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Abstract

The Grundy number of a graph $G$, denoted by $\Gamma(G)$, is the largest $k$ such that there exists a partition of $V(G)$, into $k$ independent sets $V_1, \ldots, V_k$ and every vertex of $V_i$ is adjacent to at least one vertex in $V_j$, for every $j < i$. The objects which are studied in this article are families of $r$-regular graphs such that $\Gamma(G) = r + 1$. Using the notion of independent module, a characterization of this family is given for $r = 3$. Moreover, we determine classes of graphs in this family, in particular the class of $r$-regular graphs without induced $C_4$, for $r \leq 4$. Furthermore, our propositions imply results on partial Grundy number.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph $G = (V, E)$, a proper $k$-coloring of $G$ is a surjective mapping $c : V \to \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E$; the color class $V_i$ is the set $\{u \in V | c(u) = i\}$ and a vertex $v$ has color $i$ if $v \in V_i$. A vertex $v$ of color $i$ is a Grundy vertex if $v$ is adjacent to at least one vertex colored $j$, for every $j < i$. A Grundy $k$-coloring is a proper $k$-coloring such that every vertex is a Grundy vertex. A partial Grundy $k$-coloring is a proper $k$-coloring such that every color class contains a Grundy vertex. The Grundy number (partial Grundy number, respectively) of $G$ denoted by $\Gamma(G)$ ($\partial\Gamma(G)$, respectively) is the largest $k$ such that $G$ admits a Grundy $k$-coloring (partial Grundy $k$-coloring, respectively).

Let $N(v) = \{u \in V(G) | uv \in E(G)\}$ be the neighborhood of $v$. A set $X$ of vertices is an independent module if $X$ is an independent set and all vertices

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in $X$ have the same neighborhood. The vertices in an independent module of size 2 are called false twins. Let $P_n$, $C_n$, $K_n$ and $I_n$ be respectively, the path, cycle complete and empty graph of order $n$. The concepts of Grundy $k$-coloring and domination are connected. In a Grundy coloring, $V_1$ is a dominating set. Given a graph $G$ and an ordering $\phi$ on $V(G)$ with $\phi = v_1, \ldots, v_n$, the greedy algorithm assigns to $v_i$ the minimum color that was not assigned in the set $\{v_1, \ldots, v_{i-1}\} \cap N(v_i)$. Let $\Gamma_\phi(G)$ be the number of colors used by the greedy algorithm with the ordering $\phi$ on $G$. We obtain the following result [7]:

$$\Gamma(G) = \max_{\phi \in S_n} \Gamma_\phi(G).$$

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a given graph, even for complements of bipartite graphs, is an NP-complete problem. However, for a fixed $t$, determining if a given graph has Grundy number at least $t$ is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called $t$-atoms, such that any graph with Grundy number at least $t$ contains a $t$-atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for the Grundy number [8, 15], that there exist upper bounds for $d$-degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between the products of graphs and the Grundy number [6, 1, 4]. Recently, Havet and Sampaio [9] have proven that the problem of deciding if for a given graph $G$ we have $\Gamma(G) = \Delta(G)+1$, even if $G$ is bipartite, is NP-complete. Moreover, they have proven that the dual of Grundy $k$-coloring problem is in FPT by finding an algorithm in $O(2k^{2k}|E| + 2^{2k}k^{3k+5/2})$ time.

Note that a Grundy $k$-coloring is a partial Grundy $k$-coloring, hence $\Gamma(G) \leq \partial \Gamma(G)$. Given a graph $G$ and a positive integer $k$, the problem of determining if a partial Grundy $k$-coloring exists, even for chordal graphs, is NP-complete but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the $b$-chromatic number, denoted by $\varphi(G)$, which is the largest $k$ such that there exists a proper $k$-coloring and for every color class $V_i$, there exists a vertex adjacent to at least one vertex colored $j$, for every $j$, with $j \neq i$. Note that a $b$-coloring is a partial Grundy $k$-coloring, hence $\varphi(G) \leq \partial \Gamma(G)$. The $b$-chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: A characterization of the Grundy number of every cubic graph and the following theorem: For $r \leq 4$, every $r$-regular graphs without induced $C_4$ has Grundy number $r + 1$. We conjecture that this assertion is also true for $r > 4$.

**Conjecture 1.** For any integer $r \geq 1$, every $r$-regular graph without induced $C_4$ has Grundy number $r + 1$.

Section 2 gives characterizations of some classes of graphs with Grundy number at most $k$, $2 \leq k \leq \Delta(G)$, using the notion of independent module. Section 3 contains the first main theorem: A description of the cubic graphs with Grundy number at most 3 that also allows us to prove that every cubic graph except
$K_{3,3}$ has partial Grundy number 4. This theorem implies the existence of a linear algorithm to determine the Grundy number of cubic graphs. In Section 4, we present examples of infinite families of regular graphs with Grundy number exactly or at most $k$, $3 \leq k \leq r$. To determine these families we use recursive definitions. The last section contains the second main theorem of this article: 4-regular graphs without induced $C_4$ have Grundy number 5.

2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy $k$-coloring.

Definition 2.1. Let $G$ be a graph. A Grundy partial $k$-coloring is a Grundy $k$-coloring of a subset $S$ of $V(G)$.

Observation 2.2 ([1],[6]). If $G$ admits a Grundy partial $k$-coloring, then $\Gamma(G) \geq k$.

This property has an important consequence: For a graph $G$, with $\Gamma(G) \geq t$ and any Grundy partial $t$-coloring, there exist smallest subgraphs $H$ of $G$ such that $\Gamma(H) = t$. The family of $t$-atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from Zaker’s one, insisting more on the construction of every $t$-atom.

Definition 2.3 ([15]). For any integer $t$, we define the family of $t$-atoms, denoted by $A_t$, $t = 1, \ldots$ by induction. Let the family $A_1$ contain only $K_1$. A graph $G$ is in $A_{t+1}$ if there exists a graph $G'$ in $A_t$ and an integer $m$, $m \leq |V(G')|$, such that $G$ is composed of $G'$ and an independent set $I_m$ of order $m$, adding edges between $G'$ and $I_m$ such that every vertex in $G'$ is connected to at least one vertex in $I_m$. Moreover a $t$-atom $A$ is minimal, if there is no $t$-atom included in $A$ other than itself.

Theorem 1 ([15]). For a given graph $G$, $\Gamma(G) \geq t$ if and only if $G$ contains an induced minimal $t$-atom.

We now present conditions related to the presence of modules that allows us to upper-bound the Grundy number.

Proposition 2.4 ([1]). Let $G$ be a graph and $X$ be an independent module. In every Grundy coloring of $G$, the vertices in $X$ must have the same color.

Definition 2.5. Let $G$ be an $r$-regular graph. A vertex $v$ is a $(0, \ell)$-twin-vertex if there exists an independent module of cardinality $r + 2 - \ell$ that contains $v$.

Proposition 2.6. Let $G$ be an $r$-regular graph. The color of an $(0, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$. 

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Proof. Let $v$ be a $(0, \ell)$-twin-vertex colored $\ell + 1$ in $G$. By Definition, $v$ is in an independent module $X$ of cardinality $r + 2 - \ell$ and by Proposition 2.4, every other vertex of $X$ should be colored $\ell + 1$. Let $u$ be a neighbor of $v$. There are at most $\ell - 2$ neighbors of $u$ in $V(G - X)$. Therefore, $u$ cannot be colored $\ell$. □

**Definition 2.7.** A vertex $v$ of a graph $G$ is a $(1, \ell)$-twin-vertex if $N(v)$ can be partitioned into at least $\ell - 1$ independent modules.

**Proposition 2.8.** Let $G$ be a graph. The color of an $(1, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$.

**Proof.** By Proposition 2.4, vertices of the neighborhood of $v$ can only have $\ell - 1$ different colors. Therefore, the color of $v$ is at most $\ell$. □

**Definition 2.9.** A vertex $v$ of a graph $G$ is a $(2, \ell)$-twin-vertex if $N(v)$ is independent and every vertex in $N(v)$ is a $(1, \ell)$-twin-vertex.

**Proposition 2.10.** Let $G$ be a graph. The color of an $(2, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$.

**Proof.** Let $v$ be a $(2, \ell)$-twin-vertex in $G$. Every vertex in $N(v)$ is a $(1, \ell)$-twin-vertex. If a vertex in $N(v)$ is colored $\ell$, then $v$ could only have a color at most $\ell - 1$. If the vertices in the neighborhood of $v$ have colors at most $\ell - 1$, then in every Grundy coloring of $G$, $v$ has a color at most $\ell$. □

**Corollary 2.11.** Let $G$ be a graph. If every vertex is a $(1, \ell)$-twin-vertex or a $(2, \ell)$-twin-vertex, then $\Gamma(G) \leq \ell$.

**Corollary 2.12.** Let $G$ be a regular graph. If every vertex is an $(i, \ell)$-twin-vertex, for some $i$, $0 \leq i \leq 2$, then $\Gamma(G) \leq \ell$.

**Proposition 2.13** ([1],[15]). Let $G$ be a graph. We have $\Gamma(G) \leq 2$ if and only if $G = K_{n,m}$ for some integers $n > 0$ and $m > 0$.

### 3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial $k$-colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures cannot be adjacent as it would contradict the hypothesis.

**Proposition 3.1** ([6]). Let $G$ be a connected 2-regular graph. $\partial \Gamma(G) = \Gamma(G) = 2$ if and only if $G = C_4$.

The following definition gives a construction of the cubic graphs in which every vertex is an $(i, 3)$-twin-vertex, for some $i$, $0 \leq i \leq 2$. Figure 2 gives the list of every graph of order at most 16 in this family.

**Definition 3.2.** Let $K_{2,3}$ and $K^*_3$ be the graphs from Figure 1. We define recursively the family of graphs $\mathcal{F}_3^*$ as follows:
Figure 1: The graphs $K_{2,3}$ (on the left) and $K_{3,3}^*$ (on the right).

Figure 2: The cubic graphs $G$ such that $|V(G)| < 18$ and $\Gamma(G) < 4$.

1. $K_{2,3} \in \mathcal{F}_3^*$ and $K_{3,3}^* \in \mathcal{F}_3^*$;
2. the disjoint union of two elements of $\mathcal{F}_3^*$ is in $\mathcal{F}_3^*$;
3. if $G$ is a graph in $\mathcal{F}_3^*$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most 2 is also in $\mathcal{F}_3^*$;
4. if $G$ is a graph in $\mathcal{F}_3^*$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to three vertices of degree at most 2 is in $\mathcal{F}_3^*$.

The family $\mathcal{F}_3$ is the subfamily of cubic graphs in $\mathcal{F}_3^*$.

**Proposition 3.3.** Let $G$ be a cubic graph. Every vertex of $V(G)$ is an $(i,3)$-twin vertex, for some $i$, $0 \leq i \leq 2$, if and only if $G \in \mathcal{F}_3$.

**Proof.** Every graph $G$ in $\mathcal{F}_3$ has three kind of vertices: (0,3)-twin-vertices (called also false twins), vertices where an edge is added by Point 3 and vertices
Case 1: Assume that \( x \) and \( v \) are not in an \((i,3)\)-twin-vertex and vice versa. Vertices added by Point 4 are \((1,3)\)-twin-vertices and vice versa.

**Theorem 2.** Let \( G \) be a cubic graph. \( \Gamma(G) \leq 3 \) if and only if every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

**Proof.** By Corollary 2.12, the "if" part is proven. Assume that \( G \) contains a vertex \( v \) which is not an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \) and \( \Gamma(G) < 4 \). In every configuration we want to either find a Grundy partial 4-coloring, contradicting \( \Gamma(G) < 4 \) or proving that \( v \) is an \((i,3)\)-twin-vertex, for some \( i \), with \( 0 \leq i \leq 2 \). We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases: \( v \) or a neighbor of \( v \) is in a \( C_3 \), \( v \) is in an induced \( C_4 \) and \( v \) or a neighbor of \( v \) are not in a \( C_3 \) and \( v \) is not in an induced \( C_4 \). Let \( C \) be an induced cycle of order 3 or 4 which contains \( v \) or a neighbor of \( v \) and let \( D_1 = \{ x \in V(G) | d(x, C) = 1 \} \), where \( d(x, C) \) is the distance from \( x \) to \( C \) in the graph \( G \). To simplify notation, \( D_1 \) will also denote the subgraph of \( G \) induced by \( D_1 \).

**Case 1:** Assume that \( v \) or a neighbor of \( v \) is in \( C \) and \( C = C_3 \). If \( |D_1| = 1 \), then \( G = K_4 \) and \( \Gamma(K_4) = 4 \). If \( |D_1| = 2 \) and \( D_1 = P_2 \), then \( v \) is a \((0,3)\)-twin-vertex or a \((1,3)\)-twin-vertex. If \( D_1 = I_2 \) then Figure 3.1.a yields a Grundy partial 4-coloring of \( G \). If \( |D_1| = 3 \), then we have four subcases: \( D_1 \) is \( C_3 \) or \( P_3 \) (Figure 3.1.b), \( P_2 \cup I_1 \) (Figure 3.1.c) or \( I_3 \) (Figure 3.1.d). In every case \( G \) admits a Grundy partial 4-coloring.

**Case 2:** Assume that \( v \) is in \( C \) and \( C = C_4 \). Note that for two non adjacent vertices of \( C \) who have a common neighbor in \( D_1 \), the vertex \( v \) is a \((0,3)\)-twin-vertex or a \((1,3)\)-twin-vertex. Hence, we will not consider these cases. If \( |D_1| = 2 \), then \( D_1 = P_2 \) or \( D_1 = I_2 \) (Figure 3.2.a) and in both cases, \( G \) admits a Grundy partial 4-coloring. If \( |D_1| = 3 \), Figure 3.2.b yields a Grundy partial 4-coloring of \( G \). In the case \( |D_1| = 4 \), we first assume that two adjacent vertices of \( C \) have their neighbors in \( D_1 \) adjacent (Figure 3.2.c). Afterwards, we suppose that the previous case does not happen and that two non adjacent vertices of \( C \) have their neighbors in \( D_1 \) adjacent (Figure 3.2.d). In the case \( D_1 = I_4 \), we first suppose that two vertices of \( D_1 \) which have two adjacent vertices of \( C \) as neighbor, are not adjacent to two common vertices (Figure 3.2.e) and after consider they are (Figure 3.2.f).

**Case 3:** Assume that \( v \) or a neighbor of \( v \) is not in a \( C_3 \) and \( v \) is not in an induced \( C_4 \). Firstly, suppose that a neighbor \( u \) of \( v \) is in an induced \( C_4 \). Using the coloring from the previous case, \( G \) admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of \( v \) in the \( C_4 \) have a common neighbor outside the \( C_4 \). However, this case cannot happen for every neighbor of \( v \), otherwise \( v \) would be a \((2,3)\)-twin-vertex. Assume that \( u \) is the neighbor of \( v \) not in the previous configuration. If \( u \) is in an induced \( C_4 \), then using the coloring from the previous case, \( G \).
admits a Grundy partial 4-coloring. If \( u \) is not in an induced \( C_4 \), then Figure 3.3.a yields a Grundy partial 4-coloring of \( G \). In this figure, the color 2 is given to a neighbor of \( u \) not adjacent to both \( f_1 \) and \( f_2 \). Secondly, suppose that \( v \) is in an induced \( C_5 \). Figure 3.3.b yields a Grundy partial 4-coloring of \( G \). Thirdly, if \( v \) is not in an induced \( C_5 \), then Figure 3.3.c yields a Grundy partial 4-coloring of \( G \).

Therefore, if \( \Gamma(G) \leq 3 \), then every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

Observe that if an edge is added between the two vertices of degree 2 in \( K_{3,3}^* \), then we obtain \( K_{3,3} \) which has Grundy number 2. By Proposition 3.3, in all the remaining cases, the cubic graphs which have Grundy number at most 3 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

**Corollary 3.4.** A cubic graph \( G \) does not contain any induced minimal subcubic 4-atom if and only if every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

**Corollary 3.5.** Let \( G \) be a cubic graph. If \( G \) is without induced \( C_4 \), then \( \Gamma(G) = 4 \).

**Proof.** As every graph \( G \) with \( \Gamma(G) < 4 \) is composed of copies of \( K_{2,3} \) or \( K_{3,3}^* \), the graph \( G \) always contains a square if \( \Gamma(G) < 4 \).

For a fixed integer \( t \), the largest \((t+1)\)-atom has order \( 2^t \). Thus, for a graph \( G \) of maximum degree \( t \), there exists an \( O(n^t) \)-time algorithm to determine if \( \Gamma(G) < t + 1 \) (which verifies if the graph contains an induced \((t+1)\)-atom). For a cubic graph, we obtain an \( O(n^8) \)-time algorithm, whereas our characterization yields a linear-time algorithm.

**Observation 3.6.** Let \( G \) be a cubic graph of order \( n \). There exists an \( O(n) \)-time algorithm\(^1\) to determine the Grundy number of \( G \).

**Proof.** Suppose we have a cubic graph \( G \) with its adjacency list. Verifying if \( G \) is \( K_{3,3}^* \) can be done in constant time. We suppose now that \( G \) is not \( K_{3,3}^* \). For each vertex \( v \), the algorithm verifies that \( v \) is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \). If the condition is true for all vertices, then \( \Gamma(G) = 3 \), else \( \Gamma(G) = 4 \). To determine if a vertex \( v \) is a \((0,3)\)-twin-vertex, it suffices to verify that there is a common vertex other than \( v \) in the adjacency lists of the neighbors of \( v \). To determine if a vertex \( v \) is a \((1,3)\)-twin-vertex, it suffices to verify that there are two neighbors of \( v \) which have the same adjacency list. To determine if a vertex \( v \) is a \((2,3)\)-twin-vertex, it suffices to verify that the neighborhood of \( v \) is independent and that every neighbor is a \((1,3)\)-twin-vertex. Hence, checking if a vertex is an \((i,3)\)-twin vertex can be done in constant time, so the algorithms runs in linear time.\(^1\)

\(^1\)Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.
Figure 3: Possible configurations in a cubic graph (bold vertices: Uncolored vertices, vertices with number $i$: Vertices of color $i$).
Shi et al. [13] proved that there exists a smallest integer $N_{10}$. Thus, our result is coherent with the results on the $b_r$-properties on the Grundy number of Proposition 3.7.

**Definition 4.1.** Let $G$ be a cubic connected graph. Note that if $\Gamma(G) = 4$ then $\partial \Gamma(G) = 4$. Every graph $G$ with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K^*_{3,3}$. If $G$ contains more than two copies (so it is different from $K_{3,3}$), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial \Gamma(G) = 4$.

Only $K_{3,3}$ and three other cubic graphs have $b$-chromatic number at most 3 [10]. Thus, our result is coherent with the results on the $b$-chromatic number. Shi et al. [13] proved that there exists a smallest integer $N_r$ such that every $r$-regular graph $G$ with more than $N_r$ vertices has $\partial \Gamma(G) = r + 1$. Observe that we have $N_2 = 4$ and $N_3 = 6$. It is an open question to determine $N_r$ for $r \geq 4$. However, using results on $b$-chromatic number [3], we have $N_r \leq 2r^3 - r^2 + r$.

## 4 Properties on the Grundy number of $r$-regular graphs

**Proposition 4.2.** Let $G$ be an $r$-regular graph. If $G \in \mathcal{G}_r$, then $\Gamma(G) < r + 1$.

**Proof.** Let $G$ be a cubic connected graph. Note that if $\Gamma(G) = 4$ then $\partial \Gamma(G) = 4$. Every graph $G$ with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K^*_{3,3}$. If $G$ contains more than two copies (so it is different from $K_{3,3}$), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial \Gamma(G) = 4$.

Only $K_{3,3}$ and three other cubic graphs have $b$-chromatic number at most 3 [10]. Thus, our result is coherent with the results on the $b$-chromatic number. Shi et al. [13] proved that there exists a smallest integer $N_r$ such that every $r$-regular graph $G$ with more than $N_r$ vertices has $\partial \Gamma(G) = r + 1$. Observe that we have $N_2 = 4$ and $N_3 = 6$. It is an open question to determine $N_r$ for $r \geq 4$. However, using results on $b$-chromatic number [3], we have $N_r \leq 2r^3 - r^2 + r$. 

**Definition 4.1.** Let $r \geq 2$ be an integer. We define recursively the family of graphs $\mathcal{G}_r^*$ as follows:

1. $K_{r-k,k+2} \in \mathcal{G}_r^*$, for any $k$, $0 \leq k \leq (r-2)/2$;
2. the disjoint union of two elements of $\mathcal{G}_r^*$ is in $\mathcal{G}_r^*$;
3. if $G$ is a graph in $\mathcal{G}_r^*$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most $r-1$ is also in $\mathcal{G}_r^*$;
4. if $G$ is a graph in $\mathcal{G}_r^*$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to $r$ vertices of degree at most $r-1$ is in $\mathcal{G}_r^*$.

The family $\mathcal{G}_r$ is the subfamily of $r$-regular graphs in $\mathcal{G}_r^*$.

**Proposition 4.2.** Let $G$ be an $r$-regular graph. If $G \in \mathcal{G}_r$, then $\Gamma(G) < r + 1$.

**Proof.** Let $G$ be a cubic connected graph. Note that if $\Gamma(G) = 4$ then $\partial \Gamma(G) = 4$. Every graph $G$ with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K^*_{3,3}$. If $G$ contains more than two copies (so it is different from $K_{3,3}$), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial \Gamma(G) = 4$.
color $r + 1$. Secondly, suppose there exists a vertex $u$ added by Point 4 which has color $r + 1$. As a neighbor of $u$ in an induced $K_{r-k,k+2}$ should be colored $r$, the argument is completely similar to the previous one.

**Corollary 4.3.** Let $G$ be a 4-regular graph. If $G \in \mathcal{G}_4$, then $\Gamma(G) < 5$. The reader can believe that the family of 4-regular graphs with $\Gamma(G) < 5$ contains only the family $\mathcal{G}_4$. However, there exist graphs with Grundy number $r$ which are not inside this family. For example, the power graph (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle $C_7^2$ satisfies $\Gamma(C_7^2) < 5$ and is not in $\mathcal{G}_4$.

The next proposition shows that unlike the $b$-chromatic number, $r$-regular graphs of order arbitrarily large with Grundy number $k$ can be constructed for any $r$ and any $k$, $3 \leq k \leq r + 1$.

**Proposition 4.4.** Let $r \geq 4$ and $3 \leq k \leq r + 1$ be integers. There exists an infinite family $\mathcal{H}$ of connected $r$-regular graphs such that for all $G$ in $\mathcal{H}$, $\Gamma(G) = k$.

**Proof.** Let $i \geq 2$ be a positive integer and $r_1, \ldots, r_{k-1}$ be a sequence of positive integers such that $r = r_1 + \ldots + r_{k-1}$. We construct a graph $G_{r,k,i}$ as follows: Take $2i$ copies of $K_{r_1,\ldots,r_{k-1}}$. Let $H_{j-1}$ be the copy number $j$ of $K_{r_1,\ldots,r_{k-1}}$ and $H_{j,r_1}$ be the independent $r_1$-set in $H_j$. If $j \equiv 1 \pmod{2}$, do the graph join of $H_{j} \pmod{2i}, r_1$ and $H_{j-1} \pmod{2i}, r_1$ and for an integer $l$, $1 < l < k$, do the graph join of $H_{j} \pmod{2i}, r_l$ and $H_{j+1} \pmod{2i}, r_l$. The $r$-regular graph obtained is the graph $G_{r,k,i}$. Figure 4 gives $G_{r,k,i}$, for $k = 4$ and $i \geq 2$. Note that $H_{j,r_1}$ is an independent module. Thus, every vertex is a $(0,k)$-twin-vertex. By Proposition 2.6, $\Gamma(G_{r,k,i}) \leq k$.

For an integer $l$, $1 < l < k$, color one vertex $l - 1$ in $H_{1,r_l}$ and $H_{2,r_l}$. Afterwards, color one vertex $k - 1$ in $H_{1,r_1}$ and one vertex $k$ in $H_{2,r_1}$. The given coloring is a Grundy partial $k$-coloring of $G_{r,k,i}$ for $i \geq 2$. Therefore, $\Gamma(G_{r,k,i}) = k$, for $i \geq 2$. \qed
5 Grundy number of 4-regular graphs without induced $C_4$

The following lemmas will be useful to prove the second main theorem of this paper: The family of 4-regular graphs without induced $C_4$ contains only graphs with Grundy number 5.

**Lemma 5.1.** Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains (an induced) $K_4$ then $\Gamma(G) = 5$.

*Proof.* Note that if $G = K_5$, we have $\Gamma(G) = 5$. If $G$ is not $K_5$ then every pair of neighbors of vertices of $K_4$ cannot be adjacent ($G$ would contain a $C_4$). Giving the color 1 to each neighbor of the vertices of $K_4$ and colors 2, 3, 4, 5 to the vertices of $K_4$, we obtain a Grundy partial 5-coloring of $G$. $\square$

**Lemma 5.2.** Let $G$ be a 4-regular graph without induced $C_4$ and let $W$ be the graph from Figure 5. If $G$ contains an induced $W$ then $\Gamma(G) = 5$.

*Proof.* The names of the vertices of $W$ come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of $G$ will be given using their references on Figure 5. Let $D_1$ be the set of vertices at distance 1 from vertices of $W$ in $G - W$. Suppose that two vertices of $W$ have a common neighbor in $D_1$. This two vertices could only be $u_4$ and $u_5$ or $u_3$ and $u_5$ (or $u_1$ and $u_4$, by symmetry). In the case that $u_4$ and $u_5$ have a common neighbor in $D_1$, colors will be given to neighbors of $u_3$ in $D_1$, depending if they are adjacent (Figure 5.1.a) or not (Figure 5.1.b). In the case that $u_3$ and $u_5$ have a common neighbor $w$ in $D_1$, $w$ can be adjacent with a neighbor of $u_3$ in $D_1$ (Figure 5.2.a) or not (Figure 5.2.b). Suppose now that no vertices in $W$ have a common neighbor in $D_1$. Let $w_1$ and $w_2$ be the neighbors of $u_3$ in $D_1$. We first consider that $w_1$ and $w_2$ are adjacent (Figure 5.3.a). Secondly, we consider that $w_1$ and $w_2$ are not adjacent and that $u_5$, $u_3$ and $w_1$ are in an induced $C_5$ (Figure 5.3.b). Finally, we consider that the previous configurations are impossible (Figure 5.3.c). $\square$

**Proposition 5.3.** Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains $C_3$ then $\Gamma(G) = 5$.

*Proof.* Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of $G$ in Figure 6 will be given. Let $M_i$, $i = 2$ or 3, be the graph of order $2 + i$ containing two adjacent vertices $u_1$ and $u_2$ which have exactly $i$ common neighbors, $\{v_1, \ldots, v_i\}$, that form an independent set. Let $D_1$ be the set of vertices at distance 1 from an induced $M_i$ in $G - M_i$, for $2 \leq i \leq 3$.

**Case 1:** Firstly, assume that $G$ contains an induced $M_3$ and a vertex of $M_3$ has its two neighbors in $D_1$ adjacent (Figure 6.1.a). Secondly, assume that $G$ contains an induced $M_2$ and a vertex of $M_2$ has its two neighbors in $D_1$...
The graph $W_u$. 

Figure 5: Possible configurations when $G$ contains an induced $W$.

adjacent (Figure 6.1.b). Note that these Grundy partial 5-colorings use the fact that $G$ cannot contain a $K_4$ by Lemma 5.1.

Case 2: Assume that $G$ contains an induced $M_3$ excluding the previous configuration. There are three cases: $u_1$, $v_2$ and $v_3$ are in an induced $C_5$ (Figure 6.2.a), $u_1$, $v_2$ and $v_3$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.2.b) and $u_1$, $v_2$ and $v_3$ are neither in an induced $C_5$ nor $C_6$ (Figure 6.2.c).

Case 3: Suppose that $G$ contains an induced $M_2$ excluding the previous configurations. Firstly, we suppose that $u_1$, $v_1$ and $v_2$ are in an induced $C_5$ (Figure 6.3.a). Secondly, we suppose that $u_1$, $v_3$ are in an induced $C_5$ excluding the previous case (Figure 6.3.b). Thirdly, we suppose that $u_1$, $v_2$ and $v_3$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.3.c) and finally neither in an induced $C_5$ nor $C_6$ (Figure 6.3.d).

Suppose that $G$ contains a 3-cycle $C$ and no induced $M_2$. Let $u_1$, $u_2$ and $u_3$ be the vertices of $C$. Let $w_1$ and $w_2$ be the neighbors of $u_1$ outside $C$, let $w_1'$ and $w_2'$ be the neighbors of $u_2$ outside $C$ and let $w_1''$ and $w_2''$ be the neighbors of $u_3$ outside $C$.

Case 4: Firstly, suppose that $u_1$, $u_2$, $w_1$ and $w_1'$ are in a 5-cycle and a neighbor of $u_1$, say $w_1$, has a common neighbor with $w_1'$ (Figure 6.4.a). Secondly, excluding the previous configuration, suppose that $u_1$, $u_2$, $w_1$ and $w_1'$ are in a 5-cycle; $w_1''$, $v_1$, $u_1$ and $w_1$ are in another 5-cycle and $w_1$ is in a triangle (Figure 6.4.b). We suppose that $w_1$ is not in a triangle (Figure 6.4.c). Thirdly, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of $C$ are in a 5-cycle (Figure 6.4.d). Fourthly, we suppose that two vertices of $C$ cannot be in a 5-cycle (Figure 6.4.e).

In the following two lemmas, we consider a graph $G$ of girth $g = 5$ and possibly containing an induced Petersen graph. Let $u_1$, $u_2$, $u_3$, $u_4$ and $u_5$ be
the vertices in an induced $C_5$ (or in the the outer cycle of a Petersen graph, if any). Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5$ and $v'_5$ be the remaining neighbors of respectively $u_1, u_2, u_3, u_4$ and $u_5$ (all different as $g = 5$).

Lemma 5.4. Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains the Petersen graph as induced subgraph then $\Gamma(G) = 5$.

Proof. Suppose that $v_1, v_2, v_3, v_4$ and $v_5$ form an induced $C_5$ (the inner cycle of the Petersen graph). Let $u'_2$ and $u'_5$ be the remaining neighbors of respectively $v_2$ and $v_5$. Observe that $v'_1$ can be adjacent with no more than three vertices among $v'_2, v'_3, u'_2$ and $u'_5$. Firstly, suppose that $v'_1$ is not adjacent with $v'_2$ (or $v'_4$, without loss of generality since the configuration is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$. Secondly, assume that $v'_1$ is not adjacent with $u'_5$ (or $u'_2$, without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$.

In a graph $G$, let a neighbor-connected $C_n$ be an $n$-cycle $C$ such that the set
of vertices of $G$ at distance 1 from $C$ is not independent.

**Lemma 5.5.** Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains a neighbor-connected $C_5$ as induced subgraph, then $\Gamma(G) = 5$.

**Proof.** Let $C$ be a neighbor-connected $C_5$ in $G$. By Lemma 5.4 we can suppose that the neighbors of the vertices of $C$ do not form an induced $C_5$ (otherwise a Petersen would be an induced subgraph). Hence, we can assume that the neighbors of the vertices of $C$ form a subgraph of a $C_{10}$. If there are two edges between the neighbors of the vertices of $C$, then Figure 8 illustrates Grundy partial 5-colorings of the graph $G$. Suppose that two neighbors are adjacent, say $v_1$ and $v'_3$ and the graph $G$ does not contain the previous configuration. Note that $v'_3$ can be adjacent with $v'_1$ and $v'_5$. Let $w_1$, $w_2$ and $w_3$ be the three neighbors of $v_2$ different from $u_2$. We suppose that $w_1$ can be possibly adjacent with $v'_1$ and $w_2$ can be possibly adjacent with $v'_1$. Figure 9 illustrates a Grundy partial 5-coloring of $G$ in this case. In this figure, the vertex $w_3$ can be possibly adjacent with $v'_5$ or $v_4$, but in this case we can switch the color 1 from $v'_5$ to $v_5$ or from $v'_4$ to $v_4$. \hfill \Box

**Proposition 5.6.** If $G$ is a 4-regular graph with girth $g = 5$, then $\Gamma(G) = 5$.

**Proof.** Let $C$ be a 5-cycle in $G$. Assume that two neighbors of consecutive vertices of $C$, for example $v_1$ and $v_5$, have a common neighbor $w_1$. The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph $G$. In this figure the vertex $w_1$ can be possibly adjacent with $v'_2$, $v'_3$ or $v_4$, but in this case we can switch the color 1 from $v'_2$ to $v_2$, from $v'_3$ to $v_3$ or from $v_4$ to $v'_4$. Hence, we can suppose that no neighbors of consecutive vertices of $C$ are adjacent. Among the neighbors of $v_1$, there exists one vertex $w_1$ not adjacent with both $v_4$ and $v'_4$ (otherwise $G$ would contain a $C_4$). Among the neighbor of $v'_5$, there exists one

![Figure 7: Two Grundy partial 5-colorings of a subgraph containing an induced Petersen graph.](image-url)
vertex, say \( w_2 \), not adjacent with \( w_1 \). The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph \( G \). In this figure the vertex \( w_1 \) can be possibly adjacent with \( v_4 \) and the vertex \( w_2 \) can be possibly adjacent with \( v'_2 \) or \( v_4 \), but in these cases we can switch the color 1 from \( v'_2 \) to \( v_2 \) or from \( v_4 \) to \( v'_4 \).

In the following lemma and proposition, we consider a graph \( G \) of girth \( g = 6 \). Let \( u_1, u_2, u_3, u_4, u_5 \) and \( u_6 \) be the vertices in an induced \( C_6 \). Let \( v_1, v'_1, v_2, v'_2, v'_3, v_3, v', v'_4, v_4, v'_5, v_5, v'_6, v_6 \) and \( v'_6 \) be the remaining neighbors of respectively \( u_1, u_2, u_3, u_4, u_5 \) and \( u_6 \) (all different as \( g = 6 \)).

**Lemma 5.7.** If \( G \) is a 4-regular graph with girth \( g = 6 \) which contains a neighbor-connected \( C_6 \) as induced subgraph, then \( \Gamma(G) = 5 \).

**Proof.** Firstly, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let \( w_1 \) be a neighbor of \( v'_1 \) not adjacent with \( v_4 \). The graph \( G \) admits a Grundy partial 5-coloring as the
left part of Figure 11 illustrates it. Secondly, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let $w_1$ be a neighbor of $v_3$ not adjacent with $v_2$ and let $w_2$ be a neighbor of $v'_1$ not adjacent with $w_1$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it.

Proposition 5.8. If $G$ is a 4-regular graph with girth $g = 6$, then $\Gamma(G) = 5$.

Proof. By Lemma 5.7, assume that no neighbors of the vertices of the induced $C_6$ are adjacent. Firstly, suppose that there are two neighbors at distance 4 along the cycle $C_6$, for example $v'_1$ and $v_5$, which have a common neighbor $w_1$. Let $w_2$ be a neighbor of $v_3$ not adjacent with $w_1$. $G$ admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Secondly, suppose that there
are no two neighbors at distance 4 along the cycle $C_6$ which have a common neighbor. Let $w_1$ be a neighbor of $v_1'$ not adjacent with a neighbor of $v_5$ or a neighbor of $v_3$, let $w_2$ be a neighbor of $v_3$ not adjacent with a neighbor of $v_5$, and let $w_3$ be a neighbor of $v_5$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it.

**Proposition 5.9.** If $G$ is a 4-regular graph with girth $g \geq 7$, then $\Gamma(G) = 5$.

**Proof.** Suppose that $G$ contains a 7-cycle. We denote the 5-atom which is a tree by $T_5$ (the binomial tree with maximum degree 4). It can be easily verified that $G$ contains $T_5$ where two leaves are merged (which is a 5-atom). Moreover, if $G$ does not contain a 7-cycle, then it contains $T_5$ as induced subgraph.

**Theorem 3.** Let $G$ be a 4-regular graph. If $G$ does not contain an induced $C_4$, then $\Gamma(G) = 5$.

**Proof.** Suppose that $G$ does not contain an induced $C_4$. Using Proposition 5.9 for the case $g \geq 7$, Propositions 5.6 and 5.8 for the case $g = 5, 6$, and Proposition 5.3 when $G$ contains a $C_3$ yields the desired result.

By Proposition 3.1, Corollary 3.5 and Theorem 3, any $r$-regular graph with $r \leq 4$ and without induced $C_4$ has Grundy number $r+1$. Therefore, it is natural to propose Conjecture 1.

**References**


