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On the family of $r$-regular graphs with Grundy number $r + 1$

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Abstract

The Grundy number of a graph $G$, denoted by $\Gamma(G)$, is the largest $k$ such that there exists a partition of $V(G)$ into $k$ independent sets $V_1, \ldots, V_k$ and every vertex of $V_i$ is adjacent to at least one vertex in $V_j$, for every $j < i$. The objects which are studied in this article are families of $r$-regular graphs such that $\Gamma(G) = r + 1$. Using the notion of independent module, a characterization of this family is given for $r = 3$. Moreover, we determine classes of graphs in this family, in particular the class of $r$-regular graphs without induced $C_4$, for $r \leq 4$. Furthermore, our propositions imply results on partial Grundy number.

1 Introduction

We consider only undirected connected graphs in this paper. Given a graph $G = (V, E)$, a proper $k$-coloring of $G$ is a surjective mapping $c : V \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$ for any $uv \in E$; the color class $V_i$ is the set $\{u \in V | c(u) = i\}$ and a vertex $v$ has color $i$ if $v \in V_i$. A vertex $v$ of color $i$ is a Grundy vertex if $v$ is adjacent to at least one vertex colored $j$, for every $j < i$. A Grundy $k$-coloring is a proper $k$-coloring such that every vertex is a Grundy vertex. A partial Grundy $k$-coloring is a proper $k$-coloring such that every color class contains a Grundy vertex. The Grundy number (partial Grundy number, respectively) of $G$ denoted by $\Gamma(G)$ ($\partial\Gamma(G)$, respectively) is the largest $k$ such that $G$ admits a Grundy $k$-coloring (partial Grundy $k$-coloring, respectively).

Let $N(v) = \{u \in V(G) | uv \in E(G)\}$ be the neighborhood of $v$. A set $X$ of vertices is an independent module if $X$ is an independent set and all vertices

$^*$Author partially supported by the Burgundy Council
in \( X \) have the same neighborhood. The vertices in an independent module of size 2 are called \textit{false twins}. Let \( P_n, C_n, K_n \) and \( I_n \) be respectively, the path, cycle complete and empty graph of order \( n \). The concepts of Grundy \( k \)-coloring and domination are connected. In a Grundy coloring, \( V_1 \) is a dominating set. Given a graph \( G \) and an ordering \( \phi \) on \( V(G) \) with \( \phi = v_1, \ldots, v_n \), the greedy algorithm assigns to \( v_i \) the minimum color that was not assigned in the set \( \{v_1, \ldots, v_{i-1}\} \cap N(v_i) \). Let \( \Gamma_\phi(G) \) be the number of colors used by the greedy algorithm with the ordering \( \phi \) on \( G \). We obtain the following result [7]:

\[
\Gamma(G) = \max_{\phi \in S_n} \Gamma_\phi(G).
\]

The Grundy coloring is a well studied problem. Zaker [15] proved that determining the Grundy number of a given graph, even for complements of bipartite graphs, is an NP-complete problem. However, for a fixed \( t \), determining if a given graph has Grundy number at least \( t \) is decidable in polynomial time. This result follows from the existence of a finite list of graphs, called \( t \)-atoms, such that any graph with Grundy number at least \( t \) contains a \( t \)-atom as an induced subgraph. It has been proven that there exists a Nordhaus-Gaddum type inequality for the Grundy number [8, 15], that there exist upper bounds for \( d \)-degenerate, planar and outerplanar graphs [2, 5], and that there exist connections between the products of graphs and the Grundy number [6, 1, 4].

Recently, Havet and Sampaio [9] have proven that the dual of Grundy \( k \)-coloring problem is in FPT by finding an algorithm in \( O(2^\frac{k^2}{2} \cdot |E| + 2^k k^{3k+5/2}) \) time.

Note that a Grundy \( k \)-coloring is a partial Grundy \( k \)-coloring, hence \( \Gamma(G) \leq \partial \Gamma(G) \). Given a graph \( G \) and a positive integer \( k \), the problem of determining if a partial Grundy \( k \)-coloring exists, even for chordal graphs, is NP-complete but there exists a polynomial algorithm for trees [13].

Another coloring parameter with domination constraints on the colors is the \( b \)-chromatic number, denoted by \( \phi(G) \), which is the largest \( k \) such that there exists a proper \( k \)-coloring and for every color class \( V_i \), there exists a vertex adjacent to at least one vertex colored \( j \), for every \( j \), with \( j \neq i \). Note that a \( b \)-coloring is a partial Grundy \( k \)-coloring, hence \( \phi(G) \leq \partial \Gamma(G) \). The \( b \)-chromatic number of regular graphs has been investigated in a series of papers ([11, 10, 3, 12]). Our aim is to establish similar results for the Grundy coloring. We present two main results: A characterization of the Grundy number of every cubic graph and the following theorem: For \( r \leq 4 \), every \( r \)-regular graphs without induced \( C_4 \) has Grundy number \( r + 1 \). We conjecture that this assertion is also true for \( r > 4 \).

\textbf{Conjecture 1.} For any integer \( r \geq 1 \), every \( r \)-regular graph without induced \( C_4 \) has Grundy number \( r + 1 \).

Section 2 gives characterizations of some classes of graphs with Grundy number at most \( k \), \( 2 \leq k \leq \Delta(G) \), using the notion of independent module. Section 3 contains the first main theorem: A description of the cubic graphs with Grundy number at most 3 that also allows us to prove that every cubic graph except
$K_{3,3}$ has partial Grundy number 4. This theorem implies the existence of a linear algorithm to determine the Grundy number of cubic graphs. In Section 4, we present examples of infinite families of regular graphs with Grundy number exactly or at most $k$, $3 \leq k \leq r$. To determine these families we use recursive definitions. The last section contains the second main theorem of this article: 4-regular graphs without induced $C_4$ have Grundy number 5.

2 General results

The reader has to be aware of the resemblance of name between the following notion and that of partial Grundy $k$-coloring.

**Definition 2.1.** Let $G$ be a graph. A Grundy partial $k$-coloring is a Grundy $k$-coloring of a subset $S$ of $V(G)$.

**Observation 2.2 ([1],[6]).** If $G$ admits a Grundy partial $k$-coloring, then $\Gamma(G) \geq k$.

This property has an important consequence: For a graph $G$, with $\Gamma(G) \geq t$ and any Grundy partial $t$-coloring, there exist smallest subgraphs $H$ of $G$ such that $\Gamma(H) = t$. The family of $t$-atoms corresponds to these subgraphs. This concept was introduced by Zaker [15]. The family of $t$-atoms is finite and the presence of a $t$-atom can be determined in polynomial time for a fixed $t$. The following definition is slightly different from Zaker’s one, insisting more on the construction of every $t$-atom.

**Definition 2.3 ([15]).** For any integer $t$, we define the family of $t$-atoms, denoted by $\mathcal{A}_t$, $t = 1, \ldots$ by induction. Let the family $\mathcal{A}_1$ contain only $K_1$. A graph $G$ is in $\mathcal{A}_{t+1}$ if there exists a graph $G' \in \mathcal{A}_t$ and an integer $m$, $m \leq |V(G')|$, such that $G$ is composed of $G'$ and an independent set $I_m$ of order $m$, adding edges between $G'$ and $I_m$ such that every vertex in $G'$ is connected to at least one vertex in $I_m$. Moreover a $t$-atom $A$ is minimal, if there is no $t$-atom included in $A$ other than itself.

**Theorem 1 ([15]).** For a given graph $G$, $\Gamma(G) \geq t$ if and only if $G$ contains an induced minimal $t$-atom.

We now present conditions related to the presence of modules that allows us to upper-bound the Grundy number.

**Proposition 2.4 ([1]).** Let $G$ be a graph and $X$ be an independent module. In every Grundy coloring of $G$, the vertices in $X$ must have the same color.

**Definition 2.5.** Let $G$ be an $r$-regular graph. A vertex $v$ is a $(0, \ell)$-twin-vertex if there exists an independent module of cardinality $r + 2 - \ell$ that contains $v$.

**Proposition 2.6.** Let $G$ be an $r$-regular graph. The color of an $(0, \ell)$-twin-vertex is at most $\ell$ in every Grundy coloring of $G$. 

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Proof. Let \( v \) be a \((0, \ell)\)-twin-vertex colored \( \ell + 1 \) in \( G \). By Definition, \( v \) is in an independent module \( X \) of cardinality \( r + 2 - \ell \) and by Proposition 2.4, every other vertex of \( X \) should be colored \( \ell + 1 \). Let \( u \) be a neighbor of \( v \). There are at most \( \ell - 2 \) neighbors of \( u \) in \( V(G - X) \). Therefore, \( u \) cannot be colored \( \ell \).

Definition 2.7. A vertex \( v \) of a graph \( G \) is a \((1, \ell)\)-twin-vertex if \( N(v) \) can be partitioned into at least \( \ell - 1 \) independent modules.

Proposition 2.8. Let \( G \) be a graph. The color of an \((1, \ell)\)-twin-vertex is at most \( \ell \) in every Grundy coloring of \( G \).

Proof. By Proposition 2.4, vertices of the neighborhood of \( v \) can only have \( \ell - 1 \) different colors. Therefore, the color of \( v \) is at most \( \ell \).

Definition 2.9. A vertex \( v \) of a graph \( G \) is a \((2, \ell)\)-twin-vertex if \( N(v) \) is independent and every vertex in \( N(v) \) is a \((1, \ell)\)-twin-vertex.

Proposition 2.10. Let \( G \) be a graph. The color of an \((2, \ell)\)-twin-vertex is at most \( \ell \) in every Grundy coloring of \( G \).

Proof. Let \( v \) be a \((2, \ell)\)-twin-vertex in \( G \). Every vertex in \( N(v) \) is a \((1, \ell)\)-twin-vertex. If a vertex in \( N(v) \) is colored \( \ell \), then \( v \) could only have a color at most \( \ell - 1 \). If the vertices in the neighborhood of \( v \) have colors at most \( \ell - 1 \), then in every Grundy coloring of \( G \), \( v \) has a color at most \( \ell \).

Corollary 2.11. Let \( G \) be a graph. If every vertex is a \((1, \ell)\)-twin-vertex or a \((2, \ell)\)-twin-vertex, then \( \Gamma(G) \leq \ell \).

Corollary 2.12. Let \( G \) be a regular graph. If every vertex is an \((i, \ell)\)-twin-vertex, for some \( i, 0 \leq i \leq 2 \), then \( \Gamma(G) \leq \ell \).

Proposition 2.13 ([1],[15]). Let \( G \) be a graph. We have \( \Gamma(G) \leq 2 \) if and only if \( G = K_{n,m} \) for some integers \( n > 0 \) and \( m > 0 \).

3 Grundy numbers of cubic graphs

In the following sections, the figures describe Grundy partial \( k \)-colorings. By a dashed edge we denote a possible edge. The vertices not connected by edges in the figures cannot be adjacent as it would contradict the hypothesis.

Proposition 3.1 ([6]). Let \( G \) be a connected 2-regular graph. \( \partial\Gamma(G) = \Gamma(G) = 2 \) if and only if \( G = C_4 \).

The following definition gives a construction of the cubic graphs in which every vertex is an \((i,3)\)-twin-vertex, for some \( i, 0 \leq i \leq 2 \). Figure 2 gives the list of every graph of order at most 16 in this family.

Definition 3.2. Let \( K_{2,3} \) and \( K^*_{2,3} \) be the graphs from Figure 1. We define recursively the family of graphs \( \mathcal{F}_3^* \) as follows:
Figure 1: The graphs $K_{2,3}$ (on the left) and $K_{3,3}^*$ (on the right).

Figure 2: The cubic graphs $G$ such that $|V(G)| < 18$ and $\Gamma(G) < 4$.

1. $K_{2,3} \in \mathcal{F}_3^*$ and $K_{3,3}^* \in \mathcal{F}_3^*$;
2. the disjoint union of two elements of $\mathcal{F}_3^*$ is in $\mathcal{F}_3^*$;
3. if $G$ is a graph in $\mathcal{F}_3^*$, then the graph $H$ obtained from $G$ by adding an edge between two vertices of degree at most 2 is also in $\mathcal{F}_3^*$;
4. if $G$ is a graph in $\mathcal{F}_3^*$, then the graph $H$ obtained from $G$ by adding a new vertex adjacent to three vertices of degree at most 2 is in $\mathcal{F}_3^*$.

The family $\mathcal{F}_3$ is the subfamily of cubic graphs in $\mathcal{F}_3^*$.

**Proposition 3.3.** Let $G$ be a cubic graph. Every vertex of $V(G)$ is an $(i,3)$-twin vertex, for some $i$, $0 \leq i \leq 2$, if and only if $G \in \mathcal{F}_3$.

**Proof.** Every graph $G$ in $\mathcal{F}_3$ has three kind of vertices: $(0,3)$-twin-vertices (called also false twins), vertices where an edge is added by Point 3 and vertices
added by Point 4. Vertices where an edge is added by Point 3 are (1,3)-twin-vertex and vice versa. Vertices added by Point 4 are (2,3)-twin-vertices and vice versa.

**Theorem 2.** Let $G$ be a cubic graph. $\Gamma(G) \leq 3$ if and only if every vertex is an $(i,3)$-twin-vertex, for some $i$, $0 \leq i \leq 2$.

**Proof.** By Corollary 2.12, the "if" part is proven. Assume that $G$ contains a vertex $v$ which is not an $(i,3)$-twin-vertex, for some $i$, $0 \leq i \leq 2$ and $\Gamma(G) < 4$. In every configuration we want to either find a Grundy partial 4-coloring, contradicting $\Gamma(G) < 4$ or proving that $v$ is an $(i,3)$-twin-vertex, for some $i$, with $0 \leq i \leq 2$. We will refer to a given Grundy partial 4-coloring by its reference in Figure 3. We consider three cases: $v$ or a neighbor of $v$ is in a $C_3$, $v$ is in an induced $C_4$ and $v$ or a neighbor of $v$ are not in a $C_3$ and $v$ is not in an induced $C_4$. Let $C$ be an induced cycle of order 3 or 4 which contains $v$ or a neighbor of $v$ and let $D_1 = \{x \in V(G) | d(x,C) = 1\}$, where $d(x,C)$ is the distance from $x$ to $C$ in the graph $G$. To simplify notation, $D_1$ will also denote the subgraph of $G$ induced by $D_1$.

**Case 1:** Assume that $v$ or a neighbor of $v$ is in $C$ and $C = C_3$. If $|D_1| = 1$, then $G = K_4$ and $\Gamma(K_4) = 4$. If $|D_1| = 2$ and $D_1 = P_2$, then $v$ is a $(0,3)$-twin-vertex or a $(1,3)$-twin-vertex. If $D_1 = I_2$ then Figure 3.1.a yields a Grundy partial 4-coloring of $G$. If $|D_1| = 3$, then we have four subcases: $D_1$ is $C_3$ or $P_3$ (Figure 3.1.b), $P_2 \cup I_1$ (Figure 3.1.c) or $I_3$ (Figure 3.1.d). In every case $G$ admits a Grundy partial 4-coloring.

**Case 2:** Assume that $v$ is in $C$ and $C = C_4$. Note that for two non adjacent vertices of $C$ who have a common neighbor in $D_1$, the vertex $v$ is a $(0,3)$-twin-vertex or a $(1,3)$-twin-vertex. Hence, we will not consider these cases. If $|D_1| = 2$, then $D_1 = P_2$ or $D_1 = I_2$ (Figure 3.2.a) and in both cases, $G$ admits a Grundy partial 4-coloring. If $|D_1| = 3$, Figure 3.2.b yields a Grundy partial 4-coloring of $G$. In the case $|D_1| = 4$, we first assume that two adjacent vertices of $C$ have their neighbors in $D_1$ adjacent (Figure 3.2.c). Afterwards, we suppose that the previous case does not happen and that two non adjacent vertices of $C$ have their neighbors in $D_1$ adjacent (Figure 3.2.d). In the case $D_1 = I_4$, we first suppose that two vertices of $D_1$ which have two adjacent vertices of $C$ as neighbor, are not adjacent to two common vertices (Figure 3.2.e) and after consider they are (Figure 3.2.f).

**Case 3:** Assume that $v$ or a neighbor of $v$ is not in a $C_3$ and $v$ is not in an induced $C_4$. Firstly, suppose that a neighbor $u$ of $v$ is in an induced $C_4$. Using the coloring from the previous case, $G$ admits a Grundy partial 4-coloring in every cases except in the case where two neighbors of $v$ in the $C_4$ have a common neighbor outside the $C_4$. However, this case cannot happen for every neighbor of $v$, otherwise $v$ would be a $(2,3)$-twin-vertex. Assume that $u$ is the neighbor of $v$ not in the previous configuration. If $u$ is in an induced $C_4$, then using the coloring from the previous case, $G$
admits a Grundy partial 4-coloring. If \( u \) is not in an induced \( C_4 \), then Figure 3.3.a yields a Grundy partial 4-coloring of \( G \). In this figure, the color 2 is given to a neighbor of \( u \) not adjacent to both \( f_1 \) and \( f_2 \). Secondly, suppose that \( v \) is in an induced \( C_5 \). Figure 3.3.b yields a Grundy partial 4-coloring of \( G \). Thirdly, if \( v \) is not in an induced \( C_5 \), then Figure 3.3.c yields a Grundy partial 4-coloring of \( G \).

Therefore, if \( \Gamma(G) \leq 3 \), then every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

Observe that if an edge is added between the two vertices of degree 2 in \( K^*_{3,3} \), then we obtain \( K_{3,3} \) which has Grundy number 2. By Proposition 3.3, in all the remaining cases, the cubic graphs which have Grundy number at most 3 are different from complete bipartite graphs. Therefore, they have Grundy number 3.

**Corollary 3.4.** A cubic graph \( G \) does not contain any induced minimal subcubic 4-atom if and only if every vertex is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \).

**Corollary 3.5.** Let \( G \) be a cubic graph. If \( G \) is without induced \( C_4 \), then \( \Gamma(G) = 4 \).

**Proof.** As every graph \( G \) with \( \Gamma(G) < 4 \) is composed of copies of \( K_{2,3} \) or \( K^*_{3,3} \), the graph \( G \) always contains a square if \( \Gamma(G) < 4 \).

For a fixed integer \( t \), the largest \((t+1)\)-atom has order \( 2^t \). Thus, for a graph \( G \) of maximum degree \( t \), there exists an \( O(n^2) \)-time algorithm to determine if \( \Gamma(G) = 4 \) (which verifies if the graph contains an induced \((t+1)\)-atom). For a cubic graph, we obtain an \( O(n^8) \)-time algorithm, whereas our characterization yields a linear-time algorithm.

**Observation 3.6.** Let \( G \) be a cubic graph of order \( n \). There exists an \( O(n) \)-time algorithm\(^1\) to determine the Grundy number of \( G \).

**Proof.** Suppose we have a cubic graph \( G \) with its adjacency list. Verifying if \( G \) is \( K^*_{3,3} \) can be done in constant time. We suppose now that \( G \) is not \( K^*_{3,3} \). For each vertex \( v \), the algorithm verifies that \( v \) is an \((i,3)\)-twin-vertex, for some \( i \), \( 0 \leq i \leq 2 \). If the condition is true for all vertices, then \( \Gamma(G) = 3 \), else \( \Gamma(G) = 4 \). To determine if a vertex \( v \) is a \((0,3)\)-twin-vertex, it suffices to verify that there is a common vertex other than \( v \) in the adjacency lists of the neighbors of \( v \). To determine if a vertex \( v \) is a \((1,3)\)-twin-vertex, it suffices to verify that there are two neighbors of \( v \) which have the same adjacency list. To determine if a vertex \( v \) is a \((2,3)\)-twin-vertex, it suffices to verify that the neighborhood of \( v \) is independent and that every neighbor is a \((1,3)\)-twin-vertex. Hence, checking if a vertex is an \((i,3)\)-twin vertex can be done in constant time, so the algorithms runs in linear time.

\(^1\)Independently of our work, Yahiaoui et al. [14] have established a different algorithm to determine if the Grundy number of a cubic graph is 4.
Figure 3: Possible configurations in a cubic graph (bold vertices: Uncolored vertices, vertices with number $i$: Vertices of color $i$).
**Proposition 3.7.** If $G$ is a connected cubic graph and $G \neq K_{3,3}$, then $\partial \Gamma(G) = 4$.

*Proof.* Let $G$ be a cubic connected graph. Note that if $\Gamma(G) = 4$ then $\partial \Gamma(G) = 4$. Every graph $G$ with $\Gamma(G) < 4$ is composed of copies of $K_{2,3}$ or $K^*_3$. If $G$ contains more than two copies (so it is different from $K_{3,3}$), then a vertex can be colored 4 in the first copy and a vertex can be colored 3 in the second copy. Hence, $\partial \Gamma(G) = 4$. \hfill \Box

Only $K_{3,3}$ and three other cubic graphs have $b$-chromatic number at most 3 [10]. Thus, our result is coherent with the results on the $b$-chromatic number.

Shi et al. [13] proved that there exists a smallest integer $N_b[10]$. Thus, our result is coherent with the results on the $r$-chromatic number.

**Proposition 4.1.** Let $G$ be a graph of order $n$. If $G$ has $b$-chromatic number at most 3, then $\partial \Gamma(G) \leq 4$.

**Proposition 4.2.** Let $G$ be an $r$-regular graph. If $G \in \mathcal{G}_r$, then $\Gamma(G) < r + 1$.

*Proof.* By $I_{r-k}$ and $I_{k+2}$, with $|I_{r-k}| = r - k$ and $|I_{k+2}| = k + 2$, we denote the two sets of vertices in the bipartition of an induced subgraph $K_{r-k,k+2}$ in $G$. Firstly, suppose there exists a vertex $u$ in an induced subgraph $K_{r-k,k+2}$ colored $r + 1$. Without loss of generality, suppose $u$ is in $I_{r-k}$. The $r$ neighbors of $u$ should have colors from 1 to $r$. Among the neighbors of $u$, $k + 2$ neighbors are in $I_{k+2}$. Let $v$ be the neighbor of $u$ in $I_{k+2}$ with the largest color in $I_{k+2}$. The vertex $v$ has color at least $k + 2$. Hence, there exists an integer $s \geq 0$ such that the color of $v$ is $k + 2 + s$. Note that there are $s$ vertices in $N(u) \setminus I_{k+2}$ which have colors at most $k + 2 + s$. The colors of the $s$ vertices are the only one possible remaining colors at most $k + 2 + s$ in $I_{r-k}$. Hence, as there are $k$ vertices in $N(v) \setminus I_{r-k}$, the neighbors of $v$ can only have at most $k + s$ different colors at most $k + 2 + s$. Therefore, we have a contradiction and $u$ cannot have
Figure 4: The Graph $G_{r,k,i}$, $i \geq 2$, $r = r_1 + r_2 + r_3$.

color $r + 1$. Secondly, suppose there exists a vertex $u$ added by Point 4 which has color $r + 1$. As a neighbor of $u$ in an induced $K_{r-k,k+2}$ should be colored $r$, the argument is completely similar to the previous one. □

Corollary 4.3. Let $G$ be a 4-regular graph. If $G \in \mathcal{G}_4$, then $\Gamma(G) < 5$.

The reader can believe that the family of 4-regular graphs with $\Gamma(G) < 5$ contains only the family $\mathcal{G}_4$. However, there exist graphs with Grundy number $r$ which are not inside this family. For example, the power graph (the graph where every pair of vertices at pairwise distance 2 become adjacent) of the 7-cycle $C_7^2$ satisfies $\Gamma(C_7^2) < 5$ and is not in $\mathcal{G}_4$.

The next proposition shows that unlike the $b$-chromatic number, $r$-regular graphs of order arbitrarily large with Grundy number $k$ can be constructed for any $r$ and any $k$, $3 \leq k \leq r + 1$.

Proposition 4.4. Let $r \geq 4$ and $3 \leq k \leq r + 1$ be integers. There exists an infinite family $\mathcal{H}$ of connected $r$-regular graphs such that for all $G$ in $\mathcal{H}$, $\Gamma(G) = k$.

Proof. Let $i \geq 2$ be a positive integer and $r_1, \ldots, r_{k-1}$ be a sequence of positive integers such that $r = r_1 + \ldots + r_{k-1}$. We construct a graph $G_{r,k,i}$ as follows: Take $2i$ copies of $K_{r_1,\ldots,r_{k-1}}$. Let $H_{j-1}$ be the copy number $j$ of $K_{r_1,\ldots,r_{k-1}}$ and $H_{j,r_l}$ be the independent $r_l$-set in $H_j$. If $j \equiv 1 \pmod{2}$, do the graph join of $H_{j-1} \pmod{2i},r_1$ and $H_{j-1} \pmod{2i},r_1$ and for an integer $l$, $1 < l < k$, do the graph join of $H_{j-1} \pmod{2i},r_l$ and $H_{j+1} \pmod{2i},r_l$. The $r$-regular graph obtained is the graph $G_{r,k,i}$. Figure 4 gives $G_{r,k,i}$ for $k = 4$ and $i \geq 2$. Note that $H_{j,r_l}$ is an independent module. Thus, every vertex is a $(0,k)$-twin-vertex. By Proposition 2.6, $\Gamma(G_{r,k,i}) \leq k$.

For any integer $l$, $1 < l < k$, color one vertex $l-1$ in $H_{1,r_l}$ and $H_{2,r_l}$. Afterwards, color one vertex $k-1$ in $H_{1,r_l}$ and one vertex $k$ in $H_{2,r_l}$. The given coloring is a Grundy partial $k$-coloring of $G_{r,k,i}$ for $i \geq 2$. Therefore, $\Gamma(G_{r,k,i}) = k$, for $i \geq 2$. □
5 Grundy number of 4-regular graphs without induced $C_4$

The following lemmas will be useful to prove the second main theorem of this paper: The family of 4-regular graphs without induced $C_4$ contains only graphs with Grundy number 5.

**Lemma 5.1.** Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains (an induced) $K_4$ then $\Gamma(G) = 5$.

**Proof.** Note that if $G = K_5$, we have $\Gamma(G) = 5$. If $G$ is not $K_5$ then every pair of neighbors of vertices of $K_4$ cannot be adjacent ($G$ would contain a $C_4$). Giving the color 1 to each neighbor of the vertices of $K_4$ and colors 2, 3, 4, 5 to the vertices of $K_4$, we obtain a Grundy partial 5-coloring of $G$.

**Lemma 5.2.** Let $G$ be a 4-regular graph without induced $C_4$ and let $W$ be the graph from Figure 5. If $G$ contains an induced $W$ then $\Gamma(G) = 5$.

**Proof.** The names of the vertices of $W$ come from Figure 5. Depending on the different cases that could happen, Grundy partial 5-colorings of $G$ will be given using their references on Figure 5. Let $D_1$ be the set of vertices at distance 1 from vertices of $W$ in $G - W$. Suppose that two vertices of $W$ have a common neighbor in $D_1$. This two vertices could only be $u_4$ and $u_5$ or $u_3$ and $u_5$ (or $u_1$ and $u_4$, by symmetry). In the case that $u_4$ and $u_5$ have a common neighbor in $D_1$, colors will be given to neighbors of $u_3$ in $D_1$, depending if they are adjacent (Figure 5.1.a) or not (Figure 5.1.b). In the case that $u_3$ and $u_5$ have a common neighbor $w$ in $D_1$, $w$ can be adjacent with a neighbor of $u_3$ in $D_1$ (Figure 5.2.a) or not (Figure 5.2.b). Suppose now that no vertices in $W$ have a common neighbor in $D_1$. Let $w_1$ and $w_2$ be the neighbors of $u_3$ in $D_1$. We first consider that $w_1$ and $w_2$ are adjacent (Figure 5.3.a). Secondly, we consider that $w_1$ and $w_2$ are not adjacent and that $u_5$, $u_3$ and $w_1$ are in an induced $C_5$ (Figure 5.3.b). Finally, we consider that the previous configurations are impossible (Figure 5.3.c).

**Proposition 5.3.** Let $G$ be a 4-regular graph without induced $C_4$. If $G$ contains $C_3$ then $\Gamma(G) = 5$.

**Proof.** Depending on the different cases that could happen, a reference to the Grundy partial 5-coloring of $G$ in Figure 6 will be given. Let $M_i$, $i = 2$ or 3, be the graph of order $2 + i$ containing two adjacent vertices $u_1$ and $u_2$ which have exactly $i$ common neighbors, $\{v_1, \ldots, v_i\}$, that form an independent set. Let $D_1$ be the set of vertices at distance 1 from an induced $M_i$ in $G - M_i$, for $2 \leq i \leq 3$.

**Case 1:** Firstly, assume that $G$ contains an induced $M_3$ and a vertex of $M_3$ has its two neighbors in $D_1$ adjacent (Figure 6.1.a). Secondly, assume that $G$ contains an induced $M_2$ and a vertex of $M_2$ has its two neighbors in $D_1$...
Figure 5: Possible configurations when $G$ contains an induced $W$.

adjacent (Figure 6.1.b). Note that these Grundy partial 5-colorings use the fact that $G$ cannot contain a $K_4$ by Lemma 5.1.

**Case 2:** Assume that $G$ contains an induced $M_3$ excluding the previous configuration. There are three cases: $u_1, v_2$ and $v_3$ are in an induced $C_5$ (Figure 6.2.a), $u_1, v_2$ and $v_3$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.2.b) and $u_1, v_2$ and $v_3$ are neither in an induced $C_5$ nor $C_6$ (Figure 6.2.c).

**Case 3:** Suppose that $G$ contains an induced $M_2$ excluding the previous configurations. Firstly, we suppose that $u_1, v_1$ and $v_2$ are in an induced $C_5$ (Figure 6.3.a). Secondly, we suppose that $u_1, v_1$ are in an induced $C_5$ excluding the previous case (Figure 6.3.b). Thirdly, we suppose that $u_1, v_1$ and $v_2$ are in an induced $C_6$ and not in an induced $C_5$ (Figure 6.3.c) and finally neither in an induced $C_5$ nor $C_6$ (Figure 6.3.d).

Suppose that $G$ contains a 3-cycle $C$ and no induced $M_2$. Let $u_1, u_2$ and $u_3$ be the vertices of $C$. Let $w_1$ and $w_2$ be the neighbors of $u_1$ outside $C$, let $w'_1$ and $w'_2$ be the neighbors of $u_2$ outside $C$ and let $w''_1$ and $w''_2$ be the neighbors of $u_3$ outside $C$.

**Case 4:** Firstly, suppose that $u_1, u_2, w_1$ and $w'_1$ are in a 5-cycle and a neighbor of $u_1$, say $w_1$, has a common neighbor with $w'_1$ (Figure 6.4.a). Secondly, excluding the previous configuration, suppose that $u_1, u_2, u_1$ and $w'_1$ are in a 5-cycle; $w''_1, v_1, u_1$ and $w_1$ are in another 5-cycle and $w_1$ is in a triangle (Figure 6.4.b). We suppose that $w_1$ is not in a triangle (Figure 6.4.c). Thirdly, excluding the previous configurations, we obtain a Grundy partial 5-coloring if two vertices of $C$ are in a 5-cycle (Figure 6.4.d). Fourthly, we suppose that two vertices of $C$ cannot be in a 5-cycle (Figure 6.4.e).

In the following two lemmas, we consider a graph $G$ of girth $g = 5$ and possibly containing an induced Petersen graph. Let $u_1, u_2, u_3, u_4$ and $u_5$ be
the vertices in an induced $C_5$ (or in the outer cycle of a Petersen graph, if any). Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5$ and $v'_5$ be the remaining neighbors of respectively $u_1, u_2, u_3, u_4$ and $u_5$ (all different as $g = 5$).

**Lemma 5.4.** Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains the Petersen graph as induced subgraph then $\Gamma(G) = 5$.

**Proof.** Suppose that $v_1, v_2, v_3, v_4$ and $v_5$ form an induced $C_5$ (the inner cycle of the Petersen graph). Let $u'_2$ and $u'_5$ be the remaining neighbors of respectively $v_2$ and $v_5$. Observe that $v'_1$ can be adjacent with no more than three vertices among $v'_3, v'_4, u'_2$ and $u'_5$. Firstly, suppose that $v'_1$ is not adjacent with $v'_3$ (or $v'_4$, without loss of generality since the configuration is symmetric). The left part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$. Secondly, assume that $v'_1$ is not adjacent with $u'_5$ (or $u'_2$, without loss of generality). The right part of Figure 7 illustrates a Grundy partial 5-coloring of the graph $G$. 

In a graph $G$, let a **neighbor-connected** $C_n$ be an $n$-cycle $C$ such that the set...
of vertices of $G$ at distance 1 from $C$ is not independent.

**Lemma 5.5.** Let $G$ be a 4-regular graph with girth $g = 5$. If $G$ contains a neighbor-connected $C_5$ as induced subgraph, then $\Gamma(G) = 5$.

**Proof.** Let $C$ be a neighbor-connected $C_5$ in $G$. By Lemma 5.4 we can suppose that the neighbors of the vertices of $C$ do not form an induced $C_5$ (otherwise a Petersen would be an induced subgraph). Hence, we can assume that the neighbors of the vertices of $C$ form a subgraph of a $C_{10}$. If there are two edges between the neighbors of the vertices of $C$, then Figure 8 illustrates Grundy partial 5-colorings of the graph $G$. Suppose that two neighbors are adjacent, say $v_1$ and $v'_3$ and the graph $G$ does not contain the previous configuration. Note that $v'_3$ can be adjacent with $v'_1$ and $v'_5$. Let $w_1, w_2$ and $w_3$ be the three neighbors of $v_2$ different from $u_2$. We suppose that $w_1$ can be possibly adjacent with $v'_1$ and $w_2$ can be possibly adjacent with $v'_4$. Figure 9 illustrates a Grundy partial 5-coloring of $G$ in this case. In this figure, the vertex $w_3$ can be possibly adjacent with $v'_5$ or $v_4$, but in this case we can switch the color 1 from $v'_5$ to $v_5$ or from $v'_4$ to $v_4$.

**Proposition 5.6.** If $G$ is a 4-regular graph with girth $g = 5$, then $\Gamma(G) = 5$.

**Proof.** Let $C$ be a 5-cycle in $G$. Assume that two neighbors of consecutive vertices of $C$, for example $v_1$ and $v_5$, have a common neighbor $w_1$. The left part of Figure 10 illustrates a Grundy partial 5-coloring of the graph $G$. In this figure the vertex $w_1$ can be possibly adjacent with $v'_2$, $v'_3$ or $v_4$, but in this case we can switch the color 1 from $v'_2$ to $v_2$, from $v'_3$ to $v_3$ or from $v_4$ to $v'_4$. Hence, we can suppose that no neighbors of consecutive vertices of $C$ are adjacent. Among the neighbors of $v_1$, there exists one vertex $w_1$ not adjacent with both $v_4$ and $v'_4$ (otherwise $G$ would contain a $C_4$). Among the neighbor of $v'_5$, there exists one

![Figure 7: Two Grundy partial 5-colorings of a subgraph containing an induced Petersen graph.](image-url)
vertex, say $w_2$, not adjacent with $w_1$. The right part of Figure 10 illustrates a Grundy partial 5-coloring of the graph $G$. In this figure the vertex $w_1$ can be possibly adjacent with $v_4$ and the vertex $w_2$ can be possibly adjacent with $v'_2$ or $v_4$, but in these cases we can switch the color 1 from $v'_2$ to $v_2$ or from $v_4$ to $v'_4$.

In the following lemma and proposition, we consider a graph $G$ of girth $g = 6$. Let $u_1, u_2, u_3, u_4, u_5$ and $u_6$ be the vertices in an induced $C_6$. Let $v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, v_5, v'_5, v_6$ and $v'_6$ be the remaining neighbors of respectively $u_1, u_2, u_3, u_4, u_5$ and $u_6$ (all different as $g = 6$).

**Lemma 5.7.** If $G$ is a 4-regular graph with girth $g = 6$ which contains a neighbor-connected $C_6$ as induced subgraph, then $\Gamma(G) = 5$.

**Proof.** Firstly, suppose that there are two edges which connect the neighbors in the same way than in the left part of Figure 11. Let $w_1$ be a neighbor of $v'_1$ not adjacent with $v_4$. The graph $G$ admits a Grundy partial 5-coloring as the
Figure 10: Two Grundy partial 5-colorings of a subgraph containing an induced $C_5$.

Figure 11: Two Grundy partial 5-colorings of a subgraph containing an induced neighbor-connected $C_6$.

left part of Figure 11 illustrates it. Secondly, suppose that there is one edge (or more) which connect the neighbors without the configuration from the previous case. Let $w_1$ be a neighbor of $v_3$ not adjacent with $v_2$ and let $w_2$ be a neighbor of $v'_1$ not adjacent with $w_1$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 11 illustrates it.

**Proposition 5.8.** If $G$ is a 4-regular graph with girth $g = 6$, then $\Gamma(G) = 5$.

**Proof.** By Lemma 5.7, assume that no neighbors of the vertices of the induced $C_6$ are adjacent. Firstly, suppose that there are two neighbors at distance 4 along the cycle $C_6$, for example $v'_1$ and $v_5$, which have a common neighbor $w_1$. Let $w_2$ be a neighbor of $v_3$ not adjacent with $w_1$. $G$ admits a Grundy partial 5-coloring as the left part of Figure 12 illustrates it. Secondly, suppose that there
are no two neighbors at distance 4 along the cycle $C_6$ which have a common neighbor. Let $w_1$ be a neighbor of $v'_1$ not adjacent with a neighbor of $v_5$ or a neighbor of $v_3$, let $w_2$ be a neighbor of $v_3$ not adjacent with a neighbor of $v_5$, and let $w_3$ be a neighbor of $v_5$. The graph $G$ admits a Grundy partial 5-coloring as the right part of Figure 12 illustrates it.

**Proposition 5.9.** If $G$ is a 4-regular graph with girth $g \geq 7$, then $\Gamma(G) = 5$.

**Proof.** Suppose that $G$ contains a 7-cycle. We denote the 5-atom which is a tree by $T_5$ (the binomial tree with maximum degree 4). It can be easily verified that $G$ contains $T_5$ where two leaves are merged (which is a 5-atom). Moreover, if $G$ does not contain a 7-cycle, then it contains $T_5$ as induced subgraph.

**Theorem 3.** Let $G$ be a 4-regular graph. If $G$ does not contain an induced $C_4$, then $\Gamma(G) = 5$.

**Proof.** Suppose that $G$ does not contain an induced $C_4$. Using Proposition 5.9 for the case $g \geq 7$, Propositions 5.6 and 5.8 for the case $g = 5, 6$, and Proposition 5.3 when $G$ contains a $C_3$ yields the desired result.

By Proposition 3.1, Corollary 3.5 and Theorem 3, any $r$-regular graph with $r \leq 4$ and without induced $C_4$ has Grundy number $r+1$. Therefore, it is natural to propose Conjecture 1.

**References**


