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Abstract

The Grundy number of a graph \( G \), denoted by \( \Gamma(G) \), is the largest \( k \) such that there exists a partition of \( V(G) \), into \( k \) independent sets \( V_1, \ldots, V_k \) and every vertex of \( V_i \) is adjacent to at least one vertex in \( V_j \), for every \( j < i \). The objects which are studied in this article are families of \( r \)-regular graphs such that \( \Gamma(G) = r + 1 \). Using the notion of independent module, a characterization of this family is given for \( r = 3 \).

Moreover, we determine classes of graphs in this family, in particular the class of \( r \)-regular graphs without induced \( C_4 \), for \( r \leq 4 \). Furthermore, our propositions imply results on partial Grundy number.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph \( G = (V, E) \), a proper \( k \)-coloring of \( G \) is a surjective mapping \( c : V \to \{1, \ldots, k\} \) such that \( c(u) \neq c(v) \) for any \( uv \in E \); the color class \( V_i \) is the set \( \{u \in V | c(u) = i\} \) and a vertex \( v \) has color \( i \) if \( v \in V_i \). A vertex \( v \) of color \( i \) is a Grundy vertex if \( v \) is adjacent to at least one vertex colored \( j \), for every \( j < i \). A Grundy \( k \)-coloring is a proper \( k \)-coloring such that every vertex is a Grundy vertex. A partial Grundy \( k \)-coloring is a proper \( k \)-coloring such that every color class contains a Grundy vertex. The Grundy number (partial Grundy number, respectively) of \( G \) denoted by \( \Gamma(G) \) (\( \partial \Gamma(G) \), respectively) is the largest \( k \) such that \( G \) admits a Grundy \( k \)-coloring (partial Grundy \( k \)-coloring, respectively).

Let \( N(v) = \{u \in V(G) | uv \in E(G)\} \) be the neighborhood of \( v \). A set \( X \) of vertices is an independent module if \( X \) is an independent set and all vertices

\(^*\)Author partially supported by the Burgundy Council
in $X$ have the same neighborhood. The vertices in an independent module of size 2 are called false twins. Let $P_n$, $C_n$, $K_n$ and $I_n$ be respectively, the path, cycle complete and empty graph of order $n$. The concepts of Grundy $k$-coloring and domination are connected. In a Grundy coloring, $V_1$ is a dominating set. Given a graph $G$ and an ordering $\phi$ on $V(G)$ with $\phi = v_1, \ldots, v_n$, the greedy algorithm assigns to $v_i$ the minimum color that was not assigned in the set $\{v_1, \ldots, v_{i-1}\} \cap N(v_i)$. Let $\Gamma_{\phi}(G)$ be the number of colors used by the greedy algorithm with the ordering $\phi$ on $G$. We obtain the following result [7]:

$$\Gamma(G) = \max_{\phi \in S_n} \Gamma_{\phi}(G).$$

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a given graph, even for complements of bipartite graphs, is an NP-complete problem. However, for a fixed $t$, determining if a given graph has Grundy number at least $t$ is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called $t$-atoms, such that any graph with Grundy number at least $t$ contains a $t$-atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for the Grundy number [8, 15], that there exist upper bounds for $d$-degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between the products of graphs and the Grundy number [6, 1, 4].

Recently, Havet and Sampaio [9] have proven that the dual of Grundy $k$-coloring problem is in FPT by finding an algorithm in $O(2k^{2k}.|E| + 2^{2k}k^{4k+5/2})$ time.

Another coloring parameter with domination constraints on the colors is the $b$-chromatic number, denoted by $\varphi(G)$, which is the largest $k$ such that there exists a proper $k$-coloring and for every color class $V_i$, there exists a vertex adjacent to at least one vertex colored $j$, for every $j$, with $j \neq i$. Note that a $b$-coloring is a partial Grundy $k$-coloring, hence $\varphi(G) \leq \partial \Gamma(G)$. Given a graph $G$ and a positive integer $k$, the problem of determining if a partial Grundy $k$-coloring exists, even for chordal graphs, is NP-complete but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the $b$-chromatic number, denoted by $\varphi(G)$, which is the largest $k$ such that there exists a proper $k$-coloring and for every color class $V_i$, there exists a vertex adjacent to at least one vertex colored $j$, for every $j$, with $j \neq i$. Note that a $b$-coloring is a partial Grundy $k$-coloring, hence $\varphi(G) \leq \partial \Gamma(G)$. The $b$-chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: A characterization of the Grundy number of every cubic graph and the following theorem: For $r \leq 4$, every $r$-regular graphs without induced $C_4$ has Grundy number $r + 1$. We conjecture that this assertion is also true for $r > 4$.

**Conjecture 1.** For any integer $r \geq 1$, every $r$-regular graph without induced $C_4$ has Grundy number $r + 1$.

Section 2 gives characterizations of some classes of graphs with Grundy number at most $k$, $2 \leq k \leq \Delta(G)$, using the notion of independent module. Section 3 contains the first main theorem: A description of the cubic graphs with Grundy number at most 3 that also allows us to prove that every cubic graph except...
$K_{3,3}$ has partial Grundy number 4. This theorem implies the existence of a linear algorithm to determine the Grundy number of cubic graphs. In Section 4, we present examples of infinite families of regular graphs with Grundy number exactly or at most $k$, $3 \leq k \leq r$. To determine these families we use recursive definitions. The last section contains the second main theorem of this article: 4-regular graphs without induced $C_4$ have Grundy number 5.

2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy $k$-coloring.

Definition 2.1. Let $G$ be a graph. A Grundy partial $k$-coloring is a Grundy $k$-coloring of a subset $S$ of $V(G)$.

Observation 2.2 ([1],[6]). If $G$ admits a Grundy partial $k$-coloring, then $\Gamma(G) \geq k$.

This property has an important consequence: For a graph $G$, with $\Gamma(G) \geq t$ and any Grundy partial $t$-coloring, there exist smallest subgraphs $H$ of $G$ such that $\Gamma(H) = t$. The family of $t$-atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from Zaker’s one, insisting more on the construction of every $t$-atom.

Definition 2.3 ([15]). For any integer $t$, we define the family of $t$-atoms, denoted by $\mathcal{A}_t$, $t = 1, \ldots$ by induction. Let the family $\mathcal{A}_1$ contain only $K_1$. A graph $G$ is in $\mathcal{A}_{t+1}$ if there exists a graph $G'$ in $\mathcal{A}_t$ and an integer $m$, $m \leq |V(G')|$, such that $G$ is composed of $G'$ and an independent set $I_m$ of order $m$, adding edges between $G'$ and $I_m$ such that every vertex in $G'$ is connected to at least one vertex in $I_m$. Moreover a $t$-atom $A$ is minimal, if there is no $t$-atom included in $A$ other than itself.

Theorem 1 ([15]). For a given graph $G$, $\Gamma(G) \geq t$ if and only if $G$ contains an induced minimal $t$-atom.

We now present conditions related to the presence of modules that allows us to upper-bound the Grundy number.

Proposition 2.4 ([1]). Let $G$ be a graph and $X$ be an independent module. In every Grundy coloring of $G$, the vertices in $X$ must have the same color.

Definition 2.5. Let $G$ be an $r$-regular graph. A vertex $v$ is a $(0, \ell)$-twin-vertex if there exists an independent module of cardinality $r + 2 - \ell$ that contains $v$.

Proposition 2.6. Let $G$ be an $r$-regular graph. The color of an $(0, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$. 
Proof. Let \( v \) be a \((0, \ell)\)-twin-vertex colored \( \ell + 1 \) in \( G \). By Definition, \( v \) is in an independent module \( X \) of cardinality \( r + 2 - \ell \) and by Proposition 2.4, every other vertex of \( X \) should be colored \( \ell + 1 \). Let \( u \) be a neighbor of \( v \). There are at most \( \ell - 2 \) neighbors of \( u \) in \( V(G - X) \). Therefore, \( u \) cannot be colored \( \ell \).

**Definition 2.7.** A vertex \( v \) of a graph \( G \) is a \((1, \ell)\)-twin-vertex if \( N(v) \) can be partitioned into at least \( \ell - 1 \) independent modules.

**Proposition 2.8.** Let \( G \) be a graph. The color of an \((1, \ell)\)-twin-vertex is at most \( \ell \) in every Grundy coloring of \( G \).

Proof. By Proposition 2.4, vertices of the neighborhood of \( v \) can only have \( \ell - 1 \) different colors. Therefore, the color of \( v \) is at most \( \ell \).

**Definition 2.9.** A vertex \( v \) of a graph \( G \) is a \((2, \ell)\)-twin-vertex if \( N(v) \) is independent and every vertex in \( N(v) \) is a \((1, \ell)\)-twin-vertex.

**Proposition 2.10.** Let \( G \) be a graph. The color of an \((2, \ell)\)-twin-vertex is at most \( \ell \) in every Grundy coloring of \( G \).

Proof. Let \( v \) be a \((2, \ell)\)-twin-vertex in \( G \). Every vertex in \( N(v) \) is a \((1, \ell)\)-twin-vertex. If a vertex in \( N(v) \) is colored \( \ell \), then \( v \) could only have a color at most \( \ell - 1 \). If the vertices in the neighborhood of \( v \) have colors at most \( \ell - 1 \), then in every Grundy coloring of \( G \), \( v \) has a color at most \( \ell \).

**Corollary 2.11.** Let \( G \) be a graph. If every vertex is a \((1, \ell)\)-twin-vertex or a \((2, \ell)\)-twin-vertex, then \( \Gamma(G) \leq \ell \).

**Corollary 2.12.** Let \( G \) be a regular graph. If every vertex is an \((i, \ell)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \), then \( \Gamma(G) \leq \ell \).

**Proposition 2.13** ([1],[15]). Let \( G \) be a graph. We have \( \Gamma(G) \leq 2 \) if and only if \( G = K_{n,m} \) for some integers \( n > 0 \) and \( m > 0 \).

### 3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial-\( k \)-colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures cannot be adjacent as it would contradict the hypothesis.

**Proposition 3.1** ([6]). Let \( G \) be a connected 2-regular graph. \( \partial \Gamma(G) = \Gamma(G) = 2 \) if and only if \( G = C_4 \).

The following definition gives a construction of the cubic graphs in which every vertex is an \((i, 3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \). Figure 2 gives the list of every graph of order at most 16 in this family.

**Definition 3.2.** Let \( K_{2,3} \) and \( K_{3,3}^7 \) be the graphs from Figure 1. We define recursively the family of graphs \( \mathcal{F}_3^* \) as follows:
Figure 1: The graphs $K_{2,3}$ (on the left) and $K^*_{3,3}$ (on the right).

Figure 2: The cubic graphs $G$ such that $|V(G)| < 18$ and $\Gamma(G) < 4$.

1. $K_{2,3} \in \mathcal{F}_3^*$ and $K^*_{3,3} \in \mathcal{F}_3^*$;
2. the disjoint union of two elements of $\mathcal{F}_3^*$ is in $\mathcal{F}_3^*$;
3. if $G$ is a graph in $\mathcal{F}_3^*$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most 2 is also in $\mathcal{F}_3^*$;
4. if $G$ is a graph in $\mathcal{F}_3^*$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to three vertices of degree at most 2 is in $\mathcal{F}_3^*$.

The family $\mathcal{F}_3$ is the subfamily of cubic graphs in $\mathcal{F}_3^*$.

**Proposition 3.3.** Let $G$ be a cubic graph. Every vertex of $V(G)$ is an $(i,3)$-twin vertex, for some $i$, $0 \leq i \leq 2$, if and only if $G \in \mathcal{F}_3$.

**Proof.** Every graph $G$ in $\mathcal{F}_3$ has three kind of vertices: $(0,3)$-twin-vertices (called also false twins), vertices where an edge is added by Point 3 and vertices
added by Point 4. Vertices where an edge is added by Point 3 are (1,3)-twin-vertex and vice versa. Vertices added by Point 4 are (2,3)-twin-vertices and vice versa.

Theorem 2. Let \( G \) be a cubic graph. \( \Gamma(G) \leq 3 \) if and only if every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

Proof. By Corollary 2.12, the "if" part is proven. Assume that \( G \) contains a vertex \( v \) which is not an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \) and \( \Gamma(G) < 4 \). In every configuration we want to either find a Grundy partial 4-coloring, contradicting \( \Gamma(G) < 4 \) or proving that \( v \) is an \((i,3)\)-twin-vertex, for some \( i \), with \( 0 \leq i \leq 2 \). We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases: \( v \) or a neighbor of \( v \) is in a \( C_3 \), \( v \) is in an induced \( C_4 \) and \( v \) or a neighbor of \( v \) are not in a \( C_3 \) and \( v \) is not in an induced \( C_4 \). Let \( C \) be an induced cycle of order 3 or 4 which contains \( v \) or a neighbor of \( v \) and let \( D_1 = \{ x \in V(G) | d(x,C) = 1 \} \), where \( d(x,C) \) is the distance from \( x \) to \( C \) in the graph \( G \). To simplify notation, \( D_1 \) will also denote the subgraph of \( G \) induced by \( D_1 \).

Case 1: Assume that \( v \) or a neighbor of \( v \) is in \( C \) and \( C = C_3 \). If \( |D_1| = 1 \), then \( G = K_4 \) and \( \Gamma(K_4) = 4 \). If \( |D_1| = 2 \) and \( D_1 = P_2 \), then \( v \) is a \((0,3)\)-twin-vertex or a \((1,3)\)-twin-vertex. If \( D_1 = I_2 \) then Figure 3.1.a yields a Grundy partial 4-coloring of \( G \). If \( |D_1| = 3 \), then we have four subcases: \( D_1 \) is \( C_3 \) or \( P_3 \) (Figure 3.1.b), \( P_2 \cup I_1 \) (Figure 3.1.c) or \( I_3 \) (Figure 3.1.d). In every case \( G \) admits a Grundy partial 4-coloring.

Case 2: Assume that \( v \) is in \( C \) and \( C = C_4 \). Note that for two non adjacent vertices of \( C \) who have a common neighbor in \( D_1 \), the vertex \( v \) is a \((0,3)\)-twin-vertex or a \((1,3)\)-twin-vertex. Hence, we will not consider these cases. If \( |D_1| = 2 \), then \( D_1 = P_2 \) or \( D_1 = I_2 \) (Figure 3.2.a) and in both cases, \( G \) admits a Grundy partial 4-coloring. If \( |D_1| = 3 \), Figure 3.2.b yields a Grundy partial 4-coloring of \( G \). In the case \( |D_1| = 4 \), we first assume that two adjacent vertices of \( C \) have their neighbors in \( D_1 \) adjacent (Figure 3.2.c). Afterwards, we suppose that the previous case does not happen and that two non adjacent vertices of \( C \) have their neighbors in \( D_1 \) adjacent (Figure 3.2.d). In the case \( D_1 = I_4 \), we first suppose that two vertices of \( D_1 \) which have two adjacent vertices of \( C \) as neighbor, are not adjacent to two common vertices (Figure 3.2.e) and after consider they are (Figure 3.2.f).

Case 3: Assume that \( v \) or a neighbor of \( v \) is not in a \( C_3 \) and \( v \) is not in an induced \( C_4 \). Firstly, suppose that a neighbor \( u \) of \( v \) is in an induced \( C_4 \). Using the coloring from the previous case, \( G \) admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of \( v \) in the \( C_4 \) have a common neighbor outside the \( C_4 \). However, this case cannot happen for every neighbor of \( v \), otherwise \( v \) would be a \((2,3)\)-twin-vertex. Assume that \( u \) is the neighbor of \( v \) not in the previous configuration. If \( u \) is in an induced \( C_4 \), then using the coloring from the previous case, \( G \)
admits a Grundy partial 4-coloring. If \( u \) is not in an induced \( C_4 \), then Figure 3.3.a yields a Grundy partial 4-coloring of \( G \). In this figure, the color 2 is given to a neighbor of \( u \) not adjacent to both \( f_1 \) and \( f_2 \). Secondly, suppose that \( v \) is in an induced \( C_5 \). Figure 3.3.b yields a Grundy partial 4-coloring of \( G \). Thirdly, if \( v \) is not in an induced \( C_5 \), then Figure 3.3.c yields a Grundy partial 4-coloring of \( G \).

Therefore, if \( \Gamma(G) \leq 3 \), then every vertex is an \((i,3)\)-twin-vertex, for some \( i, 0 \leq i \leq 2 \).

Observe that if an edge is added between the two vertices of degree 2 in \( K_{1,3}^* \), then we obtain \( K_{3,3}^* \) which has Grundy number 2. By Proposition 3.3, in all the remaining cases, the cubic graphs which have Grundy number at most 3 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

**Corollary 3.4.** A cubic graph \( G \) does not contain any induced minimal subcubic 4-atom if and only if every vertex is an \((i,3)\)-twin-vertex, for some \( i, 0 \leq i \leq 2 \).

**Corollary 3.5.** Let \( G \) be a cubic graph. If \( G \) is without induced \( C_4 \), then \( \Gamma(G) = 4 \).

**Proof.** As every graph \( G \) with \( \Gamma(G) < 4 \) is composed of copies of \( K_{2,3} \) or \( K_{1,3}^* \), the graph \( G \) always contains a square if \( \Gamma(G) < 4 \).

For a fixed integer \( t \), the largest \((t + 1)\)-atom has order \( 2^t \). Thus, for a graph \( G \) of maximum degree \( t \), there exists an \( O(n^2) \)-time algorithm to determine if \( \Gamma(G) < t + 1 \) (which verifies if the graph contains an induced \((t + 1)\)-atom). For a cubic graph, we obtain an \( O(n^8) \)-time algorithm, whereas our characterization yields a linear-time algorithm.

**Observation 3.6.** Let \( G \) be a cubic graph of order \( n \). There exists an \( O(n) \)-time algorithm\(^1\) to determine the Grundy number of \( G \).

**Proof.** Suppose we have a cubic graph \( G \) with its adjacency list. Verifying if \( G \) is \( K_{1,3} \) can be done in constant time. We suppose now that \( G \) is not \( K_{1,3} \). For each vertex \( v \), the algorithm verifies that \( v \) is an \((i,3)\)-twin-vertex, for some \( i, 0 \leq i \leq 2 \). If the condition is true for all vertices, then \( \Gamma(G) = 3 \), else \( \Gamma(G) = 4 \). To determine if a vertex \( v \) is a \((0,3)\)-twin-vertex, it suffices to verify that there is a common vertex other than \( v \) in the adjacency lists of the neighbors of \( v \). To determine if a vertex \( v \) is a \((1,3)\)-twin-vertex, it suffices to verify that there are two neighbors of \( v \) which have the same adjacency list. To determine if a vertex \( v \) is a \((2,3)\)-twin-vertex, it suffices to verify that the neighborhood of \( v \) is independent and that every neighbor is a \((1,3)\)-twin-vertex. Hence, checking if a vertex is an \((i,3)\)-twin vertex can be done in constant time, so the algorithm runs in linear time.

\(^1\)Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.
Figure 3: Possible configurations in a cubic graph (bold vertices: Uncolored vertices, vertices with number $i$: Vertices of color $i$).
Proposition 3.7. If $G$ is a connected cubic graph and $G \neq K_{3,3}$, then $\partial \Gamma(G) = 4$.

Proof. Let $G$ be a cubic connected graph. Note that if $\Gamma(G) = 4$ then $\partial \Gamma(G) = 4$. Every graph $G$ with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K_{3,3}^*$. If $G$ contains more than two copies (so it is different from $K_{3,3}$), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial \Gamma(G) = 4$.

Only $K_{3,3}$ and three other cubic graphs have $b$-chromatic number at most 3 [10]. Thus, our result is coherent with the results on the $b$-chromatic number. Shi et al. [13] proved that there exists a smallest integer $N$ such that every $r$-regular graph $G$ with more than $N_r$ vertices has $\partial \Gamma(G) = r + 1$. Observe that we have $N_2 = 4$ and $N_3 = 6$. It is an open question to determine $N_r$ for $r \geq 4$. However, using results on $b$-chromatic number [3], we have $N_r \leq 2r^3 - r^2 + r$.

4 Properties on the Grundy number of $r$-regular graphs

Definition 4.1. Let $r \geq 2$ be an integer. We define recursively the family of graphs $\mathcal{G}_r^*$ as follows:

1. $K_{r-k,k+2} \in \mathcal{G}_r^*$, for any $k$, $0 \leq k \leq (r-2)/2$;

2. the disjoint union of two elements of $\mathcal{G}_r^*$ is in $\mathcal{G}_r^*$;

3. if $G$ is a graph in $\mathcal{G}_r^*$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most $r-1$ is also in $\mathcal{G}_r^*$;

4. if $G$ is a graph in $\mathcal{G}_r^*$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to $r$ vertices of degree at most $r-1$ is in $\mathcal{G}_r^*$.

The family $\mathcal{G}_r$ is the subfamily of $r$-regular graphs in $\mathcal{G}_r^*$.

Proposition 4.2. Let $G$ be an $r$-regular graph. If $G \in \mathcal{G}_r$, then $\Gamma(G) < r + 1$.

Proof. By $I_{r-k}$ and $I_{k+2}$, with $|I_{r-k}| = r - k$ and $|I_{k+2}| = k + 2$, we denote the two sets of vertices in the bipartition of an induced subgraph $K_{r-k,k+2}$ in $G$. Firstly, suppose there exists a vertex $u$ in an induced subgraph $K_{r-k,k+2}$ colored $r + 1$. Without loss of generality, suppose $u$ is in $I_{r-k}$. The $r$ neighbors of $u$ should have colors from 1 to $r$. Among the neighbors of $u$, $k + 2$ neighbors are in $I_{k+2}$. Let $v$ be the neighbor of $u$ in $I_{k+2}$ with the largest color in $I_{k+2}$. The vertex $v$ has color at least $k + 2$. Hence, there exists an integer $s \geq 0$ such that the color of $v$ is $k + 2 + s$. Note that there are $s$ vertices in $N(u) \setminus I_{k+2}$ which have colors at most $k + 2 + s$. The colors of the $s$ vertices are the only one possible remaining colors at most $k + 2 + s$ in $I_{r-k}$. Hence, as there are $k$ vertices in $N(v) \setminus I_{r-k}$, the neighbors of $v$ can only have at most $k + s$ different colors at most $k + 2 + s$. Therefore, we have a contradiction and $u$ cannot have
color \( r + 1 \). Secondly, suppose there exists a vertex \( u \) added by Point 4 which has color \( r + 1 \). As a neighbor of \( u \) in an induced \( K_{r-k,k+2} \) should be colored \( r \), the argument is completely similar to the previous one.

**Corollary 4.3.** Let \( G \) be a 4-regular graph. If \( G \in \mathcal{G}_4 \), then \( \Gamma(G) < 5 \).

The reader can believe that the family of 4-regular graphs with \( \Gamma(G) < 5 \) contains only the family \( \mathcal{G}_4 \). However, there exist graphs with Grundy number \( r \) which are not inside this family. For example, the power graph (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle \( C_7^2 \) satisfies \( \Gamma(C_7^2) < 5 \) and is not in \( \mathcal{G}_4 \).

The next proposition shows that unlike the \( b \)-chromatic number, \( r \)-regular graphs of order arbitrarily large with Grundy number \( k \) can be constructed for any \( r \) and any \( k \), \( 3 \leq k \leq r + 1 \).

**Proposition 4.4.** Let \( r \geq 4 \) and \( 3 \leq k \leq r + 1 \) be integers. There exists an infinite family \( \mathcal{H} \) of connected \( r \)-regular graphs such that for all \( G \) in \( \mathcal{H} \), \( \Gamma(G) = k \).

**Proof.** Let \( i \geq 2 \) be a positive integer and \( r_1, \ldots, r_{k-1} \) be a sequence of positive integers such that \( r = r_1 + \ldots + r_{k-1} \). We construct a graph \( G_{r,k,i} \) as follows: Take \( 2i \) copies of \( K_{r_1, \ldots, r_{k-1}} \). Let \( H_{j-1} \) be the copy number \( j \) of \( K_{r_1, \ldots, r_{k-1}} \) and \( H_{j,r_{i+j}} \) be the independent \( r_i \)-set in \( H_j \). If \( j \equiv 1 \) (mod 2), do the graph join of \( H_{j-1} \) (mod 2), \( H_j \) and \( H_{j-1} \) (mod 2), \( H_j \) and for an integer \( l, 1 < l < k \), do the graph join of \( H_{j-1} \) (mod 2), \( H_j \) and \( H_{j+1} \) (mod 2), \( H_j \). The \( r \)-regular graph obtained is the graph \( G_{r,k,i} \). Figure 4 gives \( G_{r,k,i} \) for \( k = 4 \) and \( i \geq 2 \). Note that \( H_{j,r_{i+j}} \) is an independent module. Thus, every vertex is a \((0,k)\)-twin-vertex. By Proposition 2.6, \( \Gamma(G_{r,k,i}) \leq k \).

For an integer \( l, 1 < l < k \), color one vertex \( l-1 \) in \( H_{1,r_1} \) and \( H_{2,r_1} \). Afterwards, color one vertex \( k-1 \) in \( H_{1,r_1} \) and one vertex \( k \) in \( H_{2,r_1} \). The given coloring is a Grundy partial \( k \)-coloring of \( G_{r,k,i} \) for \( i \geq 2 \). Therefore, \( \Gamma(G_{r,k,i}) = k \), for \( i \geq 2 \).

**Figure 4:** The Graph \( G_{r,k,i}, i \geq 2, r = r_1 + r_2 + r_3 \).
5 Grundy number of 4-regular graphs without induced $C_4$

The following lemmas will be useful to prove the second main theorem of this paper: The family of 4-regular graphs without induced $C_4$ contains only graphs with Grundy number 5.

**Lemma 5.1.** Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains (an induced) $K_4$ then $\Gamma(G) = 5$.

**Proof.** Note that if $G = K_5$, we have $\Gamma(G) = 5$. If $G$ is not $K_5$ then every pair of neighbors of vertices of $K_4$ cannot be adjacent ($G$ would contain a $C_4$). Giving the color 1 to each neighbor of the vertices of $K_4$ and colors 2, 3, 4, 5 to the vertices of $K_4$, we obtain a Grundy partial 5-coloring of $G$. □

**Lemma 5.2.** Let $G$ be a 4-regular graph without induced $C_4$ and let $W$ be the graph from Figure 5. If $G$ contains an induced $W$ then $\Gamma(G) = 5$.

**Proof.** The names of the vertices of $W$ come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of $G$ will be given using their references on Figure 5. Let $D_1$ be the set of vertices at distance 1 from vertices of $W$ in $G - W$. Suppose that two vertices of $W$ have a common neighbor in $D_1$. This two vertices could only be $u_4$ and $u_5$ or $u_3$ and $u_5$ (or $u_1$ and $u_4$, by symmetry). In the case that $u_4$ and $u_5$ have a common neighbor in $D_1$, colors will be given to neighbors of $u_3$ in $D_1$, depending if they are adjacent (Figure 5.1.a) or not (Figure 5.1.b). In the case that $u_3$ and $u_5$ have a common neighbor $w$ in $D_1$, $w$ can be adjacent with a neighbor of $u_3$ in $D_1$ (Figure 5.2.a) or not (Figure 5.2.b). Suppose now that no vertices in $W$ have a common neighbor in $D_1$. Let $w_1$ and $w_2$ be the neighbors of $u_3$ in $D_1$. We first consider that $w_1$ and $w_2$ are adjacent (Figure 5.3.a). Secondly, we consider that $w_1$ and $w_2$ are not adjacent and that $u_5$, $u_3$ and $w_1$ are in an induced $C_5$ (Figure 5.3.b). Finally, we consider that the previous configurations are impossible (Figure 5.3.c). □

**Proposition 5.3.** Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains $C_3$ then $\Gamma(G) = 5$.

**Proof.** Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of $G$ in Figure 6 will be given. Let $M_i$, $i = 2$ or 3, be the graph of order $2 + i$ containing two adjacent vertices $u_1$ and $u_2$ which have exactly $i$ common neighbors, $\{v_1, \ldots, v_i\}$, that form an independent set. Let $D_1$ be the set of vertices at distance 1 from an induced $M_i$ in $G - M_i$, for $2 \leq i \leq 3$.

**Case 1:** Firstly, assume that $G$ contains an induced $M_3$ and a vertex of $M_3$ has its two neighbors in $D_1$ adjacent (Figure 6.1.a). Secondly, assume that $G$ contains an induced $M_2$ and a vertex of $M_2$ has its two neighbors in $D_1$.
adjacent (Figure 6.1.b). Note that these Grundy partial 5-colorings use the fact that $G$ cannot contain a $K_4$ by Lemma 5.1.

**Case 2:** Assume that $G$ contains an induced $M_3$ excluding the previous configuration. There are three cases: $u_1, v_2$ and $v_3$ are in an induced $C_5$ (Figure 6.2.a), $u_1, v_2$ and $v_3$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.2.b) and $u_1, v_2$ and $v_3$ are neither in an induced $C_3$ nor $C_6$ (Figure 6.2.c).

**Case 3:** Suppose that $G$ contains an induced $M_2$ excluding the previous configurations. Firstly, we suppose that $u_1, v_1$ and $v_2$ are in an induced $C_5$ (Figure 6.3.a). Secondly, we suppose that $u_1, v_1$ are in an induced $C_5$ excluding the previous case (Figure 6.3.b). Thirdly, we suppose that $u_1, v_1$ and $v_2$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.3.c) and finally neither in an induced $C_3$ nor $C_6$ (Figure 6.3.d).

Suppose that $G$ contains a 3-cycle $C$ and no induced $M_2$. Let $u_1, u_2$ and $u_3$ be the vertices of $C$. Let $w_1$ and $w_2$ be the neighbors of $u_1$ outside $C$, let $w_1'$ and $w_2'$ be the neighbors of $u_2$ outside $C$ and let $w_1''$ and $w_2''$ be the neighbors of $u_3$ outside $C$.

**Case 4:** Firstly, suppose that $u_1, u_2, w_1$ and $w_1'$ are in a 5-cycle and a neighbor of $u_1$, say $w_1$, has a common neighbor with $w_1'$ (Figure 6.4.a). Secondly, excluding the previous configuration, suppose that $u_1, u_2, w_1$ and $w_1'$ are in a 5-cycle; $w_1''$, $v_1$, $u_1$ and $w_1$ are in another 5-cycle and $w_1$ is in a triangle (Figure 6.4.b). We suppose that $w_1$ is not in a triangle (Figure 6.4.c). Thirdly, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of $C$ are in a 5-cycle (Figure 6.4.d). Fourthly, we suppose that two vertices of $C$ cannot be in a 5-cycle (Figure 6.4.e).

In the following two lemmas, we consider a graph $G$ of girth $g = 5$ and possibly containing an induced Petersen graph. Let $u_1, u_2, u_3, u_4$ and $u_5$ be...
the vertices in an induced $C_5$ (or in the outer cycle of a Petersen graph, if any). Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5$ and $v'_5$ be the remaining neighbors of respectively $u_1, u_2, u_3, u_4$ and $u_5$ (all different as $g = 5$).

**Lemma 5.4.** Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains the Petersen graph as induced subgraph then $\Gamma(G) = 5$.

**Proof.** Suppose that $v_1, v_2, v_3, v_4$ and $v_5$ form an induced $C_5$ (the inner cycle of the Petersen graph). Let $u'_2$ and $u'_5$ be the remaining neighbors of respectively $v_2$ and $v_5$. Observe that $v'_1$ can be adjacent with no more than three vertices among $v'_3, v'_4, u'_2$ and $u'_5$. Firstly, suppose that $v'_1$ is not adjacent with $v'_3$ (or $v'_4$, without loss of generality since the configuration is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$. Secondly, assume that $v'_1$ is not adjacent with $u'_5$ (or $u'_2$, without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$. 

In a graph $G$, let a **neighbor-connected** $C_n$ be an $n$-cycle $C$ such that the set
Figure 7: Two Grundy partial 5-colorings of a subgraph containing an induced Petersen graph.

of vertices of $G$ at distance 1 from $C$ is not independent.

**Lemma 5.5.** Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains a neighbor-connected $C_5$ as induced subgraph, then $\Gamma(G) = 5$.

**Proof.** Let $C$ be a neighbor-connected $C_5$ in $G$. By Lemma 5.4 we can suppose that the neighbors of the vertices of $C$ do not form an induced $C_5$ (otherwise a Petersen would be an induced subgraph). Hence, we can assume that the neighbors of the vertices of $C$ form a subgraph of a $C_{10}$. If there are two edges between the neighbors of the vertices of $C$, then Figure 8 illustrates Grundy partial 5-colorings of the graph $G$. Suppose that two neighbors are adjacent, say $v_1$ and $v'_3$ and the graph $G$ does not contain the previous configuration. Note that $v'_3$ can be adjacent with $v'_1$ and $v'_5$. Let $w_1$, $w_2$ and $w_3$ be the three neighbors of $v_2$ different from $u_2$. We suppose that $w_1$ can be possibly adjacent with $v'_1$ and $w_2$ can be possibly adjacent with $v'_1$. Figure 9 illustrates a Grundy partial 5-coloring of $G$ in this case. In this figure, the vertex $w_3$ can be possibly adjacent with $v'_5$ or $v_4$, but in this case we can switch the color 1 from $v'_5$ to $v_5$ or from $v'_4$ to $v_4$.

**Proposition 5.6.** If $G$ is a 4-regular graph with girth $g = 5$, then $\Gamma(G) = 5$.

**Proof.** Let $C$ be a 5-cycle in $G$. Assume that two neighbors of consecutive vertices of $C$, for example $v_1$ and $v_5$, have a common neighbor $w_1$. The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph $G$. In this figure the vertex $w_1$ can be possibly adjacent with $v'_2$, $v'_3$ or $v_4$, but in this case we can switch the color 1 from $v'_2$ to $v_2$, from $v'_3$ to $v_3$ or from $v_4$ to $v'_4$. Hence, we can suppose that no neighbors of consecutive vertices of $C$ are adjacent. Among the neighbors of $v_1$, there exists one vertex $w_1$ not adjacent with both $v_4$ and $v'_4$ (otherwise $G$ would contain a $C_4$). Among the neighbor of $v'_5$, there exists one
vertex, say $w_2$, not adjacent with $w_1$. The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph $G$. In this figure the vertex $w_1$ can be possibly adjacent with $v_4$ and the vertex $w_2$ can be possibly adjacent with $v'_2$ or $v_4$, but in these cases we can switch the color 1 from $v'_2$ to $v_2$ or from $v_4$ to $v'_4$.

In the following lemma and proposition, we consider a graph $G$ of girth $g = 6$. Let $u_1$, $u_2$, $u_3$, $u_4$, $u_5$ and $u_6$ be the vertices in an induced $C_6$. Let $v_1$, $v'_1$, $v_2$, $v'_2$, $v_3$, $v'_3$, $v_4$, $v'_4$, $v_5$, $v'_5$, $v_6$ and $v'_6$ be the remaining neighbors of respectively $u_1$, $u_2$, $u_3$, $u_4$, $u_5$ and $u_6$ (all different as $g = 6$).

**Lemma 5.7.** If $G$ is a 4-regular graph with girth $g = 6$ which contains a neighbor-connected $C_6$ as induced subgraph, then $\Gamma(G) = 5$.

**Proof.** Firstly, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let $w_1$ be a neighbor of $v'_1$ not adjacent with $v_4$. The graph $G$ admits a Grundy partial 5-coloring as the
Figure 10: Two Grundy partial 5-colorings of a subgraph containing an induced $C_5$.

Figure 11: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected $C_6$.

left part of Figure 11 illustrates it. Secondly, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let $w_1$ be a neighbor of $v_3$ not adjacent with $v_2$ and let $w_2$ be a neighbor of $v'_1$ not adjacent with $w_1$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it.

Proposition 5.8. If $G$ is a 4-regular graph with girth $g = 6$, then $\Gamma(G) = 5$.

Proof. By Lemma 5.7, assume that no neighbors of the vertices of the induced $C_6$ are adjacent. Firstly, suppose that there are two neighbors at distance 4 along the cycle $C_6$, for example $v'_1$ and $v_5$, which have a common neighbor $w_1$. Let $w_2$ be a neighbor of $v_3$ not adjacent with $w_1$. $G$ admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Secondly, suppose that there
are no two neighbors at distance 4 along the cycle $C_6$ which have a common neighbor. Let $w_1$ be a neighbor of $v'_1$ not adjacent with a neighbor of $v_5$ or a neighbor of $v_3$, let $w_2$ be a neighbor of $v_3$ not adjacent with a neighbor of $v_5$, and let $w_3$ be a neighbor of $v_5$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it.

**Proposition 5.9.** If $G$ is a 4-regular graph with girth $g \geq 7$, then $\Gamma(G) = 5$.

**Proof.** Suppose that $G$ contains a 7-cycle. We denote the 5-atom which is a tree by $T_5$ (the binomial tree with maximum degree 4). It can be easily verified that $G$ contains $T_5$ where two leaves are merged (which is a 5-atom). Moreover, if $G$ does not contain a 7-cycle, then it contains $T_5$ as induced subgraph.

**Theorem 3.** Let $G$ be a 4-regular graph. If $G$ does not contain an induced $C_4$, then $\Gamma(G) = 5$.

**Proof.** Suppose that $G$ does not contain an induced $C_4$. Using Proposition 5.9 for the case $g \geq 7$, Propositions 5.6 and 5.8 for the case $g = 5, 6$, and Proposition 5.3 when $G$ contains a $C_3$ yields the desired result.

By Proposition 3.1, Corollary 3.5 and Theorem 3, any $r$-regular graph with $r \leq 4$ and without induced $C_4$ has Grundy number $r + 1$. Therefore, it is natural to propose Conjecture 1.

**References**


