs with Grundy number 5.

Lemma 5.1. Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains (an induced) $K_4$ then $\Gamma(G) = 5$.

Proof. Note that if $G = K_5$, we have $\Gamma(G) = 5$. If $G$ is not $K_5$ then every pair of neighbors of vertices of $K_4$ cannot be adjacent ($G$ would contain a $C_4$). Giving the color 1 to each neighbor of the vertices of $K_4$ and colors 2, 3, 4, 5 to the vertices of $K_4$, we obtain a Grundy partial 5-coloring of $G$.

Lemma 5.2. Let $G$ be a 4-regular graph without induced $C_4$ and let $W$ be the graph from Figure 5. If $G$ contains an induced $W$ then $\Gamma(G) = 5$.

Proof. The names of the vertices of $W$ come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of $G$ will be given using their references on Figure 5. Let $D_1$ be the set of vertices at distance 1 from vertices of $W$ in $G - W$. Suppose that two vertices of $W$ have a common neighbor in $D_1$. This two vertices could only be $u_4$ and $u_5$ or $u_3$ and $u_5$ (or $u_1$ and $u_4$, by symmetry). In the case that $u_4$ and $u_5$ have a common neighbor in $D_1$, colors will be given to neighbors of $u_3$ in $D_1$, depending if they are adjacent (Figure 5.1.a) or not (Figure 5.1.b). In the case that $u_3$ and $u_5$ have a common neighbor in $D_1$, $w$ can be adjacent with a neighbor of $u_3$ in $D_1$ (Figure 5.2.a) or not (Figure 5.2.b). Suppose now that no vertices in $W$ have a common neighbor in $D_1$. Let $w_1$ and $w_2$ be the neighbors of $u_3$ in $D_1$. We first consider that $w_1$ and $w_2$ are adjacent (Figure 5.3.a). Secondly, we consider that $w_1$ and $w_2$ are not adjacent and that $u_5$, $u_3$, and $w_1$ are in an induced $C_5$ (Figure 5.3.b). Finally, we consider that the previous configurations are impossible (Figure 5.3.c).

Proposition 5.3. Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains $C_3$ then $\Gamma(G) = 5$.

Proof. Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of $G$ in Figure 6 will be given. Let $M_i$, $i = 2$ or 3, be the graph of order $2 + i$ containing two adjacent vertices $u_1$ and $u_2$ which have exactly $i$ common neighbors, $\{v_1, \ldots, v_i\}$, that form an independent set. Let $D_1$ be the set of vertices at distance 1 from an induced $M_i$ in $G - M_i$, for $2 \leq i \leq 3$.

Case 1: Firstly, assume that $G$ contains an induced $M_3$ and a vertex of $M_3$ has its two neighbors in $D_1$ adjacent (Figure 6.1.a). Secondly, assume that $G$ contains an induced $M_2$ and a vertex of $M_2$ has its two neighbors in $D_1$.
The graph $W$.

![Graph $W$ and configurations](image)

Figure 5: Possible configurations when $G$ contains an induced $W$.

adjacent (Figure 6.1.b). Note that these Grundy partial 5-colorings use the fact that $G$ cannot contain a $K_4$ by Lemma 5.1.

**Case 2:** Assume that $G$ contains an induced $M_3$ excluding the previous configuration. There are three cases: $u_1$, $v_2$ and $v_3$ are in an induced $C_5$ (Figure 6.2.a), $u_1$, $v_2$ and $v_3$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.2.b) and $u_1$, $v_2$ and $v_3$ are neither in an induced $C_5$ nor $C_6$ (Figure 6.2.c).

**Case 3:** Suppose that $G$ contains an induced $M_2$ excluding the previous configurations. Firstly, we suppose that $u_1$, $v_1$ and $v_2$ are in an induced $C_5$ (Figure 6.3.a). Secondly, we suppose that $u_1$, $v_1$ are in an induced $C_5$ excluding the previous case (Figure 6.3.b). Thirdly, we suppose that $u_1$, $v_1$ and $v_2$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.3.c) and finally neither in an induced $C_5$ nor $C_6$ (Figure 6.3.d).

Suppose that $G$ contains a 3-cycle $C$ and no induced $M_2$. Let $u_1$, $u_2$ and $u_3$ be the vertices of $C$. Let $w_1$ and $w_2$ be the neighbors of $u_1$ outside $C$, let $w'_1$ and $w'_2$ be the neighbors of $u_2$ outside $C$ and let $w''_1$ and $w''_2$ be the neighbors of $u_3$ outside $C$.

**Case 4:** Firstly, suppose that $u_1$, $u_2$, $w_1$ and $w'_1$ are in a 5-cycle and a neighbor of $u_1$, say $w_1$, has a common neighbor with $w'_1$ (Figure 6.4.a). Secondly, excluding the previous configuration, suppose that $u_1$, $u_2$, $w_1$ and $w'_1$ are in a 5-cycle; $w''_1$, $v_1$, $u_1$ and $w_1$ are in another 5-cycle and $w_1$ is in a triangle (Figure 6.4.b). We suppose that $w_1$ is not in a triangle (Figure 6.4.c). Thirdly, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of $C$ are in a 5-cycle (Figure 6.4.d). Fourthly, we suppose that two vertices of $C$ cannot be in a 5-cycle (Figure 6.4.e).

In the following two lemmas, we consider a graph $G$ of girth $g = 5$ and possibly containing an induced Petersen graph. Let $u_1$, $u_2$, $u_3$, $u_4$ and $u_5$ be
the vertices in an induced $C_5$ (or in the outer cycle of a Petersen graph, if any). Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5$ and $v'_5$ be the remaining neighbors of respectively $u_1, u_2, u_3, u_4$ and $u_5$ (all different as $g = 5$).

**Lemma 5.4.** Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains the Petersen graph as induced subgraph then $\Gamma(G) = 5$.

**Proof.** Suppose that $v_1, v_2, v_3, v_4$ and $v_5$ form an induced $C_5$ (the inner cycle of the Petersen graph). Let $u'_2$ and $u'_5$ be the remaining neighbors of respectively $v_2$ and $v_5$. Observe that $v'_1$ can be adjacent with no more than three vertices among $v'_3, v'_4, u'_2$ and $u'_5$. Firstly, suppose that $v'_1$ is not adjacent with $v'_3$ (or $v'_4$, without loss of generality since the configuration is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$. Secondly, assume that $v'_1$ is not adjacent with $u'_5$ (or $u'_2$, without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$. 

In a graph $G$, let a *neighbor-connected* $C_n$ be an $n$-cycle $C$ such that the set
Figure 7: Two Grundy partial 5-colorings of a subgraph containing an induced Petersen graph.

of vertices of $G$ at distance 1 from $C$ is not independent.

**Lemma 5.5.** Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains a neighbor-connected $C_5$ as induced subgraph, then $\Gamma(G) = 5$.

**Proof.** Let $C$ be a neighbor-connected $C_5$ in $G$. By Lemma 5.4 we can suppose that the neighbors of the vertices of $C$ do not form an induced $C_5$ (otherwise a Petersen would be an induced subgraph). Hence, we can assume that the neighbors of the vertices of $C$ form a subgraph of a $C_{10}$. If there are two edges between the neighbors of the vertices of $C$, then Figure 8 illustrates Grundy partial 5-colorings of the graph $G$. Suppose that two neighbors are adjacent, say $v_1$ and $v'_3$ and the graph $G$ does not contain the previous configuration. Note that $v'_3$ can be adjacent with $v'_1$ and $v'_5$. Let $w_1$, $w_2$, and $w_3$ be the three neighbors of $v_2$ different from $u_2$. We suppose that $w_1$ can be possibly adjacent with $v'_1$ and $w_2$ can be possibly adjacent with $v'_4$. Figure 9 illustrates a Grundy partial 5-coloring of $G$ in this case. In this figure, the vertex $w_3$ can be possibly adjacent with $v'_5$ or $v_4$, but in this case we can switch the color 1 from $v'_5$ to $v_5$ or from $v'_4$ to $v_4$.

**Proposition 5.6.** If $G$ is a 4-regular graph with girth $g = 5$, then $\Gamma(G) = 5$.

**Proof.** Let $C$ be a 5-cycle in $G$. Assume that two neighbors of consecutive vertices of $C$, for example $v_1$ and $v_5$, have a common neighbor $w_1$. The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph $G$. In this figure the vertex $w_1$ can be possibly adjacent with $v'_2$, $v'_3$, or $v_4$, but in this case we can switch the color 1 from $v'_2$ to $v_2$, from $v'_3$ to $v_3$ or from $v_4$ to $v'_4$. Hence, we can suppose that no neighbors of consecutive vertices of $C$ are adjacent. Among the neighbors of $v_1$, there exists one vertex $w_1$ not adjacent with both $v_4$ and $v'_4$ (otherwise $G$ would contain a $C_4$). Among the neighbor of $v'_5$, there exists one
vertex, say \( w_2 \), not adjacent with \( w_1 \). The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph \( G \). In this figure the vertex \( w_1 \) can be possibly adjacent with \( v_4 \) and the vertex \( w_2 \) can be possibly adjacent with \( v'_2 \) or \( v_4 \), but in these cases we can switch the color 1 from \( v'_2 \) to \( v_2 \) or from \( v_4 \) to \( v'_4 \).

In the following lemma and proposition, we consider a graph \( G \) of girth \( g = 6 \). Let \( u_1, u_2, u_3, u_4, u_5 \) and \( u_6 \) be the vertices in an induced \( C_6 \). Let \( v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5, v'_5, v_6 \) and \( v'_6 \) be the remaining neighbors of respectively \( u_1, u_2, u_3, u_4, u_5 \) and \( u_6 \) (all different as \( g = 6 \)).

**Lemma 5.7.** If \( G \) is a 4-regular graph with girth \( g = 6 \) which contains a neighbor-connected \( C_6 \) as induced subgraph, then \( \Gamma(G) = 5 \).

**Proof.** Firstly, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let \( w_1 \) be a neighbor of \( v'_1 \) not adjacent with \( v_4 \). The graph \( G \) admits a Grundy partial 5-coloring as the
Figure 10: Two Grundy partial 5-colorings of a subgraph containing an induced $C_5$.

Figure 11: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected $C_6$.

left part of Figure 11 illustrates it. Secondly, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let $w_1$ be a neighbor of $v_3$ not adjacent with $v_2$ and let $w_2$ be a neighbor of $v'_1$ not adjacent with $w_1$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it.

Proposition 5.8. If $G$ is a 4-regular graph with girth $g = 6$, then $\Gamma(G) = 5$.

Proof. By Lemma 5.7, assume that no neighbors of the vertices of the induced $C_6$ are adjacent. Firstly, suppose that there are two neighbors at distance 4 along the cycle $C_6$, for example $v'_1$ and $v_5$, which have a common neighbor $w_1$. Let $w_2$ be a neighbor of $v_3$ not adjacent with $w_1$. $G$ admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Secondly, suppose that there
Figure 12: Two Grundy partial 5-colorings of a subgraph containing an induced $C_6$.

are no two neighbors at distance 4 along the cycle $C_6$ which have a common neighbor. Let $w_1$ be a neighbor of $v_1'$ not adjacent with a neighbor of $v_5$ or a neighbor of $v_3$, let $w_2$ be a neighbor of $v_3$ not adjacent with a neighbor of $v_5$, and let $w_3$ be a neighbor of $v_5$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it.

Proposition 5.9. If $G$ is a 4-regular graph with girth $g \geq 7$, then $\Gamma(G) = 5$.

Proof. Suppose that $G$ contains a 7-cycle. We denote the 5-atom which is a tree by $T_5$ (the binomial tree with maximum degree 4). It can be easily verified that $G$ contains $T_5$ where two leaves are merged (which is a 5-atom). Moreover, if $G$ does not contain a 7-cycle, then it contains $T_5$ as induced subgraph.

Theorem 3. Let $G$ be a 4-regular graph. If $G$ does not contain an induced $C_4$, then $\Gamma(G) = 5$.

Proof. Suppose that $G$ does not contain an induced $C_4$. Using Proposition 5.9 for the case $g \geq 7$, Propositions 5.6 and 5.8 for the case $g = 5, 6$, and Proposition 5.3 when $G$ contains a $C_3$ yields the desired result.

By Proposition 3.1, Corollary 3.5 and Theorem 3, any $r$-regular graph with $r \leq 4$ and without induced $C_4$ has Grundy number $r + 1$. Therefore, it is natural to propose Conjecture 1.

References


