Third-Order Kalman Filter: Tuning and Steady-State Performance
Huaqiang Shu, Eric Simon, Laurent Ros

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Abstract—This letter deals with the Kalman filter (KF) based on a third-order integrated random walk model (RW3). The resulting filter, noted as RW3-KF, is well suited to track slow time-varying parameters with strong trend behaviour. We first prove that the RW3-KF in steady-state admits an equivalent structure to the third-order digital phase-locked loops (DPLL). The approximate asymptotic mean-squared-error (MSE) is obtained by solving the Riccati equations, which is given in a closed-form expression as a function of the RW3 model parameter: the state noise variance. Then, the closed-form expression of the optimum state noise variance is derived to minimize the asymptotic MSE. Simulation results are given for the particular case where the parameter to be estimated is a Rayleigh channel fading scenario. We prove that the RW3-KF in steady-state admits an equivalent structure to the third-order digital phase-locked loops (DPLL). The approximate asymptotic mean-squared-error (MSE) is obtained by solving the Riccati equations, which is given in a closed-form expression as a function of the RW3 model parameter: the state noise variance. Then, the closed-form expression of the optimum state noise variance is derived to minimize the asymptotic MSE. Simulation results are given for the particular case where the parameter to be estimated is a Rayleigh channel fading scenario.

Index Terms—Random Walk model (RW), Kalman filter (KF).

I. INTRODUCTION

Kalman filters (KF) are commonly used to track time-varying parameters. The applications of KF cover a various range of systems, like GPS systems [1], Multi-carrier systems [2], MIMO systems [3], etc. The design of KF requires a linear recursive state-space representation of the parameter to be estimated. The most used approximation model, especially for channel estimation problems, is the first-order Auto-Regressive model (AR1), combined with either a correlation matching (CM) criterion for the fast time-varying scenario [2] [4], or a minimum asymptotic variance (MAV) criterion for the slow varying scenario [5], to fix the AR coefficient. However, in certain systems, the parameter to be estimated exhibits strong trend behaviour, and the use of second-order or higher-order models is more suitable than a first-order model. For example in a satellite receiver, third-order KF as well as third-order DPLLs are often used to tackle the problem of phase tracking in the presence of time-varying Doppler frequency offset [1]. However, the tuning and performance of these estimators are most often obtained from simulation or empirical results.

In this paper, we provide analytic results about the optimal tuning and the steady-state performance of a KF based on a RW3 model. For that, we first prove that this third-order KF has the same structure in a steady-state mode as a specific equal-order DPLL, hence extending the results of [6] [7] obtained for a second-order KF.

Section II gives the approximation model and the formulae of RW3-KF. In section III, we analyze and optimize the asymptotic MSE of RW3-KF. Section IV validates the analysis and assumptions by means of MSE and BER (bit error rate) simulations, the first-order AR model-based KFs (combined with CM and MAV criterion, respectively noted as AR1_CMV-KF and AR1_MAV-KF) are selected as references.

II. STATE-SPACE MODEL AND KALMAN FILTER

Assume the parameter to be estimated \( \alpha \) is a zero-mean circular complex process with variance \( \sigma_a^2 \). The variable \( \alpha \) is supposed to be a narrow-band stationary process, with a Power Spectrum Density (PSD) \( \Gamma_\alpha(f) \) with a support limited within \( \pm f_d \). We consider the RW3 model as an approximation of the time-variation of \( \alpha \):

\[
y(n) = \alpha(n) + w(n),
\]

where \( w(n) \) is a zero-mean white noise with variance \( \sigma_w^2 \). The dynamic evolution equations (1)-(3) and the observation equation (4) compose the state-space model of \( \alpha(n) \). The on-line unbiased estimate \( \hat{\alpha}(n) \) can be carried out by KF. The MSE \( \sigma_{\hat{\alpha}}^2 \equiv E[\epsilon(n)^2] \) of the estimation error \( \epsilon(n) = \alpha(n) - \hat{\alpha}(n) \) will be investigated.

Rewrite the state-space model in the matrix form:

\[
\begin{align*}
\mathbf{a}(n) &= \mathbf{Ma}(n-1) + \mathbf{u}(n), \\
y(n) &= \mathbf{Sa}(n) + w(n),
\end{align*}
\]

with the state vector \( \mathbf{a}(n) = [\hat{\alpha}(n) \ \delta(n) \ \xi(n)]^T \), the state noise vector \( \mathbf{u}(n) = [0 \ 0 \ u(n)]^T \), the selection vector \( \mathbf{S} = [1 \ 0 \ 0] \) and the evolution matrix \( \mathbf{M} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \), the RW3-KF could then described by two-stage equations:

**Time Update Equations**

\[
\begin{align*}
\hat{\mathbf{a}}(n|n-1) &= \mathbf{Ma}(n-1|n-1), \\
\mathbf{P}(n|n-1) &= \mathbf{MP}(n-1|n-1)\mathbf{M}^T + \mathbf{U},
\end{align*}
\]

This model is adequate for many applications, e.g. it could be a flat fading channel model, \( \alpha \) is then the complex amplitude of channel; or in the vehicle tracking problem, \( \alpha \) could be in matrix form, composed by the position coordinates and velocities of vehicle, etc.
with the Kalman gain \( K(n) = \begin{bmatrix} k_1(n) & k_2(n) & k_3(n) \end{bmatrix}^T \), the state noise variance matrix \( U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \sigma_u^2 \end{bmatrix} \cdot P_{n-1}(n-1) \) and \( P_{n}(n) \) are respectively the covariance matrices of the prediction error and the estimation error.

### III. ASYMPTOTIC MSE ANALYSIS

#### A. Steady-state RW3-KF

Since the linear model ((5),(6)) is observable and controllable, an asymptotic regime is quickly reached (cf. Ch. 13.3). In other words, \( P_{n}(n), \hat{P}_{n}(n-1) \) and \( K(n) \) converge to constant values when \( n \) is large enough, i.e.,

\[
K(n) = K(n+1) = K(\infty) \equiv \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}^T,
\]

\[
P_{n}(n) = P_{n}(n+1) = P(\infty) \equiv \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix},
\]

\[
P_{n}(n-1) = P_{n}(n+1) = P'(\infty) \equiv \begin{bmatrix} P'_{11} & P'_{12} & P'_{13} \\ P'_{21} & P'_{22} & P'_{23} \\ P'_{31} & P'_{32} & P'_{33} \end{bmatrix}.
\]

Note that \( P(\infty) \) and \( P'(\infty) \) are real symmetric matrices. This can be easily verified from (8), (9), (11) if the KF starts with a real-valued matrix \( P(0-1) \). \( K(\infty) \) is also a real vector. In the steady state, from (7) and (10), the recursive equations of RW3-KF can be reduced to a time-invariant filter:

\[
\dot{\alpha}(n) = \dot{\alpha}(n-1) + \dot{\delta}(n-1) = \frac{1}{2} \ddot{\xi}(n-1) + k_1 v_e(n),
\]

\[
\ddot{\xi}(n) = \ddot{\xi}(n-1) + k_2 v_e(n),
\]

\[
\ddot{v}_e(n) = y(n) - \left( \dot{\alpha}(n-1) + \dot{\delta}(n-1) + \frac{1}{2} \ddot{\xi}(n-1) \right).
\]

Transforming (15), (16), (17) to Z-domain and substituting \( \dot{\alpha} \) and \( \ddot{\xi} \) yield:

\[
\dot{\alpha}(z)(1-z^{-1}) = \begin{bmatrix} 1 & k_2 + \frac{1}{2} k_3 z^{-1} & \frac{k_3 z^{-2}}{1-z^{-1}} \end{bmatrix} v_e(z).
\]

Combining (18), (15) and (4), and after Z-transform we have:

\[
v_e(z) = \frac{1}{1 - k_1} \cdot (\alpha(z) - \dot{\alpha}(z) + w(z)),
\]

then substitute (20) into (19), we obtain the input-output equation:

\[
\dot{\alpha}(z) = L(z) \cdot \alpha(z) + L(z) \cdot w(z),
\]

with \( L(z) \) the transfer function of steady-state RW3-KF given in (22).

In the slow fading scenario \( (f_sT \ll 1) \), we are interested in the low frequency domain part of \( L(z) \) \( (f_sT \ll 1) \), using the approximation \( 1 - z^{-1} \approx pT \), with \( z = e^{pT} \) and \( p = j2\pi f \).

With such an approximation, the steady state transfer function of the RW3-KF \( L(e^{j2\pi f}) \) is equivalent to the typical transfer function of the third-order analog PLL (cf. [19], eqns. (2), (4) and [10] eqn. (22)):

\[
L(e^{j2\pi f}) \approx \frac{(m + 2)\zeta_\omega \cdot p^2 + (1 + 2m\zeta_\omega)\omega_n^2 \cdot p + m\zeta_\omega^2}{p^3 + (m + 2)\zeta_\omega \cdot p^2 + (1 + 2m\zeta_\omega)\omega_n^2 \cdot p + m\zeta_\omega^2},
\]

with

\[
k_1 = \frac{(m + 2)\zeta_\omega T + (1 + 2m\zeta_\omega)^2(\omega_n T)^2 + m\zeta_\omega^2(\omega_n T)^2}{1 + (m + 2)\zeta_\omega T + (1 + 2m\zeta_\omega)^2(\omega_n T)^2 + m\zeta_\omega^2(\omega_n T)^2},
\]

\[
k_2 = \frac{1 + (2m\zeta_\omega^2)(\omega_n T)^2 + (1 + 2m\zeta_\omega^2)^2(\omega_n T)^2}{1 + (m + 2)\zeta_\omega T + (1 + 2m\zeta_\omega)^2(\omega_n T)^2 + m\zeta_\omega^2(\omega_n T)^2},
\]

and with \( m \) the capacitance ratio, \( \zeta_\omega \) the damping factor, \( \omega_n = 2\pi f_n \) the natural radian frequency of the loop. They are real positive physical parameters, and \( \omega_n T \ll 1 \) when assuming a slow reaction of the filter. A useful inequality could be obtained with

\[
0 < k_3 \ll k_2 \ll k_1 < 1.
\]
By solving these equations, we could find the expressions of the elements of \( P(∞) \) as a function of \( k_1, k_2, k_3, \) and \( σ_w^2 \). To this end, after some manipulations on (29), we first find:
\[
P'_{13} = P'_{31} = \frac{σ_u^2}{k_3},
\]
(30)
which enables us to find after some manipulations:
\[
P'_{11} = \frac{8k_2 + k_3(6k_1 + 3k_2 + k_3)}{2k_3^2(2k_1 + 2k_2 + k_3)}σ_w^2.
\]
(31)
Then, a relation between \( k_1, k_2 \) and \( k_3 \) is found:
\[
k_2 = 2k_1k_3.
\]
(32)
In the sequel, it is assumed that \( P'_{11} \ll σ_w^2 \), which means that the Kalman gain is low \( k_1 \ll 1 \), according to (28). Then we deduce \( k_1 \) from (28), \( k_3 \) from (28) and (30) and \( k_2 \) from (32) respectively, that is:
\[
k_1 ≈ \frac{P'_{11}}{σ_w^2}, \quad k_3 = \frac{σ_u}{\sqrt{P'_{11} + σ_w^2}} ≈ \frac{σ_u}{σ_w}, \quad k_2 ≈ \sqrt{\frac{2P'_{11}σ_u}{σ_w^2}}.
\]
(33)
To further simplify the calculation, we apply the approximation (27) on (31), yielding:
\[
P'_{11} ≈ \frac{2k_2}{k_1k_3}σ_w^2.
\]
(34)
By combining (33), (34), \( P'_{11} \) can be expressed as a function of \( σ_u \) and \( σ_w \):
\[
P'_{11} ≈ 2σ_u^4σ_w^2 = 2σ_u^2\left(\frac{σ_u}{σ_w}\right)^2,
\]
(35)
and finally,
\[
k_1 ≈ 2\left(\frac{σ_u}{σ_w}\right)^2, \quad k_2 ≈ 2\left(\frac{σ_u}{σ_w}\right)^3 = k_2^2 \frac{k_3}{2}, \quad k_3 ≈ \frac{σ_u}{σ_w} = k_3^3 \frac{k_3}{8}.
\]
(36)
Using the approximated Kalman gain relation \( 0 < k_2 \ll k_1 \ll 1 \), the transfer function of RW3-KF (22) can be simplified as:
\[
L(z) ≈ \frac{k_1(1 - z^{-1})^2 + k_3(1 - z^{-1}) + k_3}{(1 - z^{-1})^3 + k_1(1 - z^{-1})^2 + k_2(1 - z^{-1}) + k_3^3}.
\]
(37)
Comparing (37) and (23), we get \( k_1 \approx (m + 2)\zeta_0ω_nT, \)
\( k_2 \approx (1 + 2m\zeta_0^2)(ω_nT)^2, \)
\( k_3 \approx m\zeta_0(ω_nT)^3 \).

Then by using (36), we obtain \( m = 2, \zeta = 0.5 \), while its natural random frequency \( ω_nT \) can be tuned as \( \frac{k_2}{k_1} \), or eventually \( \left(\frac{σ_w}{σ_u}\right)^2 \). Thus, we can conclude that the RW3-KF is equivalent in steady-state mode and slow-tracking scenario to the third-order DPLL with fixed given parameters \( m = 2, \zeta = 0.5 \).

This conclusion generalizes to the third-order the connection between DPLL and KF established in [6] [7] for the second-order.

B. Mean Squared Error Analysis

The (unbiased) estimation error is defined by:
\[
ε(z) = α(z) - \hat{α}(z) = (1 - L(z)) \cdot α(z) - L(z) \cdot w(z)
\]
(38)
and the mean squared error is thus composed by two parts:
\[
σ_e^2 = E(ε \cdot ε^*) = σ_{αα}^2 + σ_{εw}^2,
\]
(39)
\( σ_{εw}^2 \) is the static error variance which results from the channel noise \( w \), whereas \( σ_{αα}^2 \) is the dynamic error variance, which results from the parameter \( α \) variations.

The static error variance is developed as:
\[
σ_{εw}^2 = \frac{σ_w^2}{T} \int_{-\frac{B_L}{2}}^{\frac{B_L}{2}} |L(e^{j2πfT})|^2 df
\]
\[
= \frac{5}{2} k_1 - \frac{1}{2} k_1^2 - \frac{5}{2} k_1^3 - \frac{5}{6} \frac{1}{k_1^4} σ_w^2 ≈ \frac{5}{3} k_3^3 σ_{w}^2,
\]
(40)
where the integral term \( B_L \) is the equivalent noise bandwidth. It can be calculated by the method presented in [11]. Note that we have applied the condition \( 0 < k_1 \ll 1 \) for the approximation. The dynamic error variance is developed as:
\[
σ_{αα}^2 = \int_{-\frac{1}{T}}^{1} \int_{-1}^{1} |\Gamma_λ(λ) | 1 - L(e^{j2πfT})|^2 df
\]
\[
\approx \int_{1}^{1} \int_{1}^{1} |\Gamma_λ(λ) | (2πfT)^6 df = \frac{(2π)^6}{k_3^3} S_α,
\]
(41)
where \( S_α = \int_{-1}^{1} \int_{-1}^{1} |\Gamma_λ(λ) | (fT)^6 df \) is the term which contains the PSD of \( \tilde{α} \). For the reason of simplicity, we apply \( e^{-jπfT} \approx 1 - j2πfT \) as well as \( 2πfT \ll k_3^3 \approx ω_nT \ll 1 \) to calculate \( 1 - L(e^{j2πfT})^2 \) in the slow variation channel case. The global MSE is then obtained by combining (40) and (41). After substituting the approximation (36) for \( k_3 \), the objective function to optimize is given by:
\[
σ_e^2 = \frac{5}{3} \frac{σ_w^2}{σ_u} + \frac{(2π)^6}{k_3^3} S_α,
\]
(42)
The minimization can be done by imposing the partial derivative of global MSE \( σ_e^2 \) equal to 0, yielding:
\[
σ_{αα}^2_{opt} = \left(\frac{2π}{5}\right)^{36} \left(\frac{18}{5} S_α \right)^{\frac{3}{2}} \left(\frac{2π}{k_3^3} S_α \right)^{\frac{1}{2}},
\]
(43)
and the corresponding minimized MSE is:
\[
σ_e^2_{min} = \frac{7}{9} \left(\frac{5}{π} \frac{σ_w^2}{σ_u} \right)^{\frac{1}{2}} \frac{4}{3} S_α.
\]
(44)

C. An application to the estimation of Rayleigh channel with Jakes’ Doppler spectrum

From (43) and (44), we note that the optimum parameter and the corresponding minimized MSE could be computed whatever the channel PSD is. Now we take the estimation of Rayleigh channel with Jakes’ Doppler spectrum as an example. The PSD of \( α \) is defined as:
\[
\Gamma_α(f) = \left\{ \begin{array}{ll}
\frac{σ_u^2}{\pi f_d \sqrt{1 - \frac{f^2}{f_d^2}}}, & \text{for } |f| < f_d, \\
0, & \text{for } |f| \geq f_d.
\end{array} \right.
\]
(45)
A variable change \( \cos θ = \frac{f}{f_d} \) is applied to calculate the integral \( S_α \) and we have:
\[
S_α = \int_{-f_d}^{f_d} (fT)^6 \cdot \frac{σ_u^2}{\pi f_d \sqrt{1 - \frac{f^2}{f_d^2}}} df = \frac{5}{16} (f_dT)^6 \cdot \frac{σ_u^2}{k_3^3}.
\]
(46)
The optimal \( σ_{αα}^2 \) and the corresponding minimized MSE are then obtained directly:
\[
σ_{αα}^2_{opt} = \left[ \frac{31}{218} \cdot (σ_u^2)^6 \cdot (σ_w^2) \cdot (2πf_dT)^{36} \right]^{\frac{1}{2}},
\]
(47)
\[
σ_e^2_{min} = \frac{35}{16} \cdot \frac{16}{9} \frac{f_dT \cdot \sigma_w^2}{(σ_u^2)^{\frac{1}{2}}} \cdot (σ_α^2)^{\frac{1}{2}}.
\]
(48)
IV. SIMULATION RESULTS AND CONCLUSION

The MSE analysis is verified by Monte-Carlo simulations over a Rayleigh flat fading channel. Fig. 1 shows the MSE of AR1_CM-KF [2] [3] [4], AR1_MAV-KF [5] and RW3-KF as a function of SNR, with $f_d T = 10^{-3}$. The theoretical MSE of RW3-KF as well as the online BCRB (Bayesian Cramer-Rao Bound) [12] are used as references. Fig. 2 shows the MSE of these estimators as a function of $f_d T$ with fixed SNR=20dB. From Fig. 1 and 2, we find that the theoretical and the simulation lines of RW3-KF approximately coincide. The MSE of RW3-KF is proportional to the $7/4$ power of noise variance $\sigma_n^2$ (thus inversely proportional to the SNR), and is also proportional to the $7/4$ power of $f_d T$. On the other hand, compared to the AR1_CM-KF, the AR1_MAV-KF has a much improved asymptotic performance, which means that the MAV criterion is a better choice for computing the AR1 coefficient. However it is still far from the lower bound due to the low-order filtering that causes the loss of dynamic information. Meanwhile, the RW3 model fits the real channel much better than the AR models in the slow fading case. Moreover, the MSE of RW3-KF is very close to the online BCRB.

For the BER simulation, we use QPSK transmitted symbols. The estimation is in semi-blind mode, that is, the data block is composed of 20 pilot symbols followed by 180 unknown symbols (for which the KF is in decision-directed mode). Fig.3 shows the simulation result, where we can observe that with the optimized $\sigma_n^2_{opt}$, the RW3-KF attains a performance close to the one with perfect channel knowledge.

To conclude, we have discussed in this letter the third-order modeling of the Kalman Filter for parameter estimation problems, where an application to Rayleigh fading channel with Jakes’ spectrum was also introduced. The explicit formulae of the optimum parameter and the asymptotic MSE of the RW3-KF were given, assuming the knowledge of the channel statistics. A connection between the steady-state RW3-KF and the typical third-order DPLL was established. We also conclude that, for KF-based estimators, the well-tuned third-order random walk model is more adequate compared with the first-order AR model in the low-variation context, with the resulting estimator performance very close to the BCRB. Possible future directions are to extend this work to the vectorial case for multi-path channel and/or multi-carrier modulation scenarios. Also, MSE performance of the other components of the RW3 model could be investigated.

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