Numerical methods for piecewise deterministic Markov processes with boundary

Christiane Cocozza-Thivent, Robert Eymard, Ludovic Goudenège, Michel Roussignol

Numerical methods for piecewise deterministic Markov processes with boundary

Christiane Cocozza-Thivent, Robert Eymard, Ludovic Goudenège and Michel Roussignol*

December 11, 2013

Abstract

We study the approximation of the distribution of Piecewise Deterministic Markov Processes jumping when the process reaches some boundary of the domain. We first introduce an equation to which the marginal distributions of the process are solution, which generalizes Kolmogorov equations in this case. We then prove the uniqueness of this solution, and propose a finite volume numerical scheme for its approximation. This finite volume scheme enables the approximation of the asymptotic steady problem. We then prove the convergence of this numerical scheme to the marginal distributions of the process. We conclude this paper by some properties of the marginal distributions, directly resulting from the generalized Kolmogorov equation with boundary.

KEYWORD Piecewise Deterministic Markov Process with Boundary, Approximation using Finite Volume Method, Generalized Kolmogorov Equations

1 Introduction

Piecewise Deterministic Markov Processes (PDMP) appear in many areas, such as engineering, operations research, biology, economics... One can find the definition and many properties of these processes in the founding book of M.H.A. Davis [7]. Some relations between PDMP without boundary and point processes are developed in the book of M. Jacobsen [13]. Recently, C. Cocozza-Thivent has investigated the relations between PDMP and Markov renewal theory and extended PDMP's definition [3]. In all application areas, most of interest quantities depend on the distribution of the process at each time, which means that the approximation of the marginal distributions is requested. For this purpose, Monte Carlo methods are widely used (see for instance [8, 14, 17, 18]), but it has been shown in [2, 4, 6, 9, 10, 11, 15] that Finite volume schemes could also provide an efficient approximation of these marginal distributions. These methods consist in solving numerically equations which are fulfilled by the marginal distributions, namely generalized Kolmogorov equations. The characterization of the marginal distributions by these equations is studied in [5] for a PDMP without boundary. Some of these schemes deal with PDMP with boundary from a practical point of view ([9, 15]) without a precise study of their properties. The aim of this paper is to fill up this gap and to propose a more efficient numerical scheme, handling as well transient and asymptotic steady situations.

One studies the following class of PDMP with boundary. The state space of the process is an open subset F of $\mathbb{R}^d$ and there exists a subset Γ of the topological boundary of F which will force the process to jump. The PDMP $(X_t)_{t\geq 0}$ is a jump stochastic process on F whose trajectories are deterministic between the jump times. The deterministic trajectories are determined by a flow $\phi(x,t)$: if between s and t $(s<t)$ the process does not reach the boundary and does not jump, then $X_t = \phi(X_s, t-s)$. Of course the flow has the "Markov property" $\phi(\phi(x,s),t) = \phi(x, s+t)$ as far as the boundary is not reached. In [7] the

*Laboratoire d'Analyse et de Mathématiques Appliquées (CNRS - UMR 8050) Université Paris-Est, 5 boulevard Descartes Champs sur Marne, 77454 Marne-la-Vallée Cedex 2, France

flow is supposed to be solution of the differential equation $\partial_t \phi(x,t) = \mathbf{v}(\phi(x,t))$ and $\phi(x,0) = x$ with $\mathbf{v}$ locally Lipschitz continuous. In [3] and [13] this assumption is relaxed.
Two kinds of jumps can occur. First there are stochastic jumps from a position $x \in F$ with a jump rate $\lambda(x)$ and a jump distribution $Q(x, \mathrm{d}y)$. Second when the process reaches a point x of the boundary Γ, it jumps inside F with the distribution $q(x, \mathrm{d}y)$. These two kinds of jumps have different characters. Roughly speaking, the first ones occur at random times with probability density functions while the second ones occur at times with Dirac distributions.

Let us give a simple example related to the preventive maintenance of two components. The state of two components is described by their virtual ages at each time. The system fails when one component fails and then an immediate corrective maintenance occurs. The failure distribution of a component has a probability density function. Il the failure occurs at virtual ages $x = (x_1, x_2)$, after the corrective maintenance the virtual ages are random variables with distribution $Q(x, \mathrm{d}y_1\mathrm{d}y_2)$ (the support of Q is close to $(0,0)$ and included in some square $[0, Q_1] \times [0, Q_1]$). If the virtual age of a component reaches a bound L at some state $x = (L, x_2)$ or $x = (x_1, L)$, an immediate preventive maintenance occurs. After the preventive maintenance the virtual ages are random variables with distribution $q(x, \mathrm{d}y_1\mathrm{d}y_2)$ (the support of q is also close to $(0,0)$ and included in $[0, Q_1] \times [0, Q_1]$). Of course $Q_1 < L$. The random process X_t describing the time evolution of the virtual ages is a PDMP in $F = (0, L) \times (0, L)$ with boundary $\Gamma = (0, L] \times \{L\} \cup \{L\} \times (0, L]$. The flow is $\phi((x_1, x_2), t) = (x_1 + t, x_2 + t)$.

We assume the following notations and hypotheses on the data, denoted by (H) in this paper.

1. $d \in \mathbb{N}^\star$ and $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures on $\mathbb{R}^d$ with borelian algebra, $\mathcal{P}(A)$ is the subset of $\mathcal{P}(\mathbb{R}^d)$ with support in A for any measurable subset $A \subset \mathbb{R}^d$.
2. The flow $\phi : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ is assumed to be such that:
   (a) $\phi : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$ is Lipschitz continuous with constant L_ϕ,
   (b) $\phi(x, 0) = x$ for all $x \in \mathbb{R}^d$ and
   $$\forall x \in \mathbb{R}^d, \forall t, s \in \mathbb{R}_+, \qquad \phi(\phi(x,t), s) = \phi(x, t+s)$$
   (c) Let $F \subset \mathbb{R}^d$ be a non empty open set and $G = R^d \setminus F$ the complementary of F be a non-empty closed set such that, for all $x \in \mathbb{R}^d$, there exists $t \in \mathbb{R}_+$ such that $\phi(x,t) \in G$. We then define $\alpha : \mathbb{R}^d \to \mathbb{R}_+$ by
   $$\alpha(x) = \inf\{t \geq 0 : \phi(x,t) \in G\}.$$
   Note that, for all $x \in G$, $\alpha(x) = 0$, and that, for all $x \in F$, since ϕ is continuous and G is closed, $\alpha(x) > 0$. Note that the following property holds
   $$\forall x \in F,\ \forall t \in (0, \alpha(x)), \alpha(\phi(x,t)) = \alpha(x) - t.$$
   We assume that the function α is Lipschitz continuous with constant L_α.
   (d) We then denote $\Gamma = \{\phi(x, \alpha(x)) : x \in F\}$. We have $\Gamma \subset \bar{F} \setminus F \subset G$. We cannot state whether Γ is open or closed.
3. The transition rate λ is such that $\lambda \in C_b(\overline{F}, \mathbb{R}_+)$, where $C_b(\overline{F}, \mathbb{R}_+)$ denotes the set of continuous and bounded functions from $\overline{F}$ to $\mathbb{R}_+$. We denote by $\Lambda > 0$ a bound of λ.
4. The transition probability $Q : \overline{F} \to \mathcal{P}(F)$ (we then denote by $x \mapsto Q(x, \mathrm{d}y)$ this application) is such that
   (a) there exists a function $f_Q : \mathbb{R}_+ \to \mathbb{R}_+$ such that
   $$f_Q(r) = \sup_{x \in \overline{F}} \int_{\{y \in F : |y| \geq |x| + r\}} Q(x, \mathrm{d}y)$$

and

$$\lim_{r\to\infty} f_Q(r) = 0.$$

(b) for all $\xi \in C_b(F,\mathbb{R})$, the function $x \to \int \xi(y)Q(x,\mathrm{d}y)$ is continuous from $\overline{F}$ to $\mathbb{R}$.

5. The transition probability $q : \overline{\Gamma} \to \mathcal{P}(F)$ (we then denote by $x \mapsto q(x,\mathrm{d}y)$ this application) is such that

   (a) there exists a function $f_q : \mathbb{R}_+ \to \mathbb{R}_+$ such that

   $$f_q(r) = \sup_{x\in\overline{\Gamma}} \int_{\{y\in F:|y|\geq|x|+r\}} q(x,\mathrm{d}y)$$

   and

   $$\lim_{r\to\infty} f_q(r) = 0.$$

   (b) for all $\xi \in C_b(F,\mathbb{R})$, the function $x \to \int \xi(y)q(x,\mathrm{d}y)$ is continuous from $\overline{\Gamma}$ to $\mathbb{R}$.

   (c) denoting $0 = \exp(-B\infty)$ for all $B > 0$, we assume that there exists $a_0 \in (0,1)$ and $B_0 > 0$ such that

   $$\sup_{x\in\overline{\Gamma}} \int_F e^{-B_0\alpha(y)} q(x,\mathrm{d}y) \leq 1 - a_0. \tag{1}$$

6. We assume that $\rho_{\text{ini}} \in \mathcal{P}(F)$ is given.

Following [3], let us explain how the PDMP $(X_t)_{t\geq 0}$, with parameters ϕ, λ, Q, α, q satisfying hypotheses (H), is constructed.
Let $(T_n, Y_n)_{n\geq 0}$ be a non-delayed Markov renewal process with state space F and renewal kernel N defined as follows :

$$N(x,\mathrm{d}y,\mathrm{d}s) = dF_x(s)\,\beta(x,s;\mathrm{d}y),$$

$dF_x(s)$ being the probability distribution of a random variable equals to the minimum of the constant $\alpha(x)$ and of a random variable with values in $[0,+\infty]$ and probability density function $\lambda(\phi(x,s))\,e^{-\int_0^s \lambda(\phi(x,u))\,du}$, namely

$$1_{\mathbb{R}_+}(s)\,dF_x(s) = \lambda(\phi(x,s))\,e^{-\int_0^s \lambda(\phi(x,u))\,du}\,1_{\{s<\alpha(x)\}}\,\mathrm{d}s + 1_{\{\alpha(x)<+\infty\}}\,e^{-\int_0^{\alpha(x)} \lambda(\phi(x,u))\,du}\,\delta_{\alpha(x)}(\mathrm{d}s)$$

where $1_A \in \{0,1\}$, $1_A = 1$ if and only if A is true, and

$$\beta(x,s;\mathrm{d}y) = \begin{cases} Q(\phi(x,s),\mathrm{d}y) & \text{if } s<\alpha(x), \\ q(\phi(x,\alpha(x)),\mathrm{d}y) & \text{if } s=\alpha(x). \end{cases}$$

This means that $(T_n)_{n\geq 0}$ is an increasing sequence of random variables with values in $[0,+\infty]$, $(Y_n)_{n\geq 0}$ is a sequence of random variables with values in $F\cup\{\Delta\}$ ($\Delta \notin \mathbb{R}^d$ is a cemetery point) and:

$$\mathbb{P}(T_n < +\infty, Y_n \in F) = \mathbb{P}(T_n < +\infty),$$

$$\mathbb{P}(T_{n+1} - T_n \leq t, Y_{n+1} \in A \,/\, Y_0, T_1, Y_1, \ldots, T_n < +\infty, Y_n) = N(Y_n, A\times]0,t])$$

for any $n \geq 0$, $t \geq 0$ and Borel subset A of F. Consequently, the conditional probability distribution of $T_{n+1} - T_n$ given $T_n < +\infty, Y_n = x$ is $dF_x(s)$ and the conditional probability distribution of Y_{n+1} given $T_n < +\infty, Y_n = x, T_{n+1} - T_n = s$ is $\beta(x,s;\mathrm{d}y)$.

Then the PDMP $(X_t)_{t\geq 0}$ is defined by

$$X_t = \phi(Y_n, t - T_n) \text{ if } T_n \leq t < T_{n+1}.$$

The condition $\mathbb{E}\left(\sum_{n\geq 0} 1_{\{T_n\leq t\}}\right) < +\infty$ which is assumed in the PDMP theory is satisfied thanks to Hypotheses (H.3) and (H.7) (see [3]).

Let $(P_t)_{t\geq 0}$ be the semi-group of the Markov process $(X_t)_{t\geq 0}$, namely $P_t f(x) = \mathbb{E}(f(X_t)|X_0 = x)$ for any bounded real-valued Borel function f on F. The well-known forward Kolmogorov equations can be written $P_t' f = P_t L f$ for $f \in \mathcal{D}(L)$, $(\mathcal{D}(L), L)$ being the extended generator of the PDMP, i.e.

$$\forall f \in \mathcal{D}(L), \quad \int_F f(x)\,\rho_t(\mathrm{d}x) = \int_F f(x)\,\rho_0(\mathrm{d}x) + \int_0^t \int_F Lf(x)\,\rho_s(\mathrm{d}x)\,\mathrm{d}s,$$

ρ_t being the probability distribution of X_t. These equations can be brought into the following more general form:

$$\int_F g(x,t)\,\rho_t(\mathrm{d}x) = \int_F g(x,0)\,\rho_0(\mathrm{d}x) + \int_0^t \int_F \frac{\partial g}{\partial s}(x,s)\,\rho_s(\mathrm{d}x)\,\mathrm{d}s + \int_0^t \int_F L g_s(x)\,\rho_s(\mathrm{d}x)\,\mathrm{d}s$$

for functions g which are differentiable with respect to their second variable and such that $g_s \in \mathcal{D}(L)$ where $g_s(x) = g(x,s)$.

When the flow ϕ is generated by a differential equation, M.H.A. Davis proved in [7] that $\mathcal{D}(L)$ is the set of measurable real-valued functions f on F satisfying the following conditions:

(D1) for all $x \in F$, the function $t \in [0, \alpha(x)[\to f(\phi(x,t))$ is absolutely continuous. Let $\partial_\phi f$ be such that for all $x \in F, t < \alpha(x)$, $f(\phi(x,t)) - f(x) = \displaystyle\int_0^t \partial_\phi f(\phi(x,s))\,\mathrm{d}s$,

(D2) for all $x \in F$ such that $\alpha(x) < +\infty$, $\lim_{t\to\alpha(x)} f(\phi(x,t))$ exists,

(D3) integrability conditions which are satisfied as soon as f and $\partial_\phi f$ are bounded,

(D4) for all $x \in \Gamma$, $f(x) = \displaystyle\int_F f(y)\,q(x,\mathrm{d}y)$.

Then we get:

$$\forall f \in \mathcal{D}(L), \quad Lf(x) = \partial_\phi f + \int_F (f(y) - f(x))\lambda(x)\,Q(x,\mathrm{d}y).$$

Because of Condition (D4), the domain $\mathcal{D}(L)$ is not large enough to allow efficient numerical analysis of forward Kolmogorov equations. In [3] these equations are generalized in order to overcome this difficulty. The test functions which are used are real-valued measurable functions g defined on $F \times \mathbb{R}_+$ such that for all $(x,a) \in F \times \mathbb{R}_+$, the functions $t \in [0, \alpha(x)[\to g(\phi(x,t), a+t)$ are absolutely continuous. In this paper we restrict a little the test function space to the following.

Definition 1 *Let us define, for all $T > 0$, the space*

$$\mathcal{C}_b^T = \{g \in C_b(\mathbb{R}^d \times \mathbb{R}_+),\ \forall (x,t) \in \mathbb{R}^d \times [T, +\infty[, g(x,t) = 0\}, \tag{2}$$

and let us denote by $\mathcal{C}_b^c(\mathbb{R}^d \times \mathbb{R}_+) = \bigcup_{T>0} \mathcal{C}_b^T$.

We denote by $\mathcal{T}$ the set of all functions g such that there exists $I, J \in \mathcal{C}_b^c(\mathbb{R}^d \times \mathbb{R}_+)$ with

$$\forall (x,t) \in \mathbb{R}^d \times \mathbb{R}_+,\ g(x,t) = J(\phi(x,\alpha(x)), t + \alpha(x)) - \int_0^{\alpha(x)} I(\phi(x,s), t+s)\mathrm{d}s. \tag{3}$$

We then denote $g = \mathbb{T}(I,J)$. We get $\mathcal{T} \subset \mathcal{C}_b^c(\mathbb{R}^d \times \mathbb{R}_+)$.

If $g = \mathbb{T}(I, J) \in \mathcal{T}$, it is easy to check that $g = J$ on $G \times \mathbb{R}_+$ and

$$\forall (x,a) \in F \times \mathbb{R}_+, \forall t < \alpha(x), \quad g(\phi(x,t), a+t) - g(x,a) = \int_0^t I(\phi(x,s), a+s)\,\mathrm{d}s. \tag{4}$$

Hence for $x \in F$ and $a \in \mathbb{R}_+$, $t \in [0, \alpha(x)[\to g(\phi(x,t), a+t)$ is absolutely continuous and $I = \partial_{t,\phi} g$ on $F \times \mathbb{R}_+$ where

$$\partial_{t,\phi} g(x,t) = \lim_{n\to\infty} \frac{g(\phi(x,1/n), t+1/n) - g(x,t)}{1/n}$$

The operator $\partial_{t,\phi}$ is called the derivation along the flow.
Conversely if g satisfies (4) with $I \in \mathcal{C}_b^c(\mathbb{R}^d \times \mathbb{R}_+)$, letting t tends to $\alpha(x)$ we get

$$g(x,a) = g(\phi(x, \alpha(x)), a + \alpha(x)) - \int_0^{\alpha(x)} I(\phi(x,s), a+s)\,\mathrm{d}s.$$

Moreover, for given $I, J, \widetilde{I}, \widetilde{J} \in \mathcal{C}_b^c(\mathbb{R}^d \times \mathbb{R}_+)$, such that $g = \mathbb{T}(I,J) = \mathbb{T}(\widetilde{I}, \widetilde{J})$, we get $I = \widetilde{I}$ on $\overline{F} \times \mathbb{R}_+$. We also have $J = \widetilde{J} = g$ on $G \times \mathbb{R}_+$.

Let us define the measure σ on $\Gamma \times \mathbb{R}_+$ as follows:

$$\sigma(A \times [0,t]) = \sum_{n\geq 1} \mathbb{P}(X_{T_n-} \in A, T_n \leq t, T_n - T_{n-1} = \alpha(Y_{n-1}))$$

for any Borel subset $A \subset \Gamma$ and $t \geq 0$. The measure σ is the intensity of the marked point process $(T_n, X_{T_n-})_{n\geq 1}$ restricted to the jump times T_n which occur when the process reaches the boundary. It describes the average number of times that the trajectories reach some parts of the boundary.

It is then shown in [3] that the PDMP distributions satisfy the following equation:

$$\begin{aligned}\int_F g(x,t)\,\rho_t(\mathrm{d}x) &= \int_F g(x,0)\,\rho_0(\mathrm{d}x) + \int_0^t \int_F \partial_{t,\phi} g(x,s)\,\rho_s(\mathrm{d}x)\,\mathrm{d}s \\ &+ \int_0^t \int_F \lambda(x) \int_F (g(y,s) - g(x,s))\,Q(x,\mathrm{d}y)\,\rho_s(\mathrm{d}x)\,\mathrm{d}s \\ &+ \int_{\Gamma\times]0,t]} \int_F (g(y,s) - g(x,s))\,q(x,\mathrm{d}y)\,\sigma(\mathrm{d}x,\mathrm{d}s), \quad \forall g \in \mathcal{T}.\end{aligned} \tag{5}$$

Moreover it is shown that Hypotheses (H) imply $\rho_t(\mathbb{R}^d \setminus F) = 0$.

This paper is focused on the numerical approximation of these generalized Kolmogorov equations. We propose a numerical scheme which approximates measures $\overline{\mu}$ on $\overline{F} \times \mathbb{R}_+$ and $\overline{\sigma}$ on $\overline{\Gamma} \times \mathbb{R}_+$ satisfying the equation

$$\begin{aligned}0 = &\int_F g(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x) + \int_{\overline{F}\times\mathbb{R}_+} \partial_{t,\phi} g(x,t)\,\overline{\mu}(\mathrm{d}x,\mathrm{d}t) \\ &+ \int_{\overline{F}\times\mathbb{R}_+} \lambda(x) \left(\int_F g(y,t) Q(x,\mathrm{d}y) - g(x,t)\right) \overline{\mu}(\mathrm{d}x,\mathrm{d}t) \\ &+ \int_{\overline{\Gamma}\times\mathbb{R}_+} \left(\int_F g(y,t) q(x,\mathrm{d}y) - g(x,t)\right) \overline{\sigma}(\mathrm{d}x,\mathrm{d}t), \qquad \forall g \in \mathcal{T}.\end{aligned} \tag{6}$$

Since the measures $\mu(\mathrm{d}x,\mathrm{d}t) = \rho_t(\mathrm{d}x)\mathrm{d}t$ and $\sigma(\mathrm{d}x,\mathrm{d}t)$, resulting from the above probabilistic construction, are such that equation 5 holds, they are therefore solutions to equation 6. The uniqueness of the solution $(\overline{\mu}, \overline{\sigma})$ to equation 6 will prove that $\mu = \overline{\mu}$ and $\sigma = \overline{\sigma}$.
So in Section 2, we show this uniqueness, thanks to the resolution of the adjoint problem. In Section 3 a finite volume approximation is defined and its convergence is proved. Finally a few conclusions are proposed. In an appendix we show, without using the probabilistic results, that the measure $\overline{\mu}(\mathrm{d}x,\mathrm{d}t)$ solution of equation (6) can be decomposed in $\rho_t(\mathrm{d}x)\mathrm{d}t$ and that the support of $\overline{\mu}$ is included in $F \times \mathbb{R}_+$.

Remark 1 *The results obtained in this paper can be extended to the following cases:*

1. $\mathbb{R}^d$ *can be replaced by* $\bigcup_{i=1}^{n}\{i\}\times\mathbb{R}^{d_i}$*, and then* F *by* $\bigcup_{i=1}^{n}\{i\}\times F_i$*; then* ϕ *is assumed to be stable and Lipschitz continuous on each* F_i*,* α *is assumes to be Lipschitz continuous or infinite on each* F_i*, and suitable hypotheses are made on* q *and* Q*,*

2. *Hypothesis (H.5(c)) can be generalized by assuming that there exist* $B>0$*,* $m\in\mathbb{N}^*$*,* $a\in[0,1)$ *such that*

$$\sup_{x_0\in\ F}\int_F e^{-B(\alpha(x_1)+\cdots+\alpha(x_m))}q(\phi(x_0,\alpha(x_0)),\mathrm{d}x_1)\cdots q(\phi(x_{m-1},\alpha(x_{m-1})),\mathrm{d}x_m)\le a. \qquad (7)$$

3. *Hypothesis (H.2(c)) can be relaxed: we can have* $\alpha(x)=+\infty$ *for some* $x\in F$*.*

These extended hypotheses cover a large number of interesting test cases (see [3]). We preferred focus here on the mathematical difficulties arising in the continuous model and in the convergence analysis, directly connected with the existence of a boundary.

2 Uniqueness

For simplicity, let us denote (μ,σ) solutions of equation (6) instead of $(\overline{\mu},\overline{\sigma})$.

Theorem 1 (Uniqueness) *Under hypotheses (H), there exists at most a unique couple* (μ,σ) *which is solution of equation* (6)*.*

Proof. Suppose there exist two solutions (μ_1,σ_1) and (μ_2,σ_2) to equation (6). Denote $(\bar\mu,\bar\sigma)$ the measures such that $\bar\mu=\mu_1-\mu_2$ and $\bar\sigma=\sigma_1-\sigma_2$. Then for all $g=\mathcal{T}(I,J)\in\mathcal{T}$, we have

$$\begin{aligned}0=&\int_{\overline{F}\times\mathbb{R}_+}I(x,t)\bar\mu(\mathrm{d}x,\mathrm{d}t)\\&+\int_{\overline{F}\times\mathbb{R}_+}\lambda(x)\left(\int_F g(y,t)Q(x,\mathrm{d}y)-g(x,t)\right)\bar\mu(\mathrm{d}x,\mathrm{d}t)\\&+\int_{\Gamma\times\mathbb{R}_+}\left(\int_F g(x,t)q(z,\mathrm{d}x)-g(z,t)\right)\bar\sigma(\mathrm{d}z,\mathrm{d}t).\end{aligned}$$

Let $\overline{I},\overline{J}\in\mathcal{C}_b^c(\mathbb{R}^d\times\mathbb{R}_+)$. Using lemma 1, we can find $I,J\in\mathcal{C}_b^c(\mathbb{R}^d\times\mathbb{R}_+)$ such that $g=\mathbb{T}(I,J)$ verifies

$$\forall(x,t)\in\mathbb{R}^d\times\mathbb{R}_+,\ \overline{I}(x,t)=I(x,t)+\lambda(x)\left(\int_F g(y,t)Q(x,\mathrm{d}y)-g(x,t)\right),$$

and

$$\forall(z,t)\in\mathbb{R}^d\times\mathbb{R}_+,\ \overline{J}(z,t)=\int_F g(x,t)q(z,\mathrm{d}x)-J(z,t).$$

Thus the measures $\bar\mu$ and $\bar\sigma$ verify

$$\int_{\overline{F}\times\mathbb{R}_+}\overline{I}(x,t)\bar\mu(\mathrm{d}x,\mathrm{d}t)+\int_{\Gamma\times\mathbb{R}_+}\overline{J}(z,t)\bar\sigma(\mathrm{d}z,\mathrm{d}t)=0.$$

Since this equality is verified for all $\bar I,\bar J\in\mathcal{C}_b^c(\mathbb{R}^d\times\mathbb{R}_+)$, it proves that the measures $\bar\mu$ and $\bar\sigma$ vanish. $\square$

Lemma 1 (Operator's inversion) *Under hypotheses (H), let* $\overline{I}, \overline{J} \in \mathcal{C}_b^c(\mathbb{R}^d \times \mathbb{R}_+)$. *Then there exists* $I, J \in \mathcal{C}_b^c(\mathbb{R}^d \times \mathbb{R}_+)$ *such that, setting* $g = \mathbb{T}(I, J)$, *we have*

$$\forall (x,t) \in \mathbb{R}^d \times \mathbb{R}_+,\ \overline{I}(x,t) = I(x,t) + \lambda(x)\left(\int_F g(y,t)Q(x,\mathrm{d}y) - g(x,t)\right), \tag{8}$$

and

$$\forall (z,t) \in \mathbb{R}^d \times \mathbb{R}_+,\ \overline{J}(z,t) = \int_F g(x,t)q(z,\mathrm{d}x) - J(z,t). \tag{9}$$

Proof. Let $T > 0$, and let us define $\widehat{\alpha}(x) = \min(\alpha(x), T)$ for all $x \in \mathbb{R}^d$. Then we get, from Definition 1,

$$\forall (I,J) \in (\mathcal{C}_b^T)^2, \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}_+,\ \mathbb{T}(I,J)(x,t) = J(\phi(x,\widehat{\alpha}(x)), t+\widehat{\alpha}(x)) - \int_0^{\widehat{\alpha}(x)} I(\phi(x,s), t+s)\mathrm{d}s, \tag{10}$$

and $\mathbb{T}(I,J) \in \mathcal{C}_b^T$. We define the following norm on $\mathcal{C}_b^T$ by

$$\forall g \in \mathcal{C}_b^T,\ \|g\|_{A,B} := \sup_{(x,t)\in\mathbb{R}^d\times[0,T]} \exp(B\widehat{\alpha}(x) + At) \quad |g(x,t)|, \tag{11}$$

for given $A, B > 0$ chosen later. Since the norm defined by (11) on $\mathcal{C}_b^T$ is equivalent to the L^∞ norm, the space $(\mathcal{C}_b^T, ||\cdot||_{A,B})$ is then a Banach space. We define a sequence of functions $(I^n, J^n)_{n\in\mathbb{N}}$ such that $I^0 = J^0 = 0$ and for all $n \in \mathbb{N}$

$$\forall (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, I^{n+1}(x,t) = \overline{I}(x,t) - \lambda(x)\left(\int_F \mathbb{T}(I^n,J^n)(y,t)Q(x,\mathrm{d}y) - \mathbb{T}(I^n,J^n)(x,t)\right),$$

$$\forall (z,t) \in \mathbb{R}^d \times \mathbb{R}_+,\ J^{n+1}(z,t) = \int_F \mathbb{T}(I^n,J^n)(x,t)q(z,\mathrm{d}x) - \overline{J}(z,t).$$

We thus obtain a sequence $(\mathbb{T}(I^n,J^n))_{n\in\mathbb{N}}$ of elements of $\mathcal{C}_b^T$ (the continuity can be proved straightforwardly using Hypotheses (H) (see [12] for instance) where $\mathbb{T}(I^{n+1},J^{n+1}) = \Psi(\mathbb{T}(I^n,J^n)$. Let us prove that this sequence converges. By fixed point theorem, it is sufficient to prove that there exists $k \in (0,1)$ such that for all $(I,J) \in (\mathcal{C}_b^T)^2$ and $(I',J') \in (\mathcal{C}_b^T)^2$ we have

$$\|\Psi(\mathbb{T}(I,J)) - \Psi(\mathbb{T}(I',J'))\|_{A,B} \le k\|\mathbb{T}(I,J) - \mathbb{T}(I',J')\|_{A,B}.$$

Then, using the definition of (I^{n+1}, J^{n+1}) and the Lebesgue's dominated convergence theorem, we get that the sequence $(I^n,J^n)_{n\in\mathbb{N}}$ is convergent as well and that the limit satisfies (8) and (9).

Let us set $f = \mathbb{T}(I - I', J - J')$, let $(x,t) \in \mathbb{R}^d \times [0,T]$. We have

$$\Psi(\mathbb{T}(I,J)) - \Psi(\mathbb{T}(I',J')(x,t) = T_1 - T_2 + T_3,$$

with

$$T_1 = \int_0^{\widehat{\alpha}(x)} \lambda(\phi(x,s)) \int_F f(y,t+s)Q(\phi(x,s),\mathrm{d}y)\mathrm{d}s,$$

$$T_2 = \int_0^{\widehat{\alpha}(x)} \lambda(\phi(x,s))f(\phi(x,s),t+s)\mathrm{d}s,$$

and

$$T_3 = \int_F f(y, t+\widehat{\alpha}(x))q(\phi(x,\widehat{\alpha}(x)),\mathrm{d}y).$$

We then have

$$
\begin{aligned}
|T_1| &\le \Big| \int_0^{\widehat{\alpha}(x)} \exp(-A(t+s))\lambda(\phi(x,s)) \\
&\qquad \times \int_F \exp(A(t+s))\exp(-B\widehat{\alpha}(y))\exp(B\widehat{\alpha}(y)) f(y,t+s) Q(\phi(x,s),\mathrm{d}y)\mathrm{d}s \Big| \\
&\le \int_0^{\widehat{\alpha}(x)} \exp(-A(t+s))\lambda(\phi(x,s)) \int_F \exp(-B\widehat{\alpha}(y)) \|f\|_{A,B} Q(\phi(x,s),\mathrm{d}y)\mathrm{d}s \\
&\le \|f\|_{A,B} \int_0^{\widehat{\alpha}(x)} \exp(-A(t+s))\lambda(\phi(x,s))\mathrm{d}s.
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
\exp(At + B\widehat{\alpha}(x))\Big|T_1\Big| &\le \|f\|_{A,B} \int_0^{\widehat{\alpha}(x)} \exp(-As)|\lambda(\phi(x,s))| \exp(B\widehat{\alpha}(x))\mathrm{d}s \\
&\le \frac{\|f\|_{A,B}\Lambda \exp(BT)}{A}.
\end{aligned}
$$

In the same way, assuming $A > B$ and using $\alpha(\phi(x,s)) = \alpha(x) - s$ which leads to $\widehat{\alpha}(\phi(x,s)) \ge \widehat{\alpha}(x) - s$,

$$
\begin{aligned}
\exp(At + B\widehat{\alpha}(x))\Big|T_2\Big| &= \exp(B\widehat{\alpha}(x))\Big| \int_0^{\widehat{\alpha}(x)} \exp(-As - B\widehat{\alpha}(\phi(x,s)))\lambda(\phi(x,s)) \\
&\qquad \exp(B\widehat{\alpha}(\phi(x,s)) + A(t+s)) f(\phi(x,s), t+s)\mathrm{d}s \Big| \\
&\le \exp(B\widehat{\alpha}(x))\|f\|_{A,B} \int_0^{\widehat{\alpha}(x)} \exp(-As - B(\widehat{\alpha}(x) - s))|\lambda(\phi(x,s))|\mathrm{d}s \\
&\le \Lambda \|f\|_{A,B} \int_0^{\widehat{\alpha}(x)} \exp((B-A)s)\mathrm{d}s \\
&\le \Lambda \|f\|_{A,B} \frac{1}{A-B}. \qquad (12)
\end{aligned}
$$

Finally, for $A > B$, we have

$$
\begin{aligned}
& \exp(At + B\widehat{\alpha}(x))\Big|T_3\Big| \\
\le\ & \exp(-A\widehat{\alpha}(x) + B\widehat{\alpha}(x)) \int_F \exp(-B\widehat{\alpha}(y)) \\
& \exp(A(t + \widehat{\alpha}(x)) + B\widehat{\alpha}(y)) |f(y, t + \widehat{\alpha}(x))| q(\phi(x, \widehat{\alpha}(x)), \mathrm{d}y) \\
\le\ & \exp(-(A-B)\widehat{\alpha}(x)) \int_F \exp(-B\widehat{\alpha}(y)) \|f\|_{A,B} q(\phi(x,\widehat{\alpha}(x)), \mathrm{d}y) \\
\le\ & \|f\|_{A,B} \int_F \exp(-B\widehat{\alpha}(y)) q(\phi(x,\widehat{\alpha}(x)), \mathrm{d}y) \\
\le\ & \|f\|_{A,B} \left(\int_F \exp(-B\alpha(y)) q(\phi(x,\alpha(x)), \mathrm{d}y) + \exp(-BT) \right).
\end{aligned}
$$

Finally we need to choose sufficiently large constants A and B with $A > B$ such that

$$
\sup_{x\in\mathbb{R}^d} \left(\frac{\Lambda}{A-B} + \frac{\Lambda \exp(BT)}{A} + \int_F \exp(-B\alpha(y)) q(\phi(x,\alpha(x)),\mathrm{d}y) + \exp(-BT) \right) \le k < 1.
$$

Thanks to Hypothesis (H.5(c)), setting first large B and next large A, we obtain the result. □

Since the unique solution is equal to the measures $\mu(\mathrm{d}x, \mathrm{d}t) = \rho_t(\mathrm{d}x)\mathrm{d}t$ and $\sigma(\mathrm{d}x, \mathrm{d}t)$ in Equation (5), it has the properties of these measures : $\mu(\mathbb{R}^d \setminus F \times \mathbb{R}_+) = \sigma(\mathbb{R}^d \setminus \Gamma \times \mathbb{R}_+) = 0$.

It is possible to prove directly with analysis tools that a solution of equation (6) has necessarily the properties $\mu(\mathrm{d}x, \mathrm{d}t) = \rho_t(\mathrm{d}x)\mathrm{d}t$ and $\mu(\mathbb{R}^d \setminus F \times \mathbb{R}_+) = 0$ without the probabilistic results explained in the introduction. This is done in the appendix.

3 A finite volume scheme

3.1 Definition of the scheme

We now come to the presentation of the finite volume scheme, which has been used in [9] for the approximation of a benchmark problem. Such schemes are not classically used in the framework of probabilistic studies, since they have mainly be developed by engineers, in order to approximate the solutions of balance equations. We then consider that (6) can be viewed as balance equations describing the conservation of probability. We then introduce the finite volume discretization by the following notations and definitions.

1. We define a reference measure, denoted by $\mathrm{d}x$ or $\mathrm{d}y$, on F, with respect to all borelian sets of $\mathbb{R}^d$ restricted to F.

2. An admissible mesh $\mathcal{M}$ of F is a countable partition of F, therefore such that $\cup_{K\in\mathcal{M}} K = F$ and $\forall (K, L) \in \mathcal{M}^2, K \neq L \Rightarrow K \cap L = \emptyset$.

3. $\forall K \in \mathcal{M},\ |K| := \displaystyle\int_K \mathrm{d}x > 0$.

4. $\displaystyle\sup_{K\in\mathcal{M}} \mathrm{diam}(K) < +\infty$ where $\mathrm{diam}(K) = \displaystyle\sup_{\{(x,y)\in K^2\}} |x - y|$.
   We then set $h := \displaystyle\sup_{K\in\mathcal{M}} \mathrm{diam}(K)$.

5. $\tau > 0$ and $\delta t > 0$ are given values, and we denote by $\mathcal{D} = (\mathcal{M}, \delta t, \tau)$.

The above notations and definitions are called in the following Hypotheses (HD). The value $\tau > 0$, aimed to tend to 0, is used for the definition, for all $K \in \mathcal{M}$ and $L \in \mathcal{M}$, of the flux of probability mass from K to L by

$$v_{KL} = \frac{1}{\tau}|\{x \in K : \alpha(x) > \tau \text{ and } \phi(x, \tau) \in L\}|,\ \forall K \in \mathcal{M}, \forall L \in \mathcal{M}. \tag{13}$$

We denote by

$$\begin{gathered} q_K = \frac{1}{\tau}|\{x \in K : \alpha(x) \le \tau\}|, \\ q_{KL} = \frac{1}{\tau}\int_{\{x\in K:\alpha(x)\le\tau\}} \int_L q(\phi(x, \alpha(x)), \mathrm{d}y)\mathrm{d}x,\ \forall K \in \mathcal{M}, \forall L \in \mathcal{M}. \end{gathered} \tag{14}$$

Referring to Hypothesis (H.4), we define

$$\begin{aligned} \lambda_{KL} &= \int_K \lambda(x) \int_L Q(x, \mathrm{d}y)\mathrm{d}x,\ \forall (K, L) \in \mathcal{M} \times \mathcal{M}, \\ \lambda_K &= \int_K \lambda(x)\mathrm{d}x = \sum_{L\in\mathcal{M}} \lambda_{KL},\ \forall K \in \mathcal{M}. \end{aligned} \tag{15}$$

We may now define a family $\left(p_n^{(K)}\right)_{n\in\mathbb{N},K\in\mathcal{M}}$ of real values thanks to the following finite volume scheme, which is time implicit.

$$
\begin{aligned}
&|K|\frac{p_{n+1}^{(K)}-p_n^{(K)}}{\delta t}+\sum_{L\in\mathcal{M}}\left(v_{KL}p_{n+1}^{(K)}-v_{LK}p_{n+1}^{(L)}\right)\\
&\quad+(\lambda_K+q_K)p_{n+1}^{(K)}-\sum_{L\in\mathcal{M}}p_{n+1}^{(L)}(\lambda_{LK}+q_{LK})=0,\\
&\forall K\in\mathcal{M},\ \forall n\in\mathbb{N},
\end{aligned}
\tag{16}
$$

with the initial condition

$$
|K|\,p_0^{(K)}=\int_K\rho_{\rm ini}(\mathrm{d}x),\ \forall K\in\mathcal{M},
\tag{17}
$$

Let us remark that the following property holds:

$$
\tau\left(\sum_{L\in\mathcal{M}}v_{KL}+q_K\right)=|K|,\ \forall K\in\mathcal{M}.
\tag{18}
$$

Therefore, scheme (16) may be rewritten

$$
\begin{aligned}
&\left(\left(1+\frac{\delta t}{\tau}\right)|K|+\delta t\lambda_K\right)p_{n+1}^{(K)}-\delta t\sum_{L\in\mathcal{M}}p_{n+1}^{(L)}(v_{LK}+\lambda_{LK}+q_{LK})=|K|\,p_n^{(K)},\\
&\forall K\in\mathcal{M},\ \forall n\in\mathbb{N}.
\end{aligned}
\tag{19}
$$

We then define the approximation $P_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t)$ (resp. $\sigma_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t)$) of the measure $\mu(\mathrm{d}x,\mathrm{d}t)$ on $\overline{F}\times\mathbb{R}_+$ (resp. $\sigma(\mathrm{d}x,\mathrm{d}t)$ on $\overline{\Gamma}\times\mathbb{R}_+$) by

$$
\int_{\mathbb{R}^d\times\mathbb{R}_+}f(x,t)P_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t)=\sum_{n\in\mathbb{N}}\delta t\sum_{K\in\mathcal{M}}p_{n+1}^{(K)}\int_K f(x,n\delta t)\mathrm{d}x,
\tag{20}
$$

for all bounded continuous function $f\in\mathcal{C}_b^c$, and

$$
\int_{\mathbb{R}^d\times\mathbb{R}_+}f(x,t)\sigma_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t)=\sum_{n\in\mathbb{N}}\delta t\sum_{K\in\mathcal{M}}p_{n+1}^{(K)}\frac{1}{\tau}\int_{\{x\in K:\alpha(x)\le\tau\}}f(\phi(x,\alpha(x)),n\delta t)\mathrm{d}x,
\tag{21}
$$

for all bounded continuous function $f\in\mathcal{C}_b^c$. We also define the approximation $P_{\mathcal{D}}(t)\mathrm{d}x$ of ρ_t (defined by (39)) by

$$
\int_{\overline{F}}\xi(x)P_{\mathcal{D}}(t)\mathrm{d}x=\sum_{K\in\mathcal{M}}p_{N_t+1}^{(K)}\int_K\xi(x)\mathrm{d}x,\ \forall\xi\in C_b(\mathbb{R}^d),
\tag{22}
$$

with N_t given by $N_t\delta t\le t<(N_t+1)\delta t$.
The specifications of this scheme, depending on the mesh and two parameters, τ and δt, are resulting from the following observations:

1. An explicit version of the scheme could be defined, following the ideas of [6]. But the main drawback of an explicit scheme is that it cannot provide, in the general case, an approximation of the asymptotic marginal distributions at large times, with an acceptable computing cost.

2. An implicit scheme has been provided in [10]. But in this scheme the considered flow is much more regular (it is assumed to be the solution of an EDO).

3. An explicit scheme is introduced in [2] for Lipschitz flows. The present scheme uses the value $\tau>0$ in the same way as the time step is used in [2] (where the convergence of the scheme is proved for general Lipschitz flow ϕ in the case $\delta t\to 0$ and $h/\delta t\to 0$).

4. In [9], the asymptotic states at large times have been obtained letting $\delta t \to \infty$ in an implicit scheme. Hence it is interesting to use the parameters h and τ which can tend to 0 independently of δt.

Since (16) is an infinite linear system, the existence and uniqueness of a positive solution must be first addressed.

Lemma 2 (Existence of solution) *Under Hypotheses (H) and (HD), there exists one and only one solution* $(p_n^{(K)})_{K\in\mathcal{M},n\in\mathbb{N}}$ *to Scheme* (13), (14), (15), (16), (17) *which satisfies:*

$$p_n^{(K)} \geq 0,\ \forall K \in \mathcal{M},\ \forall n \in \mathbb{N}, \tag{23}$$

$$\sum_{K\in\mathcal{M}} |K|\, p_n^{(K)} = 1,\ \forall n \in \mathbb{N}. \tag{24}$$

Proof. Let us first show the existence of a solution to the scheme. We consider the values $p_{(k)}^{(K)}$ defined, for given $n \in \mathbb{N}$ and $(p_n^{(K)})_{K\in\mathcal{M}}$ such that (23)-(24), by:

$$\begin{aligned}
&p_{(0)}^{(K)} = p_n^{(K)},\ \forall K \in \mathcal{M},\\
&\left(\left(1+\frac{\delta t}{\tau}\right)|K| + \delta t\lambda_K\right) p_{(k+1)}^{(K)} = \delta t \sum_{L\in\mathcal{M}} p_{(k)}^{(L)}(v_{LK}+\lambda_{LK}+q_{LK}) + |K|\, p_n^{(K)},\\
&\forall K \in \mathcal{M},\ \forall k \in \mathbb{N}.
\end{aligned} \tag{25}$$

Denoting, for $k \in \mathbb{N}^\star$ and $K \in \mathcal{M}$, $\widehat{p}_{(k+1)}^{(K)} = p_{(k+1)}^{(K)} - p_{(k)}^{(K)}$, we have

$$\begin{aligned}
\left(\left(1+\frac{\delta t}{\tau}\right)|K| + \delta t\lambda_K\right) \widehat{p}_{(1)}^{(K)} = {}& \delta t \sum_{L\in\mathcal{M}} p_n^{(L)}(v_{LK}+\lambda_{LK}+q_{LK})\\
&- \left(\frac{\delta t}{\tau}|K| + \delta t\lambda_K\right) p_n^{(K)},\ \forall K \in \mathcal{M},\\
\left(\left(1+\frac{\delta t}{\tau}\right)|K| + \delta t\lambda_K\right) \widehat{p}_{(k+1)}^{(K)} = {}& \delta t \sum_{L\in\mathcal{M}} \widehat{p}_{(k)}^{(L)}(v_{LK}+\lambda_{LK}+q_{LK}),\\
\forall K \in \mathcal{M},\ \forall k \in \mathbb{N}^\star.
\end{aligned}$$

We notice that, thanks to (18) and (24), we have

$$\sum_{K\in\mathcal{M}} \left(\left(1+\frac{\delta t}{\tau}\right)|K| + \delta t\lambda_K\right) |\widehat{p}_{(1)}^{(K)}| \leq 2\left(\frac{\delta t}{\tau} + \Lambda\delta t\right).$$

and

$$\sum_{K\in\mathcal{M}} \left(\left(1+\frac{\delta t}{\tau}\right)|K| + \delta t\lambda_K\right) |\widehat{p}_{(k+1)}^{(K)}| \leq \sum_{L\in\mathcal{M}} \left(\frac{\delta t}{\tau}|L| + \delta t\lambda_L\right) |\widehat{p}_{(k)}^{(L)}|,\ \forall k \in \mathbb{N}^\star.$$

Then, by induction, we get that the value $u_k = \sum_{L\in\mathcal{M}} \left(\frac{\delta t}{\tau}|L| + \delta t\lambda_L\right) |\widehat{p}_{(k)}^{(L)}|$ is positive and nonincreasing with respect to $k \in \mathbb{N}^\star$. We then deduce that the sequence $(u_k)_{k\in\mathbb{N}^\star}$ is convergent. Writing

$$\sum_{K\in\mathcal{M}} |K||\widehat{p}_{(k+1)}^{(K)}| \leq u_k - u_{k+1},$$

we deduce that $\widehat{p}_{(k+1)}^{(K)}$ is the general term of an absolutely convergent series. Therefore the sequence $(p_{(k+1)}^{(K)})_{k\in\mathbb{N}}$ satisfying

$$p_{(k+1)}^{(K)} = p_n^{(K)} + \sum_{m=0}^{k} \widehat{p}_{(m+1)}^{(K)},$$

it converges to a value denoted by $p_{n+1}^{(K)}$. Passing to the limit in (25), we obtain that these values are solution to the scheme. Moreover, they satisfy that

$$\sum_{K\in\mathcal{M}} |K| p_{(k)}^{(K)} \le 1 + u_1 - u_k \le 1 + u_1.$$

Therefore we can sum (16) on $K \in \mathcal{M}$, obtaining by induction that (24) holds for $n+1$.

Let us now turn to the uniqueness of this solution. Denoting by $\widehat{p}^{(K)}$ the difference between two solutions of (19), and using that the two solutions satisfy (23)-(24), one may write

$$\left(\left(1 + \frac{\delta t}{\tau}\right)|K| + \delta t \lambda_K\right)|\widehat{p}^{(K)}| \le \delta t \sum_{L\in\mathcal{M}} |\widehat{p}^{(L)}|(v_{LK} + \lambda_{LK} + q_{LK}),\ \forall K \in \mathcal{M},$$

which provides, summing on $K \in \mathcal{M}$,

$$\sum_{K\in\mathcal{M}} \left(\left(1 + \frac{\delta t}{\tau}\right)|K| + \delta t \lambda_K\right)|\widehat{p}^{(K)}| \le \sum_{K\in\mathcal{M}} \left(\frac{\delta t}{\tau}|K| + \delta t \lambda_K\right)|\widehat{p}^{(K)}|,$$

and therefore $\sum_{K\in\mathcal{M}} |K||\widehat{p}^{(K)}| = 0$. Hence the uniqueness proof. □

3.2 Finitness and tightness

The next lemma concerns the finitness of $\sigma_{\mathcal{D}}$ on the set $\Gamma \times [0,T]$, for all $T \in \mathbb{R}_+$.

Lemma 3 *[Finitness] Under Hypotheses (H) and (HD), let $(p_n^{(K)})_{K\in\mathcal{M},n\in\mathbb{N}}$ be the solution to Scheme* (13), (14), (15), (16), (17) *which satisfies* (23)-(24). *Then, for $T > 0$, $h < \tau$ and $\delta t \le T$, there exists $C_\sigma > 0$, only depending on T, a_0, B_0, α, such that*

$$\int_{\Gamma\times[0,T]} \sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t) = \sum_{n\in\mathbb{N}, n\delta t\le T} \delta t \sum_{K\in\mathcal{M}} q_K\, p_n^{(K)} \le C_\sigma. \tag{26}$$

Proof. Thanks to Hypothesis (H.5(c)), we choose $a_0 \in (0,1)$ and $B_0 > 0$ given by (1), and we define $\theta^{(K)} = e^{-B_0\alpha_K}$, denoting by

$$\alpha_K = \inf\{\alpha(x) \in \mathbb{R} : x \in K\},\ \forall K \in \mathcal{M}.$$

Denoting by $N \in \mathbb{N}$ such that $(N-1)\delta t < T \le N\delta t$, we then multiply (16) by $\delta t\theta^{(K)}$, sum on $n = 0, \ldots, N-1$ and $K \in \mathcal{M}$. We get

$$\sum_{K\in\mathcal{M}} |K|\,(p_N^{(K)} - p_0^{(K)})\theta^{(K)} + \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \delta t \sum_{L\in\mathcal{M}} (v_{KL} + \lambda_{KL} + q_{KL})(\theta^{(K)} - \theta^{(L)}) = 0,$$

which leads to

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)}\big(q_K\theta^{(K)} - \sum_{L\in\mathcal{M}} q_{KL}\theta^{(L)}\big) \le \sum_{K\in\mathcal{M}} |K|\,p_0^{(K)}\theta^{(K)} + \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \delta t \sum_{L\in\mathcal{M}} v_{KL}(\theta^{(L)} - \theta^{(K)})$$
$$+ \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \delta t \sum_{L\in\mathcal{M}} \lambda_{KL}\theta^{(L)}. \tag{27}$$

The first term of the right hand side of (27) is lower than 1. Let us find a bound for the second term of the right hand side of (27). Thanks to a first order expansion, we have

$$|\theta^{(K)} - \theta^{(L)}| \le \theta^{(K)} e^{B_0\,|\alpha_K - \alpha_L|} B_0 |\alpha_K - \alpha_L|.$$

For $v_{KL} \neq 0$, there exists $x_0 \in K$ such that $\phi(x_0, \tau) \in L$. Since

$$|\alpha(x_0) - \alpha_K| \leq L_\alpha h,$$

and

$$|\alpha(\phi(x_0, \tau)) - \alpha_L| \leq L_\alpha h.$$

Since $\alpha(\phi(x_0, \tau)) = \alpha(x_0) - \tau$, we get

$$|\alpha_K - \alpha_L| \leq 2L_\alpha h + \tau.$$

This implies that

$$\delta t \sum_{L\in\mathcal{M}} v_{KL}(\theta^{(L)} - \theta^{(K)}) \leq \frac{\delta t}{\tau}|K| B_0 e^{B_0(2L_\alpha h+\tau)}(2L_\alpha h + \tau).$$

The third term of the right hand side of (27) may be bounded thanks to

$$\delta t \sum_{L\in\mathcal{M}} \lambda_{KL}\theta^{(L)} \leq \delta t \Lambda |K|.$$

Let us now turn to a lower bound of the left hand side of (27). We have

$$\delta t q_K \theta^{(K)} \geq \delta t q_K e^{-B_0\tau},$$

since, if $q_K > 0$ then $\alpha_K \leq \tau$. We now have

$$\delta t \sum_{L\in\mathcal{M}} q_{KL}\theta^{(L)} = \frac{\delta t}{\tau}\int_{\{x\in K:\alpha(x)<\tau\}} \sum_{L\in\mathcal{M}} \int_L e^{-B_0\alpha_L} q(\phi(x, \alpha(x)), \mathrm{d}y)\mathrm{d}x.$$

Let us observe that, for $y \in L$,

$$e^{-B_0\alpha_L} - e^{-B_0\alpha(y)} \leq e^{-B_0\alpha(y)} B_0 L_\alpha h e^{B_0 L_\alpha h}.$$

This leads to

$$\begin{aligned}
&\delta t \sum_{L\in\mathcal{M}} q_{KL}\theta^{(L)} \\
&\leq \frac{\delta t}{\tau}(1 + B_0 L_\alpha h e^{B_0 L_\alpha h}) \int_{\{x\in K:\alpha(x)<\tau\}} \int_F e^{-B_0\alpha(y)} q(\phi(x, \alpha(x)), \mathrm{d}y)\mathrm{d}x.
\end{aligned}$$

Applying (1), we then get

$$\delta t \sum_{L\in\mathcal{M}} q_{KL}\theta^{(L)} \leq (1 - a_0)(1 + B_0 L_\alpha h e^{B_0 L_\alpha h})\delta t q_K.$$

Hence we obtain

$$\delta t \big(q_K\theta^{(K)} - \sum_{L\in\mathcal{M}} q_{KL}\theta^{(L)}\big) \geq (e^{-B_0\tau} - (1 - a_0)(1 + B_0 L_\alpha h e^{B_0 L_\alpha h}))\delta t q_K.$$

We now choose η such that, for $s \leq \eta$, we have $B_0 L_\alpha s e^{B_0 L_\alpha s} \leq a_0/4$ and $e^{-B_0 s} \geq 1 - a_0/4$. This leads, for $h \leq \tau \leq \eta$, to

$$\delta t \big(q_K\theta^{(K)} - \sum_{L\in\mathcal{M}} q_{KL}\theta^{(L)}\big) \geq \Big(1 - \frac{a_0}{4} - (1 - a_0) - \frac{a_0}{4}\Big)\,\delta t q_K = \frac{a_0}{2}\delta t q_K.$$

Therefore we get, from (27),

$$\frac{a_0}{2}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{M}}p_{n+1}^{(K)}q_K \le 1+(T+\delta t)(\Lambda+B_0\eta e^{B_0\eta(2L_\alpha+1)}(2L_\alpha+1))$$

If $\tau>\eta$, then $q_K\le\dfrac{1}{\eta}|K|$ and we have

$$\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{M}}p_{n+1}^{(K)}q_K\le\frac{1}{\eta}\sum_{n=0}^{N-1}\delta t\sum_{K\in\mathcal{M}}p_{n+1}^{(K)}|K|=\frac{1}{\eta}(T+\delta t).$$

This concludes the proof of (26). □

We now turn to the tightness proof of the family of measures $P_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t)$.

Lemma 4 (Tightness of $P_{\mathcal{D}}$) *Under hypotheses (H) and (HD), let $T>\delta t$. Let $(p_n^{(K)})_{n\in\mathbb{N},K\in\mathcal{M}}$ be the solution of Scheme* (13), (14), (15), (16), (17) *which satisfies* (23)-(24).
Then, for all $\varepsilon>0$, there exists $R>0$ only depending on T, ϕ, α, such that

$$\int_{(F\setminus B(0,R))\times[0,T]}P_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t)\le\varepsilon, \tag{28}$$

which implies the tightness of the family of probability measures, defined for the family $\mathcal{F}$ of all discretizations $\mathcal{D}$ in the sense of Hypotheses (HD), such that $\tau<1$ and $h/\tau<1$, by $(P_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t))_{\mathcal{D}\in\mathcal{F}}$ on $F\times[0,T]$.

Proof. Let $\varepsilon>0$ be given. We assume that $\tau<1$ and $h/\tau<1$. We first choose $R_0>1$ such that

$$\int_{F\setminus B(0,R_0-1)}\rho_0(\mathrm{d}x)\le\varepsilon.$$

Thanks to Hypotheses (H.4(a)) and (H.5(a)), we also denote $R_\varepsilon>2+L_\alpha$ a value such that we have

$$f_Q(r)\le\varepsilon\text{ and }f_q(r)\le\varepsilon,\ \forall r\ge R_\varepsilon-2-L_\alpha. \tag{29}$$

Let again $N\in\mathbb{N}$ be such that $(N-1)\delta t<T\le N\delta t$. Denoting by $s^+=\max(s,0)$ for all $s\in\mathbb{R}$, we define for all $n=0,\dots,N$ and $K\in\mathcal{M}$, $\theta_n^{(K)}=(T-n\delta t)^+(1-\exp(-A(R_K-R_0)^+))$, using a value $A>0$ which will be determined later as function of ε and of the data, and where R_K is defined by

$$R_K=\inf\{|x|\in\mathbb{R}:x\in K\},\ \forall K\in\mathcal{M}.$$

We then multiply (16) by $\delta t\theta_n^{(K)}$, sum on $n\in\mathbb{N}$ and $K\in\mathcal{M}$. We get, remarking that $\theta_N^{(K)}=0$,

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}}p_{n+1}^{(K)}|K|(\theta_n^{(K)}-\theta_{n+1}^{(K)})$$
$$=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}}p_{n+1}^{(K)}\delta t\sum_{L\in\mathcal{M}}(v_{KL}+\lambda_{KL}+q_{KL})(\theta_n^{(L)}-\theta_n^{(K)})+\sum_{K\in\mathcal{M}}|K|p_0^{(K)}\theta_0^{(K)}.$$

Thanks to the choice of R_0, we get that

$$\sum_{K\in\mathcal{M}}|K|p_0^{(K)}\theta_0^{(K)}\le T\varepsilon.$$

Thanks to a first order Taylor expansion, we get that

$$|\theta_n^{(L)} - \theta_n^{(K)}| \le TA\ (R_L - R_K) \le TA(|y-x| + 2h),\ \forall y \in L,\ \forall x \in K. \tag{30}$$

We then first get that

$$\begin{aligned}\sum_{L\in\mathcal{M}} v_{KL}(\theta_n^{(L)} - \theta_n^{(K)}) &= \sum_{L\in\mathcal{M}} \frac{1}{\tau}\int_{\{x\in K:\phi(x,\tau)\in L\}} (\theta_n^{(L)} - \theta_n^{(K)})\\ &\le AT\frac{1}{\tau}\int_K (|\phi(x,\tau) - x| + 2h)\mathrm{d}x \le AT\frac{1}{\tau}(L_\phi\tau + 2h)|K|,\end{aligned}$$

leading, using $N\delta t \le T + \delta t$, to

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} v_{KL}(\theta_n^{(L)} - \theta_n^{(K)}) \le (T+\delta t)AT(L_\phi + 2\frac{h}{\tau}).$$

Turning to the terms in λ, we split $\mathcal{M}$ in three subsets: $\mathcal{M}_1 = \{L \in \mathcal{M} : R_L \ge R_K + R_\varepsilon\}$, $\mathcal{M}_2 = \{L \in \mathcal{M} : R_L < R_K\}$ and $\mathcal{M}_3 = \{L \in \mathcal{M} : R_K \le R_L < R_K + R_\varepsilon\}$. We first have, using (29) and $\theta_n^{(L)} - \theta_n^{(K)} \le T$,

$$\sum_{L\in\mathcal{M}_1} \lambda_{KL}(\theta_n^{(L)} - \theta_n^{(K)}) \le T\int_K \lambda(x)\int_{\{y\in F:|y|>|x|+R_\varepsilon-2\}} Q(x,\mathrm{d}y)\mathrm{d}x \le T\Lambda|K|\varepsilon.$$

For $L \in \mathcal{M}_2$, we have $\theta_n^{(L)} < \theta_n^{(K)}$ and therefore

$$\sum_{L\in\mathcal{M}_2} \lambda_{KL}(\theta_n^{(L)} - \theta_n^{(K)}) \le 0.$$

We then use that $0 \le R_L - R_K \le R_\varepsilon$ for $L \in \mathcal{M}_3$ and therefore $\theta_n^{(L)} - \theta_n^{(K)} \le TAR_\varepsilon$, which leads to

$$\sum_{L\in\mathcal{M}_3} \lambda_{KL}(\theta_n^{(L)} - \theta_n^{(K)}) \le T\Lambda|K|AR_\varepsilon.$$

We then get that

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} \lambda_{KL}(\theta_n^{(L)} - \theta_n^{(K)}) \le (T+\delta t)\Lambda T(\varepsilon + AR_\varepsilon).$$

Similarly, using the same splitting for the study of the term $\sum_{L\in\mathcal{M}} q_{KL}(\theta_n^{(L)} - \theta_n^{(K)})$, we remark that, for any $x \in K$ such that $\alpha(x) < \tau$, we get that $|\phi(x,\alpha(x))| \le |x| + L_\alpha\tau$ and therefore, for any $y \in L$ such that $R_L \ge R_K + R_\varepsilon$, we have $|y| \ge |x| + R_\varepsilon - 2h \ge |\phi(x,\alpha(x))| + R_\varepsilon - 2h - L_\alpha\tau$. Therefore

$$\begin{aligned}\sum_{L\in\mathcal{M}_1} q_{KL}(\theta_n^{(L)} - \theta_n^{(K)}) &= \sum_{L\in\mathcal{M}_1} \frac{1}{\tau}\int_{\{x\in K:\alpha(x)<\tau\}}\int_L q(\phi(x,\alpha(x)),\mathrm{d}y)(\theta_n^{(L)} - \theta_n^{(K)})\mathrm{d}x\\ &\le T\frac{1}{\tau}\int_{\{x\in K:\alpha(x)<\tau\}}\int_{\{y\in F:|y|>|\phi(x,\alpha(x))|+R_\varepsilon-2-L_\alpha\}} q(\phi(x,\alpha(x)),\mathrm{d}y)\mathrm{d}x\\ &\le Tq_K\varepsilon.\end{aligned}$$

We again have

$$\sum_{L\in\mathcal{M}_2} q_{KL}(\theta_n^{(L)} - \theta_n^{(K)}) \le 0,$$

and
$$\sum_{L\in\mathcal{M}_3} q_{KL}(\theta_n^{(L)}-\theta_n^{(K)}) \le q_K T A R_\varepsilon.$$
It gives
$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} q_{KL}(\theta_n^{(L)}-\theta_n^{(K)}) \le C_\sigma T(\varepsilon + A R_\varepsilon).$$
We may now choose $A = \varepsilon/R_\varepsilon \le \varepsilon$, which gives the conclusion of the proof of (28), up to a rescaling of ϵ. $\square$

Let us now turn to the tightness proof of σ.

Lemma 5 (Tightness of $\sigma_{\mathcal{D}}$) *Under Hypotheses (H) and (HD), let $(p_n^{(K)})_{K\in\mathcal{M},n\in\mathbb{N}}$ be the solution to to Scheme* (13), (14), (15), (16), (17) *which satisfies* (23)-(24). *Then, for all $\varepsilon > 0$, there exists R only depending on T, α, such that*
$$\int_{(\Gamma\setminus B(0,R))\times[0,T]} \sigma_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t) \le \varepsilon, \tag{31}$$
which implies the tightness of the family of probability measures, defined for the family $\mathcal{F}$ of all discretizations $\mathcal{D}$ such that $\tau < 1$ and $h/\tau < 1$, by $\left(\sigma^{\mathcal{D}}(x,t)\mathrm{d}x\mathrm{d}t\right)_{\mathcal{D}\in\mathcal{F}}$ on $\Gamma\times[0,T]$.

Proof.
Thanks to Hypothesis (H.5(c)), we choose $a_0 \in (0,1)$ and $B_0 > 0$ given by (1).
Let $\varepsilon > 0$ be given.
Thanks to Hypothesis (H.4(a)), we let $R_Q > 0$ be such that we have
$$f_Q(r) \le \varepsilon,\ \forall r \ge R_Q - 2. \tag{32}$$
We then choose R_ρ such that
$$\int_{F\setminus B(0,R_\rho-1)} \rho_0(\mathrm{d}x) \le \varepsilon,$$
and, using Lemma 4,
$$\sum_{n\in\mathbb{N}, n\delta t\le T} \delta t \sum_{\substack{K\in\mathcal{M}\\ K\cap B(0,R_\rho-1)=\emptyset}} |K| p_n^{(K)} \le \varepsilon. \tag{33}$$
We let $R_0 = R_\rho + R_Q$.
Thanks to Hypothesis (H.5(a)), we denote $R_\varepsilon > 2 + L_\alpha$ a value such that we have
$$f_q(r) \le \varepsilon,\ \forall r \ge R_\varepsilon - 2 - L_\alpha. \tag{34}$$
Let us define $A = \varepsilon/R_\varepsilon$.

We now define $\widehat{\theta}^{(K)} = e^{-B_0\alpha_K}\theta^{(K)}$ with $\theta^{(K)} = 1 - \exp(-A(R_K - R_0)^+)$, denoting by
$$\alpha_K = \inf\{\alpha(x)\in\mathbb{R} : x \in K\},\ \forall K\in\mathcal{M}.$$
We then multiply (16) by $\delta t\widehat{\theta}^{(K)}$, sum on $n = 0,\dots,N-1$ and $K\in\mathcal{M}$. We get
$$\sum_{K\in\mathcal{M}} |K|\,(p_N^{(K)} - p_0^{(K)})\widehat{\theta}^{(K)} + \sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} (v_{KL} + \lambda_{KL} + q_{KL})(\widehat{\theta}^{(K)} - \widehat{\theta}^{(L)}) = 0.$$

This gives

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} q_{KL}(\widehat{\theta}^{(K)}-\widehat{\theta}^{(L)}) \leq$$
$$\sum_{K\in\mathcal{M}} |K|\, p_0^{(K)}\widehat{\theta}^{(K)} + \sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} (v_{KL}+\lambda_{KL})(\widehat{\theta}^{(L)}-\widehat{\theta}^{(K)}).$$

Let us define $R_1 = R_0 + \dfrac{1}{A}\log(8/a_0)$, and let us write

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{M}} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} q_{KL}(\widehat{\theta}^{(K)}-\widehat{\theta}^{(L)}) = T_3 + T_4,$$

with

$$T_4 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{M},R_K\geq R_1} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} q_{KL}(\widehat{\theta}^{(K)}-\widehat{\theta}^{(L)}),$$

and

$$T_3 = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{M},R_K< R_1} p_{n+1}^{(K)}\delta t \sum_{L\in\mathcal{M}} q_{KL}(\widehat{\theta}^{(K)}-\widehat{\theta}^{(L)}).$$

For any $K\in\mathcal{M}$ such that $\exp(-A(R_K-R_0))\leq a_0/8$ (meaning that $R_K\geq R_1 := R_0+\dfrac{1}{A}\log(8/a_0)$), we have

$$\sum_{L\in\mathcal{M}} q_{KL}\widehat{\theta}^{(K)} = q_K\widehat{\theta}^{(K)} \geq q_K e^{-B_0\tau}(1-\exp(-A(R_K-R_0)^+)) \geq (1-\frac{a_0}{8})q_K e^{-B_0\tau},$$

since, if $q_K>0$ then $\alpha_K\leq\tau$. We now have

$$\delta t \sum_{L\in\mathcal{M}} q_{KL}\widehat{\theta}^{(L)} \leq \frac{\delta t}{\tau}\int_{\{x\in K:\alpha(x)<\tau\}}\sum_{L\in\mathcal{M}}\int_L e^{-B_0\alpha_L} q(\phi(x,\alpha(x)),\mathrm{d}y)\mathrm{d}x.$$

Let us observe that, for $y\in L$,

$$e^{-B_0\alpha_L} - e^{-B_0\alpha(y)} \leq e^{-B_0\alpha(y)} B_0 L_\alpha h\exp(B_0 L_\alpha h).$$

This leads to

$$\sum_{L\in\mathcal{M}} q_{KL}\widehat{\theta}^{(L)}$$
$$\leq \frac{1}{\tau}(1+B_0L_\alpha h e^{B_0L_\alpha h})\int_{\{x\in K:\alpha(x)<\tau\}}\int_F e^{-B_0\alpha(y)} q(\phi(x,\alpha(x)),\mathrm{d}y)\mathrm{d}x.$$

We then get, using (1),

$$\sum_{L\in\mathcal{M}} q_{KL}\widehat{\theta}^{(L)} \leq (1-a_0)(1+B_0L_\alpha h e^{B_0L_\alpha h})q_K.$$

Hence we obtain, for all $K\in\mathcal{M}$ such that $R_K\geq R_1$,

$$T_4 \geq \sum_{n=0}^{N-1}\sum_{K\in\mathcal{M},R_K\geq R_1} p_{n+1}^{(K)}\delta t\left(\left(1-\frac{a_0}{8}\right)e^{-B_0L_\phi\tau} - (1-a_0)\left(1+B_0L_\alpha h e^{B_0L_\alpha h}\right)\right)q_K.$$

Let us now turn to the study of a lower bound of T_3. We have, since $q_{KL} \neq 0$ implies $\alpha_K \leq \tau$,

$$T_3 \geq \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}, R_K<R_1} p_{n+1}^{(K)} \delta t \sum_{L\in\mathcal{M}} q_{KL}(e^{-B_0\tau}\theta^{(K)} - e^{-B_0\alpha_L}\theta^{(L)}),$$

which leads to $T_3 \geq T_{31} + T_{32}$ with

$$T_{31} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}, R_K<R_1} p_{n+1}^{(K)} \delta t \sum_{L\in\mathcal{M}} q_{KL}(e^{-B_0\tau} - 1)\theta^{(K)} = (e^{-B_0\tau} - 1) \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}, R_K<R_1} p_{n+1}^{(K)} \delta t q_K \theta^{(K)},$$

and

$$T_{32} = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}, R_K<R_1} p_{n+1}^{(K)} \delta t \sum_{L\in\mathcal{M}} q_{KL} e^{-B_0\alpha_L}(\theta^{(K)} - \theta^{(L)}).$$

For any $K \in \mathcal{M}$ such that $R_K < R_1$, following the proof of Lemma 4, we again split $\mathcal{M}$ in three subsets: $\mathcal{M}_1 = \{L \in \mathcal{M} : R_L \geq R_K + R_\varepsilon\}$, $\mathcal{M}_2 = \{L \in \mathcal{M} : R_L < R_K\}$ and $\mathcal{M}_3 = \{L \in \mathcal{M} : R_K \leq R_L < R_K + R_\varepsilon\}$. We get

$$\begin{aligned}\sum_{L\in\mathcal{M}_1} q_{KL} e^{-B_0\alpha_L}(\theta^{(K)} - \theta^{(L)}) &= \sum_{L\in\mathcal{M}_1} e^{-B_0\alpha_L} \frac{1}{\tau} \int_{\{x\in K:\alpha(x)<\tau\}} \int_L q(\phi(x,\alpha(x)), \mathrm{d}y)(\theta^{(K)} - \theta^{(L)})\mathrm{d}x \\ &\geq -\frac{1}{\tau} \int_{\{x\in K:\alpha(x)<\tau\}} \int_{\{y\in F:|y|>|\phi(x,\alpha(x))|+R_\varepsilon-2-L_\alpha\}} q(\phi(x,\alpha(x)), \mathrm{d}y)\mathrm{d}x \\ &\geq -q_K\varepsilon.\end{aligned}$$

We again have

$$\sum_{L\in\mathcal{M}_2} q_{KL} e^{-B_0\alpha_L}(\theta^{(K)} - \theta^{(L)}) \geq 0,$$

and

$$\sum_{L\in\mathcal{M}_3} q_{KL} e^{-B_0\alpha_L}(\theta^{(K)} - \theta^{(L)}) \geq -q_K A R_\varepsilon.$$

It gives

$$T_{32} \geq -C_\sigma(\varepsilon + AR_\varepsilon) = -2C_\sigma\varepsilon,$$

thanks to the choice of A.
Turning to T_{31}, we easily get

$$T_{31} \geq C_\sigma(e^{-B_0\tau} - 1).$$

For any $x \in K$ and $y \in L$ such that there exists $x_0 \in K$ with $\phi(x_0, \tau) \in L$, we have

$$|x - y| \leq 2h + L_\Phi\tau.$$

Since a Lipschitz constant of the function $x \mapsto e^{-B_0\alpha(x)}$ is equal to $B_0 L_\alpha$ and that of the function $x \mapsto 1 - \exp(-A(|x| - R_0)^+)$ is A and denoting by $Y(s, r)$ the function such that $Y(s, r) = 1$ if $s > r$ and 0 if $s \leq r$, we get

$$\widehat{\theta}^{(L)} - \widehat{\theta}^{(K)} \leq (B_0L_\alpha + A)(2h + L_\Phi\tau)Y(R_K - (2h + L_\Phi\tau), R_0) \leq (B_0L_\alpha + A)(2h + L_\Phi\tau)Y(R_K - (2 + L_\Phi), R_0).$$

This implies that

$$\sum_{L\in\mathcal{M}} v_{KL}(\widehat{\theta}^{(L)} - \widehat{\theta}^{(K)}) \leq |K|(B_0L_\alpha + A)\left(2\frac{h}{\tau} + L_\Phi\right) Y(R_K - (2 + L_\Phi), R_0),$$

which leads, using (33), to

$$\sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \delta t \sum_{L\in\mathcal{M}} v_{KL}(\widehat{\theta}^{(L)} - \widehat{\theta}^{(K)}) \le \varepsilon(B_0 L_\alpha + A)(2\frac{h}{\tau} + L_\Phi).$$

We now observe that

$$\sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \sum_{L\in\mathcal{M}} \lambda_{KL}(\widehat{\theta}^{(L)} - \widehat{\theta}^{(K)}) \le \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \sum_{L\in\mathcal{M}} \lambda_{KL}\widehat{\theta}^{(L)} \le T_1 + T_2,$$

with

$$T_1 = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \sum_{L\in\mathcal{M}, R_L\ge R_K+R_Q} \lambda_{KL}\widehat{\theta}^{(L)},$$

$$T_2 = \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \sum_{L\in\mathcal{M}, R_L< R_K+R_Q} \lambda_{KL}\widehat{\theta}^{(L)}.$$

Considering T_1, we have, bounding $\widehat{\theta}^{(L)}$ by 1, using (32) and $|y| - |x| \ge R_L - R_K - 2h$ for $y \in L$ and $x \in K$, we get for $h < 1$,

$$\sum_{L\in\mathcal{M}, R_L\ge R_K+R_Q} \lambda_{KL}\widehat{\theta}^{(L)} = \int_K \lambda(x) \int_{\{y\in F: |y|\ge|x|+R_Q-2\}} Q(x, \mathrm{d}y) \le \Lambda|K|\varepsilon$$

which implies

$$T_1 \le T\Lambda\varepsilon.$$

Turning to T_2, we have, for L such that $R_L < R_K + R_Q$,

$$\widehat{\theta}^{(L)} \le (1 - \exp(-A(R_L - R_0)^+)) \le (1 - \exp(-A(R_K + R_Q - R_0)^+)) = (1 - \exp(-A(R_K - R_\rho)^+))$$

and therefore, since $\displaystyle\sum_{L\in\mathcal{M}, R_L< R_K+R_Q} \lambda_{KL} \le \lambda_K \le \Lambda|K|$,

$$T_2 \le \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \Lambda|K|(1 - \exp(-A(R_K - R_\rho)^+)) \le \Lambda\varepsilon,$$

thanks to (33).
Gathering the above results, we get

$$\sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}, R_K\ge R_1} p_{n+1}^{(K)} \delta t q_K \left(\left(1 - \frac{a_0}{8}\right) e^{-B_0 L_\phi \tau} - (1-a_0)(1 + B_0 L_\alpha h e^{B_0 L_\alpha h})\right) \le C_\sigma(1-e^{-B_0\tau}) + 2C_\sigma\varepsilon$$
$$+ \left(1 + \varepsilon(B_0 L_\alpha + A)\left(2\frac{h}{\tau} + L_\Phi\right) + \Lambda(T+1)\right).$$

Choosing η such that, for $s \le \eta$, we have $B_0 L_\alpha s e^{B_0 L_\alpha s} \le \dfrac{a_0}{4}$ and $e^{-B_0 L_\phi s} \ge 1 - \dfrac{a_0}{8}$, we get, for $h \le \tau \le \eta$,

$$\frac{a_0}{2} \sum_{n=0}^{N-1} \sum_{K\in\mathcal{M}, R_K\ge R_1} p_{n+1}^{(K)} \delta t q_K \le C_\sigma(1 - e^{-B_0\tau}) + \varepsilon(2C_\sigma + 1 + (B_0 L_\alpha + A)(2 + L_\Phi) + \Lambda(T+1)).$$

For a sequence of discretizations $(\mathcal{D}_m)_{m\in\mathbb{N}}$ such that τ_m tends to zero, we split the sequence in two parts:

1. the first one is such that $1 - e^{-B_0\tau_m} \le \varepsilon$,
2. the second one is finite, and therefore tight.

It now suffices to take a radius greater than R_1 such that the tightness of the measures $\sigma_{\mathcal{D}_m}$ is also controlled by ε for concluding the proof. □

3.3 Convergence analysis

In the proof of convergence of the numerical scheme, we need test functions which are more regular than the elements of $\mathcal{T}$. So we define a new test space $\mathcal{T}_r \subset \mathcal{T}$ which is shown to approximate the elements of $\mathcal{T}$ in Lemma 6 and such that its elements have some regularity properties (see Lemma 7).

Definition 2 *Let $C^c_{\infty,b}(\mathbb{R}^d \times \mathbb{R}_+)$ be the set of infinitely differentiable bounded functions with bounded derivatives and compact support in time.*
We denote by $\mathcal{T}_r$ the set of all functions $g = \mathbb{T}(I, J)$ such that I and J belong to $C^c_{\infty,b}(\mathbb{R}^d \times \mathbb{R}_+)$.

Lemma 6 *Let $\nu \in C_b(\mathbb{R}^d \times \mathbb{R}_-)$ be a non negative infinitely differentiable function with compact support and such that*

$$\int_{\mathbb{R}^d\times\mathbb{R}_-} \nu(x,t)\mathrm{d}x\mathrm{d}t = 1.$$

For any $f \in \mathcal{C}^c_b(\mathbb{R}^d \times \mathbb{R}_+)$ and all $n \in \mathbb{N}^\star$, let us denote by $R_n f \in \mathcal{C}^c_b(\mathbb{R}^d \times \mathbb{R}_+)$ the function defined by

$$R_n f(x,t) = n^{d+1} \int_{\mathbb{R}^d\times\mathbb{R}_+} f(y,s)\nu(n(x-y), n(t-s))\mathrm{d}y\mathrm{d}s.$$

Then

1. *If $I(x,t) = J(x,t) = 0$ for all $(x,t) \in \mathbb{R}^d \times [T, +\infty[$, then $T(I,J)(x,t) = 0$ for all $(x,t) \in \mathbb{R}^d \times [T, +\infty[$ and $||T(I,J)||_\infty \le ||J||_\infty + T||I||_\infty$.*
2. *If $I, J \in C^c_b(\mathbb{R}^d \times \mathbb{R}_+)$ are Lipschitz continuous, then $T(I,J)$ is Lipschitz continuous.*
3. *For all $f \in C^c_b(\mathbb{R}^d \times \mathbb{R}_+)$, $||R_n f||_\infty \le ||f||_\infty$, $R_n f$ is infinitely differentiable, its partial derivatives are bounded and for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$, $R_n f(x,t)$ converges to $f(x,t)$ as $n \to +\infty$.*
4. *If $I, J \in C^c_b(\mathbb{R}^d \times \mathbb{R}_+)$ then $T(R_n I, R_n J)(x,t)$ converges to $T(I,J)(x,t)$ as $n \to +\infty$.*

Proof. Assertion 1 is straightforward.
For assertion 2, we denote by L_I and L_J Lipschitz constants of I and J respectively and T such that $I(x,t) = J(x,t) = 0$ for $x \in \mathbb{R}^d; t \ge T$. We get:

$$\begin{aligned}
|T(I,J)(x_2,t_2) - T(I,J)(x_1,t_1)| &\le |J(\phi(x_2,\alpha(x_2)), t+\alpha(x_2)) - J(\phi(x_1,\alpha(x_1), t+\alpha(x_1))| \\
&+ \int_0^{\alpha(x_2)} |I(\phi(x_2,s), t_2+s) - I(\phi(x_1,s), t_1+s)|\,\mathrm{d}s + \left|\int_{\alpha(x_1)}^{\alpha(x_2)} I(\phi(x_1,s), t_1+s)\,\mathrm{d}s\right| \\
\le\ & L_J L_\phi(1+L_\alpha)|x_2-x_1| + L_J|x_2-x_1| + T L_I L_\phi |x_2-x_1| + L_I|t_2-t_1| + ||I_\infty|| L_\alpha |x_2-x_1|.
\end{aligned}$$

Assertion 3 is a standard consequence of the regularization by convolution.
Assertion 4 is a consequence of Assertion 3 and of the Lebesgue dominated convergence theorem. □

Lemma 7 *Let $g \in \mathcal{T}_r$ be given. Then*

1. *g is Lipschitz continuous on $\mathbb{R}^d \times \mathbb{R}_+$,*
2. *$\partial_t g$ is Lipschitz continuous on $\mathbb{R}^d \times \mathbb{R}_+$ (we then denote $L_{t,g}$ a Lipschitz constant for $\partial_t g$),*
3. *the function $\partial_\phi g$ defined on $\overline{F} \times \mathbb{R}_+$ by $\partial_\phi g = \partial_{t,\phi} g - \partial_t g$ is Lipschitz continuous and bounded on $\overline{F} \times \mathbb{R}_+$ and there exists $L_{\phi,g}$, only depending on g and ϕ, such that, for all $(x,t) \in \overline{F} \times \mathbb{R}^+$ and $\tau \in (0, \alpha(x)]$, then*
$$\left|\frac{1}{\tau}(g(\phi(x,\tau),t) - g(x,t)) - \partial_\phi g(x,t)\right| \le \tau L_{\phi,g}. \tag{35}$$

Proof. Item 1 is a consequence of Lemma 6. Let $I, J \in C^c_{\infty,b}(\mathbb{R}^d \times \mathbb{R}_+)$ be such that $g = \mathbb{T}(I, J)$. Item 2 results from

$$\partial_t g(x,t) = \partial_t J(\Phi(x,\alpha(x)), t+\alpha(x)) - \int_0^{\alpha(x)} \partial_t I(\Phi(x,s), t+s) ds,$$

which shows that $\partial_t g = \mathbb{T}(\partial_t I, \partial_t J)$. Since $\partial_t I$ and $\partial_t J$ belong to $C^c_{\infty,b}(\mathbb{R}^d \times \mathbb{R}_+)$ as well, we get that $\partial_t g \in \mathcal{T}_r$, and is therefore Lipschitz continuous and bounded.
Turning to Item 3, we get that $\partial_\phi g$ is the difference between two bounded Lipschitz continuous functions. We now write, for $(x,t) \in \overline{F} \times \mathbb{R}^+$ and $\tau \in [0, \alpha(x)]$,

$$g(\phi(x,\tau),t) - g(x,t) = g(\phi(x,\tau), t+\tau) - g(x,t) - (g(\phi(x,\tau), t+\tau) - g(\phi(x,\tau),t)),$$

which leads to

$$g(\phi(x,\tau),t) - g(x,t) = \int_0^\tau \partial_{t,\phi} g(\phi(x,s), t+s) ds - \int_0^\tau \partial_t g(\phi(x,\tau), t+s) \mathrm{d}s.$$

We thus obtain

$$\begin{aligned} &g(\phi(x,\tau),t) - g(x,t) - \tau \partial_\phi g(x,t) \\ &= \int_0^\tau (\partial_{t,\phi} g(\phi(x,s), t+s) - \partial_{t,\phi} g(x,t)) ds - \int_0^\tau (\partial_t g(\phi(x,\tau), t+s) - \partial_t g(x,t)) \mathrm{d}s. \end{aligned}$$

Since the functions ϕ, $\partial_{t,\phi} g$ and $\partial_t g$ are Lipschitz continuous, there exists $L_{\phi,g}$, only depending on g and ϕ, such that

$$|\partial_{t,\phi} g(\phi(x,s), t+s) - \partial_{t,\phi} g(x,t)| \le \frac{1}{2} L_{\phi,g} \tau,$$

and

$$|\partial_t g(\phi(x,\tau), t+s) - \partial_t g(x,t)| \le \frac{1}{2} L_{\phi,g} \tau,$$

which completes the proof of (35). □
We now state a first convergence result

Theorem 2 *Under hypotheses (H) and (HD), let $T > \delta t$. Let $(p_n^{(K)})_{i \in E, n \in \mathbb{N}, K \in \mathcal{M}}$ be the solution of Scheme* (13), (14), (15), (16), (17) *which satisfies* (23)-(24).
Then, assuming $\delta t \to 0$, $\tau \to 0$, and $h/\tau \to 0$, $P_\mathcal{D}(\mathrm{d}x, \mathrm{d}t)$ (resp. $\sigma_\mathcal{D}(\mathrm{d}x, \mathrm{d}t)$) tends to μ (resp. σ) unique solution of equation (6) *for the weak topology of the measures.*

Proof. Thanks to Lemma 3, we may assume that, up to the definition of a sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ that, $P_{\mathcal{D}_m}(\mathrm{d}x, \mathrm{d}t)$ (resp. $\sigma_{\mathcal{D}_m}(\mathrm{d}x, \mathrm{d}t)$) tend to some bounded measure μ (resp. σ) for the weak topology of the measures. We assume that $\tau < 1$, $h/\tau < 1$ and $\delta t < 1$. Thanks to Lemma 6, it suffices to prove that (6) holds for μ, σ and $g \in \mathcal{T}_r$.
Let L_g be a global Lipschitz constant for g and a bound of $\partial_\phi g$, let T be such that $g(x,t) = 0$ for all $t \ge T$ and let $N \in \mathbb{N}$ be such that $(N-1)\delta t \le T < N\delta t$. We multiply (16) by $\delta t \; g_n^{(K)}$, with

$$g_n^{(K)} = \frac{1}{|K|} \int_K g(x, n\delta t) \mathrm{d}x, \; \forall K \in \mathcal{M}, \; \forall n \in \mathbb{N}. \tag{36}$$

We sum on $K \in \mathcal{M}$ and $n \in \mathbb{N}$. We get $T_1^m + T_2^m + T_3^m + T_4^m + T_5^m = 0$, with

$$T_1^m = -\sum_{K \in \mathcal{M}} |K| p_0^{(K)} g_0^{(K)},$$

$$T_2^m = -\sum_{n \in \mathbb{N}} \sum_{K \in \mathcal{M}} |K| p_{n+1}^{(K)} (g_{n+1}^{(K)} - g_n^{(K)}) = -\sum_{n \in \mathbb{N}} \sum_{K \in \mathcal{M}} p_{n+1}^{(K)} \int_K \int_0^{\delta t} \partial_t g(x, n\delta t + s) ds dx,$$

$$T_3^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{M}} (v_{KL} p_{n+1}^{(K)} - v_{LK} p_{n+1}^{(L)}) g_n^{(K)},$$

$$T_4^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{M}} \lambda_{KL} p_{n+1}^{(K)} (g_n^{(K)} - g_n^{(L)}),$$

$$T_5^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{M}} q_{KL} p_{n+1}^{(K)} (g_n^{(K)} - g_n^{(L)}).$$

Study of the limit of T_1^m.

Since g is Lipschitz continuous, we get

$$|\sum_{K\in\mathcal{M}} \int_K g(x,0)\rho_{\rm ini}(\mathrm{d}x) + T_1^m| = |\sum_{K\in\mathcal{M}} \int_K (g(x,0) - g_0^{(K)})\rho_{\rm ini}(\mathrm{d}x)| \le L_g h,$$

which implies

$$\lim_{m\to\infty} T_1^m = -\int_F g(x,0)\rho_{\rm ini}(\mathrm{d}x).$$

Study of the limit of T_2^m.

Let us define T_6^m by

$$T_6^m = -\sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} |K| p_{n+1}^{(K)} \int_K \partial_t g(x, n\delta t)\mathrm{d}x = -\int_{\overline{F}\times\mathbb{R}_+} \partial_t g(x,t)\mathcal{P}_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t).$$

Since $\mathcal{P}_{\mathcal{D}}$ is tight and $\partial_t g$ is continuous and bounded, we obtain that

$$\lim_{m\to\infty} T_6^m = -\int_{\overline{F}\times\mathbb{R}_+} \partial_t g(x,t)\mu(\mathrm{d}x,\mathrm{d}t).$$

Thanks to Lemma 7, for all $s\in[0,\delta t]$, we have $|\partial_t g(x, n\delta t + s) - \partial_t g(x, n\delta t)| \le L_{t,g}\delta t$. Therefore we deduce that $\lim_{m\to\infty} |T_2^m - T_6^m| = 0$, which implies

$$\lim_{m\to\infty} T_2^m = -\int_{\overline{F}\times\mathbb{R}_+} \partial_t g(x,t)\mu(\mathrm{d}x,\mathrm{d}t).$$

Study of the limit of T_3^m.

Defining $T_7^{(K)}$ by

$$T_7^{(K)} = \sum_{L\in\mathcal{M}} v_{KL}(g_n^{(K)} - g_n^{(L)}),$$

we have

$$T_3^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} T_7^{(K)}.$$

Let us compute $T_8^{(K)} - T_7^{(K)}$ with $T_8^{(K)}$ defined by

$$T_8^{(K)} = -\int_{\{x\in K, \alpha(x)>\tau\}} \partial_\phi g(x, n\delta t)\mathrm{d}x.$$

Defining $T_9^{(K)}$ by

$$T_9^{(K)} = \int_{\{x\in K, \alpha(x)>\tau\}} \frac{1}{\tau}(g(x,n\delta t) - g(\phi(x,\tau), n\delta t))\mathrm{d}x,$$

we have

$$T_9^{(K)} = -\sum_{L\in\mathcal{M}} \int_{\{x\in K,\alpha(x)>\tau \text{ and } \Phi(x,\tau)\in L\}} \frac{1}{\tau}(g(x,n\delta t) - g(\phi(x,\tau),n\delta t))\mathrm{d}x.$$

Introducing the Lipschitz constant L_g of g, we get

$$|T_9^{(K)} - T_7^{(K)}| \le 2|K|\frac{L_g h}{\tau}.$$

Applying Lemma 7, we can write

$$|T_8^{(K)} - T_9^{(K)}| \le L_{\phi,g}\tau|K|,$$

which finally gives

$$|T_8^{(K)} - T_7^{(K)}| \le L_{\phi,g}\tau|K| + 2|K|\frac{L_g h}{\tau}.$$

Hence, defining T_{10}^m by

$$T_{10}^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} T_8^{(K)},$$

we get

$$|T_{10}^m - T_3^m| \le (2\frac{L_g h}{\tau} + L_{\phi,g}\tau)(T+\delta t).$$

We now remark that, defining $T_{11}^{(K)}$ by

$$T_{11}^{(K)} = -\int_{\{x\in K:\alpha(x)\le\tau\}} \partial_\phi g(x,n\delta t)\mathrm{d}x,$$

and T_{12}^m by

$$T_{12}^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} T_{11}^{(K)},$$

we have

$$|T_{12}^m| \le \tau L_g \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} q_K p_{n+1}^{(K)}.$$

This shows, thanks to (26), that

$$\lim_{m\to\infty} T_{12}^m = 0.$$

Now

$$T_{10}^m + T_{12}^m = -\int_{F\times\mathbb{R}_+} \partial_\phi g(x,t)\mathcal{P}_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t).$$

Thanks to the tightness of $\mathcal{P}_{\mathcal{D}}$, we have

$$\lim_{m\to\infty}(T_{10}^m + T_{12}^m) = \int_{\overline{F}\times\mathbb{R}_+} \partial_\phi g(x,t)\mu(\mathrm{d}x,\mathrm{d}t).$$

Therefore, we conclude that

$$\lim_{m\to\infty} T_3^m = -\int_{\overline{F}\times\mathbb{R}_+} \partial_\phi g(x,t)\mu(\mathrm{d}x,\mathrm{d}t).$$

Study of the limit of T_4^m.

We compare this term with T_{13}^m defined by

$$\begin{aligned} T_{13}^m &= \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} \int_K \lambda(x)\left(g(x,n\delta t) - \int_F g(y,n\delta t)Q(x,\mathrm{d}y)\right)\mathrm{d}x\ p_{n+1}^{(K)} \\ &= \int_{F\times\mathbb{R}_+} \lambda(x)\left(g(x,t) - \int_F g(y,t)Q(x,\mathrm{d}y)\right)\mathcal{P}_{\mathcal{D}}(\mathrm{d}x,\mathrm{d}t), \end{aligned}$$

which satisfies, thanks to the tightness of $\mathcal{P}_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t)$,

$$\lim_{m\to\infty} T_{13}^m = \int_{\overline{F}\times\mathbb{R}_+} \lambda(x) \left(g(x,t) - \int_F g(y,t) Q(x,\mathrm{d}y) \right) \mu(\mathrm{d}x, \mathrm{d}t).$$

We have

$$T_{13}^m - T_4^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} \int_K \lambda(x) \Big((g(x, n\delta t) - g_n^{(K)}) - \sum_{L\in\mathcal{M}} \int_L (g(y, n\delta t) - g_n^{(L)}) Q(x, \mathrm{d}y) \Big) \mathrm{d}x \; p_{n+1}^{(K)}.$$

This shows that

$$|T_{13}^m - T_4^m| \le L_g h \sum_{n=0}^{N-1} \delta t \sum_{K\in\mathcal{M}} \lambda_K p_{n+1}^{(K)} \le L_g h (T + \delta t) \; \Lambda,$$

showing that

$$\lim_{m\to\infty} |T_{13}^m - T_4^m| = 0,$$

hence proving that

$$\lim_{m\to\infty} T_4^m = \int_{\overline{F}\times\mathbb{R}_+} \lambda(x) \left(g(x,t) - \int_F g(y,t) Q(x,\mathrm{d}y) \right) \mu(\mathrm{d}x, \mathrm{d}t).$$

Study of the limit of T_5^m.

We use the measure $\sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t)$, defined by (21). We define T_{14}^m by

$$T_{14}^m = \int_{\Gamma\times\mathbb{R}_+} \left(g(x,t) - \int_F g(y,t) q(x,\mathrm{d}y) \right) \sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t),$$

which satisfies, since the sequence of measures $\sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t)$ is tight,

$$\lim_{m\to\infty} T_{14}^m = \int_{\Gamma\times\mathbb{R}_+} \left(g(x,t) - \int_F g(y,t) q(x,\mathrm{d}y) \right) \sigma(\mathrm{d}x, \mathrm{d}t).$$

We have

$$T_{14}^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \frac{1}{\tau} \int_{\{x\in K:\alpha(x)\le\tau\}} \left(g(\phi(x,\alpha(x)), n\delta t) - \int_F g(y, n\delta t) q(\phi(x,\alpha(x)), \mathrm{d}y) \right) \mathrm{d}x.$$

Therefore, we get

$$T_5^m - T_{14}^m = \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)}$$
$$\frac{1}{\tau} \int_{\{x\in K:\alpha(x)\le\tau\}} \left(g_n^{(K)} - g(\phi(x,\alpha(x)), n\delta t) - \sum_{L\in\mathcal{M}} \int_L (g_n^{(L)} - g(y, n\delta t)) q(\phi(x,\alpha(x)), \mathrm{d}y) \right) \mathrm{d}x.$$

Remarking that

$$\forall x, y \in K, \; |\phi(x,\alpha(x)) - y| \le h + L_\phi \tau,$$

this leads to

$$|T_5^m - T_{14}^m| \le (2h + L_\phi \tau) L_g \sum_{n\in\mathbb{N}} \delta t \sum_{K\in\mathcal{M}} q_K p_{n+1}^{(K)},$$

and therefore

$$\lim_{m\to\infty} |T_5^m - T_{14}^m| = 0.$$

We thus get that

$$\lim_{m\to\infty} T_5^m = \int_{\Gamma\times\mathbb{R}_+} \left(g(x,t) - \int_F g(y,t) q(x,\mathrm{d}y) \right) \sigma(\mathrm{d}x, \mathrm{d}t).$$

□

Theorem 3 *Under Hypotheses (H) and (HD), let $(p_n^{(K)})_{n\in\mathbb{N},K\in\mathcal{M}}$ be the solution of Scheme* (13)*,* (14)*,* (15)*,* (16)*,* (17) *which satisfies* (23)*-*(24)*.*
Then, assuming $\delta t \to 0$, $\tau \to 0$, and $h/\tau \to 0$, $P_{\mathcal{D}}(t)\mathrm{d}x$ (defined by (22)*) tends to ρ_t, unique solution of* (5) *(or* (41)*), for the weak topology of the measures, for almost every $t > 0$.*

Proof. Thanks to Theorem 2, we may assume that, up to the definition of a sequence $(\mathcal{D}_m)_{m\in\mathbb{N}}$ that, $P_{\mathcal{D}_m}(\mathrm{d}x, \mathrm{d}t)$ (resp. $\sigma_{\mathcal{D}_m}(\mathrm{d}x, \mathrm{d}t)$) tend to measure μ (resp. σ) for the weak topology of the measures, with $h_m/\tau_m \le 1$, $h_m \le 1$, $\tau_m \le 1$, $\delta t_m \le 1$.

Step 1

Let $g \in C_b(\mathbb{R}^d)$ be a Lipschitz continuous function, such that L_g is both a bound of g and a Lipschitz constant for g, and let $T > 0$ be given. We consider the function $f_g^m : (0,T) \to \mathbb{R}$, $f_g^m(t) = \int_{\mathbb{R}^d} g(x)P_{\mathcal{D}}(t)\mathrm{d}x$. We prolong this function by 0 outside of $(0,T)$. Let $N_T \in \mathbb{N}$ be such that $N_T\delta t \le T < (N_T+1)\delta t$. Since this function is defined by (22) as a piecewise constant function on all $]n\delta t, (n+1)\delta t[$, its $BV((0,T))$-norm is such that

$$\|f_g^m\|_{BV((0,T))} \le \sum_{n=0}^{N_T} \left| \sum_{K\in\mathcal{M}} |K|(p_{n+1}^{(K)} - p_n^{(K)})g^{(K)} \right|, \tag{37}$$

where

$$g^{(K)} = \frac{1}{|K|}\int_K g(x)\mathrm{d}x, \ \forall K \in \mathcal{M}. \tag{38}$$

Note that, if we m ultiply (16) by $\delta t\, g^{(K)}$, we get that

$$\sum_{n=0}^{N_T} \left| \sum_{K\in\mathcal{M}} |K|(p_{n+1}^{(K)} - p_n^{(K)})g^{(K)} \right| \le T_3^m + T_4^m + T_5^m,$$

with

$$T_3^m = \sum_{n=0}^{N_T} \delta t \sum_{K\in\mathcal{M}} p_{n+1}^{(K)} \sum_{L\in\mathcal{M}} v_{KL}|g^{(K)} - g^{(L)}|,$$

$$T_4^m = \sum_{n=0}^{N_T} \delta t \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{M}} \lambda_{KL} p_{n+1}^{(K)}|g^{(K)} - g^{(L)}|,$$

$$T_5^m = \sum_{n=0}^{N_T} \sum_{n=n_0+1}^{n_1} \delta t \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{M}} q_{KL} p_{n+1}^{(K)}|g^{(K)} - g^{(L)}|.$$

Since g is Lipschitz continuous, we have

$$\frac{1}{\tau}|g^{(K)} - g^{(L)}| \le L_g \frac{1}{\tau}(2h + L_\phi\tau) \le L_g(2 + L_\phi),$$

we get

$$T_3^m \le L_g(2 + L_\phi) \int_{F\times[0,T]} \mathcal{P}_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t).$$

We also have

$$T_4^m \le 2L_g \sum_{n=0}^{N_T} \delta t \sum_{K\in\mathcal{M}} \lambda_K p_{n+1}^{(K)} \le 2AL_g \int_{F\times[0,T]} \mathcal{P}_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t).$$

Finally, we may write

$$T_5^m \le 2L_g \sum_{n=0}^{N_T} \delta t \sum_{K\in\mathcal{M}} \sum_{L\in\mathcal{M}} q_{KL} p_{n+1}^{(K)} = 2L_g \int_{\overline{\Gamma}\times[0,T]} \sigma_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}t).$$

Gathering the above results and applying Lemma 3 , we get that there exists C_1 , which does not depend on the discretization data, such that

$$\|f_g^m\|_{BV((0,T))} \leq C_1 \, .$$

Applying Helly's theorem of relative compactness of bounded sets in $L^\infty \cap BV$, this proves that, up to a subsequence, f_g^m converges in L^1 to some function. This fu nction can only be $\overline{f}_g \ : \ (0,T) \to \mathbb{R}$, $\overline{f}_g(t) = \int_{\mathbb{R}^d} g(x)\rho_t(\mathrm{d}x)$: indeed, thanks to Theorem 2 and to decomposition result Lemma 8, we get that, for all continuous function ψ, the limit of $\int_0^T \psi(t) f_g^m(t)\mathrm{d}t$, easily approximated by $\int_0^T \int_{\mathbb{R}^d} \psi(t)g(x)\mathcal{P}_{\mathcal{D}}(\mathrm{d}x, \mathrm{d}s)$, is equal to $\int_0^T \int_{\mathbb{R}^d} \psi(t)g(x)\rho_t(\mathrm{d}x)\mathrm{d}t = \int_0^T \psi(t)\overline{f}_g(t)\mathrm{d}t$. Therefore the whole sequence f_g^m converges in $L^1((0,T))$ to $\overline{f}_g$.

Step 2

By density of the set of Lipschitz continuous functions in $C_b(\mathbb{R}^d)$, we get that for all $\xi \in C_b(\mathbb{R}^d)$, $f_\xi^m \ : \ (0,T) \to \mathbb{R}$, $f_\xi^m(t) = \int_{\mathbb{R}^d} \xi(x)P_{\mathcal{D}}(t)\mathrm{d}x$ converges in $L^1((0,T))$ to $\overline{f}_\xi \ : \ (0,T) \to \mathbb{R}$, $\overline{f}_\xi(t) = \int_{\mathbb{R}^d} \xi(x)\rho_t(\mathrm{d}x)$. Indeed, it suffices to consider a sequence lipschitz c ontinuou s functions uniformly converging to ξ.

Step 3

We consider a dense countable family ξ_n in the set $C_0(\mathbb{R}^d)$ (functions which tend to 0 at infinity). For $n = 1$, we extract a subsequence samely denoted, and a set $A_1 \subset (0,T)$ whose complementary in $(0,T)$ has a zero measure, such that for all $t \in A_1$ and $n \in \mathbb{N}$, $f_{\xi_1}^m(t)$ converges to $\overline{f}_{\xi_1}(t)$. From this sequence, we consider $n = 2$, and we extract a subsequence samely denoted, and a set $A_2 \subset A_1$ whose complementary in $(0,T)$ has a zero measure, such that for all $t \in A_2$ and $n \in \mathbb{N}$, $f_{\xi_2}^m(t)$ converges to $\overline{f}_{\xi_2}(t)$. Letting $A = \bigcap_{n\in\mathbb{N}}$ (whose complementary in $(0,T)$ has a zero measure), the sequence such constructed by a diagonal process is such that, for all $t \in A$ and $n \in \mathbb{N}$, $f_{\xi_n}^m(t)$ converges to $\overline{f}_{\xi_n}(t)$. By density, this achieves the proof of the convergence, for a.e. $t \in (0,t)$ and $\xi \in C_0(\mathbb{R}^d)$, of $f_\xi^m(t)$ converges to $\overline{f}_\xi(t)$ and the proof of the theorem. $\square$

4 Conclusion

Most of the practical cases require hypotheses more general than those made in this paper. In this way it is possible to rewrite the proofs in the case where F is an hybrid space and where α is infinite on some subsets (see Remark 1). Furthermore the actual form of Hypothesis (H.5(c)) is in fact restrictive since we expect that the process will be far away the boundary only after one jump at the boundary. It is possible to rewrite the proofs with a more general hypothesis including several jumps (see Remark 1). We have chosen these hypotheses in order to focus on the main ideas in the proofs, but the numerical scheme seems effective in practical cases. The case where jumps on the boundary are allowed, which is useful in some applications (see [3]) is still open.

It now remains to test the numerical method developed here for Piecewise Deterministic Markovian Processes on various practical and theoretical cases with boundaries, in comparison with the classical Monte-Carlo approach (in someway, this has already been done in [9]). Let us emphasize here the advantage of this method, which is robust with respect to large values of δt, and which therefore permits to approximate the asymptotic stationary states of a PDMP.

Appendix

In this appendix some properties of the solution of equation (6) are proved thanks to analysis tools, without using the probabilistic results provided in the introduction of this paper.

Lemma 8 (Decomposition) *Let μ and σ be a solution of equation* (6)*. Under Hypotheses (H), for any $T > 0$, let ρ_T be the finite measure defined by*

$$\int_{\mathbb{R}^d} \xi(x)\rho_T(\mathrm{d}x) = \int_F \xi(\phi(x,T))\rho_{\mathrm{ini}}(\mathrm{d}x)$$
$$+ \int_{\overline{F}\times[0,T)} \lambda(x) \left(\int_F \xi(\phi(y, T-t))Q(x, \mathrm{d}y) - \xi(\phi(x, T-t)) \right) \mu(\mathrm{d}x, \mathrm{d}t)$$
$$+ \int_{\Gamma\times[0,T)} \left(\int_F \xi(\phi(y, T-t))q(x, \mathrm{d}y) - \xi(\phi(x, T-t)) \right) \sigma(\mathrm{d}x, \mathrm{d}t),\ \forall \xi \in \mathcal{C}_b(\mathbb{R}^d). \tag{39}$$

Then $\rho_T \in \mathcal{P}(\mathbb{R}^d)$ and, for all $f \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{R}_+, \mathbb{R})$, we have

$$\int_{\overline{F}\times\mathbb{R}_+} f(x,t)\mu(\mathrm{d}x, \mathrm{d}t) = \int_{\mathbb{R}_+} \int_{\overline{F}} f(x,T)\rho_T(\mathrm{d}x)\mathrm{d}T. \tag{40}$$

Moreover, we have, for all $g \in \mathcal{T}$ with $g = \mathbb{T}(I, J)$,

$$\int_F g(\cdot, T)\rho_T(\mathrm{d}x) = \int_F g(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x) + \int_{\overline{F}\times[0,T)} I(x,t)\mu(\mathrm{d}x, \mathrm{d}t)$$
$$+ \int_{\overline{F}\times[0,T)} \lambda(x) \left(\int_F g(y,t)Q(x, \mathrm{d}y) - g(x,t) \right) \mu(\mathrm{d}x, \mathrm{d}t)$$
$$+ \int_{\Gamma\times[0,T)} \left(\int_F g(y,t)q(x, \mathrm{d}y) - g(x,t) \right) \sigma(\mathrm{d}x, \mathrm{d}t). \tag{41}$$

Proof.
Let us first show that $\rho_T \in \mathcal{P}(\mathbb{R}^d)$. Let $\varphi \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}_+)$ such that $\int_{\mathbb{R}} \varphi(x)\mathrm{d}x = 1$ and $\varphi(x) = 0$ for all $x < -1$ and $x > 0$. Next, define the family $(\psi_n)_{n\in\mathbb{N}^*}$ such that

$$\psi_n(t) = n \int_t^{+\infty} \varphi(n(x-T))\,\mathrm{d}x = \int_{n(t-T)}^{+\infty} \varphi(v)\mathrm{d}v$$

So ψ_n is decreasing such that $\psi_n(t) = 1$ for all $t \leq T - \dfrac{1}{n}$ and $\psi_n(t) = 0$ for all $t \geq T$. Moreover $\psi_n'(t) = -n\varphi(n(t-T))$.
We then have that, for all $\xi \in \mathcal{C}_b(\mathbb{R}^d)$, the function $\xi_n(x,t) = \psi_n(t)\xi(\phi(x,(T-t)^+))$ is such that $\xi_n = \mathbb{T}(I_n, J_n)$, with $I_n(x,t) = \psi_n'(t)\xi(\phi(x,(T-t)^+))$ and $J_n = \xi_n$. By dominated convergence, we get

$$\rho_T(\xi) = \lim_{n\to\infty} \rho_{n,T}(\xi)$$

with

$$\rho_{n,T}(\xi) = \int_F \psi_n(0)\xi(\phi(x,T))\rho_{\mathrm{ini}}(\mathrm{d}x)$$
$$+ \int_{\overline{F}\times[0,T)} \lambda(x)\psi_n(t) \left(\int_F \xi(\phi(y, T-t))Q(x, \mathrm{d}y) - \xi(\phi(x, T-t)) \right) \mu(\mathrm{d}x, \mathrm{d}t)$$
$$+ \int_{\Gamma\times[0,T)} \psi_n(t) \left(\int_F \xi(\phi(y, T-t))q(x, \mathrm{d}y) - \xi(\phi(x, T-t)) \right) \sigma(\mathrm{d}x, \mathrm{d}t),$$

that, using (6) with $g = \xi_n$,

$$\rho_T(\xi) = -\lim_{n\to\infty} \int_{\overline{F}\times\mathbb{R}_+} \psi_n'(t)\xi(\phi(x,(T-t)^+))\mu(\mathrm{d}x,\mathrm{d}t).$$

This proves that ρ_T is positive. Moreover its support is included in $\overline{F}$.

Remarking that, for any $g = \mathbb{T}(I, J)$, we have $\psi_n g = \mathbb{T}(\psi_n' g + \psi_n I, \psi_n J)$, we get from (6) that

$$\begin{aligned} -\int_{\overline{F}\times\mathbb{R}_+} \psi_n'(t)g(x,t)\mu(\mathrm{d}x,\mathrm{d}t) &= \int_F g(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x) + \int_{\overline{F}\times\mathbb{R}_+} \psi_n(t)I(x,t)\mu(\mathrm{d}x,\mathrm{d}t) \\ &+ \int_{\overline{F}\times[0,T)} \lambda(x)\psi_n(t)\left(\int_F g(y,t)Q(x,\mathrm{d}y) - g(x,t)\right)\mu(\mathrm{d}x,\mathrm{d}t) \\ &+ \int_{\overline{\Gamma}\times[0,T)} \psi_n(t)\left(\int_F g(y,t)q(x,\mathrm{d}y) - g(x,t)\right)\sigma(\mathrm{d}x,\mathrm{d}t). \end{aligned} \tag{42}$$

Using the regularity $g = \mathbb{T}(I, J)$ of g, for $s = T-t$, we have $g(\phi(x,(T-t)^+),T) - g(x,t) = g(\Phi(x,s),t+s) - g(x,t) = \displaystyle\int_0^s I(\phi(x,u),t+u)\mathrm{d}u$. So we have by dominated convergence

$$\begin{aligned} &\lim_{n\to\infty} \int_{\overline{F}\times\mathbb{R}_+} \psi_n'(t)(g(\phi(x,(T-t)^+),T) - g(x,t))\mu(\mathrm{d}x,\mathrm{d}t) \\ &= -\lim_{n\to\infty} \int_{\overline{F}\times\mathbb{R}_+} ns\ \varphi(1-ns)\left(\frac{1}{s}\int_0^s I(\phi(x,u),t+u)\mathrm{d}u\right)\mu(\mathrm{d}x,\mathrm{d}t) = 0. \end{aligned}$$

This shows that the left-hand side of (42) is

$$\lim_{n\to\infty} \int_{\overline{F}\times\mathbb{R}_+} \psi_n'(t)g(x,t)\mu(\mathrm{d}x,\mathrm{d}t) = \lim_{n\to\infty} \int_{\overline{F}\times\mathbb{R}_+} \psi_n'(t)g(\phi(x,(T-t)^+),T)\mu(\mathrm{d}x,\mathrm{d}t) = \rho_T(g(\cdot,T)),$$

and concludes the proof of (41).

For all $f \in \mathcal{C}_b^c$, let us define the function g by

$$g(x,t) = \int_t^\infty f(\phi(x,T-t),T)\mathrm{d}T.$$

Thanks to the properties

$$\int_0^{\alpha(x)} f(\phi(x,s),t+s)\mathrm{d}s = \int_t^{t+\alpha(x)} f(\phi(x,T-t),T)\mathrm{d}T,$$

and

$$g(\phi(x,\alpha(x)),t+\alpha(x)) = \int_{t+\alpha(x)}^\infty f(\phi(\phi(x,\alpha(x)),T-t-\alpha(x)),T)\mathrm{d}T = \int_{t+\alpha(x)}^\infty f(\phi(x,T-t),T)\mathrm{d}T,$$

we get that $g = \mathbb{T}(-f, g)$. Therefore, integrating (39) for $\xi = f(\cdot,T)$ on $T \in \mathbb{R}_+$, we get

$$\begin{aligned} \int_{\mathbb{R}_+}\int_{\overline{F}} f(x,T)\rho_T(\mathrm{d}x)\mathrm{d}T &= \int_{\mathbb{R}_+}\int_F f(\phi(x,T),T)\rho_{\mathrm{ini}}(\mathrm{d}x)\mathrm{d}T \\ + \int_{\mathbb{R}_+}\int_{\overline{F}\times[0,T)} &\lambda(x)\left(\int_F f(\phi(y,T-t),T)Q(x,\mathrm{d}y) - f(\phi(x,T-t),T)\right)\mu(\mathrm{d}x,\mathrm{d}t)\mathrm{d}T \\ + \int_{\mathbb{R}_+}\int_{\overline{\Gamma}\times[0,T)} &\left(\int_F f(\phi(y,T-t),T)q(x,\mathrm{d}y) - f(\phi(x,T-t),T)\right)\sigma(\mathrm{d}x,\mathrm{d}t)\mathrm{d}T. \end{aligned}$$

Then, using Fubini's theorem, we apply for some measures ν the equality

$$\int_{\mathbb{R}_+}\int_{\mathbb{R}^d\times[0,T)} f(\phi(x,T-t),T)\nu(\mathrm{d}x,\mathrm{d}t)\mathrm{d}T = \int_{\mathbb{R}^d\times\mathbb{R}_+}\int_t^{\infty} f(\phi(x,T-t),T)\mathrm{d}T\nu(\mathrm{d}x,\mathrm{d}t),$$

which leads to

$$\begin{aligned}\int_{\mathbb{R}_+}\int_{\overline{F}} f(x,T)\rho_T(\mathrm{d}x)\mathrm{d}T &= \int_F g(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x)\\ &+ \int_{\overline{F}\times\mathbb{R}_+}\lambda(x)\left(\int_F g(y,t)Q(x,\mathrm{d}y) - g(x,t)\right)\mu(\mathrm{d}x,\mathrm{d}t)\\ &+ \int_{\Gamma\times\mathbb{R}_+}\left(\int_F g(y,t)q(x,\mathrm{d}y) - g(x,t)\right)\sigma(\mathrm{d}x,\mathrm{d}t).\end{aligned}$$

Since μ is solution of (6), and that $g = \mathbb{T}(-f,g)$, we get

$$\begin{aligned}0 &= \int_F g(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x) - \int_{\overline{F}\times\mathbb{R}_+} f(x,t)\mu(\mathrm{d}x,\mathrm{d}t)\\ &+ \int_{\overline{F}\times\mathbb{R}_+}\lambda(x)\left(\int_F g(y,t)Q(x,\mathrm{d}y) - g(x,t)\right)\mu(\mathrm{d}x,\mathrm{d}t)\\ &+ \int_{\Gamma\times\mathbb{R}_+}\left(\int_F g(y,t)q(x,\mathrm{d}y) - g(x,t)\right)\sigma(\mathrm{d}x,\mathrm{d}t),\end{aligned}$$

which achieves the proof of (40). □

Lemma 9 *Under Hypotheses (H), the support of the measure μ solution of* (6) *is included in $F\times\mathbb{R}_+$.*

Proof. Let $T>0$ and $\psi\in\mathcal{C}_b^T$ be given. Let $0<k<1$, A and B be provided by the proof of Lemma 1 for the space $\mathcal{C}_b^T$. Now for all $C>0$ consider the function $f_C\in\mathcal{T}$ defined by

$$\begin{array}{rcl} f_C : \mathbb{R}^d\times\mathbb{R}_+ & \to & \mathbb{R}\\ (x,t) & \mapsto & \exp(-C\alpha(x))\psi(x,t).\end{array}$$

Then, by Lemma 1 with $\overline{I} = f_C$ and $\overline{J} = 0$, there exists a function $g_C = \mathbb{T}(I,J)\in\mathcal{T}$ such that (6) leads to

$$-\int_F g_C(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x) = \int_{\overline{F}\times\mathbb{R}_+} f_C(x,t)\mu(\mathrm{d}x,\mathrm{d}t) = \int_{\overline{F}\times\mathbb{R}_+}\exp(-C\alpha(x))\psi(x,t)\mu(\mathrm{d}x,\mathrm{d}t).$$

By the Lebesgue dominated convergence theorem, we get that the right hand side of the above equation converges to $\displaystyle\int_{(\overline{F}\setminus F)\times\mathbb{R}_+}\psi(x,t)\mu(\mathrm{d}x,\mathrm{d}t)$ as $C\to+\infty$. Let us now prove that the left hand side vanishes as $C\to+\infty$. Thanks to the construction of Ψ's fixed point, we can prove the following property by induction

$$\left\{\begin{array}{cl} (I^0,J^0) = (0,0), & \\ \mathbb{T}(I^{n+1},J^{n+1}) = \Psi(\mathbb{T}(I^n,J^n)) & \text{for all } n\in\mathbb{N},\\ \|\mathbb{T}(I^n,J^n)\|_{A,B} \le \|\psi\|_{A,B}\dfrac{1}{C(1-k)} & \text{for all } n\in\mathbb{N}.\end{array}\right.$$

Indeed

$$
\begin{aligned}
& \exp(B\widehat{\alpha}(x)+At)|\mathbb{T}(I^{n+1},J^{n+1})(x,t)| \\
\leq\ & \exp(B\widehat{\alpha}(x)+At)|\mathbb{T}(I^{n+1}-I^1,J^{n+1}-J^1)(x,t)| + \exp(B\widehat{\alpha}(x)+At)|\mathbb{T}(I^1,J^1)(x,t)| \\
\leq\ & k\|\mathbb{T}(I^n-I^0,J^n-J^0)\|_{A,B} + \int_0^{\alpha(x)} \exp(B\widehat{\alpha}(x)+At)|\psi(t+s)|\exp(-C\alpha(\phi(x,s)))\mathrm{d}s \\
\leq\ & k\|\psi\|_{A,B}\frac{1}{C(1-k)} + \int_0^{\alpha(x)} \|\psi\|_{A,B}\exp(-C\alpha(x)+Cs-As)\mathrm{d}s \\
\leq\ & k\|\psi\|_{A,B}\frac{1}{C(1-k)} + \|\psi\|_{A,B}\frac{1}{C} \\
\leq\ & \|\psi\|_{A,B}\frac{1}{C(1-k)}.
\end{aligned}
$$

So the bound is verified by the fixed point g_C, and since $\|\cdot\|_\infty \leq \|\cdot\|_{A,B}$, we have

$$
\liminf_{C>0} \int_F g_C(x,0)\rho_{\mathrm{ini}}(\mathrm{d}x) \leq \liminf_{C>0} \|g_C\|_\infty \rho_{\mathrm{ini}}(F) \leq \inf_{C>0} \|\psi\|_{A,B}\frac{1}{C(1-k)}\rho_{\mathrm{ini}}(F) = 0.
$$

This proves that

$$
\int_{(\overline{F}\setminus F)\times\mathbb{R}_+} \psi(x,t)\ \mu(\mathrm{d}x,\mathrm{d}t) = 0,
$$

which concludes the proof. □

References

[1] M. Benaïm, S. Le Borgne, F. Malrieu, P.-A. Zitt, (2013) Qualitative properties of certain piecewise deterministic Markov processes, arXiv:1204.4143v3.

[2] F. Bouchut, R. Eymard, A. Prignet (2011) Finite volume schemes for the approximation via characteristics of linear convection equations with irregular data, *Journal of Evolution Equations*, 11, 3 (2011) 687-724.

[3] C. Cocozza-Thivent (2013) *Renouvellement markovien et PDMP*, online at : http://perso-math.univ-mlv.fr/users/cocozza.christiane/recherche-page-perso/PresentationRMetPDMP.html (in french).

[4] C. Cocozza-Thivent, R. Eymard (2004) Approximation of the marginal distributions of a semi-Markov process using a finite volume scheme, *ESAIM: M2AN*, vol. 38, n5, p. 853-875.

[5] C. Cocozza-Thivent, R. Eymard, S. Mercier, M. Roussignol (2006) Characterization of the marginal distributions of Markov processes used in dynamic reliability, *Journal of Applied Mathematics and Stochastic Analysis*, Volume 2006, Article ID 92156, pp. 1-18.

[6] C. Cocozza-Thivent, R. Eymard, S. Mercier (2006) A finite volume scheme for dynamic reliability models, *IMA Journal of Numerical Analysis,* published online on Marsh 6, 2006, DOI:10.1093/imanum/drl001.

[7] M.H.A. Davis (1993), *Markov Models and Optimization*, Chapman & Hall, London.

[8] Y. Dutuit, E. Châtelet, J. -P. Signoret and P. Thomas (1997), Dependability modelling and evaluation by using stochastic Petri nets: application to two test cases, *Reliability Engineering & System Safety*, Volume 55, Issue 2, 117–124.

[9] R. Eymard, S. Mercier (2008), Comparison of numerical methods for the assessment of production availability of a hybrid system, *Reliability Engineering & System Safety*, Volume 93(1), 169-178.

[10] R. Eymard, S. Mercier, A. Prignet (2008) An implicit finite volume scheme for a scalar hyperbolic problem with measure data related to piecewise deterministic Markov processes, *Journal of Computational and Applied Mathematics*, 222(2), pp. 293-323.

[11] R. Eymard, S. Mercier, M. Roussignol (2011), Importance and sensitivity analysis in dynamic reliability, Methodology and Computing in Applied Probability, 13(1), pp. 75-104. DOI 10.1007/s11009-009-9122-x.

[12] C. Cocozza-Thivent, R. Eymard, L. Goudenège, M. Roussignol (2013), Numerical methods for piecewise deterministic Markov processes with boundary, ESAIM Proceedings, ISSN : 1270-900X, submitted.

[13] M. Jacobsen (2006), *Point process theory and applications: marked point and piecewise deterministic processes,* Series: Probability and Its Applications, Birkhäuser.

[14] P.E. Labeau (1996), A Monte Carlo estimation of the marginal distributions in a problem of probabilistic dynamics, *ReliabilityEngineering and System Safety*, Volume 52, Issue 1, 65–75.

[15] W. Lair, S. Mercier, M. Roussignol, R. Ziani (2011), Piecewise Deterministic Markov Processes and maintenance modelling: application to maintenance of a train air conditioning system, *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, 225(2), pp. 199-209. DOI: 10.1177/1748006XJRR347.

[16] A. Löpker, Z. Palmowski (2011), On time reversal of Piecewise Deterministic Markov Processes, arXiv:1110.3813v1

[17] M. Marseguerra, E. Zio (1996) Monte Carlo approach to PSA for dynamic process systems, *Reliability Engineering & System Safety*, Volume 52, Issue 3, 227–241.

[18] H. Zhang, F. Dufour, Y. Dutuit, K. Gonzalez (2008), Piecewise deterministic Markov processes and dynamic reliability, *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, 222(4), pp. 545551. DOI: 10.1243/1748006XJRR181.