Inward Pointing Trajectories, Normality of the Maximum Principle and the non Occurrence of the Lavrentieff Phenomenon in Optimal Control under State Constraints

Hélène Frankowska, Daniela Tonon

INWARD POINTING TRAJECTORIES, NORMALITY OF THE MAXIMUM PRINCIPLE AND THE NON OCCURRENCE OF THE LAVRENTIEFF PHENOMENON IN OPTIMAL CONTROL UNDER STATE CONSTRAINTS

HÉLÈNE FRANKOWSKA AND DANIELA TONON

Abstract. It is well known that every strong local minimizer of the Bolza problem under state constraints satisfies a constrained maximum principle. In the absence of constraints qualifications the maximum principle may be abnormal, that is, not involving the cost functions. Normality of the maximum principle can be investigated by studying reachable sets of an associated linear system under linearized state constraints. In this paper we provide sufficient conditions for the existence of solutions to such system and apply them to guarantee the non occurrence of the Lavrentieff phenomenon in optimal control under state constraints.

1. Introduction

Consider the control system

$$x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{for a.e. } t \in [0,1], \tag{1.1}$$

under state and end points constraints

$$x(t) \in K \quad \text{for all } t \in [0,1], \quad (x(0), x(1)) \in K_1, \tag{1.2}$$

where $U(\cdot)$ is a measurable set valued map from $[0,1]$ into nonempty closed subsets of a complete separable metric space $\mathcal{Z}$, $f : [0,1] \times \mathbb{R}^n \times \mathcal{Z} \to \mathbb{R}^n$, $f(\cdot, x, \cdot)$ is $\mathcal{L} \times \mathcal{B}$-measurable and $f(t, \cdot, u)$ is locally Lipschitz continuous, K and K_1 are closed subsets of $\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ respectively.

Let $S^K_{[0,1]}$ be the set of all absolutely continuous solutions of (1.1) satisfying the constraints (1.2). A pair $(x(\cdot), u(\cdot))$, with $x(\cdot)$ absolutely continuous and $u(\cdot)$ measurable, is called a *viable* (or, alternatively, *feasible*) trajectory/control pair if it satisfies (1.1) and (1.2).

The Bolza optimal control problem under state and end points constraints consists in the following minimization problem

$$\inf \left\{ \varphi(x(0), x(1)) + \int_0^1 L(t, x(t), u(t))dt \,\Big|\, x(\cdot) \in S^K_{[0,1]} \right\}, \tag{1.3}$$

where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $L : [0,1] \times \mathbb{R}^n \times \mathcal{Z} \to \mathbb{R}$ are given cost functions.

For $\lambda \in \{0,1\}$ define $H_\lambda : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and the Hamiltonian $H : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ associated to the above Bolza problem as follows

$$H_\lambda(t, x, p) := \sup_{u \in U(t)} \{\langle p, f(t, x, u)\rangle - \lambda L(t, x, u)\}, \quad H(t, x, p) := H_1(t, x, p).$$

An optimal trajectory/control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ of the above minimization problem satisfies, under some regularity assumptions on the data, the Pontryagin's maximum principle, which is a first order necessary condition for optimality. Let us first recall it in the situation when f, L and φ are smooth. It says that there exists a non trivial triple $(\lambda, p(\cdot), \psi(\cdot))$ where $\lambda \in \{0,1\}$, $p : [0,1] \to \mathbb{R}^n$ is an absolutely continuous function and $\psi : [0,1] \to \mathbb{R}^n$ is a map which belongs to the space of normalized functions with bounded variation on $[0,1]$, such that

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(i) $\psi(0) = 0$, $\psi(t) = \int_{[0,t]} \nu(s) d\mu(s)$, for all $t \in (0,1]$ for a positive finite Borel measure μ on $[0,1]$ and a Borel measurable selection $\nu(s) \in N_K(\bar{x}(s)) \cap B$ μ-almost everywhere. Here $N_K(\bar{x}(s))$ is the Clarke normal cone to K at $\bar{x}(s)$ and B is the closed unit ball in $\mathbb{R}^n$ centered at zero.

(ii) the function $p(\cdot)$ is a solution of the adjoint system

$$-p'(t) = \frac{\partial f}{\partial x}(t, \bar{x}(t), \bar{u}(t))^*(p(t) + \psi(t)) - \lambda \frac{\partial L}{\partial x}(t, \bar{x}(t), \bar{u}(t)) \quad \text{for a.e. } t \in [0,1]$$

satisfying almost everywhere the maximum principle

$$\langle p(t) + \psi(t), f(t, \bar{x}(t), \bar{u}(t)) \rangle - \lambda L(t, \bar{x}(t), \bar{u}(t)) = H_\lambda(t, \bar{x}(t), p(t) + \psi(t))$$

and the transversality condition

$$(p(0), -p(1) - \psi(1)) \in \lambda \nabla \varphi(\bar{x}(0), \bar{x}(1)) + N_{K_1}((\bar{x}(0), \bar{x}(1))).$$

A triple $(\lambda, p(\cdot), \psi(\cdot))$ is called *non-degenerate* if $\lambda + \sup_{t \in (0,1]} |p(t) + \psi(t)| \neq 0$. Moreover, when $\lambda = 1$ the above necessary conditions are called normal.

Let us underline that the transversality conditions were also derived with Clarke's normal cone replaced by smaller cones (see for instance [1], [24] and the references contained therein).

In the degenerate case the above necessary optimality condition provides no useful information about optimal controls because the maximum principle is then satisfied by every $u \in U(t)$. Considerable literature was devoted to conditions eliminating the occurrence of this phenomenon. In the abnormal case, the maximum principle does not depend on the cost functions L and φ and expresses some relations between the control system and state constraints. The normal maximum principle is a convenient tool to investigate the qualitative properties of optimal trajectories and, in particular, their Lipschitz regularity. It is also useful for deriving higher order necessary and sufficient optimality conditions. It is therefore of interest to study conditions that ensure non-degeneracy and normality of the maximum principles.

When f, L, φ are not differentiable, then, in the transversality condition and the adjoint system, the classical gradients and Jacobians are replaced by various generalized objects. Also the adjoint system of the maximum principle may take different, in general not equivalent, forms, as for instance the Hamiltonian one

$$-p'(t) \in \partial_x H_\lambda(t, \bar{x}(t), p(t) + \psi(t)),$$

where $\partial_x H_\lambda$ denotes the generalized gradient of H_λ with respect to x, or a similar inclusion involving the Pontryagin Hamiltonian, or, alternatively, an Euler-Lagrange adjoint system, see [24] for various forms of nonsmooth generalized maximum principles under state constraints. Furthermore, it may happen that, to a given optimal trajectory/control pair, correspond several, not comparable, maximum principles (cf. examples provided by Kaskosz and Lojasiewicz [18]).

In Proposition 2.6 below we show that, if in an abnormal maximum principle $p(\cdot)$ and $\psi(\cdot)$ are so that $|p'(t)| \leq k(t)|p(t) + \psi(t)|$ almost everywhere for some $k(\cdot) \in L^1$ (this is typically true under the assumption of local $k(t)-$Lipschitz continuity of $f(t, \cdot, u)$), then we can find an integrable matrix valued mapping $A(\cdot)$ such that

$$-p'(t) = A(t)^*(p(t) + \psi(t)).$$

Thus, if $p(\cdot)$ satisfies a Hamiltonian or an Euler-Lagrange inclusion, then it also satisfies an adjoint system in a more familiar form, where the matrix $A(t)$ may be unrelated to derivatives of $f(t, \cdot, \bar{u}(t))$ or $H_0(t, \cdot, p(t) + \psi(t))$, but can still be used to express sufficient conditions for normality. It could be interesting to study further properties of such matrices $A(t)$, but this is beyond the objectives of the present work.

Sufficient conditions for normality can be investigated by exploiting "inward pointing trajectories", that is solutions to a related linear control system under state constraints. For instance, for a fixed initial condition, this system has the following form

$$\begin{cases} w'(t) & \in & A(t)w(t) + \mathcal{T}(t) \text{ for a.e. } t \in [0,1] \\ w(t) & \in & \mathrm{Int}\,(C_K(\bar{x}(t))) \text{ for } t \in (0,1] \\ w(0) & = & 0, \end{cases}$$

where $C_K(y)$ denotes the Clarke tangent cone to K at $y \in K$ and $\mathcal{T}(t)$ is the closed convex cone spanned by $\overline{co}\,(f(t,\bar{x}(t),U(t))) - f(t,\bar{x}(t),\bar{u}(t))$.

Let us mention that in [11, Proposition 2.3] a related condition was stated as a generalization of the Slater "interiority" hypothesis to deduce normality. However no sufficient conditions were provided for its verification. This condition was further commented in [17] as the one difficult to check, since it involves a time-varying linear system in the presence of pathwise controls and state constraints. This question may be seen as a viability problem under time dependent state constraints. However the openness of $\mathrm{Int}\,(C_K(\bar{x}(t)))$ and the lack of upper semicontinuity of the set valued map $x \rightsquigarrow C_K(x)$ prevent us from using results of viability theory.

Existence of solutions to the above constrained differential inclusion has also further applications. For instance in [5] it was essential to study metric regularity properties of constrained control systems. In [7] it was used to prove the constrained maximum principle in a direct way. In Section 5 we apply it to approximate $\bar{x}(\cdot)$ by trajectories of (1.1), (1.2) that lie in the interior of the state constraint.

In this paper we propose a new inward pointing condition that uses tangents to the sets $f(t,x,U(t))$ at $f(t,x,u)$ for $u \in U(t)$ whenever $x \in K$ lies near the boundary of K and $f(t,x,u)$ "points toward the boundary of K" implying existence of a solution to the above differential inclusion and improving considerably the one of [14]. Even in the case when the boundary is smooth, it may happen that our condition holds true, but not the one from [14]. The adjoint system of the maximum principle does not play any role in the proofs of our results and for this reason they apply to its various formulations mentioned above.

In the difference with the previous works, where outward normals to the set of constraint were used to state inward pointing conditions, our new condition involves the reachable gradient of the oriented distance function from K. The construction of a solution to the above viability problem provided here, is an extension of the one from [14] to the case of a general closed state constraint.

Normality of the maximum principle is important for investigating Lipschitz regularity of optimal trajectories in order to avoid the so called Lavrientieff phenomenon. Indeed, in some cases, the infimum in (1.3) is strictly smaller than the infimum of the same functional taken over Lipschitz trajectories. Due to this phenomenon standard numerical methods cannot find minimizers and return a wrong optimal value. Moreover, in some physical models, this phenomenon corresponds to the occurrence of a fracture in an elastic material, thus to a meaningful physical event. The Existence of the inward pointing trajectories and normality of the maximum principle allow us to extend results on Lipschitz continuity of optimal trajectories of Frankowska and Marchini [15] and Cannarsa, Frankowska and Marchini [6] to general state constraints.

An optimal trajectory/control pair having a degenerate (resp. abnormal) maximum principle is called a *degenerate* (resp. *abnormal*) *minimizer*. It is called *strictly degenerate* (resp. *strictly abnormal*), if every maximum principle associated to it is degenerate (resp. abnormal). See [1] for an illustrating example.

There are two natural questions that can be asked at this point. The first one is : under what circumstances can we avoid the strict degeneracy (resp. strict abnormality) of the maximum principle? The second question is : when a non-degenerate maximum principle is normal? In

this work we address the second question for general, not necessarily Lipschitz, optimal trajectories. We do not derive here non-degenerate/normal maximum principles, but only investigate sufficient conditions for normality of maximum principles obtained elsewhere in a non-degenerate form.

Let us provide next a quick overview of some of the existing results in the vast literature on non-degenerate and normal maximum principles. We refer to Lopes and Fontes [19], [12] for more comprehensive descriptions of the state of art in this topic and to Arutyunov and Aseev [1, Section 6] for a detailed survey of the Russian literature on this subject.

In [11], Ferreira and Vinter studied a class of state constrained problems in which the initial state belongs to the boundary of the state constraint. This is one of the cases in which the classical maximum principle conveys no useful information, since a degenerate multiplier can be associated to every trajectory/control pair of our control system. They proposed two possible types of constraints qualifications in order to ensure the existence of non-degenerate multipliers in addition to the degenerate ones. The first one requires that, when the optimal trajectory lies on the boundary of the state constraint on a neighborhood of the initial time, then the classical inward pointing condition holds true at x_0 (i.e. the Slater type condition). The second type requires that there exists a control that pushes the trajectory away from the boundary faster than the optimal control, on a neighborhood of the initial time. The advantage of the second type of constraints qualifications is that in general it is valid for problems with less regularity of the data than the first type. Both conditions involve either optimal trajectory or optimal control.

Subsequently, several refinements of the above results were made in Ferreira, Fontes and Vinter [10]. In [20], Lopes, Fontes and de Pinho [20] restricted their attention to piecewise continuous controls and used a trajectories independent inward pointing condition to get non-degeneracy of the maximum principle. Finally, in [12], Fontes and Lopes provided a sufficient condition for normality imposing an inward pointing condition at points $\bar{x}(s)$ for all $s < \tau := \inf\{t \,|\, \mu([t,1]) = 0\}$ sufficiently close to τ. In order to avoid the use of constraints qualifications which depend on the optimal control, Rampazzo and Vinter proposed in [22] a different type of constraints qualifications, assuming existence of a continuous inward pointing feedback near the boundary of state constraints, to prove a normal maximum principle. Finally, in [21], the same authors investigated non-degeneracy under an inward pointing condition imposed on a neighborhood of the boundary of the state constraint and in the absence of end points constraints for f merely measurable with respect to time.

A different approach was pursued by Arutyunov and Aseev in [1] when $U = U(t)$ is time independent, $f(\cdot,\cdot,U)$ is Lipschitz with respect to t and x and such that $f(t,x,U)$ are convex. These authors have shown that the degeneracy phenomenon arises due to the incompleteness of the standard variants of the maximum principles for problems with state constraints which they supplemented by a jump condition on the Hamiltonian. Adding a new controllability condition on the Hamiltonian at the initial and end points of the optimal trajectory, they proved a non-degenerate maximum principle. There are no similar known results when f is merely measurable with respect to the time.

We would like to underline here that there is a qualitative difference between constrained control systems with dynamics depending on time in a Lipschitz and in a measurable way. For instance the so called neighboring feasible trajectories estimates that are valid in the Lipschitz case, do encounter counterexamples in the measurable case, see for instance [3].

The classical inward pointing condition was used in [7], [4], to prove normality in the case Lipschitz continuous optimal trajectories. In order to deal with absolutely continuous minimizers and to get sufficient conditions for the non occurrence of the Lavrentieff phenomenon, it became necessary to substitute the inward pointing condition by a new one.

It is worth to mention that some maximum principles under state constraints proved in [24] hold true for absolutely continuous optimal trajectories and for this reason, to have a complete picture, it is natural to pursue the investigation of normality to encompass also non-Lipschitz optimal trajectories.

The paper is organized as follows. Section 2 is devoted to some definitions and preliminaries. In Section 3 we discuss relations of normality of the maximum principle to controllability properties of an associated linear system under state constraints. Inward pointing conditions implying existence of "inward pointing trajectories" are stated in Section 4 with proofs postponed to Section 7. Sufficient conditions for normality are provided in Section 5. In Section 6 we apply our results to derive conditions that allow to avoid the Lavrientieff phenomenon.

2. Notations and Preliminaries

For a, $b \in \mathbb{R}$, let $a \wedge b := \min\{a, b\}$.

Given a set $K \subset \mathbb{R}^n$ we denote by $\overline{K}$ its closure, by ∂K its boundary, by $\mathrm{Int}(K)$ its interior, by $\mathrm{co}\,(K)$ its convex hull and by $\overline{\mathrm{co}}\,(K)$ its closed convex hull. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. The distance function $d_K : \mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ is defined by $d_K(x) := \inf_{y\in K} |x-y|$ for all $x \in \mathbb{R}^n$ with the convention $\inf_{y\in\emptyset} |x-y| = +\infty$.

For a metric space $\mathcal{M}$ and a family $\{K_\tau\}_{\tau\in\mathcal{M}}$ of subsets of $\mathbb{R}^n$, the *upper* and *lower limits* of $\{K_\tau\}_{\tau\in\mathcal{M}}$ when $\tau \to \bar\tau \in \mathcal{M}$ are defined respectively by

$$\operatorname*{Lim\,sup}_{\tau\to\bar\tau} K_\tau := \{x \in \mathbb{R}^n |\ \liminf_{\tau\to\bar\tau} d_{K_\tau}(x) = 0\},\ \ \operatorname*{Lim\,inf}_{\tau\to\bar\tau} K_\tau := \{x \in \mathbb{R}^n |\ \lim_{\tau\to\bar\tau} d_{K_\tau}(x) = 0\}.$$

Let K be a closed subset of $\mathbb{R}^n$ and $x \in K$. The *contingent cone*, resp. *Clarke's tangent cone* to K at x are defined by

$$T_K(x) := \operatorname*{Lim\,sup}_{h\to 0^+} \frac{K-x}{h},\ \ C_K(x) := \operatorname*{Lim\,inf}_{h\to 0^+,\, K\ni y\to x} \frac{K-y}{h}$$

and the *regular normal cone*, resp. *Clarke's normal cone* to K at x by

$$N_K^0(x) := \{p \in \mathbb{R}^n |\ \langle p, v\rangle \le 0\ \forall v \in T_K(x)\},\ \ N_K(x) := \{p \in \mathbb{R}^n |\ \langle p, v\rangle \le 0\ \forall v \in C_K(x)\}.$$

For all $x \in K$, $C_K(x)$ is a closed convex cone. It is well known that $x \in \mathrm{Int}(K)$ if and only if $C_K(x) = \mathbb{R}^n$ and that $v \in \mathrm{Int}(C_K(x))$ if and only if there exists $\varepsilon > 0$ such that $y + [0, \varepsilon]B(v, \varepsilon) \subset K$ for all $y \in K \cap B(x, \varepsilon)$. Set $N_K^1(x) := \{n \in N_K(x) |\ |n| = 1\}$.

When $Q \subset \mathbb{R}^n$ is convex, then for every $x \in Q$,

$$T_Q(x) = \overline{\cup_{\alpha\ge 0}\, \alpha(Q-x)} \tag{2.1}$$

and $v \in T_Q(x)$ if and only if $\langle n, v\rangle \le 0\ \forall n \in N_Q^0(x)$.

Consider a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and denote by $\nabla f(\cdot)$ its gradient, which is defined a.e. in $\mathbb{R}^n$. The *reachable gradient* of f at x is defined by

$$\partial^* f(x) := \operatorname*{Lim\,sup}_{y\to x}\{\nabla f(y)\}$$

and the *Clarke generalized gradient* of $f(\cdot)$ at x by

$$\partial f(x) := co\, \partial^* f(x). \tag{2.2}$$

Let $K \subset \mathbb{R}^n$ be a closed nonempty set different from $\mathbb{R}^n$. The *oriented distance* $d(\cdot) : \mathbb{R}^n \to \mathbb{R}$ from K is defined by

$$d(x) := d_K(x) - d_{\mathbb{R}^n\setminus K}(x)\ \ \forall x \in \mathbb{R}^n.$$

We set $d(\cdot) \equiv 0$ if $K = \mathbb{R}^n$.

The following proposition is well known.

Proposition 2.1. *Let $K \subset \mathbb{R}^n$ be a closed set and assume that* $\mathrm{Int}(C_K(x)) \neq \emptyset$ *for every $x \in \partial K$. Then the set valued maps $x \rightsquigarrow N_K^1(x)$ and $x \rightsquigarrow N_K(x) \cap B$ are upper semicontinuous on K, while the map $x \rightsquigarrow C_K(x)$ is lower semicontinuous on K.*

Proposition 2.2 ([16])**.** *Let $K \subset \mathbb{R}^n$ be a closed nonempty set different from $\mathbb{R}^n$ and assume that* $\mathrm{Int}(C_K(x)) \neq \emptyset$ *for every $x \in \partial K$. Let $z \in \mathbb{R}^n$ be such that $d(\cdot)$ is differentiable at z. Then there exists a unique projection y of z on ∂K and $\nabla d(z) \in N_K^1(y)$.*

In particular, for every $x \in \partial K$ we have $\partial^* d(x) \subset N_K^1(x)$.

For a set valued map $Q : X \rightsquigarrow Y$ the *domain* of $Q(\cdot)$ is $Dom(Q) = \{x \in X |\ Q(x) \neq \emptyset\}$.

By $[W^{1,1}([0,1])]^n$ we denote the space of absolutely continuous functions from $[0,1]$ into $\mathbb{R}^n$ and by $\mathcal{L}(\mathcal{A})$ the Lebesgue measure of a Lebesgue measurable set $\mathcal{A} \subset [0,1]$.

The space $[NBV([0,1])]^n$ of *normalized functions of bounded variation* on $[0,1]$ with values in $\mathbb{R}^n$ is defined as the space of functions with bounded total variation, vanishing at zero and right continuous on $(0,1)$. For any $\psi(\cdot)$ in $[NBV([0,1])]^n$ we denote by $\psi(t+)$ the right limit of $\psi(\cdot)$ at time $t \in [0,1)$ and by $\psi(t-)$ the left limit of $\psi(\cdot)$ at time $t \in (0,1]$. The total variation of $\psi(\cdot)$ on an interval $I \subset [0,1]$ is denoted by $Var(\psi(\cdot), I)$. The Stiltjes integral of a function $w(\cdot) \in [C([0,1])]^n$ with respect to $\psi(\cdot) \in [NBV([0,1])]^n$ is denoted by $\int_0^1 w(s)d\psi(s)$.

Let $M(n \times n)$ be the space of $n \times n$ matrices and for every $A \in M(n \times n)$ let $a_{i,j}$ be its elements for $i, j = 1, ..., n$. The mapping $A : [0,1] \to M(n \times n)$ is called integrable, if for all i, j, the function $a_{i,j} : [0,1] \to \mathbb{R}$ is integrable. Denote by $A(t)^*$ the transpose of $A(t)$ for $t \in [0,1]$.

Definition 2.3. A viable trajectory/control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ of (1.1)-(1.2) is called *extremal* for a triple $(\lambda, p(\cdot), \psi(\cdot))$ if $\lambda \in \{0,1\}$, $\psi(\cdot) \in [NBV([0,1])]^n$ and $p(\cdot) \in [W^{1,1}([0,1])]^n$ are such that $(\lambda, p(\cdot), \psi(\cdot)) \neq 0$ and for some integrable mappings $A : [0,1] \to M(n \times n)$, $\pi : [0,1] \to \mathbb{R}^n$, and some vectors $\pi_0, \pi_1 \in \mathbb{R}^n$ the following relations hold true:

$$-p'(t) = A(t)^*(p(t) + \psi(t)) - \lambda\pi(t) \quad \text{for a.e. } t \in [0,1], \tag{2.3}$$

$$(p(0), -p(1) - \psi(1)) \in \lambda(\pi_0, \pi_1) + N_{K_1}((\bar{x}(0), \bar{x}(1))), \tag{2.4}$$

$$\langle p(t) + \psi(t), \bar{x}'(t)\rangle - \lambda L(t, \bar{x}(t), \bar{u}(t)) = H_\lambda(t, \bar{x}(t), p(t) + \psi(t)) \quad \text{for a.e. } t \in [0,1], \tag{2.5}$$

$$\psi(0) = 0, \quad \psi(t) = \int_{[0,t]} \nu(s)d\mu(s) \ \forall t \in (0,1] \tag{2.6}$$

for a positive finite Borel measure μ on $[0,1]$ and a Borel measurable selection $\nu(t) \in N_K(\bar{x}(t)) \cap B$ for μ-a.e. $t \in [0,1]$.

A triple $(\lambda, p(\cdot), \psi(\cdot))$ is called normal if $\lambda = 1$ and *non-degenerate* if

$$\lambda + \sup_{t \in (0,1]} |p(t) + \psi(t)| \neq 0. \tag{2.7}$$

Recall that μ being a finite Borel measure on $[0,1]$, is regular.

Remark 2.4. *i*) Proposition 2.6 below implies that different forms of the adjoint inclusion in the maximum principle may be written as in Definition 2.3.

ii) Since (2.3) and (2.5) hold a.e. in $[0,1]$, they are also satisfied if we replace $\psi(\cdot)$ by $\hat{\psi}(\cdot)$ defined in the following way: $\hat{\psi}(t) := \psi(t-)$ for all $t \in (0,1)$, $\hat{\psi}(0) := 0, \hat{\psi}(1) := \psi(1)$. Notice that $\hat{\psi}(\cdot)$ has bounded total variation and $\hat{\psi}(t) = \psi(t)$ for a.e. $t \in [0,1]$, $\hat{\psi}(\cdot)$ is left continuous on $(0,1)$,

$$\hat{\psi}(t) = \int_{[0,t)} \nu(s)d\mu(s) \ \forall t \in (0,1), \quad \hat{\psi}(1) = \int_{[0,1]} \nu(s)d\mu(s).$$

Therefore, if a trajectory/control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ is extremal for a triple $(\lambda, p(\cdot), \psi(\cdot))$, then $(\lambda, p(\cdot), \hat{\psi}(\cdot))$ satisfies (2.3)-(2.5).

The following result is an immediate consequence of Proposition 2.1 and (2.6).

Proposition 2.5. *Let $K \subset \mathbb{R}^n$ be a closed set and assume that $\mathrm{Int}(C_K(x)) \neq \emptyset$ for every $x \in \partial K$. Then for any $(\bar{x}(\cdot), \bar{u}(\cdot))$ which is extremal for a triple $(\lambda, p(\cdot), \psi(\cdot))$, it holds*

$$\psi(0+) \in N_K(\bar{x}(0)) \text{ and } \psi(t) - \psi(t-) \in N_K(\bar{x}(t)) \ \forall t \in (0,1]. \tag{2.8}$$

Moreover, if $\lambda = 0$, then

$$p(t) + \psi(t) \in N_{\overline{co}(f(t,\bar{x}(t),U(t))}(\bar{x}'(t))) \quad \text{for a.e.} \quad t \in (0,1). \tag{2.9}$$

The proposition below may be used to associate to various forms of the adjoint system in the maximum principle an integrable mapping $A : [0,1] \to M(n \times n)$, as in Definition 2.3.

Proposition 2.6. *Let $q(\cdot) \in [L^1([0,1])]^n$, $g(\cdot) \in [L^\infty([0,1])]^n$ and $k : [0,1] \to \mathbb{R}_+$ be integrable and satisfying*

$$|q(t)| \leq k(t)|g(t)| \quad \text{for a.e. } t \in (0,1). \tag{2.10}$$

Then there exists an integrable $A : [0,1] \to M(n \times n)$ such that $q(t) = A(t)^* g(t)$ a.e. in $[0,1]$.

Proof. Denote by $a^*_{i,j}(t)$ the elements of the matrix $A(t)^*$ to be defined. For $t \in [0,1]$ such that $q(t) = (0, \ldots, 0)$ let

$$a^*_{i,j}(t) := 0 \quad \text{for all } i, j \in \{1, \ldots, n\}.$$

For $t \in [0,1]$ such that $q(t) \neq (0, \ldots, 0)$ and (2.10) holds true, let

$$j_0(t) = j_0 := \max\{j \in \{1, \ldots, n\} |\ |g_j(t)| = \max_{i \in \{1,\ldots,n\}} |g_i(t)|\}$$

and define for all i, j

$$a^*_{i,j}(t) := \begin{cases} 0 & j \neq j_0 \\ \frac{q_i(t)}{g_{j_0}(t)} & j = j_0. \end{cases}$$

Define $A(t) := (A(t)^*)^*$. From the measurability of $q(\cdot)$ and $g_{j_0}(\cdot)$ it follows that every element of $A(\cdot)^*$ is measurable. Moreover for any $i, j \in \{1, \ldots, n\}$ we have

$$|a^*_{i,j}(t)| \leq \left| \frac{q_i(t)}{g_{j_0}(t)} \right| \leq n \frac{|q(t)|}{|g(t)|} \leq nk(t).$$

Hence $a^*_{i,j}(\cdot)$ is integrable for any $i, j \in \{1, \ldots, n\}$ and $A(\cdot)$ is integrable.

□

In order to simplify the notation, for a fixed trajectory/control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ and for all $t \in [0,1]$ we set

$$\mathcal{T}(t) := \begin{cases} T_{\overline{co}(f(t,\bar{x}(t),U(t)))}(\bar{x}'(t)) & \text{if } \bar{x}'(t) \in f(t, \bar{x}(t), U(t)) \\ \{0\} & \text{otherwise.} \end{cases}$$

We will often refer to the following linear differential inclusion

$$w'(t) \in A(t)w(t) + \mathcal{T}(t) \text{ for a.e. } t \in [0,1],$$

for $A(\cdot)$ as in Definition 2.3.

A solution of such a differential inclusion is an absolutely continuous function $w(\cdot)$. Then

$$w'(t) = A(t)w(t) + v(t) \text{ for a.e. } t \in [0,1]$$

for an integrable selection $v(t) \in \mathcal{T}(t)$. Sometimes we will write this differential inclusion as a linear control system

$$w'(t) = A(t)w(t) + v(t), \quad v(t) \in \mathcal{T}(t) \text{ for a.e. } t \in [0,1]$$

requiring implicitly that $v(\cdot)$ must be integrable.

3. Normality and Viable Solutions of a Linear Control System

Our first result links the abnormal maximum principle to solutions of a linear system with linearized state constraints.

Lemma 3.1. *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be extremal for an abnormal triple $(0, p(\cdot), \psi(\cdot))$ and let $A(\cdot)$ be as in Definition 2.3. Then for every solution $w(\cdot)$ of the viability problem*

$$\left\{\begin{array}{rcl} w'(t) & = & A(t)w(t) + v(t), \quad v(t) \in \mathcal{T}(t) \text{ for a.e. } t \in [0,1] \\ w(t) & \in & C_K(\bar{x}(t)) \quad \forall t \in [0,1] \\ (w(0), w(1)) & \in & C_{K_1}((\bar{x}(0), \bar{x}(1))), \end{array}\right. \tag{3.1}$$

we have

$$\int_0^1 \langle p(s) + \psi(s), v(s)\rangle ds = 0, \tag{3.2}$$

$$\int_0^1 w(s) d\psi(s) = 0, \tag{3.3}$$

and

$$-\langle p(1) + \psi(1), w(1)\rangle + \langle p(0), w(0)\rangle = 0. \tag{3.4}$$

The above lemma implies that, when investigating normality, it is enough to provide sufficient conditions for one of the three terms to be strictly smaller than zero in order to reach a contradiction with the fact that $\lambda = 0$.

Proof. Since $\lambda = 0$, the trajectory/control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ and the triple $(0, p(\cdot), \psi(\cdot))$ satisfy the following relations

$$-p'(t) = A(t)^*(p(t) + \psi(t)) \quad \text{for a.e. } t \in [0,1], \tag{3.5}$$

$$(p(0), -p(1) - \psi(1)) \in N_{K_1}((\bar{x}(0), \bar{x}(1))), \tag{3.6}$$

$$\langle p(t) + \psi(t), \bar{x}'(t)\rangle = \max_{u \in U(t)} \langle p(t) + \psi(t), f(t, \bar{x}(t), u)\rangle \quad \text{for a.e. } t \in [0,1], \tag{3.7}$$

$$\psi(0) = 0, \quad \psi(t) = \int_{[0,t]} \nu(s) d\mu(s) \ \forall t \in (0,1], \ \nu(t) \in N_K(\bar{x}(t)) \cap B \quad \text{for } \mu\text{-a.e. } t \in [0,1].$$

Let $w(\cdot)$ be a solution of

$$w'(t) = A(t)w(t) + v(t), \tag{3.8}$$

for an integrable selection $v(t) \in \mathcal{T}(t)$ for a.e. $t \in [0,1]$.

Adding the two equalities below

$$\langle p(1), w(1)\rangle - \langle p(0), w(0)\rangle = \int_0^1 (p'w + pw')(s) ds$$

$$\langle \psi(1), w(1)\rangle = \int_0^1 w(s) d\psi(s) + \int_0^1 w'(s)\psi(s) ds$$

we obtain

$$\langle p(1) + \psi(1), w(1)\rangle - \langle p(0), w(0)\rangle = \int_0^1 (p'w + pw' + w'\psi)(s) ds + \int_0^1 w(s) d\psi(s). \tag{3.9}$$

From (3.8) and (3.5) we deduce that $\int_0^1 (p'w + pw' + w'\psi)(s) ds = \int_0^1 \langle p(s) + \psi(s), v(s)\rangle ds$. Hence (3.9) yields

$$\int_0^1 \langle p(s) + \psi(s), v(s)\rangle ds + \int_0^1 w(s) d\psi(s) - \langle p(1) + \psi(1), w(1)\rangle + \langle p(0), w(0)\rangle = 0. \tag{3.10}$$

Applying (2.1) to $Q = \overline{co}\,(f(t, \bar{x}(t), U(t)))$ for almost all $t \in [0,1]$ we have

$$\mathcal{T}(t) = \overline{\cup_{\alpha \geq 0}\, \alpha(\overline{co}(f(t, \bar{x}(t), U(t)) - \bar{x}'(t)))}.$$

Since $v(t) \in \mathcal{T}(t)$ for a.e. $t \in [0,1]$, equality (3.7) implies the inequality $\leq$ in (3.2) instead of equality. Moreover, since $\int_0^1 w(t)d\psi(t) = \int_{[0,1]} \langle w(t), \nu(t)\rangle d\mu(t)$, $w(t) \in C_K(\bar{x}(t))$ for all $t \in (0,1]$ and $\nu(t) \in N_K(\bar{x}(t))$ for μ-a.e. t in $[0,1]$, we deduce (3.3) again with the inequality $\leq$. Finally relations $(w(0), w(1)) \in C_{K_1}((\bar{x}(0), \bar{x}(1)))$ and (3.6) imply (3.4) again with the inequality $\leq$. Thus (3.10) is verified if and only if (3.2), (3.3) and (3.4) hold true.

□

Proposition 3.2. *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be extremal for an abnormal triple $(0, p(\cdot), \psi(\cdot))$ and let $A(\cdot)$ be as in Definition 2.3. If there exists a solution $w(\cdot)$ to the differential inclusion (3.1) satisfying*

$$w(t) \in \mathrm{Int}(C_K(\bar{x}(t))) \quad \forall t \in I, \tag{3.11}$$

where I is a subinterval of $(0,1]$ of the form (a,b) or $(a,b]$, $0 \leq a < b \leq 1$, then $\psi(t) = \psi(a+)$ for all $t \in I$.

Proof. Let $w(\cdot)$ be a solution of (3.1), (3.11). Then $w(t) \neq 0$ for all $t \in I$, such that $\bar{x}(t) \in \partial K$. By Lemma 3.1 we know that (3.3) holds true. Therefore $\mu(\{s \in I |\ \nu(s) \neq 0\}) = 0$. Indeed assuming that $\nu(s) \neq 0$ on a set of strictly positive measure in I, we get $\int_I \langle w(s), \nu(s)\rangle d\mu(s) < 0$ since $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for all $t \in I$. This contradicts (3.3). Consequently, for all $t \in I$, $\psi(t) = \psi(a+) + \int_{(a,t]} \nu(s)d\mu(s) = \psi(a+)$.

□

Remark 3.3. The above proposition implies the following sufficient condition for normality of the maximum principle. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be extremal for a triple $(\lambda, p(\cdot), \psi(\cdot))$ and $A(\cdot)$ be as in Definition 2.3. If

$$\lambda + Var(\psi(\cdot), (0,1)) \neq 0, \tag{3.12}$$

and there exists a solution $w(\cdot)$ to (3.1) satisfying $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for all $t \in (0,1)$, then $\lambda = 1$.

In the proposition and lemma below we provide other sufficient conditions for normality.

Proposition 3.4. *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be extremal for a triple $(\lambda, p(\cdot), \psi(\cdot))$ and let $A(\cdot)$ be as in Definition 2.3. Then $\lambda = 1$ whenever there exists a solution $\bar{w}(\cdot)$ of*

$$\begin{cases} w'(t) & \in & A(t)w(t) + \mathcal{T}(t) \quad \text{a.e. in } [0,1] \\ w(t) & \in & \mathrm{Int}(C_K(\bar{x}(t))) \quad \forall t \in (0,1] \\ w(0) & \in & C_K(\bar{x}(0)) \end{cases} \tag{3.13}$$

satisfying one of the following relations:

i) $\mathrm{Int}(C_K(x)) \neq \emptyset$ *for all* $x \in \partial K$, $\bar{w}(0) \in \mathrm{Int}(C_K(\bar{x}(0)))$ *and for some* $\varepsilon > 0$,

$$(\bar{w}(0), \bar{w}(1) + \varepsilon B) \subset C_{K_1}((\bar{x}(0), \bar{x}(1))).$$

ii) $(\lambda, p(\cdot), \psi(\cdot))$ *is non-degenerate and for some* $\varepsilon > 0$,

$$(\bar{w}(0), \bar{w}(1) + \varepsilon B) \subset C_{K_1}((\bar{x}(0), \bar{x}(1))). \tag{3.14}$$

iii) $\mathrm{Int}(C_K(x)) \neq \emptyset$ *for all* $x \in \partial K$, $(\bar{w}(0), \bar{w}(1)) \in C_{K_1}((\bar{x}(0), \bar{x}(1)))$, $(\lambda, p(\cdot), \psi(\cdot))$ *is non-degenerate and* $q(\cdot) \equiv 0$ *is the only solution of the adjoint system* $-q'(t) = A(t)^* q(t)$ *a.e. in* $[0,1]$, *satisfying*

$$(q(0), -q(1)) \in N_{K_1}((\bar{x}(0), \bar{x}(1))) + N_K(\bar{x}(0)) \times \{0\}.$$

Proof. By contradiction suppose that $\lambda = 0$. Then from Lemma 3.1 and Proposition 3.2 we know that (3.2) - (3.4) hold true and $\psi(t) = \psi(0+)$ for all $t \in (0,1]$. This implies that

$$0 = \int_0^1 \bar{w}(s)d\psi(s) = \langle \psi(0+), \bar{w}(0)\rangle.$$

Call $q(t) := p(t) + \psi(0+)$ for all $t \in (0,1]$. Then $-q'(t) = A(t)^* q(t)$ a.e. in $(0,1]$.

(i) Since $\bar{w}(0) \in \mathrm{Int}(C_K(\bar{x}(0)))$, by (2.8) we have $\psi(0+) = 0$. Thus $\psi(\cdot) \equiv 0$ on $[0,1]$. Thanks to (3.4), $i)$ and (3.6) we get $p(1) = 0$. Moreover the absolutely continuous function $p(\cdot)$ satisfies $-p'(t) = A(t)^* p(t)$ a.e. in $[0,1]$. Thus $p(\cdot) \equiv 0$ on $[0,1]$ and $(\lambda, p(\cdot), \psi(\cdot)) \equiv 0$, leading to a contradiction.

(ii) By (3.4), (3.6) and (3.14), $q(1) = p(1) + \psi(1) = 0$. Consequently $q(\cdot) \equiv 0$. This contradicts the non-degeneracy assumption.

(iii) (3.6) and (2.8) imply $(p(0), -p(1) - \psi(0+)) \in N_{K_1}((\bar{x}(0), \bar{x}(1)))$ and $\psi(0+) \in N_K(\bar{x}(0))$. Hence

$$(q(0), -q(1)) = (p(0) + \psi(0+), -p(1) - \psi(0+)) \in N_{K_1}((\bar{x}(0), \bar{x}(1))) + N_K(\bar{x}(0)) \times \{0\}.$$

Thus $q(\cdot) \equiv 0$, in contradiction with the non-degeneracy hypothesis.

□

Lemma 3.5. *Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be extremal for a triple $(\lambda, p(\cdot), \psi(\cdot))$ and let $A(\cdot)$ be as in Definition 2.3. Suppose that*

$$\operatorname*{ess\,inf}_{t\in(0,1)} \inf_{u\in U(t)} \langle p(t) + \psi(0+), f(t, \bar{x}(t), u) - f(t, \bar{x}(t), \bar{u}(t))\rangle < 0. \tag{3.15}$$

Then $\lambda = 1$ whenever there exists a solution $\bar{w}(\cdot)$ of (3.13) and one of the following two conditions holds true :

i) *for some $\varepsilon > 0$, $(\bar{w}(0), \bar{w}(1) + \varepsilon B) \subset C_{K_1}((\bar{x}(0), \bar{x}(1)))$,*
ii) *$\mathrm{Int}(C_K(x)) \neq \emptyset$ for all $x \in \partial K$ and every solution $w(\cdot)$ to (3.13) with $w(0) = \bar{w}(0)$ satisfies $(w(0), w(1)) \in C_{K_1}((\bar{x}(0), \bar{x}(1)))$.*

Proof. By contradiction suppose $\lambda = 0$. Then as in the proof of the previous proposition $\psi(t) = \psi(0+)$ for all $t \in (0,1]$. Moreover, call $q(t) := p(t) + \psi(0+)$ for all $t \in (0,1]$. Then $-q'(t) = A(t)^* q(t)$ a.e. in $[0,1]$.

(i) As in the proof of Proposition 3.4-$ii)$ we have $q(\cdot) \equiv 0$. This contradicts (3.15).

(ii) For $t \in [0,1]$ such that $\bar{x}'(t) = f(t, \bar{x}(t), \bar{u}(t))$, define

$$U_t := \{u \in U(t) |\ \langle p(t) + \psi(0+), f(t, \bar{x}(t), u)\rangle = \langle p(t) + \psi(0+), \bar{x}'(t)\rangle\}.$$

Then the Lebesgue measure of the set

$$\mathcal{J} := \{t \in [0,1] |\ \bar{x}'(t) = f(t, \bar{x}(t), \bar{u}(t)), f(t, \bar{x}(t), U_t) \neq f(t, \bar{x}(t), U(t))\}, \tag{3.16}$$

is strictly positive, thanks to (3.15). Moreover for all $t \in \mathcal{J}$, $p(t) + \psi(0+) \neq 0$.

From the definition of $\mathcal{J}$ and the measurable selection theorem there exists a measurable selection $u(t) \in U(t)$ defined on $\mathcal{J}$ such that

$$\langle p(t) + \psi(0+), f(t, \bar{x}(t), u(t))\rangle < \langle p(t) + \psi(0+), \bar{x}'(t)\rangle.$$

Hence for all $t \in \mathcal{J}$, $\zeta(t) := f(t, \bar{x}(t), u(t)) - \bar{x}'(t)$ satisfies $\zeta(t) \in \mathcal{T}(t)$ and $\langle p(t) + \psi(0+), \zeta(t)\rangle < 0$.

We claim that there exist $\varepsilon > 0$ and $\delta \in (0, \frac{1}{2})$, such that

$$\bar{w}(t) + \varepsilon B \subset \mathrm{Int}(C_K(\bar{x}(t))) \quad \text{for all } t \in [\delta, 1]$$

and $\mathcal{L}([\delta,1]\cap\mathcal{J})>0$. Indeed, since $0<\mathcal{L}(\mathcal{J})\leq\sum_{k>2}\mathcal{L}\left([k^{-1},1]\cap\mathcal{J}\right)$, we can find $\delta\in(0,\frac{1}{2})$ such that $\mathcal{L}([\delta,1]\cap\mathcal{J})>0$.

Consider now the set $\mathcal{K}:=\{t\in[\delta,1]|\ \bar{x}(t)\in\partial K\}$. If this set is empty, then $\mathrm{Int}(C_K(\bar{x}(t)))=\mathbb{R}^n$ for all $t\in[\delta,1]$ and the claim follows for every $\varepsilon>0$. If $\mathcal{K}\neq\emptyset$, then $\mathcal{K}$ is compact and for all $t\in\mathcal{K}$ we have $\bar{w}(t)\in\mathrm{Int}(C_K(\bar{x}(t)))$. By contradiction suppose there exists a sequence $t_i\in\mathcal{K}$, $i\geq1$ such that $\bar{w}(t_i)+\frac{1}{i}B$ is not contained in $\mathrm{Int}(C_K(\bar{x}(t_i)))$ for all $i\geq1$. Then

$$\max_{n\in N_K^1(\bar{x}(t_i))}\langle n,\bar{w}(t_i)\rangle\geq-\frac{1}{i},\tag{3.17}$$

for all $i\geq1$. Taking a subsequence and using the same notations, we can find $t_0\in\mathcal{K}$ such that $t_i\to t_0$ as $i\to+\infty$. Therefore passing to the limit in (3.17), thanks to the upper semicontinuity of $x\rightsquigarrow N_K^1(x)$, we have $\max_{n\in N_K^1(\bar{x}(t_0))}\langle n,\bar{w}(t_0)\rangle\geq0$, in contradiction with $t_0\in\mathcal{K}$. Hence, there exist $\varepsilon>0$ such that $\bar{w}(t)+\varepsilon B\subset\mathrm{Int}(C_K(\bar{x}(t)))$ for all $t\in\mathcal{K}$. Moreover, since for all $t\in[\delta,1]\setminus\mathcal{K}$, we have $\mathrm{Int}(C_K(\bar{x}(t)))=\mathbb{R}^n$, then also for such t's $\bar{w}(t)+\varepsilon B\subset\mathrm{Int}(C_K(\bar{x}(t)))$.

Consider a subset $\tilde{\mathcal{J}}\subset[\delta,1]\cap\mathcal{J}$ of strictly positive measure such that $\zeta(\cdot)$ is bounded on $\tilde{\mathcal{J}}$ and set $v(t):=\bar{w}'(t)-A(t)\bar{w}(t)$. For all $\alpha\in[0,1]$ define

$$v_\alpha(t):=\begin{cases}\alpha v(t)+(1-\alpha)\zeta(t) & t\in\tilde{\mathcal{J}}\\ v(t) & t\in[0,1]\setminus\tilde{\mathcal{J}}.\end{cases}$$

Then $v_\alpha(t)\in\mathcal{T}(t)$ for a.e. $t\in[0,1]$, $v_\alpha(\cdot)$ converges to $v(\cdot)$ uniformly on $[0,1]$ when $\alpha\to1-$ and for all $t\in\tilde{\mathcal{J}}$ and $\alpha\in[0,1)$, $\langle p(t)+\psi(0+),v_\alpha(t)\rangle<0$. Call $w_\alpha(\cdot)$ the solution of

$$\begin{cases}w_\alpha'(t) &= A(t)w_\alpha(t)+v_\alpha(t) \quad \text{a.e. in } [0,1]\\ w_\alpha(0) &= \bar{w}(0).\end{cases}$$

By the Gronwall's Lemma, $w_\alpha(\cdot)$ converges uniformly to $\bar{w}(\cdot)$ on $[0,1]$ when $\alpha\to1-$. Since $w_\alpha(\cdot)\equiv\bar{w}(\cdot)$ on $[0,\delta]$, $w_\alpha(t)\in\mathrm{Int}(C_K(\bar{x}(t)))$ for all $t\in(0,\delta]$ and $w_\alpha(0)\in C_K(\bar{x}(0))$.

Let $\eta>0$ be such that $|w_\alpha(t)-\bar{w}(t)|\leq\frac{\varepsilon}{2}$ for all $t\in[\delta,1]$ and $1-\eta\leq\alpha<1$. Then $w_\alpha(t)\in\mathrm{Int}(C_K(\bar{x}(t)))$ for all $t\in[\delta,1]$. Moreover, by *ii*), for every α, $(w_\alpha(0),w_\alpha(1))\in C_{K_1}((\bar{x}(0),\bar{x}(1)))$ since $w_\alpha(\cdot)$ is a solution of (3.13) with $w_\alpha(0)=\bar{w}(0)$.

Lemma 3.1 and Proposition 3.2 applied to $w_\alpha(\cdot)$ imply (3.2) with $v(\cdot)$ replaced by $v_\alpha(\cdot)$ and that $\psi(\cdot)\equiv\psi(0+)$ on $(0,1]$. From (2.9), the definition of $\tilde{\mathcal{J}}$ and $v_\alpha(\cdot)$

$$\int_0^1\langle p(s)+\psi(0+),v_\alpha(s)\rangle ds\leq\int_{\tilde{\mathcal{J}}}\langle p(s)+\psi(0+),v_\alpha(s)\rangle ds<0,$$

in contradiction with (3.2).

□

Remark 3.6. From Remark 2.4 one can easily deduce the following result which is similar to *ii*) of Proposition 3.4.

Let $(\bar{x}(\cdot),\bar{u}(\cdot))$ be extremal for the triple $(\lambda,p(\cdot),\psi(\cdot))$ and let $A(\cdot)$ be as in Definition 2.3. Let $w(\cdot)$ be a solution of (3.13), with $w(t)\in\mathrm{Int}(C_K(\bar{x}(t)))$ for all $t\in[0,1)$ (instead of $(0,1]$).

Suppose that for some $\varepsilon>0$, $(w(0)+\varepsilon B,w(1))\subset C_{K_1}(\bar{x}(0),\bar{x}(1))$. Then we have $\lambda=1$ whenever $(\lambda,p(\cdot),\psi(\cdot))$ is non-degenerate. Similarly, *i*) and *iii*) of Proposition 3.4 and Lemma 3.4 can be formulated in a symmetric way.

4. Existence of Inward Pointing Trajectories

Let $(\bar{x}(\cdot),\bar{u}(\cdot))$ be a viable trajectory/control pair and $A:[0,1]\to M(n\times n)$ be an integrable $(n\times n)$-matrix valued map. Denote by G^+ and G^- the sets

$$G^+:=\{t\in[0,1]|\ \max_{p\in\partial^*d(\bar{x}(t))}\langle p,f(t,\bar{x}(t),\bar{u}(t))\rangle\geq0\},$$

$$G^- := \{t \in [0,1] | \min_{p \in \partial^* d(\bar{x}(t))} \langle p, f(t, \bar{x}(t), \bar{u}(t))\rangle \leq 0\}.$$

We say that $(\bar{x}(\cdot), \bar{u}(\cdot))$ satisfies the *inward pointing condition* on $I \subset [0,1]$ if

$$\left\{\begin{array}{l} \exists M > 0, \rho > 0 \text{ and a set } \Gamma \subset [0,1] \text{ of zero Lebesgue measure} \\ \text{such that } \forall t \in I \text{ with } \bar{x}(t) \in \partial K,\ \exists\, \delta_t > 0 \text{ satisfying} \\ \min_{v \in \mathcal{T}(s) \cap MB} \max_{p \in \partial^* d(\bar{x}(t))} \langle p, v\rangle \leq -\rho,\ \ \forall\, s \in (t-\delta_t, t+\delta_t) \cap (G^+ \backslash \Gamma). \end{array}\right. \tag{4.1}$$

We say that $(\bar{x}(\cdot), \bar{u}(\cdot))$ satisfies the *outward pointing condition* on I if

$$\left\{\begin{array}{l} \exists M > 0, \rho > 0 \text{ and a set } \Gamma \subset [0,1] \text{ of zero Lebesgue measure} \\ \text{such that } \forall t \in I \text{ with } \bar{x}(t) \in \partial K,\ \exists\, \delta_t > 0 \text{ satisfying} \\ \max_{v \in \mathcal{T}(s) \cap MB} \min_{p \in \partial^* d(\bar{x}(t))} \langle p, v\rangle \geq \rho,\ \ \forall\, s \in (t-\delta_t, t+\delta_t) \cap (G^- \backslash \Gamma). \end{array}\right. \tag{4.2}$$

Define the set of the reachable points at time $\bar{t} \in [0,1]$ from $w_0 \in \mathbb{R}^n$ (respectively $w_1 \in \mathbb{R}^n$) by trajectories of the constrained differential inclusion

$$\left\{\begin{array}{lcll} w'(t) & \in & A(t)w(t) + \mathcal{T}(t) & \text{a.e.} \\ w(t) & \in & \mathrm{Int}(C_K(\bar{x}(t))) & \forall\, t \end{array}\right. \tag{4.3}$$

by

$$\mathcal{R}_0(\bar{t}, w_0) := \{w(\bar{t}) |\ w(\cdot) \text{ is a solution of (4.3) on } [0, \bar{t}],\ w(0) = w_0\}$$

(resp.

$$\mathcal{R}_1(\bar{t}, w_1) := \{w(\bar{t}) |\ w(\cdot) \text{ is a solution of (4.3) on } [\bar{t}, 1],\ w(1) = w_1\})$$

and set

$$\tilde{\mathcal{R}}_0(\bar{t}) := \{w(\bar{t}) |\ w(\cdot) \text{ is a solution of (4.3) on } (0, \bar{t}],\ w(0) = 0\},$$
$$\tilde{\mathcal{R}}_1(\bar{t}) := \{w(\bar{t}) |\ w(\cdot) \text{ is a solution of (4.3) on } [\bar{t}, 1),\ w(1) = 0\}.$$

We state next two theorems on existence of inward pointing trajectories whose proofs are postponed to Section 7.

Theorem 4.1. *Assume that* $\mathrm{Int}(C_K(x)) \neq \emptyset$ *for every* $x \in \partial K$. *Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be a viable trajectory/control pair for which (4.1) holds on* $[0, \bar{t}]$ *for some* $0 < \bar{t} \leq 1$ *(resp. (4.2) holds on* $[\bar{t}, 1]$ *for some* $0 \leq \bar{t} < 1$*). Assume that*

$$\mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0)) \neq \emptyset,$$
$$(\textit{resp. } \mathrm{Int}(C_K(\bar{x}(1))) \cap C_{Q_1}(\bar{x}(1)) \neq \emptyset.)$$

Then, for any integrable $(n \times n)$*-matrix valued map* $A : [0,1] \to M(n \times n)$ *and any* $w_0 \in \mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0))$*, we have* $\mathcal{R}_0(\bar{t}, w_0) \neq \emptyset$
(resp. $\mathcal{R}_1(\bar{t}, w_1) \neq \emptyset$ *for any* $w_1 \in \mathrm{Int}(C_K(\bar{x}(1))) \cap C_{Q_1}(\bar{x}(1))$*).*

Theorem 4.2. *Assume that* $\mathrm{Int}(C_K(x)) \neq \emptyset$ *for every* $x \in \partial K$. *Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be a viable trajectory/control pair for which (4.1) holds on* $[0, \bar{t}]$ *for some* $0 < \bar{t} \leq 1$ *(resp. (4.2) holds on* $[\bar{t}, 1]$ *for some* $0 \leq \bar{t} < 1$*). Then, for any integrable* $(n \times n)$*-matrix valued map* $A : [0,1] \to M(n \times n)$*,* $\tilde{\mathcal{R}}_0(\bar{t}) \neq \emptyset$ *(resp.* $\tilde{\mathcal{R}}_1(\bar{t}) \neq \emptyset$*).*

Remark 4.3. Theorems 4.1 and 4.2 augmented by Remark 3.3, Proposition 3.4 and Lemma 3.5 can be used to deduce normality of the maximum principle.

Naturally, the inward pointing condition can be stated to be independent from $(\bar{x}, \bar{u})$: For all $t \in [0,1]$, $x \in K$ denote by $G^+(t,x)$ and $G^-(t,x)$ the sets

$$G^+(t,x) := \{f(t,x,u) |\ u \in U(t),\ \max_{p \in \partial^* d(x)} \langle p, f(t,x,u)\rangle \geq 0\},$$

$$G^-(t,x) := \{f(t,x,u) |\ u \in U(t),\ \min_{p \in \partial^* d(x)} \langle p, f(t,x,u)\rangle \leq 0\}.$$

Then the following inward pointing condition implies (4.1) :

$$
(4.4)\qquad \begin{cases} \forall R>0, \exists M_R>0, \rho_R>0 \text{ and a set } \Gamma_R\subset[0,1] \text{ of zero Lebesgue measure} \\ \forall (t,x)\in[0,1]\ \times(\partial K\cap RB),\ \exists\,\delta>0 \text{ such that} \\ \forall\ (s,y)\in([0,1]\backslash\Gamma_R)\times K \text{ with } |(s,y)-(t,x)|<\delta,\ \forall\, f(s,y,u)\in G^+(s,y) \\ \exists\ v\in T_{\overline{co}(f(s,y,U(s)))}(f(s,y,u))\cap M_RB \text{ satisfying } \max_{p\in\partial^* d(x)}\langle p,v\rangle\le-\rho_R \end{cases}
$$

and the following outward pointing condition implies (4.2) :

$$
(4.5)\qquad \begin{cases} \forall R>0, \exists M_R>0, \rho_R>0 \text{ and a set } \Gamma_R\subset[0,1] \text{ of zero Lebesgue measure} \\ \forall (t,x)\in[0,1]\times(\partial K\cap RB),\ \exists\,\delta>0 \text{ such that} \\ \forall\ (s,y)\in([0,1]\backslash\Gamma_R)\times K \text{ with } |(s,y)-(t,x)|<\delta,\ \forall\, f(s,y,u)\in G^-(s,y) \\ \exists\ v\in T_{\overline{co}(f(s,y,U(s)))}(f(s,y,u))\cap M_RB \text{ satisfying } \min_{p\in\partial^* d(x)}\langle p,v\rangle\ge\rho_R. \end{cases}
$$

Both conditions (4.4) and (4.5) imply that $\mathrm{Int}(C_K(x))\neq\emptyset$ for every $x\in\partial K$.

Remark 4.4. *i*) Assume that K has a C^1 boundary and the set valued map $(t,x)\rightsquigarrow f(t,x,U(t))$ is continuous and has compact images. Then it is not difficult to verify that (4.4) is satisfied with $\Gamma_R=\emptyset$ if and only if the following relaxed classical inward pointing condition holds true:

$$
\forall\, x\in\partial K,\ \exists\, v_x\in\overline{co}\,(f(t,x,U(t)))\text{ such that }\langle n_x,v_x\rangle<0,
$$

where n_x denotes the unit outward normal to K at $x\in\partial K$.

ii) The following example shows that our condition is more general than the earlier ones even when boundary of K is smooth. Let

$$
U:=U(t)=\{-1,1\},\ K=\left\{(x_1,x_2)\in\mathbb{R}^2\mid x_1>0,\ x_2\le 1-\frac{1}{x_1}\right\},
$$

$f(t,(x_1,x_2),u)=(1-x_1,ux_1)$ for $x_1\ge1$ and $f(t,(x_1,x_2),u)=(0,ux_1)$ whenever $x_1<1$. Then $T_{\overline{co}(f(t,(x_1,x_2),U))}(f(t,(x_1,x_2),1))=\{0\}\times\mathbb{R}_-$ and it is not difficult to check that (4.4) holds true, while the inward pointing condition proposed in [14] is not verified because $\lim_{x_1\to0+}ux_1=0$.

5. Sufficient Conditions for Normality

In this section we restrict our attention to the case $K_1:=Q_0\times Q_1$, where Q_i is a closed subset of $\mathbb{R}^n$, for $i\in\{0,1\}$. Let $(\bar x(\cdot),\bar u(\cdot))$ be a viable trajectory/control pair and $A:[0,1]\to M(n\times n)$ be an integrable $(n\times n)$-matrix valued map.

Definition 5.1. The linear system

$$
(5.1)\qquad w'(t)=A(t)w(t)+v(t)\quad v(t)\in\mathcal{T}(t)
$$

is said to be *controllable* at time $\bar t\in[0,1]$, if there exists $\delta>0$ such that for all $t_1,t_2\in[\bar t-\delta,\bar t+\delta]\cap[0,1]$ with $t_1<t_2$ and for all $w_1,w_2\in\mathbb{R}^n$ there exists a solution $w(\cdot)$ to the system (5.1) defined on $[t_1,t_2]$, which satisfies $w(t_1)=w_1$ and $w(t_2)=w_2$.

Theorem 5.2. *Assume that* $\mathrm{Int}(C_K(x))\neq\emptyset$ *for every* $x\in\partial K$. *Let* $(\bar x(\cdot),\bar u(\cdot))$ *be extremal for a triple* $(\lambda,p(\cdot),\psi(\cdot))$ *and* $A(\cdot)$ *be as in Definition 2.3. Assume there exists* $0<\bar t<1$ *such that* $\bar x(\bar t)\in\mathrm{Int}(K)$, *the system (5.1) is controllable at* $\bar t$, *(4.1) is satisfied on* $[0,\bar t]$ *and (4.2) is satisfied on* $[\bar t,1]$. *Then* $\lambda=1$ *whenever*

$$
\mathrm{Int}(C_K(\bar x(0)))\cap C_{Q_0}(\bar x(0))\neq\emptyset\ \ \textit{and}\ \ \mathrm{Int}(C_K(\bar x(1)))\cap\mathrm{Int}(C_{Q_1}(\bar x(1)))\neq\emptyset.
$$

Proof. It is enough to find a solution to (3.13) that satisfies $i)$ of Proposition 3.4.

Let $w_0 \in \mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0))$ and $w_1 \in \mathrm{Int}(C_K(\bar{x}(1))) \cap \mathrm{Int}(C_{Q_1}(\bar{x}(1)))$. By Theorem 4.1 we can find a solution $w_0(\cdot)$ of (4.3) on $[0, \bar{t}]$, with $w_0(0) = w_0$, and a solution $w_1(\cdot)$ of (4.3) on $[\bar{t}, 1]$, with $w_1(1) = w_1$. Moreover thanks to the controllability assumption we can find $\delta > 0$ such that, for any $t_1, t_2 \in [\bar{t} - \delta, \bar{t} + \delta] \cap (0, 1)$, $t_1 < t_2$, there exists a solution $\bar{w}(\cdot)$ of (5.1) defined on $[t_1, t_2]$, such that $\bar{w}(t_1) = w_0(t_1)$ and $\bar{w}(t_2) = w_1(t_2)$. Since $\bar{x}(\bar{t}) \in \mathrm{Int}(K)$, t_1, t_2 can be chosen such that $\bar{x}([t_1, t_2]) \subset \mathrm{Int}(K)$, hence, we have $\bar{w}(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for all $t \in [t_1, t_2]$. Define

$$w(t) := \begin{cases} w_0(t) \text{ for } t \in [0, t_1] \\ \bar{w}(t) \text{ for } t \in [t_1, t_2] \\ w_1(t) \text{ for } t \in [t_2, 1], \end{cases}$$

then $w(\cdot)$ is as required.

□

Theorem 5.3. *Assume that* $\mathrm{Int}(C_K(x)) \neq \emptyset$ *for every* $x \in \partial K$. *Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be extremal for a triple* $(\lambda, p(\cdot), \psi(\cdot))$ *that satisfies (3.12) and* $A(\cdot)$ *be as in Definition 2.3. Assume there exists* $0 < \bar{t} < 1$ *such that* $\bar{x}(\bar{t}) \in \mathrm{Int}(K)$, *the system (5.1) is controllable at* $\bar{t}$, *(4.1) is satisfied on* $[0, \bar{t}]$ *and (4.2) is satisfied on* $[\bar{t}, 1]$. *Then* $\lambda = 1$.

Proof. Since $(\bar{x}(\cdot), \bar{u}(\cdot))$ is extremal for a triple $(\lambda, p(\cdot), \psi(\cdot))$ that satisfies (3.12), Remark 3.3 implies that it is enough to find a solution to (3.1) satisfying $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for $t \in (0, 1)$.

By Theorem 4.2 we can find a solution $w_0(\cdot)$ of (4.3) on $(0, \bar{t}]$, with $w_0(0) = 0$ and a solution $w_1(\cdot)$ of (4.3) on $[\bar{t}, 1)$, with $w_1(1) = 0$. The proof ends by applying exactly the same arguments as in the proof of Theorem 5.2.

□

Results of Sections 3 and 4 immediately yield the following theorems.

Theorem 5.4. *Assume that either* $i)$ *or* $ii)$ *below hold true:*

$i)$ $K_1 = Q_0 \times \mathbb{R}^n$, *where* Q_0 *is a closed subset of* $\mathbb{R}^n$, *(4.4) and* $\mathrm{Int}(C_K(z)) \cap C_{Q_0}(z) \neq \emptyset$, $\forall\, z \in \partial K \cap \partial Q_0$;

$ii)$ $K_1 = \mathbb{R}^n \times Q_1$, *where* Q_1 *is a closed subset of* $\mathbb{R}^n$, *(4.5) and* $\mathrm{Int}(C_K(z)) \cap C_{Q_1}(z) \neq \emptyset$, $\forall\, z \in \partial K \cap \partial Q_1$.

If $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is extremal for a triple* $(\lambda, p(\cdot), \psi(\cdot))$, *then* $\lambda = 1$.

If the set $Q_0 = \{x_0\}$ for some $x_0 \in \mathbb{R}^n$, then $C_{Q_0}(x_0) = \{0\}$. Hence the condition $0 \in \mathrm{Int}(C_K(x_0))$ cannot be verified whenever $x_0 \in \partial K$. A similar situation occurs when $Q_1 = \{x_1\}$ for some $x_1 \in \mathbb{R}^n$. We then have the following result.

Theorem 5.5. *Assume that either* $i)$ *or* $ii)$ *below hold true:*

$i)$ $K_1 = \{x_0\} \times \mathbb{R}^n$ *for some* $x_0 \in \mathbb{R}^n$ *and (4.4);*

$ii)$ $K_1 = \mathbb{R}^n \times \{x_1\}$ *for some* $x_1 \in \mathbb{R}^n$ *and (4.5).*

If $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is extremal for a non-degenerate triple* $(\lambda, p(\cdot), \psi(\cdot))$, *then* $\lambda = 1$.

In the presence of both endpoints constraints, i.e. $K_1 = Q_0 \times Q_1$ where Q_i is a proper closed subset of K, for $i \in \{0, 1\}$, we are able to provide sufficient conditions for the normality of the maximum principle only for trajectories ending in ∂K.

Theorem 5.6. *Assume that* $K_1 = Q_0 \times Q_1$ *where* Q_i *is a closed subset of* K, *for* $i \in \{0, 1\}$ *and either* $i)$ *or* $ii)$ *below hold true:*

$i)$ *(4.4),* $\mathrm{Int}(C_K(z)) \cap C_{Q_0}(z) \neq \emptyset$, $\forall\, z \in \partial K \cap \partial Q_0$ *and* $C_K(y) \subset C_{Q_1}(y)$ *for all* $y \in \partial K \cap \partial Q_1$;

$ii)$ *(4.5),* $\mathrm{Int}(C_K(z)) \cap C_{Q_1}(z) \neq \emptyset$, $\forall\, z \in \partial K \cap \partial Q_1$ *and* $C_K(y) \subset C_{Q_0}(y)$ *for all* $y \in \partial K \cap \partial Q_0$.

If $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is extremal for a triple* $(\lambda, p(\cdot), \psi(\cdot))$ *and either* $\bar{x}(1) \in \partial K$ *if* $i)$ *holds true or* $\bar{x}(0) \in \partial K$ *if* $ii)$ *holds true, then* $\lambda = 1$.

Theorem 5.7. *Assume that either i) or ii) below hold true:*
i) $Q_0 = \{x_0\}$, *(4.4) and* $C_K(y) \subset C_{Q_1}(y)$ *for every* $y \in \partial K \cap \partial Q_1$*;*
ii) $Q_1 = \{x_1\}$, *(4.5) and* $C_K(y) \subset C_{Q_0}(y)$ *for every* $y \in \partial K \cap \partial Q_0$.
If $(\bar{x}(\cdot), \bar{u}(\cdot))$ *is extremal for a non-degenerate triple* $(\lambda, p(\cdot), \psi(\cdot))$ *and* $\bar{x}(1) \in \partial K$ *if i) holds true or* $\bar{x}(0) \in \partial K$ *if ii) holds true, then* $\lambda = 1$.

6. Non Occurrence of the Lavrentieff Phenomenon

Results on normality and existence of the inward pointing trajectories of linear systems obtained in the previous sections, can be applied to guarantee the absence of the Lavrientieff phenomenon and to generalize some results from [15] and [6], on the Lipschitz regularity of optimal trajectories for the Bolza problem.

In this section we consider the set $K_1 = Q_0 \times \mathbb{R}^n$, where Q_0 is a compact subset of $\mathbb{R}^n$.

Let us first address the case of the Bolza optimal control problem under the Tonelli's type growth condition that is for L having a superlinear growth with respect to f. When this happens and the Hamiltonian H, defined as in the introduction, is locally bounded from below, then the Lipschitz regularity of optimal trajectories, satisfying the normal maximum principle, can be shown through a very simple argument, see the proof of the following proposition from [15].

Proposition 6.1 (Proposition 1 of [15])**.** *Assume that*

(G) *there exists a function* $\phi : \mathbb{R} \to \mathbb{R}$ *satisfying* $\lim_{r\to+\infty} \frac{\phi(r)}{r} = +\infty$ *and* $L(t,x,u) \geq \phi(|f(t,x,u)|)$, *for all* $(t,x,u) \in [0,1] \times \mathbb{R}^n \times \mathcal{Z}$.

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be an extremal for which the maximum principle is normal. If the Hamiltonian is locally bounded from below, then* $\bar{x}(\cdot)$ *is Lipschitzian. Moreover, if* $\mathcal{Z}$ *is a separable Banach space, then* $\bar{u}(\cdot)$ *is essentially bounded whenever*

$$\lim_{\|u\|_{\mathcal{Z}}\to\infty} \inf_{t\in[0,1]} |f(t,\bar{x}(t),u)| = \infty. \tag{6.1}$$

Remark 6.2. Note that the Lipschitzianity of f, L and φ, required in subsection 3.1 of [15], is not necessary in the proof of the above proposition.

Theorem 5.4 provides sufficient conditions for normality of the maximum principle. Thus, as a consequence of the above proposition, once the hypotheses of this result are satisfied, we obtain the following corollary.

Corollary 6.3. *Assume (G), (4.4), that the Hamiltonian* H *is locally bounded from below and*

$$\mathrm{Int}(C_K(x)) \cap C_{Q_0}(x) \neq \emptyset, \ \forall x \in \partial K \cap \partial Q_0.$$

Then, for every extremal $(\bar{x}(\cdot), \bar{u}(\cdot))$, $\bar{x}(\cdot)$ *is Lipschitz continuous. Moreover, if* $\mathcal{Z}$ *is a separable Banach space and (6.1) holds, then* $\bar{u}(\cdot)$ *is essentially bounded.*

We present now a theorem, which says that, under suitable assumptions, the Lavrientieff phenomenon cannot occur for the minimization problem (1.3) when $Q_1 = \mathbb{R}^n$. We need the following assumptions.

Assumption (H1):

i) $\mathcal{Z} = \mathbb{R}^m$, f is continuous and L and φ are continuous, non-negative functions;
ii) for every $R > 0$, $\exists\, C_R > 0$ such that, for any $t \in [0,1]$, $x_1, x_2, y_1, y_2 \in RB \cap K$ and any $u \in U(t)$,
  *ii*1) $|\varphi(x_1,y_1) - \varphi(x_2,y_2)| \leq C_R(|x_1 - x_2| + |y_1 - y_2|)$,
  *ii*2) $|L(t,x_1,u) - L(t,x_2,u)| \leq C_R|x_1 - x_2|[1 + L(t,x_1,u) \wedge L(t,x_2,u)]$,
  *ii*3) $|f(t,x_1,u) - f(t,x_2,u)| \leq C_R|x_1 - x_2|[1 + |f(t,x_1,u)| \wedge |f(t,x_2,u)| + L(t,x_1,u) \wedge L(t,x_2,u)]$;

iii) for all $t \in [0,1]$ and $x \in K$, the set $F(t,x) := \{(L(t,x,u)+\eta, f(t,x,u))|\ u \in U(t), \eta \geq 0\}$ is closed and convex;

iv) $\mathrm{Int}(C_K(x)) \cap C_{Q_0}(x) \neq \emptyset,\ \forall x \in \partial K \cap \partial Q_0$.

Remark 6.4. The functions L and φ are supposed to be nonnegative just to simplify notations in the proofs. If instead, for some $c \geq 0$, L and φ are bounded from the below by $-c$, then replacing these functions by $L+c$ and $\varphi+c$ we get nonnegative mappings for which the associated Bolza problem has the same optimal trajectories.

Theorem 6.5. *Assume (G), (H1), (4.4), that $U(\cdot)$ is lower semicontinuous and the infimum in the Bolza problem (1.3) is finite. Then it is attained and every optimal trajectory is Lipschitz continuous. Furthermore every optimal trajectory/control pair $(\bar{x}, \bar{u})$ is a normal extremal.*

Moreover, if $\forall R > 0$

$$\liminf_{||u||_{\mathcal{Z}} \to \infty} \operatorname*{ess\,inf}_{t \in [0,1]} \inf_{x \in RB} |f(t,x,u)| = +\infty,$$

then every optimal control $\bar{u}(\cdot)$ is essentially bounded.

Remark 6.6. A similar regularity result was obtained in [15] for less general state constraints. A standard hypothesis, ensuring that the infimum in the Bolza problem (1.3) is finite, is hypothesis (H2)*ii)* below.

From the provided proof it follows that every optimal trajectory/control pair verifies the normal maximum principle of Theorem 9.3.1 of [24].

Proof. Thanks to hypothesis (G) and (H1) we can use Theorem 11.4.i of Cesari [8] to obtain the existence of an optimal solution for the minimization problem, over trajectories which belong to $S^K_{[0,1]}$.

Consider an optimal trajectory/control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ for this problem. Then, from [24, Theorem 9.3.1], using the oriented distance $d(\cdot)$ in the place of the state constraint function $h(\cdot)$, we obtain that $(\bar{x}(\cdot), \bar{u}(\cdot))$ satisfies a form of Pontryagin's maximum principle that can be easily reduced to the one of Definition 2.3. Note that, thanks to the continuity of f and L, the lower semicontinuity of $U(\cdot)$ and the growth condition, the corresponding Hamiltonian is lower semicontinuous, hence, locally bounded from below.

Normality of the extremal $(\bar{x}(\cdot), \bar{u}(\cdot))$ follows from Theorem 5.4 (i) and Proposition 6.1 completes the proof. □

Since in many cases of interest the Lagrangian fails to verify the Tonelli's growth condition, it is useful to find results on Lipschitz regularity even when this condition is not satisfied. This situation is studied in [6]. The general strategy is to consider a sequence of penalized problems with super linear growth, to which the direct method of Tonelli can be applied. Then, through an approximation result and a structural assumption on the Hamiltonian it is possible to pass to the limit obtaining the existence of a trajectory/control pair $(x^*(\cdot), u^*(\cdot))$ of the original problem such that $x^*(\cdot)$ is Lipschitz and $L(\cdot, x^*(\cdot), u^*(\cdot))$ is essentially bounded.

To apply this strategy, taking advantage of our results on normality of the maximum principle, we impose again assumptions (H1), but we replace (4.4) by the following inward pointing condition

$$\begin{cases} \forall R > 0, \exists M_R > 0, \rho_R > 0 \text{ and a set } \Gamma_R \subset [0,1] \text{ of zero Lebesgue measure} \\ \forall (t,x) \in [0,1] \ \times (\partial K \cap RB),\ \exists\, \delta > 0 \text{ such that} \\ \forall\ (s,y) \in ([0,1] \backslash \Gamma_R) \times K \text{ with } |(s,y)-(t,x)| < \delta,\ \forall\, f(s,y,u) \in G^+(s,y), \\ \exists\ (\nu, v) \in T_{F(s,y)}((L(s,y,u), f(s,y,u))) \\ \ \text{satisfying } |\nu| + |v| \leq M_R \ \text{ and } \ \max_{p \in \partial^* d(x)} \langle p, v \rangle \leq -\rho_R. \end{cases} \tag{6.2}$$

Assumption (H2):

i) Assumption (H1) is verified with (H1)-*ii*3) replaced by (H1)-*ii*3)*bis* : for some $\alpha \geq 2$,

$$|f(t,x_1,u)-f(t,x_2,u)| \leq C_R|x_1-x_2|[1+|f(t,x_1,u)|\wedge|f(t,x_2,u)|+(L(t,x_1,u)\wedge L(t,x_2,u))^{\frac{1}{\alpha}}];$$

ii) there exists a measurable selection $\tilde{u}(t) \in U(t), \forall t \in [0,1]$, and $v(\cdot) \in L^\alpha([0,1])$ such that for a.e. $t \in [0,1]$, $|f(t,x,\tilde{u}(t))| \leq v(t)(1+|x|)$ and $\forall R > 0$, $\exists m_R(\cdot) \in L^1([0,1])$ satisfying

$$L(t,x,\tilde{u}(t)) \leq m_R(t),\ \forall x \in RB \cap K \ \text{ a.e. in } [0,1];$$

iii) for all $(t,x) \in [0,1] \times \mathbb{R}^n$, $L(t,x,\cdot)$ is locally Lipschitz and $f(t,x,\cdot)$ is differentiable;

iv) $f(t,\cdot,u)$ and $L(t,\cdot,u)$ are differentiable for all $u \in U(t)$ and $t \in [0,1]$.

For any $t \in [0,1]$, $(x,u,p) \in \mathbb{R}^n \times U(t) \times \mathbb{R}^n$, let

$$\mathcal{H}(t,x,u,p) := \langle p, f(t,x,u)\rangle - L(t,x,u),$$

$$P(t,x,u) := \{p \in \mathbb{R}^n |\ \frac{\partial f}{\partial u}(t,x,u)^* p \in \partial_u L(t,x,u) + N_{U(t)}(u)\}.$$

Lemma 6.7. *Assume (H2) i), iii), iv) and (6.2). Let $\varepsilon > 0$ and $(\bar{x}(\cdot),\bar{u}(\cdot))$ be a viable trajectory/control pair. Then there exists a trajectory/control pair $(x_\varepsilon(\cdot),u_\varepsilon(\cdot))$ satisfying*

$$x_\varepsilon([0,1]) \subset \mathrm{Int}(K), \quad \|x_\varepsilon - \bar{x}\|_\infty \leq \varepsilon \quad \text{and} \quad J(x_\varepsilon(\cdot),u_\varepsilon(\cdot)) < J(\bar{x}(\cdot),\bar{u}(\cdot)) + \varepsilon.$$

Proof. Let us reduce our Bolza problem to the Mayer problem by considering the control system

$$\left\{\begin{array}{l} z'(t) = L(t,x(t),u(t)) + \eta(t) \\ x'(t) = f(t,x(t),u(t)) \end{array}\right. \qquad (\eta(t),u(t)) \in \mathbb{R}_+ \times U(t) \text{ for a.e. } t \in [0,1] \tag{6.3}$$

under the state and initial point constraints

$$(z(t),x(t)) \in \mathbb{R} \times K \text{ for all } t \in [0,1],\ (z(0),x(0)) \in \{0\} \times Q_0. \tag{6.4}$$

Let $S^{\mathbb{R}\times K}_{[0,1]}$ be the set of all absolutely continuous solutions to (6.3) satisfying (6.4). The associated Mayer problem is

$$\inf\{\varphi(x(0),x(1)) + z(1)|\ (z(\cdot),x(\cdot)) \in S^{\mathbb{R}\times K}_{[0,1]}\}.$$

Define $\bar{z}(t) := \int_0^t L(s,\bar{x}(s),\bar{u}(s))ds$. Then $((\bar{z}(\cdot),\bar{x}(\cdot)),(0,\bar{u}(\cdot)))$ is a viable trajectory/control pair of the Mayer problem and $(\bar{z}'(t),\bar{x}'(t)) = (L(t,\bar{x}(t),\bar{u}(t)), f(t,\bar{x}(t),\bar{u}(t)))$ a.e.

Consider the linearized problem

$$\left\{\begin{array}{l} (\tilde{w}'(t),w'(t)) \in (L_x(t,\bar{x}(t),\bar{u}(t))w(t), f_x(t,\bar{x}(t),\bar{u}(t))w(t)) + T_{F(t,\bar{x}(t))}((\bar{z}'(t),\bar{x}'(t))) \\ (\tilde{w}(t),w(t)) \in \mathrm{Int}(C_{\mathbb{R}\times K}((\bar{z}(t),\bar{x}(t)))) \quad t \in [0,1] \\ (\tilde{w}(0),w(0)) \in C_{\{0\}\times Q_0}((\bar{z}(0),\bar{x}(0))). \end{array}\right. \tag{6.5}$$

Note that the hypothesis (H1)-*iv*) is equivalent to

$$\mathrm{Int}(C_{\mathbb{R}\times K}((z,x))) \cap C_{\{0\}\times Q_0}((z,x)) \neq \emptyset,\ \forall (z,x) \in (\mathbb{R}\times\partial K) \cap (\{0\}\times\partial Q_0).$$

Moreover, thanks to the fact that the oriented distance from $\mathbb{R}\times K$, which will be denoted by $d^{\mathbb{R}\times K}(\cdot)$, satisfies $\partial^* d^{\mathbb{R}\times K}(z,x) = \{0\}\times\partial^* d(x)$ for all $(z,x) \in \mathbb{R}\times\mathbb{R}^n$, hypothesis (6.2) is in fact assumption (4.4) for the system (6.3). By (H1)-*iii*), $F(t,\bar{x}(t)) = \overline{co}(F(t,\bar{x}(t)))$ for all $t \in [0,1]$. Thus we can apply Theorem 4.1 to get a solution $(\tilde{w},w)(\cdot)$ of (6.5), such that

$$(\tilde{w},w)(0) \in \mathrm{Int}(C_{\mathbb{R}\times K}((\bar{z},\bar{x})(0))) \cap C_{\{0\}\times Q_0}((\bar{z},\bar{x})(0)) = \{0\}\times(\mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0))).$$

Thanks to the hypotheses (H2) *i), iii), iv)*, by the same arguments as those in the proofs of Proposition 5.26 of [2] and Theorem 4.2 of [13], it can be shown that there exist $(z_h(\cdot),x_h(\cdot)) \in S^{\mathbb{R}\times K}_{[0,1]}$, with

$$z_h(0) = \bar{z}(0),\ \ x_h(0) = \bar{x}(0) + hw_h(0),$$

for some $w_h(0) \to w(0)$ when $h \to 0+$ satisfying $\bar{x}(0) + hw_h(0) \in Q_0$, such that $(\tilde{w}_h, w_h)(\cdot) := \frac{(z_h,x_h)(\cdot)-(\bar{z},\bar{x})(\cdot)}{h}$ converges uniformly to $(\tilde{w}, w)(\cdot)$ as $h \to 0$. Therefore $(z_h, x_h)(\cdot)$ converges uniformly to $(\bar{z}, \bar{x})(\cdot)$.

Fix $t \in [0,1]$, then from $(\tilde{w}, w)(t) \in \mathrm{Int}(C_{\mathbb{R}\times K}((\bar{z}, \bar{x})(t)))$ and the continuity of $(\bar{z}, \bar{x})(\cdot)$ we obtain that there exists $\varepsilon_t > 0$ such that for all $s \in [t - \varepsilon_t, t + \varepsilon_t] \cap [0,1]$,

$$(\bar{z}(s), \bar{x}(s)) + [0, \varepsilon_t]B((\tilde{w}, w)(t), \varepsilon_t) \subset \mathbb{R} \times K.$$

Moreover from the continuity of $(\tilde{w}, w)(\cdot)$, eventually making ε_t smaller, we obtain that for all $s \in [t - \varepsilon_t, t + \varepsilon_t] \cap [0,1]$,

$$(\bar{z}(s), \bar{x}(s)) + [0, \varepsilon_t]B((\tilde{w}, w)(s), \varepsilon_t) \subset \mathbb{R} \times K.$$

Hence, we can cover $[0,1]$ by a finite number of intervals $[t_i - \varepsilon_i, t_i + \varepsilon_i] \cap [0,1]$, for $i = 1, \dots, m$, over which the above inclusion holds. Thus, taking $\varepsilon = \min\{\varepsilon_i |\ i = 1, \dots, m\}$, we can say that for all $t \in [0,1]$, $(\bar{z}(t), \bar{x}(t)) + [0, \varepsilon]B((\tilde{w}, w)(t), \varepsilon) \subset \mathbb{R} \times K$. Thanks to the uniform convergence of $(\tilde{w}_h, w_h)(\cdot)$ to $(\tilde{w}, w)(\cdot)$, it follows that, for all small $h > 0$,

$$x_h(t) = \bar{x}(t) + hw_h(t) \in \bar{x}(t) + [0, \varepsilon]B(w(t), \varepsilon) \subset \mathrm{Int}(K) \quad \forall\, t \in [0,1].$$

Let $\eta_h(\cdot)$ and a selection $u_h(t) \in U(t), \forall t \in [0,1]$ be measurable controls corresponding to $(z_h, x_h)(\cdot)$. Since $z_h(1) = \int_0^1 (L(s, x_h(s), u_h(s)) + \eta_h(s))ds$ converges to $\bar{z}(1) = \int_0^1 L(s, \bar{x}(s), \bar{u}(s))ds$ we deduce that $J(x_h(\cdot), u_h(\cdot)) < J(\bar{x}(\cdot), \bar{u}(\cdot)) + \varepsilon$ for $h > 0$ sufficiently small.

□

For a trajectory/control pair $(x(\cdot), u(\cdot))$ define

$$J(x(\cdot), u(\cdot)) := \varphi(x(0), x(1)) + \int_0^1 L(s, x(s), u(s))ds.$$

From the above lemma, we can deduce, in the same way as in [6], the following corollary.

Corollary 6.8. *Assume (H2) and (6.2). Then for α as in hypothesis (H1)-ii3)bis*

$$\inf\left\{J(x(\cdot), u(\cdot))|\ x(\cdot) \in S^K_{[0,1]}\right\} = \inf\left\{J(x(\cdot), u(\cdot))|\ x(\cdot) \in S^K_{[0,1]} \cap [W^{1,\alpha}([0,1])]^n\right\}.$$

Theorem 6.9. *Assume (H2) and (6.2). Suppose that there exists a trajectory/control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ satisfying $\bar{x}'(\cdot) \in [L^\alpha([0,1])]^n$ and*

$$\inf\left\{J(x(\cdot), u(\cdot))|\ x(\cdot) \in S^K_{[0,1]}\right\} < J(\bar{x}(\cdot), \bar{u}(\cdot)) < \infty.$$

Moreover, assume that there exists $k > 0$ such that:

1) *for any trajectory/control pair $(x(\cdot), u(\cdot))$ such that $J(x(\cdot), u(\cdot)) < J(\bar{x}(\cdot), \bar{u}(\cdot))$ we have*

$$\|x\|_\infty \le k \quad \textit{and} \quad \operatorname*{ess\,inf}_{t\in[0,1]} |f(t, x(t), u(t))| \le k;$$

2) *for any $t \in [0,1]$*

$$\sup_{\substack{|x| \le k, |f(t,x,u)| \le k \\ x \in K, p \in P(t,x,u)}} \mathcal{H}(t,x,u,p) < \liminf_{c\to+\infty} \inf_{\substack{|x| \le k, |f(t,x,u)| \ge c \\ x \in K, p \in P(t,x,u)}} \mathcal{H}(t,x,u,p)$$

Then,

a) *problem (1.3) has an optimal solution $(x^*(\cdot), u^*(\cdot))$ such that $x^*(\cdot)$ is Lipschitz and $L(\cdot, x^*(\cdot), u^*(\cdot)) \in L^\infty([0,1])$;*
b) *$(x^*(\cdot), u^*(\cdot))$ is a normal extremal with a costate $p(\cdot) \in [W^{1,\infty}([0,1])]^n$.*

The proof of Theorem 6.9 can be done similarly to the one presented in [6], using Lemma 6.7 and Corollary 6.8. The main difference is that results of Section 5 must be used in order to deduce normality. Moreover this proof implies that every optimal trajectory/control pair verifies the normal maximum principle of Theorem 9.3.1 of [24].

7. Construction of Inward Pointing Trajectories

This section is devoted to the proofs of Theorems 4.1 and 4.2. We first recall some notions of nonsmooth analysis. Let $\partial_- d_K(x)$ be the subdifferential of $d_K(\cdot)$ at x, given by

$$\partial_- d_K(x) := \{v \in \mathbb{R}^n | \liminf_{y\to x} \frac{d_K(y) - d_K(x) - \langle v, y-x\rangle}{|y-x|} \geq 0\}.$$

For all $x \in \mathbb{R}^n$, define

$$\hat{\partial} d_K(x) := \operatorname{Lim\,sup}_{y\to x} \partial_- d_K(y).$$

Then $\partial d_K(x) = co(\hat{\partial} d_K(x))$ for all $x \in \mathbb{R}^n$.

Denote by $N_K^{lim}(x)$ the *limiting normal cone* to K at x: $N_K^{lim}(x) := \operatorname{Lim\,sup}_{K\ni y\to x} N_K^0(y)$.

Thanks to Example 8.53 in [23], we have $\hat{\partial} d_K(x) = N_K^{lim}(x) \cap B$ for every $x \in K$. Hence $N_K^{lim}(x) \cap B \subset \partial d_K(x)$ for all $x \in K$. Moreover, by [9, Proposition 2.4 and Corollary 2.5],

$$\partial d_K(x) = co\{0, \lim_{i\to\infty} \nabla d_K(x_i) |\ x_i \notin K\ \forall i \in \mathbb{N} \text{ and } \lim_{i\to\infty} x_i = x\}.$$

If $\operatorname{Int}(C_K(x)) \neq \emptyset$ for every $x \in \partial K$, then, thanks to Proposition 2.2, $|\nabla d(x)| = 1$ for every $x \in K$ at which $d(\cdot)$ is differentiable, hence, by convexity of $\partial d_K(x) \subset B$,

$$N_K^{lim}(x) \cap S^{n-1} \subset \partial d_K(x) \cap S^{n-1} \subset \partial^* d(x) \subset N_K^1(x).$$

Lemma 7.1. *Assume that* $\operatorname{Int}(C_K(x)) \neq \emptyset$ *for every* $x \in \partial K$. *Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be a viable trajectory/control pair and* $0 \leq \tau_1 < \tau_2 \leq 1$ *be such that* $\bar{x}([\tau_1, \tau_2]) \subset \partial K$. *Then for every* $s \in [\tau_1, \tau_2]$ *such that* $\bar{x}'(s)$ *exists we have*

$$\max_{p\in\partial^* d(\bar{x}(s))} \langle p, \bar{x}'(s)\rangle \geq 0$$

Proof. We first prove that for all $t \in [\tau_1, \tau_2]$ such that $\bar{x}'(t)$ exists we have

$$\max_{p\in N_K^{lim}(\bar{x}(t))\cap S^{n-1}} \langle p, \bar{x}'(t)\rangle \geq 0.$$

Indeed let $t \in [\tau_1, \tau_2]$ be such that $\bar{x}(\cdot)$ is differentiable at t. Assume by contradiction that

$$\max_{p\in N_K^{lim}(\bar{x}(t))\cap S^{n-1}} \langle p, \bar{x}'(t)\rangle < 0.$$

Then, by Theorem 6.36 from [23], we have $\bar{x}'(t) \in \operatorname{Int}(C_K(\bar{x}(t)))$. Therefore there exists an $\varepsilon > 0$ such that $\bar{x}(t) + [0, \varepsilon]B(\bar{x}'(t), \varepsilon) \subset K$, implying that $\bar{x}(t) + h\bar{x}'(t) + h\varepsilon B \subset K$ for all $h \in [0, \varepsilon]$. Hence for h small enough $\bar{x}(t+h) + h\frac{\varepsilon}{2}B \subset K$ and $\bar{x}(t+h) \in \operatorname{Int}(K)$, in contradiction with $\bar{x}([\tau_1, \tau_2]) \subset \partial K$.

Thus, for all $t \in [\tau_1, \tau_2]$ such that $\bar{x}'(t)$ exists,

$$0 \leq \max_{p\in N_K^{lim}(\bar{x}(t))\cap S^{n-1}} \langle p, \bar{x}'(t)\rangle \leq \max_{p\in\partial d_K(\bar{x}(t))\cap S^{n-1}} \langle p, \bar{x}'(t)\rangle \leq \max_{p\in\partial^* d(\bar{x}(t))} \langle p, \bar{x}'(t)\rangle.$$

□

Lemma 7.2. *Assume* $\operatorname{Int}(C_K(x)) \neq \emptyset$ *for every* $x \in \partial K$. *Let* $(\bar{x}(\cdot), \bar{u}(\cdot))$ *be a viable trajectory/control pair and let* $0 \leq \tau_1 < \tau_2 \leq 1$ *be such that* $d(\bar{x}(\tau_1)) \leq 0$, $d(\bar{x}(\tau_2)) = 0$. *Then the Lebesgue measure of the set*

$$\{s \in [\tau_1, \tau_2] |\ \bar{x}'(s) \text{ exists and } \max_{p\in\partial^* d(\bar{x}(s))} \langle p, \bar{x}'(s)\rangle \geq 0\}$$

is strictly positive.

Proof. If $\bar{x}([\tau_1,\tau_2]) \subset \partial K$ then we can apply Lemma 7.1 to get our conclusion.

Let $\tau_1 \le \tau_0 < \tau_3 \le \tau_2$ be such that $\bar{x}(\tau_0) \in \mathrm{Int}(K)$ and $\bar{x}(\tau_3) \in \partial K$, $\bar{x}([\tau_0,\tau_3)) \subset \mathrm{Int}(K)$.

Denote by $d^0(x)(v)$ the directional derivative of $d(\cdot)$ at x in the direction v defined by

$$d^0(x)(v) := \limsup_{y\to x, h\to 0^+} \frac{d(y+hv)-d(y)}{h}$$

for any $x, v \in \mathbb{R}^n$. It is well known that $d^0(x)(\cdot)$ is a convex continuous function.

Define the absolutely continuous function $\phi(\cdot)$ by $\phi(t) := d(\bar{x}(t))$ for all $t \in [\tau_0,\tau_3]$. Then for all $t \in [\tau_0,\tau_3]$ such that $\bar{x}'(t)$ and $\phi'(t)$ do exist

$$\phi'(t) = \lim_{h\to 0^+} \frac{d(\bar{x}(t+h))-d(\bar{x}(t))}{h} = \lim_{h\to 0^+} \frac{d(\bar{x}(t)+h\bar{x}'(t))-d(\bar{x}(t))}{h} \le d^0(\bar{x}(t))(\bar{x}'(t)).$$

Since $\partial d(\bar{x}(t)) = \{p \in \mathbb{R}^n |\ \langle p, v\rangle \le d^0(\bar{x}(t))(v),\ \forall\, v \in \mathbb{R}^n\}$,

$$0 < d(\bar{x}(\tau_3)) - d(\bar{x}(\tau_0)) = \int_{\tau_0}^{\tau_3} \phi'(t)dt \le \int_{\tau_0}^{\tau_3} d^0(\bar{x}(t))(\bar{x}'(t))dt = \int_{\tau_0}^{\tau_3} \max_{p\in\partial d(\bar{x}(t))} \langle p, \bar{x}'(t)\rangle dt.$$

Moreover (2.2) applied to $d(\cdot)$ implies

$$0 < \int_{\tau_0}^{\tau_3} \max_{p\in\partial d(\bar{x}(t))} \langle p, \bar{x}'(t)\rangle dt = \int_{\tau_0}^{\tau_3} \max_{p\in\partial^* d(\bar{x}(t))} \langle p, \bar{x}'(t)\rangle dt.$$

□

From now, till the end of this section, we assume that $\mathrm{Int}(C_K(x)) \neq \emptyset$ for every $x \in \partial K$. Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be extremal for a triple $(\lambda, p(\cdot), \psi(\cdot))$ and $A(\cdot)$ be as in Definition 2.3. Assume hypotesis (4.1) holds for $I = [0,1]$ and let M, ρ, Γ be as in (4.1).

For any $\tau \in [0,1]$ denote by $Y(\cdot,\tau)$ the matrix valued solution of

$$\begin{cases} X'(t) &= A(t)X(t) \quad \text{on } [\tau,1] \\ X(\tau) &= \mathbb{I}. \end{cases}$$

Then there exists $\tilde{\varepsilon} > 0$ independent from τ such that for any $0 \le \tau \le s < t \le 1$, with $t - s < \tilde{\varepsilon}$,

$$\|Y(t,\tau)Y^{-1}(s,\tau) - \mathbb{I}\| \le \frac{\rho}{2M}.$$

Set $L := \max_{0\le\tau\le t\le 1} \|Y(t,\tau)\|$ and define the sets

$$\mathcal{D} := \{s \in [0,1]\backslash\Gamma |\ \bar{x}'(s) = f(s,\bar{x}(s),\bar{u}(s)) \text{ and } \max_{p\in\partial^* d(\bar{x}(s))} \langle p, \bar{x}'(s)\rangle \ge 0\} \subset G^+ \setminus \Gamma,$$

$$\mathcal{K} := \{t \in [0,1] |\ \bar{x}(t) \in \partial K\}.$$

Without any loss of generality we may assume that (4.1) holds true with ρ replaced by 2ρ. Let $t \in \mathcal{K}$. Then, thanks to the upper semicontinuity of $\partial^* d(\cdot)$, for some $\varepsilon > 0$ (depending on t) and for all $s, \tau \in [t-\varepsilon, t+\varepsilon]$ such that $s \in G^+\backslash\Gamma$ we have

$$\min_{\zeta\in\mathcal{T}(s)\cap MB} \max_{p\in\partial^* d(\bar{x}(\tau))} \langle p, \zeta\rangle \le -\rho.$$

Hence, by the measurable selection theorem, we can find a measurable $v(\cdot)$, such that for $s \in [t-\varepsilon, t+\varepsilon]$ satisfying $s \in G^+\backslash\Gamma$, we have $v(s) \in \mathcal{T}(s) \cap MB$ and

$$\max_{p\in\partial^* d(\bar{x}(\tau))} \langle p, v(s)\rangle \le -\rho.$$

Define $v(s) := 0$ for all $s \in ([t-\varepsilon, t+\varepsilon] \setminus G^+) \cup \Gamma$.

Then we can cover $\mathcal{K}$ by a finite number of open intervals $(s_i - \varepsilon_i, s_i + \varepsilon_i)$, $0 \le s_i \le 1, \varepsilon_i > 0$, for $i \in \{0,\dots,l\}$, for some $l \in \mathbb{N}$. Moreover, for every $i \in \{0,\dots,l\}$ a measurable function $v_i(\cdot)$, as above, is well-defined on $(s_i - \varepsilon_i, s_i + \varepsilon_i)$.

Removing some elements of such a covering, it is not restrictive to assume that for every i we have $(s_i-\varepsilon_i, s_i+\varepsilon_i)\setminus\bigcup_{j\neq i}(s_j-\varepsilon_j, s_j+\varepsilon_j)\neq\emptyset$. Reordering and keeping the same notations we may assume that for every $i=1,...,l$ we have $s_{i-1}+\varepsilon_{i-1}<s_i+\varepsilon_i$.

Then, for every $i=0,...,l-1$, either $s_i+\varepsilon_i\in(s_{i+1}-\varepsilon_{i+1}, s_{i+1}+\varepsilon_{i+1})$ or

$$(s_i-\varepsilon_i, s_i+\varepsilon_i)\cap\bigcup_{j\geq i+1}(s_j-\varepsilon_j, s_j+\varepsilon_j)=\emptyset.$$

Let $v_i(\cdot)$ be the corresponding measurable functions.

Set $[a_0,b_0]:=[s_0-\varepsilon_0, s_0+\varepsilon_0]\cap[0,1]$. We define next for all $i=1,...,l$ the intervals $[a_i,b_i]$ in the following way : if $s_{i-1}+\varepsilon_{i-1}\in(s_i-\varepsilon_i, s_i+\varepsilon_i)$, then set $[a_i,b_i]:=[s_{i-1}+\varepsilon_{i-1}, s_i+\varepsilon_i]\cap[0,1]$, otherwise set $[a_i,b_i]:=[s_i-\varepsilon_i, s_i+\varepsilon_i]\cap[0,1]$. If for some i, $b_i=1$, then let $m\leq i$ be the smallest index such that $b_m=1$. Otherwise set $m=l$. Observe that the open intervals $\{(a_i,b_i)\}_{i=0}^m$ are mutually disjoint.

For all $i=0,...,m$ define $v(t)=v_i(t)$ for all $t\in(a_i,b_i)$ and let $v(t):=0$ for all $t\in[0,1]\setminus\cup_{i=0}^m(a_i,b_i)$.

Set $I_i:=[a_i,b_i]$. To simplify the notation, when we refer to a single interval I_i for $i\in\{0,\dots,m\}$, we will rename $a_i=a$ and $b_i=b$ writing $I_i=[a,b]$.

We need the following lemmas.

Lemma 7.3. *For any $I_i=[a,b]$, $i\in\{0,\dots,m\}$, let $t_1\in[a,b)$ be such that $\bar{x}(t_1)\in\partial K$ and $w_{t_1}\in\mathrm{Int}(C_K(\bar{x}(t_1)))$. Define $t_2:=\min\{b,t_1+\tilde{\varepsilon}\}$. Then, there exists a solution to the differential inclusion*

$$\left\{\begin{array}{rcl} w'(t) & \in & A(t)w(t)+\mathcal{T}(t) \ \ a.e.\ in\ [t_1,t_2] \\ w(t) & \in & \mathrm{Int}(C_K(\bar{x}(t))) \quad for\ all\ t\in[t_1,t_2] \\ w(t_1) & = & w_{t_1}. \end{array}\right. \tag{7.1}$$

Proof. If $\bar{x}((t_1,t_2])\subset\mathrm{Int}(K)$ then it is enough to consider the solution $w(\cdot)$ to

$$\left\{\begin{array}{rcl} w'(t) & = & A(t)w(t) \quad t\in[t_1,t_2] \\ w(t_1) & = & w_{t_1}. \end{array}\right.$$

Indeed, since $0\in\mathcal{T}(t)$ for all $t\in[t_1,t_2]$ and $\mathrm{Int}(C_K(\bar{x}(t)))=\mathbb{R}^n$ for all $t\in(t_1,t_2]$, $w(\cdot)$ is also a solution to (7.1).

We next consider the case $\bar{x}((t_1,t_2])\cap\partial K\neq\emptyset$. Let $t_1<\tau_0\leq t_2$ be such that $\langle p, Y(t,t_1)w_{t_1}\rangle\leq 0$ for all $t\in[t_1,\tau_0]$ and $p\in\partial^* d(\bar{x}(t))$. Such τ_0 exists because, $\partial^* d(\bar{x}(\cdot))$ is upper semicontinuous and because, by Proposition 2.2, $\partial^* d(\bar{x}(t_1))\subset N_K^1(\bar{x}(t_1))$ implying that $\max_{p\in\partial^* d(\bar{x}(t_1))}\langle p, \mathbb{I}w_{t_1}\rangle<0$.

CASE 1. $\bar{x}(\tau)\in\partial K$ for some $t_1<\tau\leq\tau_0$.

By Lemma 7.2 the Lebesgue measure of the set $\mathcal{A}:=[t_1,\tau]\cap\mathcal{D}$ is strictly positive. Consider now the solution $w(\cdot)$ to

$$\left\{\begin{array}{rcl} w'(t) & = & A(t)w(t)+\frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}v(t) \quad \text{a.e. in } [t_1,t_2] \\ w(t_1) & = & w_{t_1}. \end{array}\right.$$

Then

$$w(t)=Y(t,t_1)w_{t_1}+\frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\int_{t_1}^t Y(t,t_1)Y^{-1}(s,t_1)v(s)ds.$$

We claim that for any $t\in[t_1,\tau]$ we have $w(t)\in\mathrm{Int}(C_K(\bar{x}(t)))$. This follows immediately if $\bar{x}(t)\in\mathrm{Int}(K)$. Let $t\in[t_1,\tau]$ be such that $\bar{x}(t)\in\partial K$. By Lemma 7.2, $\mathcal{L}([t_1,t]\cap\mathcal{A})>0$. Then,

for any $n \in N_K^{lim}(\bar{x}(t)) \cap S^{n-1} \subset \partial^* d(\bar{x}(t))$ we have

$$\begin{aligned}\langle n, w(t)\rangle &= \langle n, Y(t,t_1)w_{t_1}\rangle + \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\int_{t_1}^t \langle n, Y(t,t_1)Y^{-1}(s,t_1)v(s)\rangle ds \\ &\leq \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\int_{t_1}^t \langle n, v(s)\rangle ds + \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\int_{t_1}^t \|Y(t,t_1)Y^{-1}(s,t_1) - \mathbb{I}\|\ |v(s)|ds \\ &\leq -\frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\rho\mathcal{L}([t_1,t]\cap\mathcal{A}) + \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\frac{\rho}{2M}M\mathcal{L}([t_1,t]\cap\mathcal{A}) < 0.\end{aligned}$$

Thus, by Theorem 6.36 from [23], $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for all $t \in [t_1, \tau]$.

On the other hand, for any $t \in [\tau, t_2]$ such that $\bar{x}(t) \in \partial K$ we have $\mathcal{L}([t_1,t]\cap\mathcal{D}) \geq \mathcal{L}(\mathcal{A})$ and therefore and for any $n \in N_K^{lim}(\bar{x}(t)) \cap S^{n-1}$,

$$\begin{aligned}\langle n, w(t)\rangle &= \langle n, Y(t,t_1)w_{t_1}\rangle + \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\int_{t_1}^t \langle n, Y(t,t_1)Y^{-1}(s,t_1)v(s)\rangle ds \\ &\leq L|w_{t_1}| + \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\int_{t_1}^t \langle n, v(s)\rangle ds + \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\int_{t_1}^t \|Y(t,t_1)Y^{-1}(s,t_1) - \mathbb{I}\||v(s)|ds \\ &\leq L|w_{t_1}| - \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\rho\mathcal{L}([t_1,t]\cap\mathcal{D}) + \frac{4L|w_{t_1}|}{\rho\mathcal{L}(\mathcal{A})}\frac{\rho}{2M}M\mathcal{L}([t_1,t]\cap\mathcal{D}) \\ &\leq L|w_{t_1}| - 2L|w_{t_1}|\frac{\mathcal{L}([t_1,t]\cap\mathcal{D})}{\mathcal{L}(\mathcal{A})} \leq -L|w_{t_1}| < 0.\end{aligned}$$

Thus $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for all $t \in [\tau, t_2]$.

CASE 2. $\bar{x}((t_1, \tau_0]) \subset \mathrm{Int}(K)$. Let $\tau := \min\{t \in [\tau_0, t_2] |\ \bar{x}(t) \in \partial K\}$.

Call $\mathcal{A} := [t_1, \tau] \cap \mathcal{D}$. As before, the Lebesgue measure of $\mathcal{A}$ is strictly positive. Define $w(\cdot)$ in the same way as in the Case 1 for this $\mathcal{A}$.

Then for all $t \in (t_1, \tau)$ we have $\bar{x}(t) \in \mathrm{Int}(K)$, thus $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$. Moreover, for any $t \in [\tau, t_2]$ such that $\bar{x}(t) \in \partial K$, we can use the inequality obtained in the second part of CASE 1 to show that for any $n \in N_K^{lim}(\bar{x}(t)) \cap S^{n-1}$ it holds $\langle n, w(t)\rangle < 0$. □

Lemma 7.4. *For any $I_i = [a,b]$, $i \in \{0,\dots,m\}$, let $t_1 \in [a,b)$ be such that $\bar{x}(t_1) \in \mathrm{Int}(K)$, $\bar{x}([t_1,b]) \cap \partial K \neq \emptyset$ and $0 \neq w_{t_1} \in \mathbb{R}^n$. Define $t_2 := \min\{t \in [t_1,b] |\ \bar{x}(t) \in \partial K\}$. Then there exists a solution to the differential inclusion (7.1). In particular $w(t_2) \neq 0$.*

Proof. Let $t_1 < \tau < t_2$ be such that $t_2 - \tau < \tilde{\varepsilon}$. Note that for all $t \in [t_1, t_2)$ we have $\bar{x}(t) \subset \mathrm{Int}(K)$. Consider the solution $w(\cdot)$ to

$$\begin{cases} w'(t) &= A(t)w(t) \quad t \in [t_1, \tau] \\ w(t_1) &= w_{t_1}. \end{cases}$$

Call $w_\tau := w(\tau) \neq 0$. As before, the Lebesgue measure of $\mathcal{A} := [\tau, t_2] \cap \mathcal{D}$ is strictly positive. Let $w(\cdot)$ be the solution to

$$\begin{cases} w'(t) &= A(t)w(t) + \frac{4L|w_\tau|}{\rho\mathcal{L}(\mathcal{A})}v(t) \quad \text{a.e. in } [\tau, t_2] \\ w(\tau) &= w_\tau. \end{cases}$$

We have to show that $w(t_2) \in \mathrm{Int}(C_K(\bar{x}(t_2)))$. For any $n \in N_K^{lim}(\bar{x}(t_2)) \cap S^{n-1}$, we have

$$\begin{aligned}
\langle n, w(t_2)\rangle &= \langle n, Y(t_2,\tau)w_\tau\rangle + \frac{4L|w_\tau|}{\rho\mathcal{L}(\mathcal{A})}\int_\tau^{t_2}\langle n, Y(t_2,\tau)Y^{-1}(s,\tau)v(s)\rangle ds \\
&\le L|w_\tau| + \frac{4L|w_\tau|}{\rho\mathcal{L}(\mathcal{A})}\int_\tau^{t_2}\langle n, v(s)\rangle ds + \frac{4L|w_\tau|}{\rho\mathcal{L}(\mathcal{A})}\int_\tau^{t_2}\|Y(t_2,\tau)Y^{-1}(s,\tau)-\mathbb{I}\||v(s)|ds \\
&\le L|w_\tau| - \frac{4L|w_\tau|}{\rho\mathcal{L}(\mathcal{A})}\rho\mathcal{L}(\mathcal{A}) + \frac{4L|w_\tau|}{\rho\mathcal{L}(\mathcal{A})}\frac{\rho}{2M}M\mathcal{L}(\mathcal{A}) \\
&\le L|w_\tau| - 2L|w_\tau| = -L|w_\tau| < 0.
\end{aligned}$$

□

Lemma 7.5. *For any $I_i = [a,b]$, $i \in \{0,\dots,m\}$, let $t_1 \in [a,b)$ be such that $\bar{x}((t_1,b]) \cap \partial K \neq \emptyset$. Then there exist $t_1 < \theta \le b$ with $\bar{x}(\theta) \in \partial K$ and a solution to*

$$\begin{cases} w'(t) & \in & A(t)w(t) + \mathcal{T}(t) \ \text{ a.e. in } [t_1,\theta] \\ w(t) & \in & \mathrm{Int}(C_K(\bar{x}(t))) \quad \text{for all } t \in (t_1,\theta] \\ w(t_1) & = & 0. \end{cases}$$

In particular $w(\theta) \neq 0$.

Proof. Let $t_2 := \min\{s \in [t_1,b] |\ \bar{x}(s) \in \partial K\}$.

CASE 1. If $t_2 > t_1$ then $\bar{x}([t_1,t_2)) \subset \mathrm{Int}(K)$. Fix a $t_1 < \tau < t_2$ such that $t_2 - \tau \le \tilde{\varepsilon}$.

Then $w(\cdot) \equiv 0$ is the solution to $w'(t) = A(t)w(t)$, $w(t_1) = 0$ and $w(t) \in \mathrm{Int}(C_K(\bar{x}(t))) = \mathbb{R}^n$ for all $t \in [t_1,\tau]$. By Lemma 7.2, the Lebesgue measure of $\mathcal{A} := [\tau,t_2] \cap \mathcal{D}$ is strictly positive. Consider now the solution $w(\cdot)$ to

$$\begin{cases} w'(t) & = & A(t)w(t) + v(t), \quad \text{a.e. in } [\tau,t_2] \\ w(\tau) & = & 0. \end{cases}$$

For any $t \in [\tau,t_2)$ we have $\bar{x}(t) \in \mathrm{Int}(K)$. Hence $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$, while for any $n \in N_K^{lim}(\bar{x}(t_2)) \cap S^{n-1}$ we have

$$\begin{aligned}
\langle n, w(t_2)\rangle &= \int_\tau^{t_2}\langle n, Y(t_2,\tau)Y^{-1}(s,\tau)v(s)\rangle ds \\
&\le \int_\tau^{t_2}\langle n, v(s)\rangle ds + \int_\tau^{t_2}\|Y(t,\tau)Y^{-1}(s,\tau)-\mathbb{I}\|\ |v(s)|ds \\
&\le -\rho\mathcal{L}(\mathcal{A}) + \frac{\rho}{2M}M\mathcal{L}(\mathcal{A}) < 0.
\end{aligned}$$

Thus $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for all $t \in [\tau,t_2]$ and $w(t_2) \neq 0$. Therefore it is enough to set $\theta := t_2$.

CASE 2.1 If $t_2 = t_1$ and for some $\varepsilon > 0$, $\bar{x}((t_1,t_1+\varepsilon]) \subset \mathrm{Int}(K)$, then the previous argument can be applied to $\theta := \min\{t \in [t_1+\varepsilon,b] |\ \bar{x}(t) \in \partial K\}$ instead of t_2 and $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for all $t \in (t_1,\theta]$.

CASE 2.2 If $t_2 = t_1$ and for every $\varepsilon > 0$, $\bar{x}((t_1,t_1+\varepsilon]) \cap \partial K \neq \emptyset$, then let $t_1 < \theta < \min\{1, t_1+\tilde{\varepsilon}\}$ be such that $\bar{x}(\theta) \in \partial K$. We can apply Lemma 7.2 to deduce that the Lebesgue measure of $\mathcal{A} := [t_1,\theta] \cap \mathcal{D}$ is strictly positive. Consider the system

$$\begin{cases} w'(t) & = & A(t)w(t) + v(t) \quad \text{a.e. in } [t_1,\theta] \\ w(t_1) & = & 0. \end{cases}$$

Then

$$w(t) = \int_{t_1}^t Y(t,0)Y^{-1}(s,0)v(s)ds.$$

For any $t \in (t_1, \theta)$ such that $\bar{x}(t) \in \mathrm{Int}(K)$ we have $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$, while for any $t \in (t_1, \theta]$ such that $\bar{x}(t) \in \partial K$, $\mathcal{L}([t_1, t] \cap \mathcal{A}) > 0$ and for any $n \in N_K^{lim}(\bar{x}(t)) \cap S^{n-1}$

$$\begin{aligned}\langle n, w(t)\rangle &= \int_{t_1}^{t} \langle n, Y(t,t_1)Y^{-1}(s,t_1)v(s)\rangle ds \\ &\leq \int_{t_1}^{t} \langle n, v(s)\rangle ds + \int_{t_1}^{t} \|Y(t,t_1)Y^{-1}(s,t_1) - \mathbb{I}\| \, |v(s)| ds \\ &\leq -\rho\mathcal{L}([t_1,t] \cap \mathcal{A}) + \frac{\rho}{2M} M\mathcal{L}([t_1,t] \cap \mathcal{A}) < 0.\end{aligned}$$

Thus $w(t) \in \mathrm{Int}(C_K(\bar{x}(t)))$ for all $t \in (t_1, \theta]$.

□

Lemma 7.6. *For any $I_i = [a,b]$, $i \in \{0, \dots, m\}$, let $w_a \in \mathrm{Int}(C_K(\bar{x}(a)))$. Then, there exists a solution to (4.3) on $[a,b]$, with $w(a) = w_a$.*

Proof. Fix $w_a \in \mathrm{Int}(C_K(\bar{x}(a)))$. We will construct the solution to (4.3) on $[a,b]$, with $w(a) = w_a$, subdividing $[a,b]$ on small intervals of time and using Lemmas 7.3, 7.4 and 7.5.

CLAIM 1 For all $t_1 \in [a,b)$ such that $\bar{x}([t_1,b]) \cap \partial K \neq \emptyset$ and $w_{t_1} \in \mathrm{Int}(C_K(\bar{x}(t_1)))$ there exists $t_1 < \delta(t_1) \leq b$ such that the differential inclusion (7.1) has a solution for $t_2 = \delta(t_1)$ and either $\delta(t_1) = b$ or $\bar{x}(\delta(t_1)) \in \partial K$, $\delta(t_1) - t_1 \geq \tilde{\varepsilon}$.

Indeed, if $\bar{x}(t_1) \in \partial K$ apply Lemma 7.3 and let $w(\cdot)$, t_2 be as in its claim. If $t_2 = b$, then set $\delta(t_1) = b$. Consider next the case $t_2 = t_1 + \tilde{\varepsilon}$. If $\bar{x}(t_2) \in \partial K$, then set $\delta(t_1) = t_2$.

If $\bar{x}(t_2) \in \mathrm{Int}(K)$ and $\bar{x}((t_2,b]) \cap \partial K = \emptyset$, then extend $w(\cdot)$ on $[t_2,b]$ by the solution to $w'(t) = A(t)w(t)$ starting at $w(t_2)$ at time t_2 and set $\delta(t_1) = b$.

If $\bar{x}(t_2) \in \mathrm{Int}(K)$ and $\bar{x}((t_2,b]) \cap \partial K \neq \emptyset$, then by Lemma 7.4 or Lemma 7.5 there exist $\delta(t_1) > t_2$ such that $\bar{x}(\delta(t_1)) \in \partial K$ and a solution $w(\cdot)$ to the differential inclusion (7.1) on $[t_2, \delta(t_1)]$ taking value $w(t_2)$ at time t_2. Clearly $\delta(t_1) - t_1 > \tilde{\varepsilon}$.

It remains to consider the case $\bar{x}(t_1) \in \mathrm{Int}(K)$. Then by Lemma 7.4 or Lemma 7.5 there exist $b \geq t_2 > t_1$ such that $\bar{x}(t_2) \in \partial K$ and $w(\cdot)$ solving the differential inclusion (7.1) on $[t_1, t_2]$. If $\bar{x}((t_2,b]) \cap \partial K = \emptyset$, then set $\delta(t_1) = b$ and extend $w(\cdot)$ on $[t_2,b]$ by the solution to $w'(t) = A(t)w(t)$ starting at $w(t_2)$ at time t_2. If $\bar{x}((t_2,b]) \cap \partial K \neq \emptyset$, then by the previous arguments there exists $\delta(t_1) > t_2$ such that either $\delta(t_1) = b$ or $\delta(t_1) - t_2 \geq \tilde{\varepsilon}$, $\bar{x}(\delta(t_1)) \in \partial K$ and a solution $w(\cdot)$ to differential inclusion (7.1) on $[t_2, \delta(t_1)]$ taking value $w(t_2)$ at time t_2.

We construct a finite sequence $a = \tau_0 < \tau_1 < \cdots < \tau_s = b$ such that $\tau_{i+1} - \tau_i \geq \tilde{\varepsilon}$ for all $i \leq s-2$, $\bar{x}(\tau_i) \in \partial K$ for $1 \leq i \leq s-1$ and $w(\cdot)$ as in the claim of our lemma using an induction argument. Set $t_1 = a = \tau_0$. CLAIM 1 implies the existence of $a < \tau_1 \leq b$ and a solution $w(\cdot)$ to (7.1) on $[a, \tau_1]$ such that $w(a) = w_a$ and either $\tau_1 = b$ or $\tau_1 - \tau_0 \geq \tilde{\varepsilon}$ and $\bar{x}(\tau_1) \in \partial K$. Assume that the solution $w(\cdot)$ is already defined on $[\tau_0, \tau_i]$ for some $\tau_0 < \tau_1 < \dots < \tau_i < b$ satisfying $\tau_j - \tau_{j-1} \geq \tilde{\varepsilon}$ for $j \in \{1, \dots, i\}$ and $\bar{x}(\tau_i) \in \partial K$. If $\bar{x}((\tau_i, b]) \cap \partial K = \emptyset$, then consider the solution to $w'(t) = A(t)w(t)$ on $[\tau_i, b]$ taking value $w(\tau_i)$ at time τ_i. Otherwise, using again CLAIM 1 we can find $\tau_{i+1} > \tau_i$ such that either $\tau_{i+1} = b$ or $\tau_{i+1} - \tau_i \geq \tilde{\varepsilon}$ and extend the solution $w(\cdot)$ to (7.1) on the time interval $[\tau_i, \tau_{i+1}]$. Since $\tau_{i+1} - \tau_i \geq \tilde{\varepsilon}$ for all i satisfying $\tau_{i+1} \neq b$, our construction ends in a final number of steps. □

Proof of Theorem 4.1. Let $w_0 \in \mathrm{Int}(C_K(\bar{x}(0))) \cap C_{Q_0}(\bar{x}(0))$. We provide the proof when $\bar{t} = 1$. The proof when $\bar{t} < 1$ is similar.

We will construct the solution to (4.3) on $[0,1]$, with $w(0) = w_0$, using the finite covering $\{I_i\}_{i \in \{0,\dots,m\}}$.

If $a_0 = 0$, then thanks to Lemma 7.6 we can construct a solution $w(\cdot)$ to (4.3) on $[0, b_0]$, with $w(0) = w_0$.

If $a_0 > 0$, then, since $v(t) = 0$ for $t \in [0, a_0)$ and $\bar{x}(t) \in \mathrm{Int}(K)$ for all $t \in [0, a_0]$, it is enough to consider the solution $w(\cdot)$ of $w'(t) = A(t)w(t)$, $w(0) = w_0$ on $[0, a_0]$. Call $w(a_0) := w_{a_0}$. Then using Lemma 7.6 we can extend the solution to (4.3) on $[a_0, b_0]$, imposing $w(a_0) = w_{a_0}$.

For every $i \in 1, \dots, m$ let $w(\cdot)$ be the constructed solution to (4.3) on $[0, b_{i-1}]$, with $w(0) = w_0$. Call $w(b_{i-1}) := w_{b_{i-1}}$.

If $a_i = b_{i-1}$, then thanks to Lemma 7.6 we can extend the solution $w(\cdot)$ to (4.3) on $[a_i, b_i]$, with $w(a_i) = w_{b_{i-1}}$.

If $a_i > b_{i-1}$, then using that $v(t) = 0$ on (b_{i-1}, a_i) and $\bar{x}(t) \in \mathrm{Int}(K)$ for all $t \in [b_{i-1}, a_i]$, we extend $w(\cdot)$ on the time interval $[b_{i-1}, a_i]$, by taking the solution of $w'(t) = A(t)w(t)$, $w(b_{i-1}) = w_{b_{i-1}}$ on $[b_{i-1}, a_i]$. Call $w_{a_i} := w(a_i)$. Then using Lemma 7.6 we can further extend the solution to (4.3) on $[a_i, b_i]$, imposing $w(a_i) = w_{a_i}$.

Thus, in a finite number of steps, we obtain a solution $w(\cdot)$ to (4.3) on $[0, b_m]$, with $w(0) = w_0$. Call $w(b_m) := w_{b_m}$.

If $b_m = 1$, the proof ends. If $b_m < 1$, then, using that $v(t) = 0$ on $(b_m, 1]$ and $\bar{x}(t) \in \mathrm{Int}(K)$ for all $t \in [b_m, 1]$, we extend the solution $w(\cdot)$ to (4.3) on $[b_m, 1]$ in the same way as before.

(Assume next that (4.2) holds true. We restrict our attention to the case $\bar{t} = 0$. The proof in the case $\bar{t} > 0$ is similar. Define $\tilde{x} : [0, 1] \to K$ by $\tilde{x}(t) := \bar{x}(1 - t)$, $\tilde{U} : [0, 1] \to \mathcal{Z}$ by $\tilde{U}(t) := U(1 - t)$, $\tilde{u} : [0, 1] \to \mathcal{Z}$ by $\tilde{u}(t) := \bar{u}(1 - t)$.

If (4.2) holds for $(\bar{x}(\cdot), \bar{u}(\cdot))$, then (4.1) holds for $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ and f replaced by $-f$. The previous proof can be used to construct a solution $\tilde{w}(\cdot)$ of (4.3) on [0,1] with A replaced by $-A$, f by $-f$ and $\tilde{w}(0) = w_1$. Hence $w(\cdot)$, defined as $w(t) := \tilde{w}(1 - t)$ for all $t \in [0, 1]$, belongs to $\mathcal{R}_1(0, w_1)$.)

□

Proof of Theorem 4.2. We consider only $\bar{t} = 1$. By Theorem 4.1 it is enough to investigate the case $\bar{x}(0) \in \partial K$.

If $\bar{x}((0, b_0]) \cap \partial K = \emptyset$, then consider a solution to $w'(t) = A(t)w(t)$, $w(0) = 0$ and set $\theta = b_0$. Otherwise applying Lemma 7.5 we can find $\theta \in (0, b_0]$ such that $\bar{x}(\theta) \in \partial K$ and $w(\cdot)$ as in Lemma 7.5. Using a time shift, we apply Theorem 4.1 on the time interval $[\theta, 1]$ instead of $[0, 1]$ for $Q_0 = \mathbb{R}^n$.

□

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Hélène Frankowska
Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, case 247, 4 place Jussieu, 75252 Paris, France,
Phone : +33144277187 Fax : +33144272724
E-mail address: frankowska@math.jussieu.fr

Daniela Tonon
Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, case 247, 4 place Jussieu, 75252 Paris, France,
Phone +33 144279863
E-mail address: tonondaniela83@gmail.com