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Nonlinear Randomized Urn Models: a Stochastic Approximation Viewpoint

Sophie Laruelle * Gilles Pagès †

May 11, 2018

Abstract

This paper extends the link between stochastic approximation (SA) theory and randomized urn models developed in [31], and their applications to clinical trials introduced in [2, 3, 4]. We no longer assume that the drawing rule is uniform among the balls of the urn (which contains d colors), but can be reinforced by a function f. This is a way to model risk aversion. Firstly, by considering that f is concave or convex and by reformulating the dynamics of the urn composition as an SA algorithm with remainder, we derive the $a.s.$ convergence and the asymptotic normality (Central Limit Theorem, CLT) of the normalized procedure by calling upon the so-called ODE and SDE methods. An in-depth analysis of the case $d = 2$ exhibits two different behaviors: A single equilibrium point when f is concave, and when f is convex, a transition phase from a single attracting equilibrium to a system with two attracting and one repulsive equilibrium points. The last setting is solved using results on non-convergence toward noisy and noiseless "traps" in order to deduce the $a.s.$ convergence toward one of the attracting points. Secondly, the special case of a Pólya urn (when the addition rule is the I_d matrix) is analyzed, still using result from SA theory about "traps". Finally, these results are applied to a function with regular variation and to an optimal asset allocation in Finance.

Keywords *Stochastic approximation, extended Pólya urn models, reinforcement, non-homogeneous generating matrix, strong consistency, asymptotic normality, bandit algorithms.*

2010 AMS classification: 62L20, 62E20, 62L05 secondary: 62F12, 62P10.

1 Introduction

In this paper, we introduce and study in-depth a class of generalized Pólya urns (with d colors) characterized by their nonlinear drawing rules. These models appear as a generalization of randomized urn models originally devised for clinical trials which takes into account the risk aversion attitude of the agent. Randomized urn models have been extensively investigated by various authors (see [2, 3, 4]) during the last twenty years based on *ad hoc* martingale arguments to solve $a.s.$ convergence as well as

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its rate of convergence. In a recent paper [31] (see also [32, 37]), we revisited, unified and often extended these results by showing that they can be established by relying on the main results of Stochastic Approximation (SA) theory, especially $a.s.$ convergence and weak convergence rate (Central Limit Theorem (CLT)). Although this analysis is more demanding than for urn models with linear drawing rules, this SA "toolbox" turns out to be very efficient (see also [5] and the references therein). SA deals with the asymptotic behavior of zero search stochastic recursive procedures and goes back to the seminal paper by Robbins & Monro in the 1950's. Since then, this theory has been developed extensively by many authors (see [26, 27, 9, 17, 18] and the references therein for an overview and historical notes) and has been applied in various directions (Automatic Control, Mathematical Psychology, Artificial Neural Networks, Statistics, Stochastic Control, Numerical Probability, etc.).

Considering nonlinear drawing rules leads, once written as a recursive stochastic algorithm, to dynamics for the (normalized) urn composition having several *equilibrium points*. By equilibrium point, which belongs to the SA terminology adopted in the sequel, we mean a zero of the mean field of the stochastic algorithm which in practice corresponds to a potential asymptotic composition of the urn. These equilibrium points include not only local attractors, but also "parasitic" ones (repeller, saddle points, etc). This is a major difference with the linear case investigated in [31] since, this time, we will need to call upon the whole machinery of SA, in particular "second order" results about these parasitic noisy and noiseless equilibrium points, sometimes called "traps" in the SA literature (see [12, 17, 30], see also [34, 35, 6]). Taking advantage of these results, we will establish the $a.s.$ convergence, or strong consistency, of the (normalized) urn composition even in presence of multiple attractors. Then, we will analyze its weak convergence rate.

Let us be more precise on the urn model under consideration in this paper. We consider an urn containing balls of (at most) d different types (or colors). All random variables involved in the model are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote by $Y_0 = (Y_0^i)_{i=1,\ldots,d} \in \mathbb{R}_+^d \setminus \{0\}$ the initial composition of the urn, where Y_0^i is the number of balls of type $i \in \{1, \ldots, d\}$ (of course a more natural, though not mandatory, assumption would be $Y_0 \in \mathbb{N}^d \setminus \{0\}$). The urn composition at draw n is denoted by $Y_n = (Y_n^i)_{i=1,\ldots,d}$. At the n^{th} stage, one draws randomly (according to a law defined further on) a ball from the urn with instant replacement. If the drawn ball is of type j, then the urn composition is updated by adding D_n^{ij} balls of type i, for every $i \in \{1, \ldots, d\}$. The procedure is then iterated. The urn composition at stage n, modeled by an $\mathbb{R}^d$-valued vector Y_n, satisfies the following recursive updating rule between times n and $n+1$:

$$Y_{n+1} = Y_n + D_{n+1} X_{n+1}, \quad n \geq 0, \quad Y_0 \in \mathbb{R}_+^d \setminus \{0\}, \tag{1.1}$$

where $D_n = (D_n^{ij})_{1 \leq i,j \leq d}$ is the *addition rule matrix* and $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to \{e^1, \cdots, e^d\}$ models the type of the drawn ball at time n ($\{e^1, \cdots, e^d\}$ denotes the canonical basis of $\mathbb{R}^d$ with e^j standing for type j). We assume that there is no extinction *i.e.* $Y_n \in \mathbb{R}_+^d \setminus \{0\}$ $a.s.$ for every $n \geq 1$: This is always the case if all the entries D_n^{ij} are $a.s.$ non-negative (see [31]). The filtration of the model is defined by $\mathcal{F}_n = \sigma(Y_0, X_k, D_k, 1 \leq k \leq n)$, $n \geq 0$.

The *generating matrices* are defined as the $\mathcal{F}_n$-compensator of the additions rule sequence *i.e.*

$$H_n = \left[\mathbb{E}\left(D_n^{ij} \,|\, \mathcal{F}_{n-1}\right)\right]_{1 \leq i,j \leq d}, \; n \geq 1. \tag{1.2}$$

We will also assume that the sequence of generating matrices $a.s.$ converges toward a limiting generating matrix denoted by H.

We will mostly investigate the asymptotic behavior of skewed drawing rules of the form

$$\forall i \in \{1,\dots,d\}, \quad \mathbb{P}\big(X_{n+1}=e^i\big|\mathcal{F}_n\big)=\frac{f\big(Y_n^i/(n+\mathbf{w}(Y_0))\big)}{\sum_{j=1}^d f\big(Y_n^j/(n+\mathbf{w}\big(Y_0)\big)\big)}, \quad n\geq 0, \tag{1.3}$$

where $\mathbf{w}(y)=y^1+\cdots+y^d$ denotes the *weight* of a vector $y=(y^1,\dots,y^d)^t\in\mathbb{R}_+^d$ and the function

$$f:\mathbb{R}_+\to\mathbb{R}_+ \text{ is non-decreasing with } f(0)=0,\ f(1)=1. \tag{1.4}$$

The function f usually satisfies an *additional convexity or concavity property*. Such a rule will be called *normalized f-skewed empirical frequency based drawing rule*, f being the *skewing function*, non-trivial when $f\neq \mathrm{Id}_{\mathbb{R}_+}$. When $f=\mathrm{Id}_{\mathbb{R}_+}$, we retrieve a more standard drawing rule based on the regular empirical frequency of the types in the urn (see [2, 31] among others).

In fact, we will see that the result obtained for this family of frequency-based drawing rules allows us to elucidate a second way to skew the drawing, called *normalized f-skewed distribution*, based this time on the *number* of balls of each type in the urn, namely

$$\forall i \in \{1,\dots,d\}, \quad \mathbb{P}\big(X_{n+1}=e^i\,\big|\,\mathcal{F}_n\big)=\frac{f(Y_n^i)}{\sum_{j=1}^d f(Y_n^j)}, \quad n\geq 0, \tag{1.5}$$

where the function

$$f:\mathbb{R}_+\to\mathbb{R}_+ \text{ is non-decreasing, with } f(0)=0 \text{ and is regularly varying with index } \alpha>0 \tag{1.6}$$

(*i.e.* for every $t>0$, $\frac{f(tx)}{f(x)}\underset{x\to+\infty}{\longrightarrow} t^\alpha$).

Moreover, we will make the assumption that D_n and X_n are conditionally independent given $\mathcal{F}_{n-1}$ (see (**A2**) further on). Such a drawing procedure can be performed by using an exogenous i.i.d. sequence $(U_n)_{n\geq 1}$ of random variables with uniform distribution on the unit interval, independent of the sequence $(D_n)_{n\geq 1}$, to simulate the above conditional probabilities.

Let us remark that, when $f=\mathrm{Id}_{\mathbb{R}_+}$, both updating rules (1.3) and (1.5) coincide. In fact, the normalized f-skewed distribution drawing rule will appear as a by-product of the first one (see Section 6.1) by noting that, if f is bounded on every interval $(0,M]$ and regularly varying with index $\alpha>0$, then $\frac{f(tx)}{f(x)}\underset{x\to\infty}{\longrightarrow} t^\alpha$ uniformly in t on every interval $(0,T]$, $0<T<+\infty$ (see Theorem 1.5.2 p.22 in [10]). Thus, if $\frac{Y_n}{n+\sum_{i=1}^d Y_0^i}$ lies in a compact set, then

$$\max_{1\leq i\leq d}\left|\frac{f(Y_n^i)}{f(n+\sum_{i=1}^d Y_0^i)}-\left(\frac{Y_n}{n+\sum_{i=1}^d Y_0^i}\right)^\alpha\right|\underset{n\to+\infty}{\longrightarrow} 0.$$

Then, we will conclude by applying the result related to the f-skewed empirical frequency based drawing rule to the functions $x\mapsto x^\alpha$, $\alpha>0$.

In this paper we both randomize a single urn in the sense that the addition rule matrix (defined by (1.2)) itself can be random (as introduced in [24, 3, 4] and studied with SA theory in [31]) and investigate wide non-parametric classes of convex and concave drawing rules (see (1.3) and (1.5)). *A.s.* convergence of the urn composition and that of the drawing rule is proved as well as their convergence rate, either in a weak or strong sense, depending on the structure of the updating rules. In the particular

case of Pólya's urns (*i.e.* when the addition rule matrix D_n is equal to identity), but implemented here with a convex skewed drawing rule (for generalized Pólya urn, see for example [23, 38, 16]), "noiseless traps" may appear (*i.e.* unstable equilibria, noiseless since they lie at the boundary of the state space). To determine whether they are parasitic, we develop a dedicated approach, close in spirit to that introduced in [30] and [28] the analysis of adaptive bandit algorithms where the authors establish a kind of "oracle" inequality relying on a specified martingale (see Lemma 5.1 further on). These results highlight the efficiency of SA theory, even in presence of noiseless repulsive equilibrium points.

Recently, a system of Pólya's urns with graph based interactions and a "power drawing" rule has also been investigated using SA techniques in [7, 15] and the $a.s.$ convergence of the normalized urn composition is established.

Generalized Pólya Urn models (GPU) have been widely studied in the literature with different points of view: Martingale method (see *e.g.* [21]), algebraic approach (see *e.g.* [36]), reinforcement process (see *e.g.* [36]), branching process (see *e.g.* [24]), stochastic approximation (see for example [5, 33]), contraction method (see [25]). These models also have applications to many areas: Biology, random walks and clinical trials, statistics and learning, computer science, psychology, economics or finance for instance (see [38]).

In these adaptive models, the key point is the updating rules of the urn composition after each drawing given here by (1.3) and (1.5). Basically, we will show that (a normalized version of) this urn composition can be formulated as a classical recursive stochastic algorithm with step $\gamma_n = \frac{1}{n+\mathbf{w}(Y_0)}$ where $\mathbf{w}(Y_0)$ denotes the number of balls in the urn at time 0. Doing so, we will be in position to first establish the $a.s.$ convergence of the procedure by calling upon the so-called Ordinary Differential Equation method (ODE method) toward a finite set of equilibrium points (but usually not reduced to a single point). As a second step, we will rely on $a.s.$ non-convergence results toward traps (see [12, 17]) and on $a.s.$ convergence in presence of multiple targets (see [8, 17, 19]). As a third step, we entirely elucidate the rate of convergence (namely a weak rate through a CLT or an $a.s.$ rate) by using the Stochastic Differential Equation method (SDE method, see *e.g.* [18, 9]). The three main theoretical results from SA are recalled in a self-contained form in the Appendix. Proofs of such results can be found in classical textbooks on SA ([9, 17, 18, 27]). As for the CLT, they go back to [26] and [11], see also [37] more recent results. We will verify once again how powerful these general theorems are to solve such questions, sparing tedious computations and repetitive proofs.

Among many fields of application, we present in Section 6 an adaptive asset allocation procedure based on a reinforcement principle relying on non-linear randomized urns, illustrated by a first numerical test. One may also consider a similar procedure as a strategy to update the composition of a portfolio or even a whole fund, based on the (recent) past performances of the assets.

The paper is organized as follows. Section 2 presents the framework of skewed randomized urn models with the required assumptions on both the addition rule matrices and the generating matrices. After rewriting the dynamics of the urn composition as an SA procedure in Section 2.2, we analyze in Section 2.4 the equilibrium points and their stability for the associated ODE when the f-skewed drawing rule is convex/concave. An in-depth analysis of the 2-color urn is carried out in Section 3. We exhibit several kinds of behaviors: When f is concave, there is always a unique stable equilibrium point and when f is convex three generic situations may occur with one, tow or three equilibrium points (one being parasitic in the last two settings). By calling upon SA result on traps, we prove the $a.s.$ convergence towards one of the attracting equilibrium points; then we derive from the SDE method all the possible rates of convergence. In Section 5, we study the case of Pólya urns – urn

dynamics whose addition rule matrix equals to identity – updated by a skewed drawing rule. We rely on methods borrowed from the analysis of adaptive bandit algorithms to prove the convergence towards the "targeted urn composition" and the non-convergence towards traps. Finally, in Section 6, we first transfer our results to the second type of drawing rule and we conclude by an application to portfolio allocation.

NOTATIONS. For $u = (u^i)_{i=1,\dots,d} \in \mathbb{R}^d$, $(\cdot \,|\, \cdot)$ denote the Euclidean inner product and $\|u\|$ its related norm of the column vector $u \in \mathbb{R}^d$, $\mathbf{w}(u) = \sum_{k=1}^d u^k$ denotes its "weight", u^t denotes its transpose, $u \otimes v = [u^i v^j]_{i,j=1,\dots,d}$; $|\!|\!| A |\!|\!|$ denotes the operator norm of the matrix $A \in \mathcal{M}_{d,q}(\mathbb{R})$ with d rows and q columns with respect to the two canonical Euclidean norms. When $d = q$, $\mathrm{Sp}(A)$ denotes the set of eigenvalues of A. $\mathbf{1} = (1 \cdots 1)^t$ denotes the unit column vector in $\mathbb{R}^d$, I_d denotes the $d \times d$ identity matrix, $\mathrm{diag}(u) = [\delta_{ij} u_i]_{1 \le i,j \le d}$, where δ_{ij} stands for the Kronecker symbol and $\mathcal{S}_d = \left\{ u \in \mathbb{R}_+^d : \sum_{i=1}^d u^i = 1 \right\}$ denotes the canonical simplex. $\mathcal{E}_d = \{ y \in \mathcal{S}_d : h(y) = 0 \}$ denotes the set of zeros of h called equilibrium points in reference to the ODE $\dot{y} = -h(y)$.

2 Skewed randomized urn models

2.1 Main assumptions and definitions

With the notations and definitions described in the introduction, we are in position to formulate the main assumptions needed to establish the $a.s.$ convergence of the urn composition.

$$(\mathbf{A1}) \equiv \begin{cases} (i) & \textit{Addition rule matrix:} \text{ For every } n \ge 1, \text{ the matrix } D_n \ a.s. \text{ has non-negative entries.} \\ (ii) & \textit{Generating matrix:} \text{ For every } n \ge 1, \text{ the generating matrix } H_n = (H_n^{ij})_{1 \le i,j \le d} \\ & a.s. \text{ satisfies:} \qquad \forall\, j \in \{1, \dots, d\}, \quad \displaystyle\sum_{i=1}^d H_n^{ij} = c > 0. \\ (iii) & \textit{Starting value:} \text{ The starting urn composition vector } Y_0 \in \mathbb{R}_+^d \setminus \{0\}. \end{cases}$$

The constant c is known as the *balance* of the urn. In fact, we may assume without loss of generality, up to a renormalization of Y_n, that $c = 1$. As a matter of fact, we set for every $n \ge 0$, $\widehat{Y}_n = \frac{Y_n}{c}$ and $\widehat{D}_{n+1} = \frac{D_{n+1}}{c}$ where (Y_n, X_n, D_n) satisfies (1.1), then the couple $(\widehat{Y}_n, X_n, \widehat{D}_n)_{n \ge 1}$, still satisfies the dynamics (1.1), namely

$$\widehat{Y}_{n+1} = \widehat{Y}_n + \widehat{D}_{n+1} X_{n+1}, \quad n \ge 0, \quad \widehat{Y}_0 \in \mathbb{R}_+^d \setminus \{0\},$$

whereas $\widehat{H}_n = \mathbb{E}[\widehat{D}_n | \mathcal{F}_{n-1}]$ satisfies now $(\mathbf{A1})$-(iii) with $c = 1$ *i.e.* $\widehat{H}_n$ is *co-stochastic* (in the sense that its transpose is a stochastic matrix). Assumptions $(\mathbf{A1})$-$(i)\&(iii)$ combined with the drawing rule (1.1) ensure that $Y_n \in \mathbb{R}_+^d \setminus \{0\}$, for every $n \ge 0$.

From now on, throughout the paper, we will consider this normalized balanced version, still denoted by Y_n and D_n for convenience.

$$(\mathbf{A2}) \equiv \begin{cases} (i) & \text{The addition rule } D_n \text{ and the drawing procedure } X_n \text{ are conditionally independent} \\ & \quad \text{given } \mathcal{F}_{n-1}. \\ (ii) & \forall j \in \{1, \dots, d\}, \quad \sup_{n \ge 1} \mathbb{E}\left[\left\| D_n^{\cdot j} \right\|^2 \,|\, \mathcal{F}_{n-1} \right] < +\infty \quad a.s. \\ & \qquad\qquad \Longleftrightarrow \forall\, i, j \in \{1, \dots, d\}, \quad \sup_{n \ge 1} \mathbb{E}\left[(D_n^{ij})^2 \,|\, \mathcal{F}_{n-1} \right] < +\infty \quad a.s. \end{cases}$$

where $D_n^{\cdot j} = (D_n^{ij})_{i=1,\dots,d}$ (column vector).

(**A3**) There exists an irreducible $d \times d$ matrix H (with non-negative entries) such that

$$H_n \xrightarrow[n\to+\infty]{a.s.} H \quad \text{and} \quad \sum_{n\geq 1} \|H_n - H\|^2 < +\infty \quad a.s. \tag{2.7}$$

H is called the *limiting generating matrix*.

The central object of interest of this paper will be the *(quasi-)normalized composition* of the urn at time n defined by

$$\widetilde{Y}_n = \frac{Y_n}{n + \mathbf{w}(Y_0)} \tag{2.8}$$

(where the weight function is defined in the introduction). The reason for introducing such a renormalization factor is that, as established further on in the next subsection,

$$\forall n \geq 0, \quad \mathbb{E}\left[\mathbf{w}(Y_n)\right] = n + \mathbf{w}(Y_0).$$

Then, one may guess that $\widetilde{Y}_n$ is close to the simplex $\mathcal{S}_d = \left\{u \in \mathbb{R}_+^d : \sum_{i=1}^d u^i = 1\right\}$ and will possibly asymptotically lie in it. Even note that when the matrices D_n are themselves co-stochastic, $\widetilde{Y}_n$ is $\mathcal{S}_d$-valued. Therefore, this is a natural deterministic way to normalize the urn composition vector.

Definition 2.1 (Skewing functions and skewed drawing rules)**.** (*a*) *A function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$f \text{ non-decreasing, convex or concave, } f(0) = 0 \text{ and } f(1) = 1 \text{ and } \{f > 0\} = (0, +\infty) \tag{2.9}$$

is called a skewing function.

(*b*) *Assuming* (**A1**)-(*i*) *&* (*iii*)*, the f-*skewed drawing rule $(X_n)_{n\geq 1}$ *induced by a skewing function f is defined by*

$$\forall\, i \in \{1, \ldots, d\}, \quad \mathbb{P}\big(X_{n+1} = e^i \big| \mathcal{F}_n\big) = \frac{f(\widetilde{Y}_n^i)}{\sum_{j=1}^d f(\widetilde{Y}_n^j)}, \quad n \geq 0, \tag{2.10}$$

where $\widetilde{Y}_n$ is defined by (2.8)*, and $(Y_n, X_n, D_n)_{n\geq 1}$ satisfies* (1.1)*.*

Of course, such a drawing rule is really *skewed* only if $f \not\equiv \mathrm{Id}_{\mathbb{R}_+}$. Note that this definition is consistent under (**A1**)-(*i*) & (*iii*) since $Y_n \in \mathbb{R}_+^d \setminus \{0\}$, for every $n \geq 0$ so that $\widetilde{Y}_n \in \mathbb{R}_+^d \setminus \{0\}$.

We will extensively use that a skewing function f always satisfies that the function $\xi \in \mathbb{R}_+ \mapsto \frac{f(\xi)}{\xi}$ is monotonic (non-decreasing if f is convex, non-increasing if f is concave) and strictly monotonic if the concavity/convexity of f is itself strict.

2.2 Representation as a stochastic algorithm

The starting point, like in [31], is to reformulate the dynamics (1.1)-(1.3) as a recursive stochastic algorithm in order to take advantage of classical results from *SA* Theory to elucidate the asymptotic properties (*a.s.* convergence) of both the urn composition Y_n and the ball drawing rule X_n. To do so, we start from (1.1) with $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$. For every $n \geq 0$, we note that

$$Y_{n+1} = Y_n + D_{n+1}X_{n+1} = Y_n + \mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right] + \Delta M_{n+1}, \tag{2.11}$$

where

$$\Delta M_{n+1} := D_{n+1}X_{n+1} - \mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right] \tag{2.12}$$

is an $\mathcal{F}_n$-local martingale increment (integrability follows from (**A2**)-(ii)). By the definition (1.2) of the generating matrix H_n, we have, owing to the conditional independence assumption (**A2**)-(i),

$$\begin{aligned}\mathbb{E}\left[D_{n+1}X_{n+1} \,|\, \mathcal{F}_n\right] &= \sum_{i=1}^{d} \mathbb{E}\left[D_{n+1}\mathbb{1}_{\{X_{n+1}=e^i\}} \,|\, \mathcal{F}_n\right] e^i = \sum_{i=1}^{d} \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right] \mathbb{P}\left(X_{n+1} = e^i \,|\, \mathcal{F}_n\right) e^i \\ &= H_{n+1} \sum_{i=1}^{d} \frac{f(\widetilde{Y}_n^i)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} e^i = H_{n+1} \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\end{aligned}$$

where, for every $y = (y^1, \ldots, y^d)^t \in \mathbb{R}_+^d \setminus \{0\}$,

$$\widetilde{f}\big((y^1, \ldots, y^d)^t\big) = \big(f(y^i)\big)_{1\le i\le d} \in \mathbb{R}_+^d \setminus \{0\} \tag{2.13}$$

is a column vector, so that

$$Y_{n+1} = Y_n + H_{n+1} \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} + \Delta M_{n+1}. \tag{2.14}$$

Now we can derive a stochastic approximation for the normalized urn composition $\widetilde{Y}_n = \dfrac{Y_n}{n + \mathbf{w}(Y_0)}$, $n \ge 0$. First, we have, for every $n \ge 0$,

$$\frac{Y_{n+1}}{n+1+\mathbf{w}(Y_0)} = \frac{Y_n}{n+\mathbf{w}(Y_0)} + \frac{1}{n+1+\mathbf{w}(Y_0)} \left(H_{n+1} \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} - \frac{Y_n}{n+\mathbf{w}(Y_0)} \right) + \frac{\Delta M_{n+1}}{n+1+\mathbf{w}(Y_0)}. \tag{2.15}$$

Consequently, the sequence $(\widetilde{Y}_n)_{n\ge 0}$, satisfies the canonical recursive stochastic approximation procedure starting from $\widetilde{Y}_0 \in \mathbb{R}_+^d \setminus \{0\}$,

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n + \frac{1}{n+1+\mathbf{w}(Y_0)} \left(H_{n+1} \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} - \widetilde{Y}_n \right) + \frac{1}{n+1+\mathbf{w}(Y_0)} \Delta M_{n+1} \tag{2.16}$$

or, equivalently,

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \gamma_{n+1} h(\widetilde{Y}_n) + \gamma_{n+1} \left(\Delta M_{n+1} + r_{n+1} \right) \tag{2.17}$$

where the *mean field* $h : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}^d$ of the procedure is defined by

$$h(y) := \mathbb{E}\left[\widetilde{Y}_n - H \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \,\middle|\, \widetilde{Y}_n = y \right] = y - H \frac{\widetilde{f}(y)}{\mathbf{w}\big(\widetilde{f}(y)\big)}, \tag{2.18}$$

$\gamma_n := \frac{1}{n+\mathbf{w}(Y_0)}$ is its step parameter and

$$r_{n+1} := (H_{n+1} - H) \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \quad \text{is an } \mathcal{F}_n\text{-adapted remainder term.} \tag{2.19}$$

2.3 Boundedness of the normalized urn composition

Our first task is to establish the *a.s.* boundedness of the sequence $(\widetilde{Y}_n)_{n\geq 0}$. By summing up the components of $\widetilde{Y}_n$ in (2.14), we obtain

$$\mathbf{w}(Y_{n+1}) = \mathbf{w}(Y_n) + \frac{\mathbf{w}(H_{n+1}\widetilde{f}(\widetilde{Y}_n))}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} + \mathbf{w}(\Delta M_{n+1}).$$

Using that the transpose of the generating matrix H_{n+1} is a stochastic matrix by (**A1**)-(*ii*) (with $c = 1$), we obtain

$$\mathbf{w}(H_{n+1}\widetilde{f}(\widetilde{Y}_n)) = \sum_{i=1}^{d}(H_{n+1}\widetilde{f}(\widetilde{Y}_n))_i = \sum_{i=1}^{d}\sum_{j=1}^{d} H_{n+1}^{ij} f(\widetilde{Y}_n^j) = \sum_{j=1}^{d}\left(\sum_{i=1}^{d} H_{n+1}^{ij}\right) f(\widetilde{Y}_n^j) = \mathbf{w}(\widetilde{f}(\widetilde{Y}_n)).$$

Consequently, for every $n \geq 0$,

$$\mathbf{w}(Y_{n+1}) = \mathbf{w}(Y_n) + 1 + \mathbf{w}(\Delta M_{n+1}). \tag{2.20}$$

Let $N_0 = 0$ and $N_n := \sum_{k=1}^{n} X_k$, $n \geq 1$, denote the number of times each type of ball was drawn between draws 1 and n. For every $n \geq 0$,

$$\begin{aligned} N_{n+1} &= N_n + X_{n+1} = N_n + \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} + \Delta\widetilde{M}_{n+1}, \\ \text{where} \qquad \Delta\widetilde{M}_{n+1} &:= X_{n+1} - \mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right] = X_{n+1} - \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \end{aligned}$$

is an $\mathcal{F}_n$-martingale increment. Thus, $\widetilde{N}_n := \dfrac{N_n}{n}$ satisfies, still for every $n \geq 0$,

$$\widetilde{N}_{n+1} = \widetilde{N}_n - \frac{1}{n+1}\left(\widetilde{N}_n - \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right) + \frac{1}{n+1}\Delta\widetilde{M}_{n+1}.$$

Proposition 2.1. *Let $(Y_n)_{n\geq 0}$ be the urn composition sequence defined by (1.1)-(1.3).*
(a) Under the assumptions **(A1)** *and* **(A2)**,

$$\frac{\mathbf{w}(Y_n)}{n + \mathbf{w}(Y_0)} \xrightarrow[n\to+\infty]{a.s.} 1.$$

(b) If the addition rule matrices D_n themselves are co-stochastic, then $\mathbf{w}(Y_n) = n + \mathbf{w}(Y_0)$, and the sequence $(\widetilde{Y}_n)_{n\geq 0}$ lives in the simplex $\mathcal{S}_d$.

Proof. *(a)* We derive from the identity

$$D_{n+1}X_{n+1} = \sum_{j=1}^{d} D_{n+1}^{\cdot j}\mathbb{1}_{\{X_{n+1}=e^j\}}, \quad n \geq 0,$$

that

$$\|D_{n+1}X_{n+1}\|^2 = \sum_{j=1}^{d}\left\|D_{n+1}^{\cdot j}\right\|^2 \mathbb{1}_{\{X_{n+1}=e^j\}}.$$

Hence, owing to (**A2**)-(ii),

$$\begin{aligned}\mathbb{E}\left[\|D_{n+1}X_{n+1}\|^2 \,|\, \mathcal{F}_n\right] &= \sum_{j=1}^{d} \mathbb{E}\left[\left\|D^{\cdot j}_{n+1}\right\|^2 \,|\, \mathcal{F}_n\right] \mathbb{P}\left(X_{n+1} = e^j \,|\, \mathcal{F}_n\right) \\ &\leq \sup_{n\geq 0}\sup_{1\leq j\leq d} \mathbb{E}\left[\left\|D^{\cdot j}_{n+1}\right\|^2 \,|\, \mathcal{F}_n\right] < +\infty \quad a.s.\end{aligned}$$

Consequently $\sup_{n\geq 1}\mathbb{E}\left[\|\Delta M_{n+1}\|^2 \,|\, \mathcal{F}_n\right] < +\infty$ $a.s.$ and thanks to the strong law of large numbers for conditionally L^2-bounded local martingale increments, we have $\frac{M_n}{n} \underset{n\to+\infty}{\longrightarrow} 0$ $a.s.$ Finally, it follows from (2.20) that

$$\frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} = 1 + \frac{\mathbf{w}(M_n)}{n+\mathbf{w}(Y_0)} \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} 1.$$

(b) In this case $\mathbf{w}(M_n) = 0$, consequently for every $n \geq 0$, $\mathbf{w}(\widetilde{Y}_n) = 1$. □

2.4 Existence of equilibrium points

As written in (2.17), the urn dynamics appears as a recursive zero search algorithm with mean field $h : [0,1]^d \setminus \{0\} \to \mathbb{R}^d$ whose potential limiting points lies in the the canonical simplex $\mathcal{S}_d$ defined by

$$\mathcal{S}_d = \mathbf{w}^{-1}\{1\} = \left\{y \in \mathbb{R}^d_+ \,|\, \mathbf{w}(y) = 1\right\}.$$

More generally, since the components of $\widetilde{Y}_n = \frac{Y_n}{n+\mathbf{w}(Y_0)}$ are non-negative by construction and – under (**A1**)-(**A2**) – $\mathbf{w}(\widetilde{Y}_n) = \frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} \underset{n\to+\infty}{\longrightarrow} 1$ $a.s.$, it is clear that $\mathbb{P}(d\omega)$-$a.s.$, the sequence $(\widetilde{Y}_n(\omega))_{n\geq 0}$ is bounded and that the set $\mathcal{Y}_\infty(\omega)$ of its limiting values lies in the simplex $\mathcal{S}_d$. Consequently, we search $\mathcal{S}_d$-valued equilibrium points *i.e.* points $y \in \mathcal{S}_d$ such that $h(y) = 0$ where h is given by (2.18).

Throughout this section f denotes a skewing function in the sense of (2.9) and H is a *deterministic* (co-stochastic) matrix, intended to be a limiting generating matrix as soon as it is irreducible.

Proposition 2.2. (a) *Let H be a deterministic co-stochastic matrix. The function $\varphi_H : [0,1]^d \setminus \{0\} \to \mathcal{S}_d$ defined by $\varphi_H(y) = H\frac{\widetilde{f}(y)}{\mathbf{w}\left(\widetilde{f}(y)\right)}$ has at least one fixed point. As a consequence h has at least one zero y^* and $\{h = 0\} \subset \mathcal{S}_d$. (When H is bi-stochastic, $y(d) := \frac{1}{d}\mathbf{1}$ is such a zero of h.)*
(b) *If, furthermore, for every $i, j \in \{1,\ldots,d\}$, $H^{ij} > 0$, then for every zero y^* of h in $\mathcal{S}_d$,*

$$\forall\, i \in \{1,\ldots,d\}, \quad \min_{1\leq j\leq d} H^{ij} \leq y^{*,i} \leq \max_{1\leq j\leq d} H^{ij}.$$

Proof. (a) The function φ_H is well-defined on $[0,1]^d \setminus \{0\}$ since $\mathbf{w}\left(\widetilde{f}(y)\right) > 0$ on $[0,1]^d \setminus \{0\}$ owing to the fact that $f > 0$ on $(0,1)$. The function $\varphi : y \mapsto \frac{\widetilde{f}(y)}{\mathbf{w}\left(\widetilde{f}(y)\right)}$ clearly maps $[0,1]^d \setminus \{0\}$ into $\mathcal{S}_d$. So does φ_H since H is co-stochastic and subsequently maps $\mathcal{S}_d$ into $\mathcal{S}_d$. Since $h(y) = y - \varphi_H(y)$ by (2.18), it follows that $\{h = 0\} \subset \mathcal{S}_d$.

Now, both functions $y \mapsto \widetilde{f}(y)$ and $y \mapsto \mathbf{w}\left(\widetilde{f}(y)\right)$ are continuous on $\mathcal{S}_d$. Moreover $\mathbf{w}\left(\widetilde{f}(y)\right) \geq f\left(\frac{1}{d}\right) > 0$ since, for any $y = (y^1,\ldots,y^d)^t \in \mathcal{S}_d$, there exists $i_0 \in \{1,\ldots,d\}$ such that $y^{i_0} \geq \frac{1}{d}$ so

that $\mathbf{w}\big(\widetilde{f}(y)\big) \geq f(y^{i_0}) \geq f\left(\frac{1}{d}\right) > 0$. Therefore, $\varphi_{_H}$ is continuous and maps $\mathcal{S}_d$ into $\mathcal{S}_d$. Then, by Brouwer's Theorem, φ_H has at least one fixed point *i.e.* $\{h = 0\} \neq \varnothing$. The last claim is obvious since $\varphi(y(d)) = y(d)$ and $H\mathbf{1} = \mathbf{1}$.

(*b*) Let $i \in \{1, \ldots, d\}$. It follows from the identity $\displaystyle\sum_{j=1}^{d} H^{ij} \frac{f(y^{*,j})}{\mathbf{w}(\widetilde{f}(y^*))} = y^{*,i}$, that $\min\limits_{1\leq j\leq d} H^{ij} \leq y^{*i} \leq \max\limits_{1\leq j\leq d} H^{ij}$ since f is non-negative. $\square$

Proposition 2.3 (Bi-stochastic case)**.** *Let $\mathcal{E}_d := \{h = 0\} = \big\{y \in \mathbb{R}^d_+ \setminus \{0\} : h(y) = 0\big\} \subset \mathcal{S}_d$ denote the (non-empty) set of equilibrium points.*

(*a*) *If H is bi-stochastic, then $y(d) := \frac{1}{d}\mathbf{1} \in \mathcal{E}_d$.*

(*b*) *If $H = I_d$, then $\big\{\widetilde{e}_{_I},\, I \subset \{1, \ldots, d\}, I \neq \varnothing\big\} \subset \mathcal{E}_d$, where $\widetilde{e}_{_I} = \frac{1}{|I|}\sum_{i\in I} e^i$ with $(e^i)_{1\leq i\leq d}$ the canonical basis of $\mathbb{R}^d$.*

(*c*) *If $H = I_d$ and f is a strictly convex or strictly concave skewing function, then*

$$\mathcal{E}_d = \big\{\widetilde{e}_{_I}, I \subset \{1, \ldots, d\}, I \neq \varnothing\big\}.$$

(*d*) *If H is bi-stochastic and irreducible, then $\mathcal{E}_d \subset \overset{\circ}{\mathcal{S}}_d = \Big\{y \in (0,1)^d : \sum_{i=1}^d y^i = 1\Big\}$.*

(*e*) *If, furthermore, f is strictly concave, then $\mathcal{E}_d = \big\{\frac{1}{d}\mathbf{1}\big\}$.*

Proof. (*a*) Set $y(d) = \frac{1}{d}\mathbf{1} \in \mathcal{S}_d$. Thus, $\frac{f(\frac{1}{d})}{d\cdot f(\frac{1}{d})} = \frac{1}{d}$ since $f\left(\frac{1}{d}\right) > 0$ and, consequently, $\displaystyle\sum_{j=1}^{d} H^{ij}\frac{1}{d} = 1 \times \frac{1}{d}$ for every $i \in \{1, \ldots, d\}$, therefore $h\big(y(d)\big) = y(d) - y(d) = 0$.

(*b*) Let $I \neq \varnothing$. Then

$$f(\widetilde{e}^{\,i}_{_I}) = \begin{cases} 0 & \text{if} \quad i \notin I \\ f\left(\frac{1}{|I|}\right) > 0 & \text{if} \quad i \in I \end{cases}.$$

Hence, $\mathbf{w}(\widetilde{f}(\widetilde{e}_{_I})) = |I| f\left(\frac{1}{|I|}\right)$ as well. Consequently $h(\widetilde{e}_{_I}) = 0$.

(*c*) Let $y^* \in \mathcal{E}_d$. Assume that there exists y^{*,i_0}, y^{*,i_1} such that $0 < y^{*,i_0} < y^{*,i_1} \leq 1$. Then $y^{*,i_0} = \frac{f(y^{*,i_0})}{\mathbf{w}(\widetilde{f}(y^*))}$ and $y^{*,i_1} = \frac{f(y^{*,i_1})}{\mathbf{w}(\widetilde{f}(y^*))}$ so that $\frac{f(y^{*,i_0})}{y^{*,i_0}} = \frac{f(y^{*,i_1})}{y^{*,i_1}} = \mathbf{w}(\widetilde{f}(y^*))$. Now, if f or $-f$ is strictly convex, then the function $\xi \in \mathbb{R}_+ \mapsto \frac{f(\xi)}{\xi}$ is strictly monotonic since $f(0) = 0$. This yields a contradiction.

As a consequence, there exists $\xi_0 \in (0, 1]$ such that $y^{*,i} \in \{0, \xi_0\}$, $i = 1, \ldots, d$. Consequently, if $I_{\xi_0} = \{i : y^{*,i} = \xi_0\}$, $\mathbf{w}(\widetilde{f}(y^*)) = |I_{\xi_0}| f(\xi_0)$ and, for every $i \in I_{\xi_0}$, $\frac{f(y^{*,i})}{\mathbf{w}(\widetilde{f}(y^*))} = \frac{f(\xi_0)}{|I_{\xi_0}| f(\xi_0)} = \frac{1}{|I_{\xi_0}|}$, *i.e.* $h(y^*) = \frac{1}{|I_{\xi_0}|}\sum_{i\in I_{\xi_0}} e^i$. Hence, $y^* = \frac{1}{|I_{\xi_0}|}\sum_{i\in I_{\xi_0}} e^i$ so that $\xi_0 = 1$ and $y^* \in \big\{\tilde{e}_I,\, I \subset \{1, \ldots, d\},\, I \neq \varnothing\big\}$.

(*d*) Let i_0 be such that $y^{*,i_0} = \min_i y^{*,i}$. If $y^* \notin \overset{\circ}{\mathcal{S}}_d$, then $y^{*,i_0} = 0$ so that

$$\sum_{j=1}^{d} H^{i_0 j} \frac{f(y^{*,j})}{\mathbf{w}(\widetilde{f}(y^*))} = 0$$

and $i_0 \in I_0^* = \{i : y^{*,i} = 0\} \neq \varnothing$. Let $I^*_{>0} = I \setminus I_0^* = \{j : y^{*,j} > 0\} \neq \varnothing$ since $y^* \in \mathcal{S}_d$. Let us show by induction that $I^*_{>0} = \{j : y^{*,j} > 0\}$ and $I_0^* = \{j = y_j^{*,i} = 0\}$ are not connected by any power of H

which will contradict the irreducibility of H. Let $i_0 \in I_0^*$ and $j_0 \in I_{>0}^*$, then the above equality implies $H^{i_0 j_0} = 0$ since $f(y^{*,j_0}) > 0$. Now assume that $H_{ij}^k = 0$ for every $(i,j) \in I_0^* \times I_{>0}^*$. Then

$$H_{i_0 j_0}^k = \sum_{\ell=1}^d H_{i_0 \ell} H_{\ell j_0}^{k-1} = \sum_{\ell \in I_{>0}^*} H_{i_0 \ell} H_{\ell j_0}^{k-1} + \sum_{\ell \in I_0^*} H_{i_0 \ell} H_{\ell j_0}^{k-1} = \sum_{\ell \in I_0^*} H_{i_0 \ell} H_{\ell j_0}^{k-1} = 0.$$

This contradicts the irreducibility of H since i_0 and j_0 are not connected through a power of H.
(e) We know that $\frac{1}{d}\mathbf{1} \in \mathcal{E}_d$ and that, any $y^* \in \mathcal{E}_d$, $\min_i y^{*,i} > 0$. Let i_0 be such that $y^{*,i_0} = \min_i y^{*,i}$. Then

$$\sum_{j=1}^d H^{i_0 j} f(y^{*,j}) \geq \sum_{j=1}^d H^{i_0 j} f(y^{*,i_0}) = f(y^{*,i_0})$$

since H is stochastic. On the other hand, using the concavity of f, we derive that

$$\forall\, y \in \mathcal{S}_d, \quad \mathbf{w}\big(\widetilde{f}(y)\big) = d \sum_{j=1}^d \frac{1}{d} f(y_j) \leq d \cdot f\Big(\frac{1}{d} \sum_{j=1}^d y_j\Big) = d \cdot f\Big(\frac{1}{d}\Big).$$

As a consequence

$$y^{*,i_0} = \frac{\sum_{j=1}^d H^{i_0 j} f(y^{*,j})}{\mathbf{w}(\widetilde{f}(y^*))} \geq \frac{f(y^{*,i_0})}{d \cdot f\left(\frac{1}{d}\right)},$$

which can be rewritten as $\frac{f(y^{*,i_0})}{y^{*,i_0}} \leq \frac{f(1/d)}{1/d}$. As $\xi \mapsto \frac{f(\xi)}{\xi}$ is (strictly) decreasing since f is strictly concave and $f(0) = 0$, it implies that $y^{*,i_0} \geq \frac{1}{d}$ which in turn implies that $y^* = \frac{1}{d}\mathbf{1}$ since $y^* \in \mathcal{S}_d$. $\square$

Remark. When H is not bi-stochastic but simply co-stochastic, we have no closed form for an $\overset{\circ}{\mathcal{S}}_d$-valued equilibrium and we could not manage to establish uniqueness even if f is (strictly) concave.

Note that, as f and $\mathbf{w}$ are defined on $[0,1]^d \setminus \{0\}$, $\widetilde{f}\big([0,1]^d \setminus \{0\}\big) \subset [0,1]^d \setminus \{0\}$ and $\mathbf{w}\big(\widetilde{f}(y)\big) > 0$ on $[0,1]^d \setminus \{0\}$ since $f > 0$ on $(0,1)$. Hence on may define the function

$$\varphi : y \mapsto \frac{\widetilde{f}(y)}{\mathbf{w}(\widetilde{f}(y))} \quad \text{on} \quad [0,1]^d \setminus \{0\}. \tag{2.21}$$

Lemma 2.1. *If f is differentiable on in the neighbourhood of $\frac{1}{d}$, then the functions $\widetilde{f}$, $\mathbf{w} \circ \widetilde{f}$ and φ defined on $[0,1]^d \setminus \{0\}$ are bounded on $(0,1]^d$ and the last two functions are differentiable in the neighbourhood of $y(d) = \frac{1}{d}\mathbf{1}$. Moreover, $\varphi(y(d)) = y(d)$ and the Jacobian of φ at $y(d)$ is given by*

$$J_\varphi(y(d)) = \begin{pmatrix} a & b & \dots & \dots & b \\ b & a & b & \dots & b \\ \vdots & b & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & b \\ b & b & \dots & b & a \end{pmatrix} \quad \textit{with} \quad a = \frac{f'(1/d)(d-1)}{d^2 f(1/d)} \quad \textit{and} \quad b = -\frac{f'(1/d)}{d^2 f(1/d)} = -\frac{a}{d-1}.$$

As a symmetric matrix, $J_\varphi(y(d))$ is diagonalizable in the orthogonal group $\mathcal{O}(d,\mathbb{R})$ with eigenvalues 0 (associated to the eigenvector $\mathbf{1}$) and $\frac{f'(1/d)}{d\, f(1/d)}$ with eigenspace the hyperplane $\mathbf{1}^\perp = \Big\{u \in \mathbb{R}^d : \sum_{i=1}^d u^i = 0\Big\}$.

Proof. If $y \in (0,1)^d$ and f is differentiable in the neighborhood of $y^1, \dots, y^d$, then φ is differentiable at y and

$$\frac{\partial \varphi^i}{\partial y^j}(y) = -\frac{f(y^i)f'(y^j)}{\mathbf{w}\big(\widetilde{f}(y)\big)^2} + \delta_{ij}\frac{f'(y^i)}{\mathbf{w}\big(\widetilde{f}(y)\big)},\ 1 \le i,j \le d. \tag{2.22}$$

The form of $J_\varphi(y(d))$ follows. Elementary and classical computations show that, for every real numbers a, b,

$$\det \begin{pmatrix} a & b & \dots & \dots & b \\ b & a & b & \dots & b \\ \vdots & b & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & b \\ b & b & \dots & b & a \end{pmatrix} = (a-b)^{d-1}(a+b(d-1)),$$

which in turn implies that the characteristic polynomial of $J_\varphi(y(d))$ is given by $(a-b-\lambda)^{d-1}(a+b(d-1)-\lambda)$ so that the eigenvalues are

$$\lambda_0 = a+b(d-1) = 0 \text{ (order 1)} \quad \text{and} \quad \lambda_1 = a-b = \frac{f'(1/d)}{df(1/d)} \text{ (order } d-1).$$

The eigenspace associated to λ_0 is clearly $\mathbb{R}\mathbf{1}$ and that associated to λ_1 is $\mathbf{1}^\perp = \Big\{u \in \mathbb{R}^d : \sum_{i=1}^d u^i = 0\Big\}$, so that $J_\varphi(y(d))_{|\mathbf{1}^\perp} = \lambda_1 I_{d|\mathbf{1}^\perp}$. □

To go beyond and provide convergence results, especially in the bi-stochastic case, we will rely on an important tool in SA theory, the ODE method. This method makes the connection between the asymptotic behaviour of the stochastic algorithm with mean field h with that of $ODE_h \equiv \dot{\theta} = -h(\theta)$ (for some key results on this theory, we refer to the Appendix).

Proposition 2.4. *(a) From any $\xi \in \mathcal{S}_d$, there exists at least one $\mathcal{S}_d$-valued solution $(y(\xi,t))_{t\in\mathbb{R}_+}$ to ODE_h starting from ξ.*

(b) From any $\xi \in \overset{\circ}{\mathcal{S}}_d = \mathcal{S}_d \cap (0,1]^d$, there exists a unique solution to ODE_h starting from ξ, denoted $(\Phi(\xi,t))_{t\in\mathbb{R}_+}$ is $\overset{\circ}{\mathcal{S}}_d$-valued.

Proof. *(a)* Let $\eta_0 > 0$. As h is continuous $\mathcal{S}_d \cap [\eta_0,1]^d \to \mathbf{1}^\perp = \{\sum_{i=1}^d u^i = 0\}$, it is clear that, on the one hand, the Euler scheme of ODE_h with step $\frac{1}{n}$ starting from $\xi \in \mathcal{S}_d$ defined by $\bar{y}^n_{\frac{k+1}{n}} = \bar{y}^n_{\frac{k}{n}} - \frac{1}{n}h(\bar{y}^n_{\frac{k}{n}})$ is $\mathcal{S}_d$-valued since $\mathbf{w}\big(h(\bar{y}^n_{\frac{k}{n}})\big)$ is constant equal to $\mathbf{w}(y) = 1$, and, on the other hand, converges owing to Peano's theorem to a solution of ODE_h.

(b) The function $u \mapsto f(u)$ is Lipschitz continuous on $[\eta_0,1]$, with Lipschitz coefficient $f'_\ell(\eta_0) \le 1$ since f' is concave. Hence the (canonical extensions of) the functions $\widetilde{f}$ and $\mathbf{w}(\widetilde{f})$ on $[\eta_0,1]^d$ are Lipschitz as well. Both are also bounded. Moreover $\mathbf{w}\big(\widetilde{f}(y)\big) \ge f\left(\frac{1}{d}\right) > 0$ because for any $y = (y^1,\dots,y^d)^t \in \mathcal{S}_d$, there exists $i_0 \in \{1,\dots,d\}$ such that $y^{i_0} \ge \frac{1}{d}$ so that $\mathbf{w}\big(\widetilde{f}(y)\big) \ge f(y^{i_0}) \ge f\left(\frac{1}{d}\right) > 0$. As a consequence, $\mathbf{w}(\widetilde{f}) \ge \frac{1}{2}f\left(\frac{1}{d}\right)$ on a neigbourhood of $\mathcal{S}_d$ in $[0,1]^d$. Therefore, the functions $\varphi_H : y \mapsto H\frac{\widetilde{f}(y)}{\mathbf{w}\left(\widetilde{f}(y)\right)}$ from $\mathcal{S}_d \cap [\eta_0,1]^d$ to $\mathcal{S}_d$ and $h = I_d - \varphi_H$ from $\mathcal{S}_d \cap [\eta_0,1]^d$ to $\{\sum_{i=1}^d u^i = 0\}$ are Lipschitz continuous too in that neighbourhood. This guarantees the existence of an $\mathcal{S}_d$-valued flow for ODE_h on $\mathcal{S}_d \cap (0,1]^d = \overset{\circ}{\mathcal{S}}_d$.

Definition 2.2 (Stable equilibrium point)**.** *An element y^* of $\mathcal{S}_d$ is a* stable equilibrium *for $ODE_h \equiv \dot{y} = -h(y)$ on $\mathcal{S}_d$ if there is (compact) neighborhood $\mathcal{K}^*$ of y^* in $\mathcal{S}_d$ such that*

$$\lim_{t\to+\infty} \sup\Big\{ \big|y(\xi,t) - y^*\big|,\ \xi \in \mathcal{K}^*,\ y(\xi,\cdot)\ \mathcal{S}_d\text{-valued solution to } ODE_h,\ y(\xi,0) = 0 \Big\} = 0$$

with a slight abuse of notation since possibly several solutions may start from $\xi \in \partial\mathcal{S}_d$. See also Theorem A.1(b).

Remarks. • If the $\mathcal{S}_d$ -valued flow of ODE_h is well-defined in the neighbourhhood of y^* in $\mathcal{S}_d$, this boils down to showing that $y(\xi,t)$ is $\mathcal{S}_d$-valued and converges to y^* as $t \to +\infty$, uniformly with respect to ξ in a (compact) neighbourhood of y^* in $\mathcal{S}_d$.

• Since h is differentiable, an equilibrium y^* is attractive if all the eigenvalues of $J_h(y^*)_{|\mathbf{1}^\perp} = \big(I_d - HJ_\varphi(y^*)\big)_{|\mathbf{1}^\perp}$ have (strictly) positive real parts, see [6]. If one of these eigenvalues has a negative real part then the equilibrium is *unstable* and if all eigenvalues have negative real parts the equilibrium is called a *repeller*.

Proposition 2.5. *Assume H be a bi-stochastic matrix (keep in mind that $y(d)$ is a zero of h). Let $\lambda_1 = \frac{f'(1/d)}{d\cdot f(1/d)}$ and let $\mu_{\max}$ be the eigenvalue of $H_{|\mathbf{1}^\perp}$ with the highest real part.*

(a) If $\Re e(\mu_{\max}) < \frac{1}{\lambda_1}$, then $y(d) = \mathbf{1}/d$ is always a stable equilibrium of $ODE_h \equiv \dot{y} = -h(y)$. Thus, if

$$\Big(\lambda_1 \le 1 \ \text{ and } 1 \notin \mathrm{Sp}(H_{|\mathbf{1}^\perp})\Big) \text{ or } (\lambda_1 < 1)\,,$$

then $y(d)$ is always a stable equilibrium.

In particular the above left condition is satisfied if H is irreducible and f is concave, whereas the right one is always fulfilled if f is strictly concave over $(0, 1/d)$.

(b) If $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$, then $y(d)$ is unstable (it is even a repeller when $H = I_d$). Note that if f is convex (resp. strictly over $(0, \frac{1}{d})$), then $\lambda_1 \ge 1$ (resp. > 1).

Remark. • If $\lambda_1 > 1$ (*e.g.* since f is strictly convex), then $\frac{1}{\lambda_1} < 1$ and the two opposite situations $\Re e(\mu_{\max}) < \frac{1}{\lambda_1}$ and $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$ may occur a priori *i.e.* $y(d)$ may switch from uniform stability to instability (see Section 3 for the two-type case: $d = 2$).

• Note that if f is strictly convex and $1 \in \mathrm{Sp}(H_{|\mathbf{1}^\perp})$, then $y(d)$ is always unstable.

Proof. (*a*) It follows from Lemma 2.1 that $J_\varphi(y(d)) = \lambda_1 I_{d|\mathbf{1}^\perp}$. Hence

$$J_h(y(d))_{|\mathbf{1}^\perp} \ = \ I_{d|\mathbf{1}^\perp} - \big(HJ_\varphi(y(d))\big)_{|\mathbf{1}^\perp} = I_{d|\mathbf{1}^\perp} - H\lambda_1 I_{d|\mathbf{1}^\perp} = (I_d - \lambda_1 H)_{|\mathbf{1}^\perp}.$$

Hence $\mathrm{Sp}(J_h(y(d))_{|\mathbf{1}^\perp}) = \big\{1 - \lambda_1\mu, \mu \in \mathrm{Sp}(H_{|\mathbf{1}^\perp})\big\} \subset \mathbb{C}$.

Every $\mu \in \mathrm{Sp}(H)$ satisfies $|\mu| \le 1$ since H is stochastic. If $\lambda_1 < 1$, then $|\lambda_1\mu_{\max}| < 1$ and consequently $\Re e(1-\lambda_1\mu_{\max}) > 0$ which ensures that $y(d)$ is attracting. The other case follows likewise.

When H is irreducible, the eigenvalue 1 is simple owing to the Perron-Frobenius Theorem so that $1 \notin \mathrm{Sp}(H_{|\mathbf{1}^\perp})$.

(*b*) If f is convex (and $f > 0$ on $(0,1)$), then $0 < f(1/d) \le f'(1/d)/d$ since $f(0) = 0$ so that $\lambda_1 \ge 1$. This inequality is strict if f is strictly convex over $(0, \frac{1}{d})$. □

Proposition 2.6. *(a) If H is bi-stochastic and irreducible and f is strictly concave, then ODE_h has at least one solution starting from every $\xi \in \mathcal{S}_d$. The flow of $ODE_h \equiv \dot{y} = -h(y)$, denoted by $\big(y(\xi,t)\big)_{\xi\in\overset{\circ}{\mathcal{S}}_d}$, $t \geq 0$, exists on $\overset{\circ}{\mathcal{S}}_d$ and is $\mathcal{S}_d$-valued. Furthermore (still with $y(d) = \frac{\mathbf{1}}{d}$),*

$$\lim_{t\to+\infty} \sup\Big\{\Big|y(\xi,t) - y(d)\Big|,\ \xi \in \mathcal{S}_d,\ y(\xi,.) \text{ solution to } ODE_h,\ y(\xi,0) = 0\Big\} = 0.$$

(b) If H is simply bi-stochastic (and possibly not irreducible), then the flow, denoted by $y(\xi,t)$, converges toward $y(d)$ for every $\xi \in \overset{\circ}{\mathcal{S}}_d$.

Proof. (a) Let us denote by $y(\xi,t)$ an $\mathcal{S}_d$-valued solution starting from $\xi \in \mathcal{S}_d$.

Let $\xi \in \mathcal{S}_d\backslash\{y(d)\}$ and let $i(t)$ be the right continuous function such that $y^{i(t)}(\xi,t) = \min_j y^j(\xi,t) \in [0, \frac{1}{d}]$. We know that $y^{i(0)}(\xi,0) = \xi^{i(0)} < \frac{1}{d}$ since $\xi \neq y(d)$. Note that the function $g_i : t \mapsto y^{i(t)}(\xi,t)$ is continuous and right differentiable with a right derivative given by

$$(g_i)'_r(\xi,t) = \sum_{j=1}^d H^{i(t)j} \frac{f(y^j(\xi,t))}{\mathbf{w}(\widetilde{f}(y(\xi,t)))} - y^{i(t)}(\xi,t) \geq 1 \times \frac{f(y^{i(t)}(\xi,t))}{\mathbf{w}(\widetilde{f}(y(\xi,t)))} - y^{i(t)}(\xi,t).$$

Note that $\mathbf{w}(\widetilde{f}(y(\xi,t))) \leq d \cdot f\left(\frac{1}{d}\right)$ since f is concave. It follows that

$$(g_i)'_r(\xi,t) \geq \frac{f(y^{i(t)}(\xi,t))}{d \cdot f\left(\frac{1}{d}\right)} - y^{i(t)}(\xi,t) \geq 0, \tag{2.23}$$

since $u \mapsto \frac{f(u)}{u}$ is non-increasing (f is concave and $f(0) = 0$) and $y^{i(t)}(\xi,t) \leq \frac{1}{d}$ for every $t \geq 0$.

If $\xi^{i(0)} = 0$, then $y^{i(t)}(\xi,t) \geq 0$, for every $t \geq 0$. Assume there exists $\varepsilon_0 > 0$ such that $y^{i(t)}(\xi,t) = 0$ for every $t \in (0,\varepsilon_0]$, then one derives from the integrated form of ODE_h that

$$\int_0^{\varepsilon_0} \left(\sum_{j=1}^d H^{i(t)j} \frac{f(y^j(\xi,s))}{\mathbf{w}(\widetilde{f}(y(\xi,s)))}\right) ds = 0,$$

so that, as the above integrand is non-negative and continuous, $s \mapsto \sum_{j=1}^d H^{i(s)j} \frac{f(y^j(\xi,s))}{\mathbf{w}(\widetilde{f}(y(\xi,s)))} \equiv 0$ on $[0,\varepsilon_0]$. Let $I_0 = \{i \text{ s.t. } y^i(\xi,\varepsilon_0) = 0,\ 1 \leq i \leq d\}$ and $I_1 = I_0^c$. Then, it follows that $i(\varepsilon_0) \in I_0$ and $H^{i(\varepsilon_0)j} = 0$, for every $j \in I_1$, which is not empty since $\sum_j y^j(\xi,t) = 1$. Hence, reasoning like in Proposition 2.3(d), $i(\varepsilon_0)$ (and more generally any element of I_0) and I_1 are not H-connected which contradicts the irreducibility. Consequently, $y^{i(t)}(\xi,t) > 0$, $t \in (0,\varepsilon_0]$, *i.e.* $y(\xi,t)$ lives in $\overset{\circ}{\mathcal{S}}_d$ and, consequently, lives in $\overset{\circ}{\mathcal{S}}_d$ for every $t \geq 0$ by monotonicity of $y^{i(t)}(\xi,t)$.

If $\xi^{i(0)} > 0$, the same conclusion follows simply from the monotonicity of $y^{i(t)}(\xi,t)$.

Finally, for every $t > 0$, $y^{i(t)}(\xi,t)$ is positive and increasing. We can integrate (2.23) between a fixed $\varepsilon > 0$ and $t > \varepsilon$ as follows

$$-\log\left(\frac{d}{\xi^{i(0)}(\varepsilon)}\right) \geq \log\left(\frac{y^{i(t)}(\xi,t)}{\xi^{i(0)}(\varepsilon)}\right) \geq \int_\varepsilon^t \left(\underbrace{\frac{f(y^{i(s)}(\xi,s))}{y^{i(s)}(\xi,s)} - \frac{f\,(1/d)}{1/d}}_{\geq 0}\right) ds \geq 0$$

and the latter integral is increasing in t as long as $y^{i(t)}(\xi,t) < \frac{1}{d}$. As the left hand side of the above string of inequalities is finite, it follows that $\int_0^{+\infty} \left(\frac{f(y^{i(s)}(\xi,s))}{y^{i(s)}(\xi,s)} - \frac{f(1/d)}{1/d} \right) ds < +\infty$. Then, combining that $u \mapsto \frac{f(u)}{u}$ is decreasing (by strict concavity) on $(0,+\infty)$ with the fact that $y^{i(t)}(\xi,t)$ is increasing and positive, we derive that the function $t \mapsto \frac{f(y^{i(t)}(\xi,t))}{y^{i(t)}(\xi,s)} - \frac{f(1/d)}{1/d}$ is decreasing. The finiteness of the above integral implies that $\dfrac{f\,(1/d)}{1/d} - \dfrac{f(y^{i(t)}(\xi,t))}{y^{i(t)}(\xi,t)} \to 0$ as $t \to +\infty$ or, equivalently, that

$$y^{i(t)}(\xi,t) = \min_{1\le i\le d} y^i(\xi,t) \to \frac{1}{d} \quad \text{as} \quad t \to +\infty.$$

The convergence of every component $y^i(\xi,t)$ follows since $y(\xi,t) \in \mathcal{S}_d$. It remains to prove that the convergence holds uniformly in the starting value (the existence of the flow *i.e* uniqueness starting from the boundary of the simplex is not mandatory). First note that, h being bounded, the family of all possible solutions $((y(\xi,t)_{t\ge 0})_{\xi\in\mathcal{S}_d})$ of ODE_h are $\|h\|_\infty$-Lipschitz continuous since

$$y(\xi,t) = \xi - \int_0^t h\big(y(\xi,s)\big)ds.$$

Assume there exists $\xi_n \to \xi_\infty$ and $t_n \to +\infty$ such that $\big|y(\xi_n,t_n) - y(d)\big| \ge \varepsilon_0 \ge 0$. By Proposition 2.1, $\frac{\mathbf{1}}{d}$ is an attractor of ODE_h, hence there exists $\eta_0 > 0$ such that $\sup_{|\xi - y(d)|\le\eta_0} \big|y(\xi,t) - y(d)\big| \le \varepsilon_0$, $t \ge t_0$ (the flow does exist in the neighbourhood of $y(d)$ so that $y(\xi,\cdot)$ is unique). Consequently, for every $t \in [0, t_n - t_0]$ (at least for n large enough), $|y(\xi_n,t) - y(d)| > \eta_0$. By Ascoli's Theorem, up to an extraction, one may assume that $y(\xi_n,\cdot)$ converges on compact sets of $\mathbb{R}_+$ toward a solution $y(\xi_\infty,\cdot)$ of ODE_h since h is continuous. Then, letting n go to infinity shows that this solution satisfies $|y(\xi_\infty,t) - y(d)| \ge \eta_0$ for every $t \ge 0$. This contradicts the fact that any solution starting from $\mathcal{S}_d$ converges toward $y(d)$. We can conclude that all the solutions of ODE_h starting from $\mathcal{S}_d$ converge uniformly toward $y(d)$.

(b) is obvious given the above proof. □

Application (I): *A.s.* **convergence of the algorithm when** f **is concave.** When f is concave and H is bi-stochastic, the mean point $y(d) = \frac{\mathbf{1}}{d}$ of the simplex is the unique *a.s.* target of the urn composition.

Proposition 2.7 (When f is concave $y(d)$ is the target)**.** *If H is (deterministic and) bi-stochastic, irreducible and f is a strictly concave skewing function, then*

$$\widetilde{Y}_n \xrightarrow[n\to+\infty]{a.s.} y(d) = \frac{\mathbf{1}}{d}.$$

Proof. It follows from Proposition 2.3(e) that the flow of ODE_h uniformly converges to $y(d)$. Hence, Theorem A.1(b) from the Appendix implies that the set Θ^∞ of the limiting values of ODE_h is reduced to $\{\frac{1}{d}\mathbf{1}\}$ which completes the proof. □

Application (II): The mean point $y(d)$ **may be a noisy trap when** f **is convex.** We now give a (partial) result in the convex setting (a more precise one is provided in Section 5 devoted to randomized Pólya's urns, that is when $H = I_d$). We keep the notations introduced Propositions 2.2 to 2.5: $\lambda_1 = \frac{f'(1/d)}{df(1/d)}$ and $\mu_{\max}$ the eigenvalues of $H_{|\mathbf{1}^\perp}$ with the highest real part.

As H is (bi-)stochastic $\Re e(\mu_{\max}) \leq 1$. If H is also irreducible, by the Perron-Frobenius Theorem, we know that $\Re e(\mu_{\max}) < 1$ since the eigenspace of the eigenvalue 1 is one-dimensional, hence equal to $\mathbb{R}\mathbf{1}$. On the other hand, when f is convex then $\lambda_1 \geq 1$ and even > 1 if f is *e.g.* strictly convex over $(0, 1/d)$. Consequently there exist matrices H for which the inequality $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$ is satisfied, implying by Proposition 2.5 that $y(d)$ is unstable for ODE_h. We show that $y(d)$ may be a noisy trap.

Proposition 2.8 (When $y(d)$ becomes a trap)**.** *Assume that f is convex and H is a bi-stochastic matrix satisfying $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$ so that $y(d)$ is unstable. Assume that $\sup_{n\geq 1} \mathbb{E}\left[\|\!|D_{n+1}|\!\| \,|\, \mathcal{F}_{n-1}\right] \leq L \in \mathbb{R}_+$.*

(a) If, for every $v \in \mathbb{R}^d$ with $\mathbf{w}(v) = 0$ and $\|v\| = 1$,

$$\liminf_n \mathbb{E}\big(\|D_{n+1}v\|^2 \,|\, \mathcal{F}_n\big) > 0,$$

then $\mathbb{P}\big(\widetilde{Y}_n \to y(d)\big) = 0$.

(b) Assume that H is also symmetric. Let v_μ be a unitary eigenvector attached to an eigenvalue μ of $H_{|\mathbf{1}^\perp}$ such that $\Re e(\mu) > \frac{1}{\lambda_1}$. If

$$\liminf_n \mathbb{E}\left[\|D_{n+1}v_\mu\|^2 \,|\, \mathcal{F}_n\right] > 0,$$

then $\mathbb{P}\big(\widetilde{Y}_n \to y(d)\big) = 0$.

Proof. We will focus on claim (b) (claim (a) relies on the same formal proof applied to any nonzero vector v with 0 weight). We want to apply Theorem A.2 from the Appendix. The first point to be checked is that the Jacobian J_h is locally Lipschitz continuous in the neighbourhood of $y(d)$. Note that $h = I_d - H\varphi$ where φ is defined in (2.21). This follows from the expression (2.22) for $\frac{\partial \varphi_i}{\partial x_j}$ having in mind that f' is non-decreasing by convexity of f. Then, we have to check Assumption (A.44). Elementary computations show that

$$\mathbb{E}\left[(\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\right] = v_\mu^t \left[\mathbb{E}\left[D_{n+1}\mathrm{diag}(\varphi_{_H}(\widetilde{Y}_n))D_{n+1}^t \,|\, \mathcal{F}_n\right] - 2\varphi_{_H}(\widetilde{Y}_n) \otimes \widetilde{Y}_n + \widetilde{Y}_n^{\otimes 2}\right] v_\mu.$$

On the event $\big\{\widetilde{Y}_n \to y(d)\big\}$, we derive, owing to Assumption (**A2**), the continuity of $\varphi_{_H}$ and $\varphi_{_H}(y(d)) = y(d)$, that

$$\mathbb{E}\left[(\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\right] = v_\mu^t \mathbb{E}\left[D_{n+1}\mathrm{diag}\big(y(d)\big)D_{n+1}^t \,|\, \mathcal{F}_n\right] v_\mu - (v_\mu|y(d))^2 + o(1).$$

Now, $\mathrm{diag}(y(d)) = \frac{1}{d} I_d$, and $v_\mu \in \mathbf{1}^\perp$ since $(v_\mu|y(d))^2 = 0$. Consequently,

$$\liminf_n \mathbb{E}\left[(\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\right] = \frac{1}{d} \liminf_n \mathbb{E}\left[\|D_{n+1}v_\mu\|^2 \,|\, \mathcal{F}_n\right] > 0.$$

Now, we note that

$$\|D_{n+1}v_\mu\| \geq \frac{\|D_{n+1}v_\mu\|^2}{|\!|\!|D_{n+1}|\!|\!|} \geq \frac{1}{L},$$

so that $\liminf_n \mathbb{E}\big[|(\Delta M_{n+1}|v_\mu)| \,|\, \mathcal{F}_n\big] > 0$. Now, as H is symmetric and leaves $\mathbf{1}^\perp$ stable, H is self-adjoint on $\mathbf{1}^\perp$. Hence, there exists an orthonormal basis of $\mathbf{1}^\perp$ containing v_μ so that,

$$\liminf_n \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \,|\, \mathcal{F}_n\right] \geq \liminf_n \mathbb{E}\left[(\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\right] > 0. \qquad \square$$

3 Bi-dimensional non linear randomized urn model

When the skewing function f is convex the situation becomes much more involved: Thus, if H is bi-stochastic and irreducible, then $y(d)$ is an equilibrium of h and $\mathcal{E}_d \subset \overset{\circ}{\mathcal{S}}_d$ (cf. Proposition 2.3(d)), but $\mathcal{E}_d$ is not reduced to $y(d)$ (which is a unstable for ODE_h when $H = I_d$).

To start elucidating this case, we limit ourselves in this paper to a two-type urn ($d = 2$) and an irreducible matrix H. We will see, as expected, that the asymptotic behavior of the urn is much more involved since a phase transition appears.

In such a simplified setting, the irreducible co-stochastic generating matrix H can be written as follows

$$H = \begin{pmatrix} p_1 & 1-p_2 \\ 1-p_1 & p_2 \end{pmatrix}, \qquad 0 < p_i < 1, \quad i = 1, 2,$$

and the mean function h of the model is still given by (2.18).

In this section, we assume for simplicity that f *is differentiable on* $(0, 1]$, so that the mean field h is differentiable too on $\overset{\circ}{\mathcal{S}}_2$. We saw that analyzing the $a.s.$ convergence properties essentially boils down to elucidate the behavior of ODE_h on $\mathcal{S}_2$ which in turn can be reduced to a one dimension differential system since the simplex $\mathcal{S}_2$ can be parametrized by $(u, 1-u)$, $u \in [0, 1]$, and h is $\mathbf{1}^\perp = \{z : z^1 + z^2 = 0\}$-valued on $\overset{\circ}{\mathcal{S}}_2$ so that $h^2 = 1 - h^1$. Thus, the asymptotic analysis of ODE_h is equivalent to that of

$$ODE_{h_0} \equiv \dot{u} = -h_0(u) \quad \text{with} \quad h_0(u) = h^1(u, 1-u), \quad u \in [0, 1].$$

Elementary computations show that, for every $u \in [0, 1]$,

$$h_0(u) = u - \frac{p_1 f(u) + (1-p_2) f(1-u)}{f(u) + f(1-u)} \tag{3.24}$$

and

$$h_0'(u) = 1 - (p_1 + p_2 - 1) \frac{f'(u) f(1-u) + f(u) f'(1-u)}{(f(u) + f(1-u))^2}. \tag{3.25}$$

We will now determine the combinatorics and the nature of the equilibrium points depending on the parameters p_1 and p_2.

Note that H is bi-stochastic if and only if $p_1 = p_2$. Note also that $h_0\left(\frac{1}{2}\right) = \dfrac{p_2 - p_1}{2}$ whatever the skewing function f is.

3.1 Counting equilibrium points

It follows from (3.24) that the equation $h_0(u) = 0$ reads

$$(p_1 - u) f(u) + (1 - p_2 - u) f(1-u) = 0. \tag{3.26}$$

As preliminary remarks, note that:

– if $p_1 = 1 - p_2$, then $u^* = p_1$ is the unique solution of (3.26) because $f > 0$ on $(0, 1]$ and $h_0'(u^*) = 1 > 0$ so that u^* is a stable equilibrium.

– h_0' being symmetric w.r.t. $\frac{1}{2}$, h_0'' is antisymmetric w.r.t. $\frac{1}{2}$. Hence $h_0''\left(\frac{1}{2}\right) = 0$.

– When the generating matrix H is bi-stochastic (if and only if $p_1 = p_2$), then $u^* = \frac{1}{2}$ is always solution to (3.26). Thus, if $h_0'\left(\frac{1}{2}\right) = 0$, as $h_0''\left(\frac{1}{2}\right) = 0$, the status of the equilibrium u^* requires to investigate higher order (see Example in Section 3.2 devoted to the bi-stochastic case).

Let u^* be a solution of (3.26). From Proposition 2.2(b), we have that any zero u^* of h_0 lies in $I^* := [p_1 \wedge (1-p_2), p_1 \vee (1-p_2)]$ (or in its interior $\overset{\circ}{I^*}$).

Proposition 3.1. *Let p_1, $p_2 \in (0,1)$ and f a skew-function.*

(a) *If $p_1 + p_2 \leq 1$, then (3.26) has a unique solution u^* lying in I^* (and in $\overset{\circ}{I^*}$ if $p_1 + p_2 < 1$).*

(b) *If $p_1 + p_2 > 1$ and f is concave, then (3.26) has a unique solution u^* lying in $\overset{\circ}{I^*}$.*

(c) *If $p_1 + p_2 > 1$ and f is strictly convex, then (3.26) may have one, two or three solutions lying in $\overset{\circ}{I^*}$. However, when $f'(1) \leq \frac{1}{p_1+p_2-1}$, then h_0 is increasing and (3.26) subsequently has a unique zero u^* lying in $\overset{\circ}{I^*}$.*

Proof. (a) As f is increasing and non-negative, $h_0' > 0$ when $p_1 + p_2 \leq 1$, therefore h_0 is increasing with $h_0(0) = p_2 - 1 < 0$ and $h_0(1) = 1 - p_1 > 0$, so h_0 has a unique zero lying in I^*.

(b) Assume that $p_1 + p_2 > 1$. Then it is obvious that p_1 and $1 - p_2$ are not solutions of (3.26) which can be rewritten as

$$g_1(u) = g_2(u) \quad \text{on} \quad \mathcal{J} = [0,1] \setminus \{p_1, 1-p_2\},$$

where

$$g_1(u) = \frac{f(1-u)}{p_1 - u}, \quad u \in [0,1] \setminus \{p_1\} \quad \text{and} \quad g_2(u) = \frac{f(u)}{u - 1 + p_2}, \quad u \in [0,1] \setminus \{1-p_2\}. \tag{3.27}$$

Let us compute the first derivative of these functions: We obtain, for $u \in \mathcal{J}$,

$$g_1'(u) = \frac{f(1-u) - f'(1-u)(p_1 - u)}{(p_1 - u)^2} \quad \text{and} \quad g_2'(u) = \frac{f'(u)(u - 1 + p_2) - f(u)}{(u - 1 + p_2)^2}.$$

As f is concave non-negative and $f(0) = 0$, $f(u) - uf'(u) \geq 0$, $u \in [0,1]$. Let us show that, as $0 < p_1, p_2 < 1$, $g_1' > 0$ and $g_2' < 0$ on I^*. In fact,

$$g_2'(u) = \frac{uf'(u) - f(u) - (1-p_2)f'(u)}{(u - 1 + p_2)^2} \leq 0 \quad \text{since} \quad f'(u) \geq 0,$$

and if $uf'(u) - f(u) - (1-p_2)f'(u) = 0$, then $(1-p_2)f'(u) \leq 0$. Hence $f'(u) = 0$ since $p_2 < 1$ so that $f = 0$ on $[0,u]$ which contradicts the positivity of f on $(0,1)$ as a skew-function. This shows that $g_2' < 0$. One shows likewise that $g_1' > 0$. Hence (3.26) has a unique solution $u^* \in \overset{\circ}{I^*}$.

(c) If $p_1 + p_2 > 1$ and f is strictly convex, we set

$$\varphi_1(u) = (p_1 - u)f(u) \quad \text{and} \quad \varphi_2(u) = (u - 1 + p_2)f(1-u).$$

Then we have three possibilities: When f is strictly convex, the functions φ_1 and φ_2 have a maximum on I^*. Let us set $u_1, u_2 \in I^*$ such that $\varphi_1'(u_1) = 0$ and $\varphi_2'(u_2) = 0$. As $p_1 + p_2 - 1 > 0$ and f is strictly convex, $u_2 < u_1$ and, depending on the relative position of $\varphi_1(u_2)$ and $\varphi_2(u_2)$, we obtain the three announced results. □

Example of phase transition (I) : Convex power skewing function. Let us illustrate the strictly convex setting by considering the family of strictly convex functions $f(u) = f_\alpha(u) = u^\alpha$, $\alpha > 1$.

Then Equation $h_0(u) = 0$ (see (3.26)) can be rewritten as

$$\varphi_1(u) = \varphi_2(u) \quad \text{on} \quad I^*,$$

where

$$\varphi_1(u) = (p_1 - u)f(u) = (p_1 - u)u^\alpha \quad \text{and} \quad \varphi_2(u) = (u - 1 + p_2)f(1 - u) = (u - 1 + p_2)(1 - u)^\alpha.$$

Then, the first derivatives of these functions read

$$\varphi_1'(u) = \big(\alpha p_1 - (\alpha + 1)u\big)u^{\alpha-1} \quad \text{and} \quad \varphi_2'(u) = \big((\alpha + 1)(1 - u) - \alpha p_2\big)(1 - u)^{\alpha-1}.$$

Each of these two derivatives φ_i' has only one zero lying in I^* which are the maximum of each function φ_i, $i = 1, 2$: $u_1 = \frac{\alpha p_1}{\alpha+1}$ for φ_1' and $u_2 = 1 - \frac{\alpha p_2}{\alpha+1}$ for φ_2'. As $p_1 + p_2 - 1 > 0$, we have that $1 - \frac{\alpha p_2}{\alpha+1} < \frac{\alpha p_1}{\alpha+1}$. Besides, we have $\varphi_1(1 - p_2) = (p_1 + p_2 - 1)(1 - p_2)^\alpha > 0$, $\varphi_1(p_1) = 0$, $\varphi_2(1 - p_2) = 0$ and $\varphi_2(p_1) = (p_1 + p_2 - 1)(1 - p_1)^\alpha > 0$. Therefore we have three possibilities (of course consistent with claim (c) of the above proposition):

- If $\varphi_1\left(1 - \frac{\alpha p_2}{\alpha+1}\right) > \varphi_2\left(1 - \frac{\alpha p_2}{\alpha+1}\right)$, then (3.26) has a unique solution lying in I^*.
- If $\varphi_1\left(1 - \frac{\alpha p_2}{\alpha+1}\right) = \varphi_2\left(1 - \frac{\alpha p_2}{\alpha+1}\right)$, then (3.26) has two solutions lying in I^*.
- If $\varphi_1\left(1 - \frac{\alpha p_2}{\alpha+1}\right) < \varphi_2\left(1 - \frac{\alpha p_2}{\alpha+1}\right)$, then (3.26) has three solutions lying in I^*.

In this parametric setting, as $f'(1) = \alpha$, it follows from Proposition 3.1 that the simpler criterion for the existence of a unique attractive equilibrium reads

$$1 < p_1 + p_2 \leq 1 + \frac{1}{\alpha}.$$

To exhibit a transition phase with three equilibrium points, this condition need to be violated. However it is not a sufficient condition. This is illustrated by Figure 3.1 below: Choose p_1 and p_2 such that $p_1 + p_2 > 1$. The critical α_0 is clearly strictly larger than the barrier $1/(p_1 + p_2 - 1)$ suggested by the criterion. For example, for $p_1 = 0.7$ and $p_2 = 0.75$, $\alpha_0 \approx 3.09 > 1/(p_1 + p_2 - 1) = \frac{20}{9}$.

3.2 Stability of equilibrium points of ODE_{h_0}

Once elucidated the existence and the number of equilibrium points u^* of ODE_{h_0}, we have to determine their nature *i.e.* the sign of $h_0'(u^*)$.

Let $u^* \in \{h_0 = 0\} \subset I^*$. We deduce from (3.24) that the condition $h_0(u^*) = 0$ reads

$$(u - p_1)f(u) + (u - 1 - p_2)f(1 - u) = 0.$$

Plugging this equality into the expression (3.25) for $h_0'(u^*)$ shows that, for such equilibrium points u^*,

$$h_0'(u^*) = 1 - \frac{f'(u^*)(p_1 - u^*) + f'(1 - u^*)(u^* - 1 + p_2)}{f(u^*) + f(1 - u^*)}. \tag{3.28}$$

The function h_0 has at most three zeros owing to Proposition 3.1. Therefore, each such zero u^* of h_0 has three possible "status" (see *e.g.* Figure 3.2):

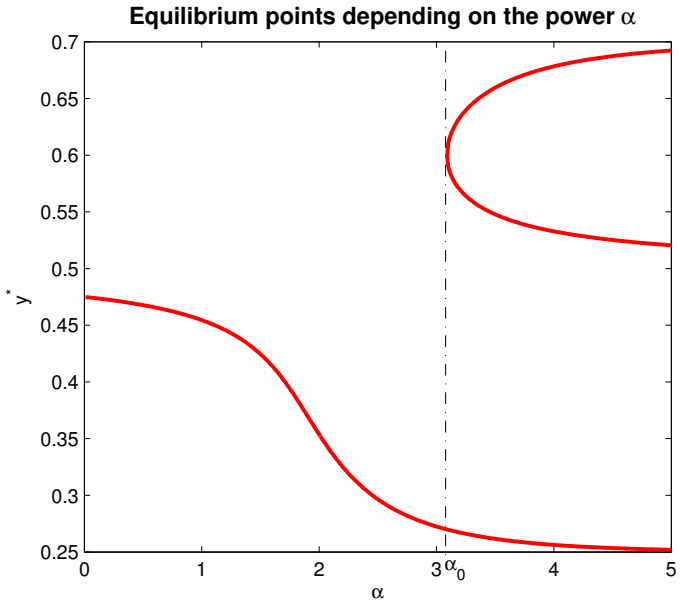


Figure 3.1: *Equilibrium points for* $f(u) = u^\alpha$ *depending on* $\alpha \in (0, 5]$ *with* $p_1 = 0.7$ *and* $p_2 = 0.75$. *Bifurcation appears at* $\alpha_0 \approx 3.09$.

- If there exists $\varepsilon_0 > 0$ such that $h_{0|(u^*-\varepsilon_0,u^*)} < 0$ and $h_{0|(u^*,u^*+\varepsilon_0)} > 0$, then u^* is a *stable* equilibrium point. This is the case *e.g.* if $h_0'(u^*) > 0$ or, more generally, if the first non zero derivative at u^* has an odd order and is positive.

- If there exists $\varepsilon_0 > 0$ such that $h_{0|(u^*-\varepsilon_0,u^*)} > 0$ and $h_{0|(u^*,u^*+\varepsilon_0)} < 0$, then u^* is an *unstable* equilibrium point (or a repeller). This is the case *e.g.* if $h_0'(u^*) < 0$ or if the first non-zero derivative at u^* has an odd order and is negative.

- If h_0 has a constant sign over an interval $(u^* - \varepsilon_0, u^* + \varepsilon_0)$, for some $\varepsilon_0 > 0$, then u^* is a *semi-stable equilibrium point* (one-sided stable and one-sided unstable). This is the case if $h_0''(u^*) = 0$ and the first non-zero derivative at u^* occurs at an even order. The simplest case is when $h_0''(u^*)$ exists and is non-zero.

Then it is an elementary exercise to derive the following proposition, if one keeps in mind that $h_0(0) = -(1 - p_2) < 0$ and $h_0(1) = 1 - p_1 > 0$.

Proposition 3.2. *Let* $p_1, p_2 \in (0, 1)$.

(*i*) *If* h_0 *has a unique equilibrium point, then it is stable.*

(*ii*) *If* h_0 *has two equilibrium points, then one is stable and one is semi-stable.*

(*iii*) *If* h_0 *has three equilibrium points, then either*

- *the lowest and the highest ones are stable and the one in the middle is unstable.*

  or

- *the one in the middle is stable and the other two are semi-stable.*

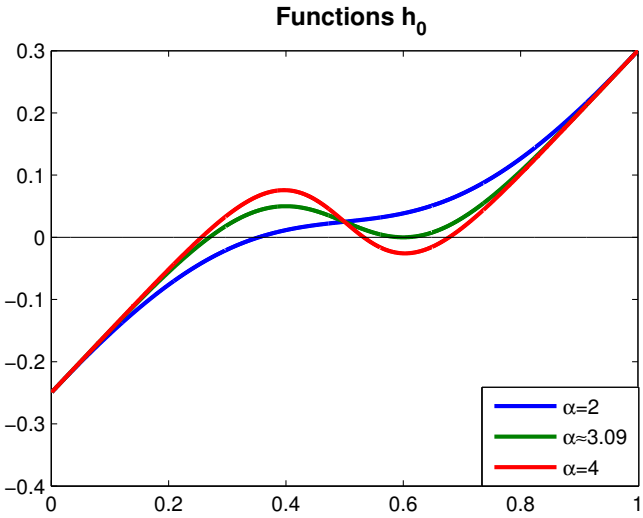


Figure 3.2: *Examples of function* h_0*:* $p_1 = 0.7$ *and* $p_2 = 0.75$ *and* $f(u) = u^\alpha$*. One zero for* $\alpha = 2$*, two zeros for* $\alpha \approx 3.09$*, three zeros for* $\alpha = 4$*.*

Remark. The second setting in item (*iii*) does not correspond to a "generic" situation but rather to a limiting case of a situation with 5 equilibrium points and 3 extrema. Thus, it never occurs with the family $f_\alpha(y) = y^\alpha$, $\alpha > 0$, of skewing functions. That is why we will not consider this configuration in what follows.

Example of phase transition (II). We still consider the family $f(u) = u^\alpha$, $\alpha > 1$. The "bifurcation" has been established in part I. Now we want to elucidate what happens at α_0. The critical two equilibrium case occurs at $\alpha_0 \approx 3.09$. By continuity of $(\alpha, u) \mapsto u^\alpha$ and monotony in α and u, it is clear that at least one – in fact exactly one – of the two equilibrium point u^* satisfies $h_0(u^*) = 0$ and h_0 does not change its sign around u^* so that $h_0'(u^*) = 0$. To determine its status we have to look at the sign of the second derivative $h_0''(u^*)$. In fact elementary computations show that , for any $u \in [0, 1]$ such that $h_0(u) = h_0'(u) = 0$,

$$h_0''(u) = (1-p_1-p_2)\alpha u^{\alpha-2}(1-u)^{\alpha-2}\frac{(\alpha-1)(1-2u)(u^\alpha+(1-u)^\alpha) - 2\alpha u(1-u)(u^{\alpha-1}-(1-u)^{\alpha-1})}{(u^\alpha+(1-u)^\alpha)^3} > 0.$$

Hence, u^* is semi-stable (stable on the right and unstable on the left).

It remains to show that the algorithm does not converge towards the repulsive equilibrium point, denoted by $\hat{u}$ in what follows. To show that there is an excitation in the repulsive direction, we have to prove that assumption (A.44) holds (see Theorem A.2 in the Appendix).

Proposition 3.3. *Let* $\hat{u}$ *be an unstable equilibrium point for* h_0 *satisfying* $h_0'(\hat{u}) < 0$*. Then*

$$\mathbb{P}\big(\widetilde{Y}_n \to (\hat{u}, 1-\hat{u})\big) = 0.$$

Proof. It follows from (3.25) that h_0' is locally Lipschitz continuous. We know that $\{\widetilde{Y}_n \to (\hat{u}, 1-\hat{u})\} = \{\widetilde{Y}_n^1 \to \hat{u}\}$, so we may focus on the first component. We rely on Theorem A.2 in the Appendix and adopt its notation, namely

$$\Delta M_{n+1}^{(r)} = \Delta M_{n+1}^1.$$

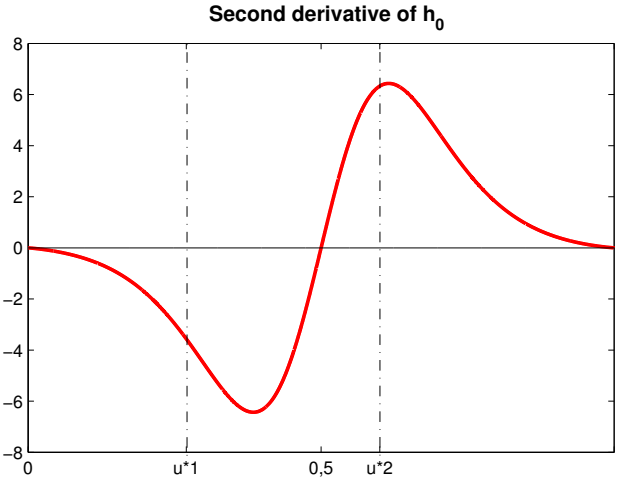


Figure 3.3: Second derivative of h_0 for $f(u) = u^{3.09}$ with $p_1 = 0.7$ and $p_2 = 0.75$.

Using Assumption **(A1)**, we obtain

$$\Delta M_{n+1}^{(r)} = D_{n+1}^{11} X_{n+1}^1 + D_{n+1}^{12} X_{n+1}^2 - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}.$$

Therefore

$$\begin{aligned}
\mathbb{E}\left[\left\|\Delta M_{n+1}^{(r)}\right\| \left|\mathcal{F}_n\right.\right] &= \mathbb{E}\left[\left|\Delta M_{n+1}^{1}\right| \left|\mathcal{F}_n\right.\right] \\
&= \frac{f(\widetilde{Y}_n^1)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \mathbb{E}\left[\left| D_{n+1}^{11} - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \right| \left|\mathcal{F}_n\right.\right] \\
&\quad + \frac{f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \mathbb{E}\left[\left| D_{n+1}^{12} - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \right| \left|\mathcal{F}_n\right.\right].
\end{aligned}$$

By Jensen's inequality applied to both conditional expectations in the right hand side, we obtain

$$\begin{aligned}
\mathbb{E}\left[\left\|\Delta M_{n+1}^{(r)}\right\| \left|\mathcal{F}_n\right.\right] &\geq \frac{f(\widetilde{Y}_n^1)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \left| H_{n+1}^{11} - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \right| \\
&\quad + \frac{f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \left| H_{n+1}^{12} - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \right|.
\end{aligned}$$

Owing to $\mathbf{(A5)}_v$, $H_{n+1}^{ij} \xrightarrow[n\to+\infty]{a.s.} H^{ij}$, where $H^{ii} = p_i$ and $H^{ij} = 1 - p_j$, $1 \leq i, j \leq 2$. Furthermore, on $\hat{\mathcal{Y}}_\infty = \left\{\omega : \widetilde{Y}_n^1(\omega) \to \hat{u}\right\}$,

$$\frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \xrightarrow[n\to+\infty]{a.s.} \hat{u}.$$

Consequently,

$$\frac{f(\widetilde{Y}_n^1)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \left| H_{n+1}^{11} - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \right| \xrightarrow[n\to+\infty]{a.s.} \frac{f(\hat{u})}{f(\hat{u}) + f(1-\hat{u})} |p_1 - \hat{u}| > 0$$

and

$$\frac{f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\left|H_{n+1}^{12}-\frac{H_{n+1}^{11}f(\widetilde{Y}_n^1)+H_{n+1}^{12}f(\widetilde{Y}_n^2)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right| \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} \frac{f(\hat{u})}{f(\hat{u})+f(1-\hat{u}))}\,|1-p_2-\hat{u}|>0$$

since $\hat{u}\in I^*$. Thus (A.44) is satisfied. Then, by using (2.7) and by applying Theorem A.2 in the Appendix, $\mathbb{P}(\hat{\mathcal{Y}}_\infty)=0$. □

Example (bi-stochastic generating matrix H). If $p_1=p_2=p\in(0,1)$ then (H is bi-stochastic) $(\frac{1}{2};\frac{1}{2})$ is an equilibrium and $h_0'(1/2)=1-\frac{f'(1/2)}{2f(1/2)}(2p-1)$.

– If f is convex and $2p-1\le 0$, then $h_0'\left(\frac{1}{2}\right)>0$ and the equilibrium is unique (since h_0' is increasing).

– If f is convex and $2p-1>0$, then we have three possibilities:

- If $p>\frac{1}{2}+\frac{f(1/2)}{f'(1/2)}$, then $h_0'(1/2)<0$, so that $\mathbb{P}(\hat{\mathcal{Y}}_\infty)=0$.
- If $p=\frac{1}{2}+\frac{f(1/2)}{f'(1/2)}$, then $h_0'(1/2)=0$. But $h_0''(1/2)=0$, so we need to investigate higher order.
- If $p<\frac{1}{2}+\frac{f(1/2)}{f'(1/2)}$, then $h_0'(1/2)>0$, so u^* is stable.

If $f(u)=u^\alpha$, the first case also reads $\alpha>\frac{1}{2p-1}$. For the second case, higher order analysis leads to $h_0^{(3)}(1/2)>0$, so that the equilibrium is stable.

3.3 *A.s.* convergence

Theorem 3.1. *Let $(Y_n)_{n\ge0}$ be the urn composition sequence defined by (1.1)-(1.3). Under the assumptions* **(A1)**, **(A2)**, **(A3)** *and the f-skewed drawing rule,*

(a) $\dfrac{Y_n}{\mathbf{w}(Y_n)} \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} (u^*,1-u^*)\in\mathcal{E}_2$ *(i.e. $u^*\in\{h_0=0\}$ and u^* is not unstable for ODE_{h_0}).*

(b) $\widetilde{N}_n=\dfrac{1}{n}\displaystyle\sum_{k=1}^n X_k \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} \begin{pmatrix}\frac{f(u^*)}{f(u^*)+f(1-u^*))}\\ \frac{f(u^*)}{f(u^*)+f(1-u^*))}\end{pmatrix}$.

Proof. First, we will prove that $(a)\Rightarrow(b)$, then (a).

$(a)\Rightarrow(b)$. For every $n\ge1$, we have

$$\mathbb{E}\left[X_n\,|\,\mathcal{F}_{n-1}\right]=\sum_{i=1}^d\frac{f(\widetilde{Y}_{n-1}^i)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_{n-1}))}e^i=\frac{\widetilde{f}(\widetilde{Y}_{n-1})}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_{n-1}))}$$

and, by construction $\|X_n\|^2=1$, so that $\mathbb{E}\left[\|X_n\|^2\,|\,\mathcal{F}_{n-1}\right]=1$. Hence, the martingale

$$\widetilde{M}_n=\sum_{k=1}^n\frac{X_k-\mathbb{E}\left[X_k\,|\,\mathcal{F}_{k-1}\right]}{k} \underset{n\to+\infty}{\overset{a.s.\&\,L^2}{\longrightarrow}} \widetilde{M}_\infty\in L^2.$$

Finally, it follows from the Kronecker Lemma that

$$\frac{1}{n}\sum_{k=1}^n X_k-\frac{1}{n}\sum_{k=1}^n\frac{\widetilde{f}(\widetilde{Y}_{k-1})}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_{k-1}))} \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} 0.$$

This proves the announced implication owing to the Césaro Lemma since $\frac{\widetilde{f}}{\mathbf{w}(\widetilde{f})}$ is continuous at every point of $\mathcal{S}_2$.

(a) The algorithm is bounded by construction since it is $[0,1]$-valued. Assumption **(A2)** implies that $\sup_{n\geq 0}\mathbb{E}\left[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n\right] < +\infty$ *a.s.* and Assumption **(A3)** implies that $r_n \xrightarrow[n\to+\infty]{a.s.} 0$. The set $\{h=0\}$ is finite hence $\widetilde{Y}_n$ *a.s.* converges toward a zero of h (see Theorem A.1(c) in the appendix). Moreover, it follows from Proposition 3.2 devoted to attractiveness that this zero cannot be a repulsive (a point at which h_0' is negative).

4 Weak rate of convergence

To establish a *CLT* for the sequence $(\widetilde{Y}_n)_{n\geq 0}$ on a convergence event $\left\{\widetilde{Y}_n \to y^*\right\}$, we need to make the following additional assumptions:

(A4) The addition rules D_n *a.s.* satisfy on the event $\left\{\widetilde{Y}_n \to y^*\right\}$

$$\forall k\in\{1,\dots,d\},\quad \left\{\begin{array}{ll}(i) & \sup_{n\geq 1}\mathbb{E}\left[\|D_n^{\cdot k}\|^{2+\delta}\,|\,\mathcal{F}_{n-1}\right]\leq\kappa<+\infty \quad \text{for a } \delta>0,\\ (ii) & \mathbb{E}\left[D_n^{\cdot k}(D_n^{\cdot k})^t\,|\,\mathcal{F}_{n-1}\right]\underset{n\to+\infty}{\longrightarrow} C_{y^*}^k,\end{array}\right.$$

where $C_{y^*}^k=(C_{y^*,ij}^k)_{1\leq i,j\leq d}$, $k=1,\dots,d$, are $d\times d$ positive definite matrices.

Note that **(A4)** $\Rightarrow$**(A2)** since $\mathbb{E}\left[\|D_n^{\cdot k}\|^2\,|\,\mathcal{F}_{n-1}\right]\leq\left(\mathbb{E}\left[\|D_n^{\cdot k}\|^{2+\delta}\,|\,\mathcal{F}_{n-1}\right]\right)^{\frac{2}{2+\delta}}$.

Also note that, if the matrices D_n are *deterministic* (hence co-stochastic) then $C^k=\lim_n D_n^{\cdot k}\otimes D_n^{\cdot k}$ so that $\mathbf{1}^tC^k\mathbf{1}=1$ for every $k\in\{1,\dots,d\}$.

Let $v=(v_n)_{n\geq 1}$ be a sequence of positive real numbers.

(A5)$_v$ The matrices H_n and H satisfy on the event $\left\{\widetilde{Y}_n\to y^*\right\}$

$$n\,v_n\,\mathbb{E}\left[|\!|\!|H_n-H|\!|\!|^2\right]\underset{n\to+\infty}{\longrightarrow}0. \tag{4.29}$$

To establish the weak rate of convergence we cannot restrict ourselves to simplex as we essentially did for the convergence result since we must take into account the rate of convergence of the algorithm $\widetilde{Y}_n$ toward the simplex which is itself non trivial in general. Actually, we saw in (2.20) that

$$\mathbf{w}\big(\widetilde{Y}_n\big)-1=\frac{\mathbf{w}(M_n)}{n+\mathbf{w}(Y_0)}.$$

Elementary computations show, under Assumptions **(A1)**, **(A3)**, **(A4)**-(i) and if $\widetilde{Y}_n\to y^*$, that

$$\mathbb{E}\left[\mathbf{w}(M_n)^2\,|\,\mathcal{F}_{n-1}\right]\to\sigma^2(y^*)=\mathbf{1}^t\frac{\sum_{k=1}^d f(y_*^k)C_{y^*}^k}{\mathbf{w}(\widetilde{f}(y^*))}\mathbf{1}-1.$$

Note that, if the matrices D_n are deterministic then $\sigma^2(y^*)=0$ which is expected since in that case $\widetilde{Y}_n$ is $\mathcal{S}_d$-valued. Owing to Condition **(A4)** Lindeberg's *CLT* for arrays of martingale applies to $\big(\frac{\mathbf{w}(M_\ell)}{\sqrt{n}}\big)_{1\leq\ell\leq n}$ (see *e.g.* Corollary 3.1, p.58 in [22]) if one keeps in mind that or that **(A4)**-(ii) implies the usual condition as a straightforward application of Markov inequality). Consequently

$$\sqrt{n}\Big(\mathbf{w}\big(\widetilde{Y}_n\big)-1\Big)=\frac{n}{n+\mathbf{w}(Y_0)}\frac{\mathbf{w}(M_n)}{\sqrt{n}}\overset{\mathcal{L}_{stably}}{\longrightarrow}\mathcal{N}\big(0;\sigma^*(y^*)\big)\ \text{ on the event }\left\{\widetilde{Y}_n\to y^*\right\}.$$

4.1 Strictly concave case with irreducible bi-stochastic limiting generating matrix

Assume H is bi-stochastic and irreducible and the skew-function is strictly concave. We know from Proposition 2.7 that $\widetilde{Y}_n \to y(d) = \frac{1}{d}$ *a.s.* On the other hand we know from (the proof of) Lemma 2.1 that $J_\varphi(y(d)) = \lambda_1 I_{d|\mathbf{1}^\perp}$ so that

$$J_h\big(y(d)\big) = I_d - \lambda_1 H_{|\mathbf{1}^\perp} \quad \text{still with} \quad \lambda_1 = \frac{f'(1/d)}{d \cdot f(1/d)} < 1.$$

As H is (co-)stochastic, we know that 1 is its eigenvalue with the highest real part so that the eigenvalue of $J_h(y(d))$ with the lowest real part is $1 - \lambda_1 > 0$.

Proposition 4.1. *Assume that the skewed drawing rule f is strictly concave and H is irreducible and bi-stochastic. Assume* **(A1)**, **(A3)**, **(A4)** *and* $\mathbf{(A5)}_v$ *hold.*
(a) If $f'(1/d) < \frac{d}{2} f(1/d)$ and $\mathbf{(A5)}_v$ holds with $v_n = 1$, then

$$\sqrt{n}\big(\widetilde{Y}_n - y(d)\big) \overset{\mathcal{L}_{stably}}{\longrightarrow} \mathcal{N}\big(0; \Sigma^*\big) \quad \text{with} \quad \Sigma^* = \int_0^{+\infty} e^{-u\left(J_h(y^*) - \frac{I_2}{2}\right)^t} \Gamma^* e^{-u\left(J_h(y^*) - \frac{I_2}{2}\right)} du$$

with

$$\Gamma^* = \frac{\sum_{k=1}^d f(y^{*,k}) C_{y^*}^k}{\mathbf{w}\big(\widetilde{f}(y^*)\big)} - y^*(y^*)^t. \tag{4.30}$$

(b) If $f'(1/d) = \frac{d}{2} f(1/d)$ and $\mathbf{(A5)}_v$ holds with $v_n = \log n$, then

$$\sqrt{\frac{n}{\log n}}\big(\widetilde{Y}_n - y(d)\big) \overset{\mathcal{L}_{stably}}{\longrightarrow} \mathcal{N}\big(0; \Sigma^*\big)$$

with

$$\Sigma^* = \lim_n \frac{1}{n} \int_0^n e^{-u\left(J_h(y^*) - \frac{I_d}{2}\right)^t} \Gamma^* e^{-u\left(J_h(y^*) - \frac{I_d}{2}\right)} du. \tag{4.31}$$

(c) If $f'(1/d) > \frac{d}{2} f(1/d)$ and $\mathbf{(A5)}_v$ holds with $v_n = n^{1-2\lambda_1+\eta}$ for some $\eta > 0$, then

$$n^{1-\lambda_1}\big(\widetilde{Y}_n - y(d)\big) \text{ converges toward a finite random variable.}$$

Proof. First note that $\widetilde{Y}_n \to y(d)$ *a.s.* under the assumptions. We will check the three assumptions of Theorem A.3 (*CLT* for *SA* algorithms) recalled in the Appendix. The parameter Λ which rules the regime of the rate is given here by $\Lambda = 1 - \lambda_1$ which justifies the above three cases. Secondly Assumption **(A4)** ensures that Condition (A.46) is satisfied since

$$\sup_{n \geq 1} \mathbb{E}\left[\|\Delta M_n\|^{2+\delta} \,|\, \mathcal{F}_{n-1}\right] < +\infty \quad a.s. \quad \text{and} \quad \mathbb{E}\left[\Delta M_n \Delta M_n^t \,|\, \mathcal{F}_{n-1}\right] \underset{n \to +\infty}{\overset{a.s.}{\longrightarrow}} \Gamma^*.$$

To be more precise on the convergence on the right-hand side

$$\begin{aligned}
\mathbb{E}\left[\Delta M_{n+1} \Delta M_{n+1}^t \,|\, \mathcal{F}_n\right] &= \sum_{k=1}^d \mathbb{P}(X_{n+1} = e^k \,|\, \mathcal{F}_n)\Big(\mathbb{E}\left[D_{n+1}^{\cdot k} (D_{n+1}^{\cdot k})^t \,|\, \mathcal{F}_n\right] \\
&\quad - \mathbb{E}\left[D_{n+1} X_{n+1} \,|\, \mathcal{F}_n\right] \mathbb{E}\left[D_{n+1} X_{n+1} \,|\, \mathcal{F}_n\right]^t\Big) \\
&= \sum_{k=1}^d \frac{f(\widetilde{Y}_n^q)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))} \mathbb{E}\left(D_{n+1}^{\cdot k} (D_{n+1}^{\cdot k})^t \,|\, \mathcal{F}_n\right) - \left(H_{n+1} \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right)\left(H_{n+1} \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}\right)^t \\
&\underset{n \to +\infty}{\overset{a.s.}{\longrightarrow}} \Gamma^* = \frac{\sum_{k=1}^d f(y^{*k}) C_{y^*}^k}{\mathbf{w}(\widetilde{f}(y^*))} - y^*(y^*)^t.
\end{aligned}$$

Finally, using $\mathbf{(A5)}_v$ with the appropriate sequence $(v_n)_{n\geq 1}$, one proves in the three cases that the remainder sequence $(r_n)_{n\geq 1}$ defined by (2.19) satisfies (A.47) since one checks that $\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}$ is bounded. □

4.2 Back to the "convex" bi-dimensional case

As for the weak rate, we need to deal with the original 2-dimensional algorithm on its whole.

Theorem 4.1. *Assume* **(A1)**, **(A3)**, **(A4)** *hold. Every equilibrium point* $y^* \in \mathcal{S}_2$ *is of the form* $y^* = (u^*, 1-u^*)$ *where* u^* *is solution to* (3.26) *and lies in* $I^* = [p_1 \wedge (1-p_2), p_1 \vee (1-p_2)]$, *and*

$$Sp\big(J_h(y^*)\big) = \big\{1, 1-\rho^*\big\},$$

where

$$\rho^* = \rho(y^*) = \frac{f'(u^*)(p_1-u^*) + f'(1-u^*)(u^*-1+p_2)}{f(u^*)+f(1-u^*)}.$$

(a) If $p_1+p_2 \leq 1$ *and* $\mathbf{(A5)}_v$ *holds with* $v_n = 1$, $n \geq 1$, *then* y^* *is unique, stable,* $Y_n \to y^*$ *a.s. owing to Theorem 3.1 (a) and*

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \overset{\mathcal{L}_{stably}}{\underset{n\to+\infty}{\longrightarrow}} \mathcal{N}\left(0, \Sigma^*\right) \quad \text{with} \quad \Sigma^* = \int_0^{+\infty} e^{-u(J_h(y^*)-\frac{I_2}{2})^t} \Gamma^* e^{-u(J_h(y^*)-\frac{I_2}{2})} du$$

$$\text{and} \quad \Gamma^* = \frac{f(u^*)C^1 + f(1-u^*)C^2}{\mathbf{w}(\widetilde{f}(y^*))} - y^*(y^*)^t.$$

(b) If $p_1+p_2 > 1$, *we have three possible rates of convergence on an event* $\big\{\widetilde{Y}_n \to y^*\big\}$, *where* $y^* \in \mathcal{E}_2$ *is not unstable, depending on* $\rho^* = \lambda(y^*)$*:*

(i) If $0 < \rho^* < \frac{1}{2}$ *and* $\mathbf{(A5)}_v$ *holds with* $v_n = 1$, $n \geq 1$, *then*

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \overset{\mathcal{L}_{stably}}{\underset{n\to+\infty}{\longrightarrow}} \mathcal{N}\left(0, \Sigma^*\right) \quad \text{on} \quad \big\{\widetilde{Y}_n \to y^*\big\},$$

where Σ^* *is formally defined like in item (a).*

(ii) If $\rho^* = \frac{1}{2}$ *and* $\mathbf{(A5)}_v$ *holds with* $v_n = \log n$, $n \geq 1$, *then*

$$\sqrt{\frac{n}{\log n}}\left(\widetilde{Y}_n - y^*\right) \overset{\mathcal{L}}{\underset{n\to+\infty}{\longrightarrow}} \mathcal{N}\left(0, \Sigma^*\right) \quad \text{where } \Sigma^* \text{ is given by (4.31).}$$

(iii) If $\frac{1}{2} < \rho^* < 1$ *and* $\mathbf{(A5)}_v$ *holds with* $v_n = n^{1-2\rho^*+\eta}$, $\eta > 0$, *then* $n^{\rho^*}\left(\widetilde{Y}_n - y^*\right) \to \Upsilon$ *a.s. converges as* $n \to +\infty$ *where* Υ *is a finite random variable.*

Remark. • The condition $0 < \rho^* < \frac{1}{2}$ is satisfied as soon as

$$\begin{cases} \dfrac{(f'(1-p_1)+f'(1-p_2))(p_1+p_2-1)}{f(1-p_1)+f(1-p_2)} < \dfrac{1}{2} & \text{if } f \text{ is concave,} \\[2ex] \dfrac{(f'(p_1)+f'(p_2))(p_1+p_2-1)}{f(1-p_1)+f(1-p_2)} < \dfrac{1}{2} & \text{if } f \text{ is convex,} \end{cases}$$

by using the monotonicity of f and f' and that $u^* \in (1-p_2, p_1)$.

• If $f(y) = y$, then y^* is unique, is given by

$$y^* = \left(\frac{1-p_2}{2-p_1-p_2}, \frac{1-p_1}{2-p_1-p_2}\right) \quad \text{and} \quad \rho^* = p_1 + p_2 - 1.$$

Thus, if $p_1 + p_2 < \frac{3}{2}$, then the recursive procedure (2.17) satisfies a regular CLT; if $p_1 + p_2 = \frac{3}{2}$, (2.17) satisfies Theorem 4.1(b)-(ii); and if $p_1 + p_2 > \frac{3}{2}$, (2.17) admits an $a.s.$-rate of convergence.

• In [13, 14], the properties of the random variable Υ are deeply investigated in the more standard framework of Pólya's urn with deterministic addition rule matrix. It is shown to be solution to a smoothing equation obtained by a smart decomposition of the urn into canonical components. Thus, it is proved that its distribution is characterized by its moments. It is clear that such results are out of reach of standard SA techniques although it would be challenging to check whether similar results about Υ in the randomized and nonlinear framework are true.

Proof. We again rely on Theorem A.3 from the Appendix. Elementary though tedious computations yield the following formula for the Jacobian $J_h(y)$ of h at $y \in \mathbb{R}_+^d \setminus \{0\}$. We obtain

$$J_h(y) = \begin{pmatrix} 1 + \frac{f'(y^1)}{f(y^1)+f(y^2)}\left(\frac{p_1 f(y^1)+(1-p_2)f(y^2)}{f(y^1)+f(y^2)} - p_1\right) & \frac{f'(y^2)}{f(y^1)+f(y^2)}\left(\frac{p_1 f(y^1)+(1-p_2)f(y^2)}{f(y^1)+f(y^2)} - (1-p_2)\right) \\ \frac{f'(y^1)}{f(y^1)+f(y^2)}\left(\frac{(1-p_1)f(y^1)+p_2 f(y^2)}{f(y^1)+f(y^2)} - (1-p_1)\right) & 1 + \frac{f'(y^2)}{f(y^1)+f(y^2)}\left(\frac{(1-p_1)f(y^1)+p_2 f(y^2)}{f(y^1)+f(y^2)} - p_2\right) \end{pmatrix}.$$

As all equilibrium points y^* lie in the simplex $\mathcal{S}_2$, we have $y^{*2} = 1 - y^{*1}$. Combined with the constraint $h(y^*) = 0$ (see (3.26)), we finally obtain the following formula only true at equilibrium points:

$$J_h(y^*) = \begin{pmatrix} 1 + \frac{f'(y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(y^{*1} - p_1\right) & \frac{f'(1-y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(y^{*1} - (1-p_2)\right) \\ \frac{f'(y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(p_1 - y^{*1}\right) & 1 + \frac{f'(1-y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(1 - p_2 - y^{*1}\right) \end{pmatrix}.$$

Then, one easily checks that the spectrum of $J_h(y^*)$ is real given by

$$\mathrm{Sp}\left(J_h(y^*)\right) = \{1, 1-\rho^*\}.$$

(a) When $p_1 \leq 1 - p_2$, we know from Proposition 3.1(a) that the equilibrium point y^* is unique and u^* lies in I^*. Hence $\rho^* \leq 0$ so that 1 is the lowest eigenvalue of $J_h(y^*)$. Consequently, we are in the "regular" case of the CLT for SA (Theorem A.3(a) in the Appendix) since $1 > \frac{1}{2}$. Then, following the lines of the proof of Proposition 4.1, we check that Assumption **(A4)** ensures that Condition (A.46) is satisfied with $\Gamma^* = \dfrac{f(y^{*1})C^1 + f(1-y^{*1})C^2}{\mathbf{w}(\widetilde{f}(y^*))} - y^*(y^*)^t$. Finally, using $\mathbf{(A5)}_v$, the remainder sequence $(r_n)_{n\geq 1}$ defined by (2.19) satisfies (A.47) since $\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}$ is bounded.

(b) If $p_1 + p_2 > 1$, then $\rho^* > 0$ which explains the three cases. The rest of the proof is the same as above, given the convergence event $\{\widetilde{Y}_n \to y^*\}$. □

5 Pólya urn with concave reinforced drawing rule: a bandit approach

By *Pólya urn*, we mean in this section that the matrices D_n involved in the drawing rule all satisfy $D_n = I_d$, $n \geq 1$. Moreover we assume that the drawing rue is still skewed following (2.10) where the function f is concave/convex and that the initial urn composition vector $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$. Note that when $f(u) = u$, then the urn dynamics is that of a regular Pólya urn with d colors.

In such a framework, $H = H_n = I_d$, $n \geq 1$, therefore H is no more irreducible and we cannot use the results proved in Sections 2 and 3.

We still normalize Y_n by setting $\widetilde{Y}_n := \frac{Y_n}{n+\mathbf{w}(Y_0)}$, $n \geq 0$. The sequence $(\widetilde{Y}_n)_{n\geq 0}$ satisfies the following recursive stochastic algorithm (obvious consequence of (2.15))

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1+\mathbf{w}(Y_0)}\left(\widetilde{Y}_n - \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}\left(\widetilde{f}(\widetilde{Y}_n)\right)}\right) + \frac{1}{n+1+\mathbf{w}(Y_0)}\Delta M_{n+1}, \quad n \geq 1, \tag{5.32}$$

where

$$\Delta M_{n+1} := X_{n+1} - \mathbb{E}\left[X_{n+1} \,|\, \mathcal{F}_n\right] \tag{5.33}$$

is a true $(\mathcal{F}_n)_{n\geq 0}$ martingale increment. Let us remark that, in this setting,

$$\frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} = \frac{n+\mathbf{w}(Y_0)}{n+\mathbf{w}(Y_0)} = 1, \quad n \geq 0,$$

so that the sequence

$$\widetilde{Y}_n \in \mathcal{S}_d, \quad \text{for every } n \geq 0,$$

since it has non-negative components.
The special case of a linear drawing rule $f(y) = y$ is entirely elucidated by the celebrated Athreya theorem recalled below for completeness.

Theorem 5.1 (Athreya's Theorem, see [1])**.** *Let $(Y_n)_{n\geq 0}$ be the urn composition sequence defined by (1.1) and (1.3) with $D_n = I_d$, $n \geq 1$, and a linear drawing rule (i.e. (2.10) with $f(u) = u$). Then, if $Y_0 \in \mathbb{R}^d \setminus \{0\}$ is deterministic, there exists a random vector $\widetilde{Y}_\infty$ having values in the simplex $\mathcal{S}_d$ such that*

$$\widetilde{Y}_n = \frac{Y_n}{\mathbf{w}(Y_n)} \xrightarrow[n\to+\infty]{a.s.} \widetilde{Y}_\infty \quad a.s.$$

Furthermore,

(i) $\widetilde{Y}_\infty$ has a Dirichlet distribution with parameter Y_0.

(ii) In particular, if $d = 2$, $\widetilde{Y}_\infty^1$ has a Beta distribution with parameters Y_0^1 and Y_0^2 (in particular, $\widetilde{Y}_\infty^1$ has a uniform distribution on $[0,1]$ if $Y_0^1 = Y_0^2 = 1$).

Now, we investigate the case $f \neq \mathrm{Id}_{\mathbb{R}_+}$ by borrowing tools to adaptive bandit models analysis (see [30, 28, 29]). First note that, as $\widetilde{Y}_n$ lives in the simplex $\mathcal{S}_d$, *the function f only needs to be defined on $[0,1]$*. Moreover, we will no longer ask f to be convex or concave on $(0,1)$ but require finiteness of the derivatives at 0 and 1.

f is continuous, non-decreasing, $f(0)=0$, $f(1)=1$, $f>0$ on $(0,1]$, with finite right and left derivatives at 0 and 1.

A typical example could be $f(u)=4\left(u-\frac{1}{2}\right)^3+\frac{1}{2}$, $u\in[0,1]$. We will see that our study requires new tools, especially a method to avoid noiseless repulsive equilibrium points (sometimes called *noiseless traps* in the SA literature).

Theorem 5.2. *(a) Let $I\subsetneq\{1,\dots,d\}$ be non-empty. If f satisfies $f'_r(0)>|I|f\left(\frac{1}{|I|}\right)$, then, for every deterministic initial value such that $Y_0^j>0$ for some $j\notin I$,*

$$\mathbb{P}\left(\widetilde{Y}_\infty=\widetilde{e}_I\right)=0.$$

(b) If $d=2$, the above conclusion still holds if $f'_r(0)=1$ and $f'_l(1)+\frac{f''_r(0)}{2}>1$.
(c) If f is strictly concave then $\mathcal{E}_d=\{\widetilde{e}_I, I\subset\{1,\dots,d\}, I\neq\varnothing\}$ by Proposition 2.3(c). Then, for every starting value $Y_0\in(0,+\infty)^d$,

$$\widetilde{Y}_n\overset{a.s.}{\longrightarrow}\widetilde{e}_{\{1,\dots,d\}}=y(d)\quad as\quad n\to+\infty.$$

Remarks. • In claim (b), if $f(0)=0$, $f(1)=1$, $f'_r(0)=1$ and f is convex or concave, then $f=\mathrm{Id}$, so this case is interesting only out of the concave/convex framework.

• If $Y_0^i=0$ for some $i\in\{1,\dots,d\}$, then, as proved below, $Y_n^i=0$ for every $n\geq 0$. So, as soon as $Y_0\in\mathbb{R}_+^d\setminus\{0\}$, one may apply the above result (c) to the urn restricted to $I'=\{i\in I,\ Y_0^i>0\}$ to prove that $\widetilde{Y}_n\to\widetilde{e}_{I'}$ as $n\to+\infty$.

Proof. (a)-(b) It follows from (1.1) and the fact that $D_n\equiv I_d$ that, if $Y_0^i=0$, then, for every instant $n\geq 0$, $Y_n^i=0$. So, up to a reduction of the dimension d, we may always assume that all components $Y_0^i>0$.

As a consequence we may assume that, for every $n\geq 0$, $\min_i\widetilde{Y}_n^i>0$. Our aim is to prove that $\mathbb{P}\left(\widetilde{Y}_\infty^j=0\right)=0$, for every $j\notin I$. To this end, we will show that $\{\widetilde{Y}_\infty^j=0\}\subset\{\widetilde{L}_\infty=0\}$ where $\widetilde{L}_\infty$ is the terminal value of a non-negative martingale. Then we will apply an "oracle" inequality to this martingale. Without loss of generality, we may assume that, up to a permutation, $1\notin I$ and $j=1$ in what follows.

Step 1: First, we define the function $\widetilde{h}$ by

$$\widetilde{h}(y)=1-\frac{f(y^1)}{y^1\mathbf{w}(\tilde{f}(y))}\mathbf{1}_{\{y^1\neq 0\}},\ y\in\mathcal{S}_d,$$

which satisfies $\widetilde{h}(y)<1$, for every $y\in\mathcal{S}_d\setminus\{y:y^1=0\}$.

Starting from the dynamics of $\widetilde{Y}_n^1$ given by (5.32), we have, for every $n\geq 0$,

$$\begin{aligned}\widetilde{Y}_{n+1}^1&=\widetilde{Y}_n^1-\frac{1}{n+1+\mathbf{w}(Y_0)}\left(\widetilde{Y}_n^1-\frac{f(\widetilde{Y}_n^1)}{\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big)}\right)+\frac{1}{n+1+\mathbf{w}(Y_0)}\Delta M_{n+1}^1\\&=\widetilde{Y}_n^1\left(1-\frac{1}{n+1+\mathbf{w}(Y_0)}\widetilde{h}\big(\widetilde{Y}_n^1\big)\right)+\frac{1}{n+1+\mathbf{w}(Y_0)}\Delta M_{n+1}^1.\end{aligned}$$

We derive that the (non-negative) sequence

$$\widetilde{L}_n := \frac{\widetilde{Y}_n^1}{\prod_{k=1}^n \left(1 - \frac{1}{k+\mathbf{w}(Y_0)}\widetilde{h}(\widetilde{Y}_{k-1}^1)\right)}, \quad n \geq 0, \tag{5.34}$$

is a non-negative martingale satisfying the recursive equation $\widetilde{L}_0 = \widetilde{Y}_0^1$ and

$$\widetilde{L}_{n+1} = \widetilde{L}_n + \frac{1}{n+1+\mathbf{w}(Y_0)} \frac{\Delta M_{n+1}^1}{\prod_{k=1}^{n+1} \left(1 - \frac{1}{k+\mathbf{w}(Y_0)}\widetilde{h}(\widetilde{Y}_{k-1}^1)\right)}, \quad n \geq 0.$$

• If $f_r'(0) > |I| f\left(\frac{1}{|I|}\right)$, then $\widetilde{h}(y) \longrightarrow \kappa := 1 - \frac{f_r'(0)}{|I| f\left(\frac{1}{|I|}\right)} < 0$ as $y \to \widetilde{e}_I$. Therefore, on the event $\left\{\widetilde{Y}_n \to \widetilde{e}_I\right\}$, $\widetilde{Y}_n^1 \to 0$ so that $\widetilde{h}\left(\widetilde{Y}_{n-1}^1\right) \overset{a.s.}{\sim} \kappa < 0$, which in turn implies $\prod_{k=1}^n \left(1 - \frac{1}{k+\mathbf{w}(Y_0)}\widetilde{h}(\widetilde{Y}_{k-1}^1)\right) \xrightarrow{a.s.} +\infty$. It follows from its definition in (5.34) that $\widetilde{L}_n \xrightarrow{a.s.} 0$ on $\left\{\widetilde{Y}_n \to \widetilde{e}_I\right\}$ since $0 \leq \widetilde{Y}_n^1 \leq \frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} \xrightarrow{a.s.} 1$. Consequently,

$$\left\{\widetilde{Y}_n \to \widetilde{e}_I\right\} \subset \left\{\widetilde{L}_n \to 0\right\}.$$

• (Case $d = 2$) If $f_r'(0) = 1$ and $f_l'(1) + \frac{f_r''(0)}{2} > 1$, then $\widetilde{h}(y) < 0$ for y in the neighborhood of 0. So, with in mind $\widetilde{e}_{\{2\}} = e^2$, we still have $\left\{\widetilde{Y}_n \to e^2\right\} = \left\{\widetilde{Y}_n^1 \to 0\right\} \subset \left\{\widetilde{L}_n \to 0\right\}$.

Step 2: The end of the proof is based on the following (short) "oracle" lemma (see [30]) reproduced here for the reader's convenience.

Lemma 5.1 (Oracle inequality)**.** *Let $(M_n)_{n\geq 0}$ be a non-negative martingale. Then*

$$\forall n \geq 0, \quad \mathbb{P}\big(M_\infty = 0 \,\big|\, \mathcal{F}_n\big) \leq \frac{\mathbb{E}\left[\Delta \langle M \rangle_{n+1}^\infty \,\middle|\, \mathcal{F}_n\right]}{M_n^2}.$$

Proof of Lemma 5.1. It is sufficient to observe that, for every $n \geq 0$,

$$\mathbb{P}\big(M_\infty = 0 \,\big|\, \mathcal{F}_n\big) = \frac{\mathbb{E}\left[\mathbb{1}_{\{M_\infty = 0\}} M_n^2 \middle| \mathcal{F}_n\right]}{M_n^2} \leq \frac{\mathbb{E}\left[(M_\infty - M_n)^2 \,\middle|\, \mathcal{F}_n\right]}{M_n^2} = \frac{\mathbb{E}\left[\Delta \langle M \rangle_{n+1}^\infty \,\middle|\, \mathcal{F}_n\right]}{M_n^2}. \quad \square$$

First we note that

$$\mathbb{E}\left[(\Delta \widetilde{L}_{n+1})^2 \middle| \mathcal{F}_n\right] = \left(\frac{1}{n+1+\mathbf{w}(Y_0)}\right)^2 \frac{\mathbb{E}\left[(\Delta M_{n+1}^1)^2 \middle| \mathcal{F}_n\right]}{\left(\prod_{k=1}^{n+1}\left(1 - \frac{\widetilde{h}(\widetilde{Y}_{k-1}^1)}{k+\mathbf{w}(Y_0)}\right)\right)^2}$$

and

$$\mathbb{E}\left[\left(\Delta M_{n+1}^1\right)^2 \middle| \mathcal{F}_n\right] = \frac{f(\widetilde{Y}_n^1)\big(\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big) - f(\widetilde{Y}_n^1)\big)}{\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big)^2} \leq \frac{d-1}{f(1/d)^2}$$

since $\mathbf{w}(f(y)) \geq f\big(\max_i y_i\big) \geq f(1/d)$ and $\widetilde{Y}_n \in \mathcal{S}_d$. As a consequence

$$\mathbb{E}\left[(\Delta\widetilde{L}_{n+1})^2 \,\middle|\, \mathcal{F}_n\right] = \frac{1}{(n+1+\mathbf{w}(Y_0))^2 \left(\prod_{k=1}^{n+1}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{k-1})}{k+\mathbf{w}(Y_0)}\right)\right)^2} \frac{f(\widetilde{Y}^1_n)\big(\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big) - f(\widetilde{Y}^1_n)\big)}{\Big(\mathbf{w}\big(\widetilde{f}(\widetilde{Y}_n)\big)\Big)^2}.$$

Then, applying Lemma 5.1 to the non-negative martingale $(\widetilde{L}_n)_{n\geq 1}$ yields

$$\begin{aligned}\mathbb{P}\big(\widetilde{L}_\infty = 0 \,\big|\, \mathcal{F}_n\big) &\leq \frac{\mathbb{E}\left[\Delta\langle\widetilde{L}\rangle^\infty_{n+1}\middle|\,\mathcal{F}_n\right]}{\widetilde{L}_n^2} \\ &= \frac{1}{\widetilde{L}_n^2}\mathbb{E}\left[\sum_{k=n+1}^{\infty} \frac{F(\widetilde{Y}^1_{k-1})}{(k+\mathbf{w}(Y_0))^2\left(\prod_{\ell=1}^{k}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right)\right)^2}\middle|\mathcal{F}_n\right],\end{aligned}$$

where the function F, defined by $F(y) := \frac{f(y^1)\big(\mathbf{w}(\widetilde{f}(y))-f(y^1)\big)}{\big(\mathbf{w}(\widetilde{f}(y))\big)^2}$, $y \in (0,1]\times[0,1]^{d-1}$, is clearly non-negative and bounded by $\kappa_d := (d-1)/f(1/d)^2$. Consequently,

$$\begin{aligned}&\mathbb{P}\big(\widetilde{L}_\infty = 0 \,\big|\, \mathcal{F}_n\big) \\ &\leq \frac{\kappa_d}{\widetilde{L}_n^2}\sum_{k=n+1}^{\infty}\frac{1}{(k+\mathbf{w}(Y_0))^2}\mathbb{E}\left[\underbrace{\frac{\widetilde{Y}^1_{k-1}}{\prod_{\ell=1}^{k-1}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right)}}_{=\widetilde{L}_{k-1}} \frac{\left(1-\frac{1}{k+\mathbf{w}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right)^{-1}}{\prod_{\ell=1}^{k}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right)}\,\middle|\,\mathcal{F}_n\right].\end{aligned}$$

Let $\widetilde{h}_+ := \max(\widetilde{h}, 0)$, so that $\widetilde{h} \leq \widetilde{h}_+ \leq \big\|\widetilde{h}_+\big\|_\infty$. First note that $\widetilde{h}_+(y) \leq 1 - \frac{f(y^1)}{d\cdot y^1} < 1$, for every $y \in \mathcal{S}_d \setminus \{y^1 = 0\}$ since $\mathbf{w}(\tilde{f}(y)) \leq d$ and we assumed $f(y^1) > 0$ on $(0,1]$. Moreover, $\limsup_{y^1\to 0}\widetilde{h}(y) \leq 1 - f_r'(0)/d < 1$ since $f_r'(0) > 0$ by assumption. As a consequence $\big\|\widetilde{h}_+\big\|_\infty < 1$.

In 2-dimensions, under the additional assumption in the critical case ($f_r'(0) = 1$), the extension of the function h over $[0,1]$ is negative so that $\widetilde{h}_+ \equiv 0$.

Finally, we obtain

$$\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{k-1})}{k+\mathbf{w}(Y_0)}\right)^{-1} \leq \left(1-\frac{\big\|\widetilde{h}_+\big\|_\infty}{k+\mathbf{w}(Y_0)}\right)^{-1}, \quad k \geq 1.$$

Then, as $\mathbb{E}\left[\widetilde{L}_{k-1}\middle|\mathcal{F}_n\right]=\widetilde{L}_n$ for every $k\geq n+1$, since $\widetilde{M}$ is a $(\mathbb{P},\mathcal{F}_n)$-martingale,

$$\begin{aligned}
&\mathbb{P}(\widetilde{L}_\infty=0|\mathcal{F}_n)\\
&\qquad\leq\frac{\kappa_d}{\widetilde{L}_n^2}\sum_{k=n+1}^{\infty}\frac{1}{(k+\mathbf{w}(Y_0))^2}\frac{\widetilde{L}_n}{\left(1-\frac{\|\widetilde{h}_+\|_\infty}{k+\mathbf{w}(Y_0)}\right)\prod_{\ell=1}^{n}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right)\prod_{\ell=n+1}^{k}\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right)}\\
&\qquad=\frac{\kappa_d}{\widetilde{L}_n\prod_{\ell=1}^{n}\left(1-\frac{\widetilde{h}(\widetilde{Y}^1_{\ell-1})}{\ell+\mathbf{w}(Y_0)}\right)}\sum_{k=n+1}^{\infty}\frac{1}{\left(1-\frac{\|\widetilde{h}_+\|_\infty}{k+\mathbf{w}(Y_0)}\right)(k+\mathbf{w}(Y_0))^2\prod_{\ell=n+1}^{k}\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right)}\\
&\qquad\leq\frac{\kappa_d}{C_0\widetilde{Y}^1_n}\sum_{k=n+1}^{\infty}\frac{1}{(k+\mathbf{w}(Y_0))^2}\exp\left(-\sum_{\ell=n+1}^{k}\log\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right)\right)
\end{aligned}\tag{5.35}$$

where $C_0=\frac{\mathbf{w}(Y_0)}{1+\mathbf{w}(Y_0)}\in(0,1)$ since $1-\frac{\|\widetilde{h}_+\|_\infty}{k+\mathbf{w}(Y_0)}>1-\frac{1}{k+\mathbf{w}(Y_0)}\geq C_0$.

Note that $\log(1-u)\geq-\frac{u}{1-u_0}$, $u\in(0,u_0)$. Applying this inequality with $u_0=\frac{1}{n+1+\mathbf{w}(Y_0)}$ yields, for every $\ell\geq n+1$,

$$\log\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right)\geq-\left(1+\frac{1}{n+\mathbf{w}(Y_0)}\right)\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}.$$

Hence

$$\begin{aligned}
\sum_{\ell=n+1}^{k}\log\left(1-\frac{\|\widetilde{h}_+\|_\infty}{\ell+\mathbf{w}(Y_0)}\right)&\geq-\|\widetilde{h}_+\|_\infty\left(1+\frac{1}{n+\mathbf{w}(Y_0)}\right)\sum_{\ell=n+1}^{k}\frac{1}{\ell+\mathbf{w}(Y_0)}\\
&\geq-\|\widetilde{h}_+\|_\infty\left(1+\frac{1}{n+\mathbf{w}(Y_0)}\right)\int_n^k\frac{du}{u+\mathbf{w}(Y_0)}\\
&=-\|\widetilde{h}_+\|_\infty\left(1+\frac{1}{n+\mathbf{w}(Y_0)}\right)\log\left(\frac{k+\mathbf{w}(Y_0)}{n+\mathbf{w}(Y_0)}\right).
\end{aligned}$$

Plugging this inequality into (5.35) and using that $\widetilde{Y}^1_n=\frac{Y^1_n}{n+\mathbf{w}(Y_0)}$,

$$\mathbb{P}\big(\widetilde{L}_\infty=0\,|\,\mathcal{F}_n\big)\leq\frac{\kappa_d(n+\mathbf{w}(Y_0))}{C_0Y^1_n}\sum_{k=n+1}^{+\infty}\frac{e^{(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty\log\left(\frac{k+\mathbf{w}(Y_0)}{n+\mathbf{w}(Y_0)}\right)}}{(k+\mathbf{w}(Y_0))^2}.$$

As $\|\widetilde{h}_+\|_\infty<1$, there exists n_0 be such that for every $n\geq n_0$, $1-(1+\frac{1}{n+\mathbf{w}(Y_0)})\|\widetilde{h}_+\|_\infty>0$. Hence

$$
\begin{aligned}
\mathbb{P}\big(\widetilde{L}_\infty = 0 \,|\, \mathcal{F}_n\big) &\leq \frac{\kappa_d(n+\mathbf{w}(Y_0))}{C_0 Y_n^1} \sum_{k=n+1}^{\infty} \frac{(n+\mathbf{w}(Y_0))^{-(1+\frac{1}{n+\mathbf{W}(Y_0)})\|\widetilde{h}_+\|_\infty}}{(k+\mathbf{w}(Y_0))^{2-(1+\frac{1}{n+\mathbf{W}(Y_0)})\|\widetilde{h}_+\|_\infty}} \\
&= \frac{\kappa_d(n+\mathbf{w}(Y_0))^{1-(1+\frac{1}{n+\mathbf{W}(Y_0)})\|\widetilde{h}_+\|_\infty}}{C_0 Y_n^1} \sum_{k=n+1}^{\infty} \frac{1}{(k+\mathbf{w}(Y_0))^{2-(1+\frac{1}{n+\mathbf{W}(Y_0)})\|\widetilde{h}_+\|_\infty}} \\
&\leq \frac{\kappa_d(n+\mathbf{w}(Y_0))^{1-(1+\frac{1}{n+\mathbf{W}(Y_0)})\|\widetilde{h}_+\|_\infty}}{C_0 Y_n^1} \int_n^{+\infty} \frac{du}{(u+\mathbf{w}(Y_0))^{2-(1+\frac{1}{n+\mathbf{W}(Y_0)})\|\widetilde{h}_+\|_\infty}} \\
&\leq \frac{\kappa_d}{C_0 Y_n^1} \frac{1}{1-(1+\frac{1}{n+\mathbf{W}(Y_0)})\|\widetilde{h}_+\|_\infty}.
\end{aligned}
$$

Now, it remains to prove that $Y_n^1 \xrightarrow{a.s.} +\infty$. Let $Y_\infty^1 := \lim_n Y_n^1$ (the components of Y_n are non-decreasing). One checks that

$$
\big\{Y_\infty^1 < +\infty\big\} = \bigcup_{n\geq 0} \bigcap_{k>n} \left\{ U_k > \frac{f(\frac{Y_n^1}{k-1+\mathbf{W}(Y_0)})}{f(\frac{Y_n^1}{k-1+\mathbf{W}(Y_0)}) + f(1-\frac{Y_n^1}{k-1+\mathbf{W}(Y_0)})} \right\},
$$

then

$$
\forall n \in \mathbb{N}, \quad \mathbb{P}\left(Y_\infty^1 < +\infty \,|\, Y_n^1 = y\right) = \prod_{k>n} \left(1 - \frac{f(y/(k-1+\mathbf{w}(Y_0)))}{f(y/(k-1+\mathbf{w}(Y_0))) + f(1-y/(k-1+\mathbf{w}(Y_0)))}\right) = 0,
$$

since $\displaystyle\sum_{k\geq 1} \frac{f(y/k+\mathbf{w}(Y_0))}{f(y/k+\mathbf{w}(Y_0)) + f(1-y/k+\mathbf{w}(Y_0))} = +\infty$ because $f_r'(0) > 0$. Therefore $Y_\infty^1 = +\infty$ *a.s.*

On the other hand, the closed martingale $\mathbb{P}\big(\widetilde{L}_\infty = 0 | \,\mathcal{F}_n\big) \to \mathbf{1}_{\{\widetilde{L}_\infty = 0\}}$ *a.s.* and in L^1 so that

$$
\mathbf{1}_{\{\widetilde{L}_\infty = 0\}} = 0 \quad a.s. \quad \text{i.e.} \quad \mathbb{P}(\widetilde{L}_n \to 0) = \mathbb{P}(\widetilde{L}_\infty = 0) = 0
$$

which in turn implies that $\mathbb{P}\left(\widetilde{Y}_\infty^1 = 0\right)$ since it was proved in STEP 1 that $\{\widetilde{Y}_n^1 \to 0\} \subset \{\widetilde{L}_n \to 0\}$.

(*c*) As $Y_0^i > 0$, $i \in \{1, \ldots, d\}$, we derive from what precedes that $\mathbb{P}\big(\widetilde{Y}_n \to \partial\mathcal{S}_d\big) = 0$. As a consequence, following Theorem A.1, $\mathbb{P}(d\omega)$-*a.s.*, the compact connected flow invariant set $\Theta^\infty(\omega)$ of limiting values of $(\widetilde{Y}_n(\omega))_{n\geq 0}$ is a minimal connected attractor of ODE_h in $\overset{\circ}{\mathcal{S}}_d$. We know from Proposition 2.5(*b*) that $y(d) = \frac{1}{d}\mathbf{1}$ is a uniformly attracting point ODE_h and from Proposition 2.6(*b*) that the flow of ODE_h $(y(y_0,t))_{t\geq 0, y_0 \in \overset{\circ}{\mathcal{S}}_d}$ converges toward $\frac{1}{d}\mathbf{1}$, so it converges uniformly with respect to $y_0 \in \Theta^\infty(\omega)$. Consequently, one concludes by Theorem A.1 that $\Theta^\infty(\omega) = \{y(d)\}$ (otherwise it would have an internal attractor). Hence $\widetilde{Y}_n \xrightarrow{a.s.} y(d)$. □

6 Applications

6.1 A drawing rule based on a function with regular variation

Let define the law of the drawings as follows

$$
\forall 1 \leq i \leq d, \quad \mathbb{P}(X_{n+1} = e^i \,|\, \mathcal{F}_n) = \frac{f(Y_n^i)}{\sum_{j=1}^d f(Y_n^j)}, \quad n \geq 0, \tag{6.36}
$$

where f has regular variation with index $\alpha > 0$ in the sense that for every $t > 0$, $\dfrac{f(tx)}{f(x)} \underset{x\to\infty}{\longrightarrow} t^\alpha$ and f is bounded on each interval $(0, M]$. Then, by applying Theorem 1.5.2 p.22 in [10], $\dfrac{f(tx)}{f(x)} \underset{x\to\infty}{\longrightarrow} t^\alpha$ uniformly in t on each $(0, b]$, $0 < b < \infty$.

We can reformulate the dynamics (1.1)-(1.3) into a recursive stochastic algorithm like in the Section 2.2, and we obtain the following recursive procedure satisfied by the sequence $(\widetilde{Y}_n)_{n\geq 0}$:

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \gamma_{n+1}\left(\widetilde{Y}_n - H\frac{\widetilde{Y}_n^\alpha}{\mathbf{w}(\widetilde{Y}_n^\alpha)}\right) + \gamma_{n+1}\left(\Delta M_{n+1} + \widehat{r}_{n+1}\right), \tag{6.37}$$

with the step $\gamma_n = \frac{1}{n+\mathbf{w}(Y_0)}$, $\widetilde{Y}_n^\alpha = \left((\widetilde{Y}_n^i)^\alpha\right)_{1\leq i\leq d}$ and an $\mathcal{F}_n$-measurable remainder term given by

$$\widehat{r}_{n+1} := H_{n+1}\frac{\widetilde{f}(Y_n)}{\mathbf{w}(\widetilde{f}(Y_n))} - H\frac{\widetilde{Y}_n^\alpha}{\mathbf{w}(\widetilde{Y}_n^\alpha)}. \tag{6.38}$$

Notice that, in the skewing case, the remainder term was $r_{n+1} = (H_{n+1} - H)\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathbf{w}(\widetilde{f}(\widetilde{Y}_n))}$, therefore assumption **(A3)** implied directly that $r_n \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} 0$. Here we have to use the uniform convergence of the regular variation to prove the required assumption on $\widehat{r}_{n+1}$.

By the same arguments used in Section 2.2, $\mathbf{w}(Y_n)$ satisfies (2.20). Moreover, for the quantity $\widetilde{N}_n := \frac{1}{n}\sum_{k=1}^n X_k$, we also devise a stochastic recursive procedure in the same way as before, namely

$$\widetilde{N}_{n+1} = \widetilde{N}_n - \frac{1}{n+1}\left(\widetilde{N}_n - \frac{\widetilde{Y}_n^\alpha}{\mathbf{w}(\widetilde{Y}_n^\alpha)}\right) + \frac{1}{n+1}\left(\Delta\widetilde{M}_{n+1} + \widetilde{r}_{n+1}\right),$$

where $\widetilde{r}_{n+1} = \dfrac{\widetilde{f}(Y_n)}{\mathbf{w}(\widetilde{f}(Y_n))} - \dfrac{\widetilde{Y}_n^\alpha}{\mathbf{w}(\widetilde{Y}_n^\alpha)}$, thus $\widetilde{r}_{n+1} \in \mathcal{F}_n$.

Theorem 6.1. *Let $d = 2$. Assume that* **(A1)**, **(A2)** *and* **(A3)** *hold.*

(a) If $0 < \alpha \leq 1$, then h has a unique zero $y^* \in I^*$ and

$$\frac{\mathbf{w}(Y_n)}{n+\mathbf{w}(Y_0)} \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} 1, \quad \frac{Y_n}{\mathbf{w}(Y_n)} \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} y^* \quad \text{and} \quad \widetilde{N}_n \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} \frac{(y^*)^\alpha}{\mathbf{w}((y^*)^\alpha)}.$$

(b) If $\alpha > 1$, then h has a unique zero $y^* \in I^*$ or ODE_h has two attracting equilibrium points in I^* (as we have established in Section 2.3). Thus, the stochastic recursive procedure a.s. converges to one of the possible limit values.

Proof. By the same arguments like in Section 2.3, $\mathbf{w}(Y_n)$ satisfies (2.20), therefore Proposition 2.1 holds. Consequently, $\widetilde{Y}_n$ lies in a compact of $\mathbb{R}_+$, thus

$$\max_{1\leq i\leq d}\left|\frac{f(Y_n^i)}{f(n+\mathbf{w}(Y_0))} - \left(\frac{Y_n}{n+\mathbf{w}(Y_0)}\right)^\alpha\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

Set $a_n^i = \frac{f(Y_n^i)}{f(n+\mathbf{w}(Y_0))}$ and $b_n^i = (\widetilde{Y}_n^i)^\alpha$, $i \in \{1, \ldots, d\}$. Then, for every $i \in \{1, \ldots, d\}$,

$$\frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)} = \frac{a_n^i - b_n^i}{\mathbf{w}(b_n)} + \frac{a_n^i}{\mathbf{w}(a_n)}\left(1 - \frac{\mathbf{w}(a_n)}{\mathbf{w}(b_n)}\right).$$

But

$$\mathbf{w}(b_n) = \sum_{i=1}^{d} (\widetilde{Y}_n^i)^\alpha \geq \begin{cases} \left(\sum_{i=1}^d \widetilde{Y}_n^i\right)^\alpha = \mathbf{w}(\widetilde{Y}_n)^\alpha & \text{if} \quad \alpha \in [0,1] \\ d^{1-\alpha}\mathbf{w}(\widetilde{Y}_n)^\alpha & \text{if} \quad \alpha > 1 \end{cases},$$

therefore

$$\mathbf{w}(b_n) \geq \frac{\mathbf{w}(\widetilde{Y}_n)^\alpha}{d^{(\alpha-1)_+}} \underset{a.s.}{\sim} \frac{(n+\mathbf{w}(\widetilde{Y}_0))^\alpha}{d^{(\alpha-1)_+}}.$$

Consequently, for every $i \in \{1, \ldots, d\}$,

$$\frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)} \leq \frac{\max_{1\leq i\leq d} |a_n^i - b_n^i| + \sum_{j=1}^d |a_n^j - b_n^j|}{\mathbf{w}(b_n)},$$

i.e.

$$\max_{1\leq i\leq d} \left| \frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)} \right| \leq \frac{d+1}{\mathbf{w}(b_n)} \max_{1\leq i\leq d} |a_n^i - b_n^i| \xrightarrow[n\to+\infty]{a.s.} 0.$$

Thus

$$|\widehat{r}_{n+1}| \leq |\!|\!| H |\!|\!| \max_{1\leq i\leq d} \left| \frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)} \right| + |\!|\!| H_{n+1} - H |\!|\!| \xrightarrow[n\to+\infty]{a.s.} 0,$$

and in the same way $\widetilde{r}_{n+1} \xrightarrow[n\to+\infty]{a.s.} 0$. Consequently claim (a) follows from Proposition 3.1(a) and Theorem 3.1.

We have to check the assumption on the remainder term to apply result on traps for SA. We have that

$$\max_{1\leq i\leq d} \left| \frac{a_n^i}{\mathbf{w}(a_n)} - \frac{b_n^i}{\mathbf{w}(b_n)} \right| \lesssim \frac{(d+1)d^{(\alpha-1)_+}}{(n+\mathbf{w}(Y_0))^\alpha} \max_{1\leq i\leq d} |a_n^i - b_n^i| = o(n^{-\alpha}). \tag{6.39}$$

So, $\alpha > 1$, under assumption **(A3)** on the generating matrices,

$$\sum_{n\geq 0} \|r_{n+1}\|^2 < +\infty.$$

The end of the proof follows from Proposition 3.1(b)&(c) and Theorem 3.1. □

To establish a CLT for the sequence $(\widetilde{Y}_n)_{n\geq 0}$ we need that the remainder term $(r_n)_{n\geq 1}$ satisfies (A.47). Then we will assume that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy **(A1)**-(ii) to ensure that $(\widetilde{Y}_n)_{n\geq 0}$ lies in the simplex (which implies that the rate in (6.39) is no more $a.s.$) and we assume also that $\alpha > 1/2$.

Theorem 6.2. *Assume that the index of regular variation $\alpha > 1/2$, that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy* **(A1)**-(ii), **(A3)**, **(A4)** *and that* **(A5)**$_v$ *holds. We have*

$$Sp\left(J_h(y^*)\right) = \{1, 1-\rho^*\},$$

where

$$\rho^* = \frac{\alpha(y^{*1})^{\alpha-1}(p_1 - y^{*1}) + \alpha(1-y^{*1})^{\alpha-1}(y^{*1} - 1 + p_2)}{(y^{*1})^\alpha + f(1-y^{*1})^\alpha}.$$

(a) *If $p_1 + p_2 - 1 \leq 0$ and $v_n = 1$, $n \geq 1$, then*

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \xrightarrow[n\to+\infty]{\mathcal{L}_{stably}} \mathcal{N}(0, \Sigma) \qquad \textit{with} \qquad \Sigma = \int_0^{+\infty} e^{u(J_h(y^*) - \frac{I}{2})} \Gamma e^{u(J_h(y^*) - \frac{I}{2})^t} du$$

$$\text{and} \quad \Gamma = \frac{(y^{*1})^\alpha C^1 + (1-y^{*1})^\alpha C^2}{\mathbf{w}((y^*)^\alpha)} - y^*(y^*)^t = a.s.\text{-}\lim_{n\to+\infty} \mathbb{E}\left[\Delta M_n \Delta M_n^t \,|\, \mathcal{F}_{n-1}\right]. \tag{6.40}$$

(b) If $p_1 + p_2 - 1 > 0$, we have three possible rate of convergence depending on the second eigenvalue:
(i) If $0 < \rho^* < \frac{1}{2}$ and $v_n = 1$, $n \geq 1$, then

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \overset{\mathcal{L}_{stably}}{\underset{n\to+\infty}{\longrightarrow}} \mathcal{N}(0,\Sigma).$$

(ii) If $\rho^* = \frac{1}{2}$ and $v_n = \log n$, $n \geq 1$, then

$$\sqrt{\frac{n}{\log n}}\left(\widetilde{Y}_n - y^*\right) \underset{n\to+\infty}{\overset{\mathcal{L}}{\longrightarrow}} \mathcal{N}(0,\Sigma) \quad \text{where} \quad \Sigma = \lim_n \frac{1}{\log n}\int_0^{\log n} e^{u(J_h(y^*)-\frac{I}{2})}\Gamma e^{u(J_h(y^*)-\frac{I}{2})^t} du.$$

(iii) If $\frac{1}{2} < \rho^* < 1$ and $v_n = n^{1-2\rho^*+\eta}$, $\eta > 0$, then $n^{\rho^*}\left(\widetilde{Y}_n - y^*\right)$ a.s. converges as $n \to +\infty$ towards a positive finite random variable Υ.

This result follows from Theorem A.3 and Theorem 4.1.

6.2 An application to Finance: Adaptive asset allocation

Such urn based recursive procedures can be applied to adaptive portfolio allocation by an asset manager or a trader, or to optimal split across liquidity pools. Indeed the first setting has already been done in [30] and successfully implemented with multi-armed bandit procedure. We develop in this section the adaptive portfolio allocation, but the optimal split across liquidity pools can be implemented in the same way, by considering that the different colors represent the different liquidity pools, and the trader want to optimally split a large volume of a single asset among the different possible destinations.

Imagine an asset manager who deals with a portfolio of d tradable assets. To optimize the yield of her portfolio, she can modify the proportions invested in each asset. She starts with the initial allocation vector Y_0. At stage n, she chooses a tradable asset according to the distribution (1.3) or (1.5) of X_n, then evaluates its performance over one time step and modifies the portfolio composition accordingly (most likely virtually) and proceeds. Thus the normalized urn composition $\widetilde{Y}_n$ represents the allocation vector among the assets and the addition rule matrices D_n model the successive reallocations depending on the past performances of the different assets. The evaluation of the asset performances can be carried out recursively with an estimator like with multi-arm clinical trials (see [4, 31]). In practice, it can be used to design the addition rule matrices D_n. For example, we may consider $(T_n^i)_{n\geq 1}$, $i \in \{1,\ldots,d\}$, a *success indicator*, namely d independent sequences of i.i.d. $\{0,1\}$-valued Bernoulli trials with respective parameter p_i (with convention $T_n^i = 1$ if the return of the i^{th} asset in the n^{th} reallocation is positive and $T_n^i = 0$ otherwise).

Let $N_n^i := \sum_{k=1}^n X_k^i$ be the number of times the i^{th} asset is selected among the first n stages with $N_0^i = 0$, $i \in \{1,\ldots,d\}$, and let S_n be the d dimensional vector defined by

$$S_n^i = S_{n-1}^i + T_n^i X_n^i, \quad n \geq 1, \quad S_0^i = 1, \quad i \in \{1,\ldots,d\},$$

denoting the number of successes of the i^{th} asset among these N_n^i reallocations. Define Π_n an estimator of the vector of success probabilities, namely $\Pi_n^i = \frac{S_n^i}{N_n^i}$, $i \in \{1,\ldots,d\}$. We can prove that $\Pi_n \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}}$

$p := (p^1, \dots, p^d)^t$ (see [4, 31]). Then we build the following addition rule matrices

$$D_{n+1} = \begin{pmatrix} T^1_{n+1} & \frac{\Pi^1_n(1-T^2_{n+1})}{\sum_{j\neq 2}\Pi^j_n} & \dots & \frac{\Pi^1_n(1-T^d_{n+1})}{\sum_{j\neq d}\Pi^j_n} \\ \frac{\Pi^2_n(1-T^1_{n+1})}{\sum_{j\neq 1}\Pi^j_n} & T^2_{n+1} & \dots & \frac{\Pi^2_n(1-T^d_{n+1})}{\sum_{j\neq d}\Pi^j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Pi^d_n(1-T^1_{n+1})}{\sum^d_{j\neq 1}\Pi^j_n} & \frac{\Pi^d_n(1-T^2_{n+1})}{\sum^d_{j\neq 2}\Pi^j_n} & \dots & T^d_{n+1} \end{pmatrix}, \tag{6.41}$$

i.e. at stage $n+1$, if the return of the j^{th} asset is positive, then one ball of type j is added in the urn. Otherwise, $\frac{\Pi^i_n}{\sum_{k\neq j}\Pi^k_n}$ (virtual) balls of type i, $i \neq j$, are added. This addition rule matrix clearly satisfies **(A1)**-(i) and **(A2)**. Then, one easily checks that the generating matrices $H_n = \mathbb{E}\left[D_{n+1} \,|\, \mathcal{F}_n\right]$ satisfy **(A1)**-(ii) and, as soon as $Y_0 \in \mathbb{R}^d_+ \setminus \{0\}$, $H_n \xrightarrow{a.s.} H$ (see [4, 31]), where

$$H_{n+1} = \begin{pmatrix} p_1 & \frac{\Pi^1_n(1-p_2)}{\sum_{j\neq 2}\Pi^j_n} & \dots & \frac{\Pi^1_n(1-p_d)}{\sum_{j\neq d}\Pi^j_n} \\ \frac{\Pi^2_n(1-p_1)}{\sum_{j\neq 1}\Pi^j_n} & p_2 & \dots & \frac{\Pi^2_n(1-p_d)}{\sum_{j\neq d}\Pi^j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Pi^d_n(1-p_1)}{\sum_{j\neq 1}\Pi^j_n} & \frac{\Pi^d_n(1-p_2)}{\sum_{j\neq 2}\Pi^j_n} & \dots & p_d \end{pmatrix}, \quad H = \begin{pmatrix} p^1 & \frac{p^1(1-p^2)}{\sum_{j\neq 2}p^j} & \dots & \frac{p^1(1-p^d)}{\sum_{j\neq d}p^j} \\ \frac{p^2(1-p^1)}{\sum_{j\neq 1}p^j} & p^2 & \dots & \frac{p^2(1-p^d)}{\sum_{j\neq d}p^j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p^d(1-p^1)}{\sum_{j\neq 1}p^j} & \frac{p^d(1-p^2)}{\sum_{j\neq 2}p^j} & \dots & p^d \end{pmatrix}.$$

Let remark that H is $\mathbb{R}$-diagonalizable since it is symmetric with respect to its invariant measure. Therefore, the number of each asset in the portfolio Y_n follows the dynamics (1.1) and the distribution of the portfolio in each asset follows the dynamics (2.17) or (6.37) depending on the drawing rule.

Here the components of the limiting generating matrix H can be interpreted as constraints on the composition of the portfolio. Indeed, in presence of two assets (or colors), we prove that the first component of the allocation vector y^{*1} lies in I^* (see Proposition 3.1), therefore the portfolio will contain at least $p_1 \vee (1-p_2)\%$ and no more than $p_1 \wedge (1-p_2)\%$ of the first asset. Such rules may be prescribed by the regulation, the bank policy or the bank customer, and our approach is a natural way to have them satisfied (at least asymptotically).

The idea of reinforcing the drawing rule (instead of considering the uniform drawing) like in (1.3) or (1.5) can be interpreted as a way to take into account the risk aversion of the trader or the customer. Indeed, if f is concave the equilibrium point will be in the middle of the simplex (see Theorem 3.2 and Theorem 3.3), so the trader prefers to have diversification in her portfolio. On the contrary, if f is convex, the equilibrium points will lie on the boundary of the set of constraints induced by the limiting generating matrix H, so she prefers to take advantage of the most money-making asset (like in a "winner take all" or a "0-1" strategy).

Numerical experiments. We present some numerical experiments for the drawing rule defined by (1.3), firstly with a concave function $f : y \mapsto \sqrt{y}$ and secondly with a convex function $f : y \mapsto y^4$. Therefore we have a unique equilibrium point in the first setting and two attracting targets in the second framework. We consider an asset manager who deal with a portfolio of 2 tradable assets. We model the addition rule matrices like in the multi-arm clinical trials, namely D_n is defined by (6.41). We use the same success probabilities, namely $p_1 = 0.7$ and $p_2 = 0.75$, and the initial urn composition is chosen randomly in the simplex $\mathcal{S}_2$.

▷ *Convergence of the portfolio allocation with concave drawing rule.*

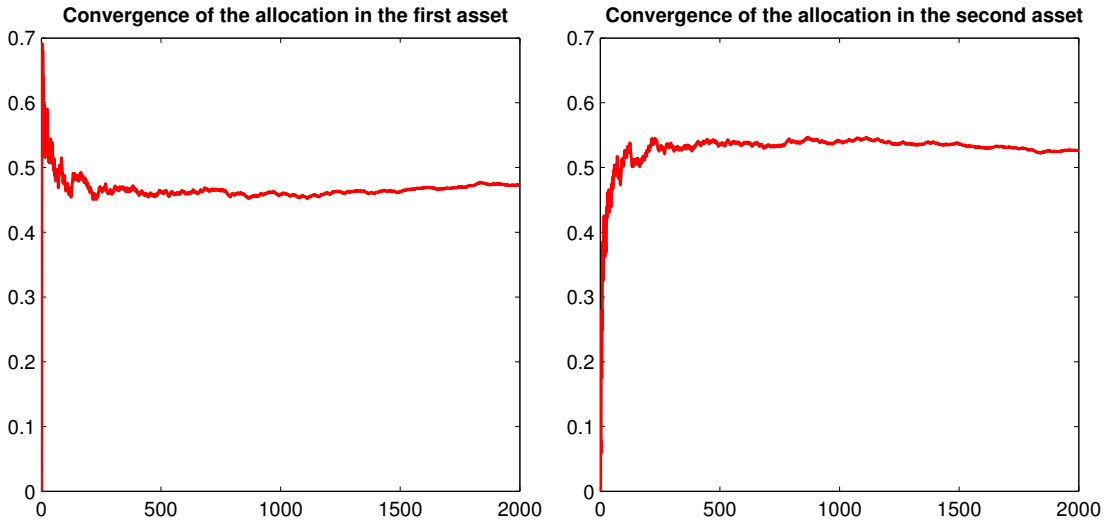


Figure 6.4: Convergence of $\widetilde{Y}_n$ toward y^* for $f(y) = \sqrt{y}$ with $p_1 = 0.7$ and $p_2 = 0.75$.

We have that $y^{*1} \in (0.25, 0.7)$ and y^{*1} and y^{*2} are close to $\frac{1}{2}$, so the portfolio is diversified because in this case the investor is risk adverse.

▷ *Convergence of the portfolio allocation with convex drawing rule.*

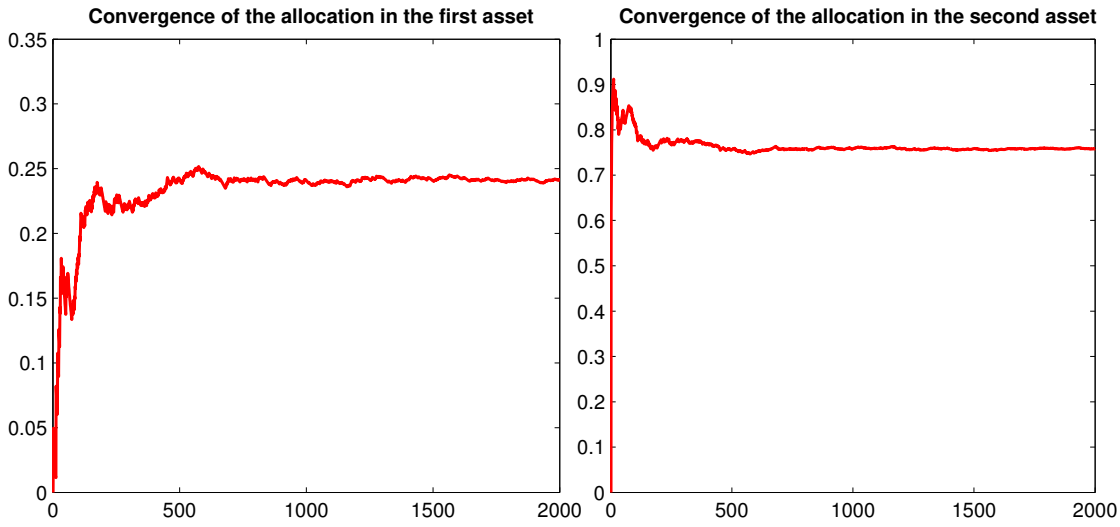


Figure 6.5: Convergence of $\widetilde{Y}_n$ toward y^* for $f(y) = y^4$ with $p_1 = 0.7$ and $p_2 = 0.75$.

In the convex framework, we have two possible strategies and they are close to the boundaries defined by regulation. Moreover the distribution of the portfolio between the two assets is more

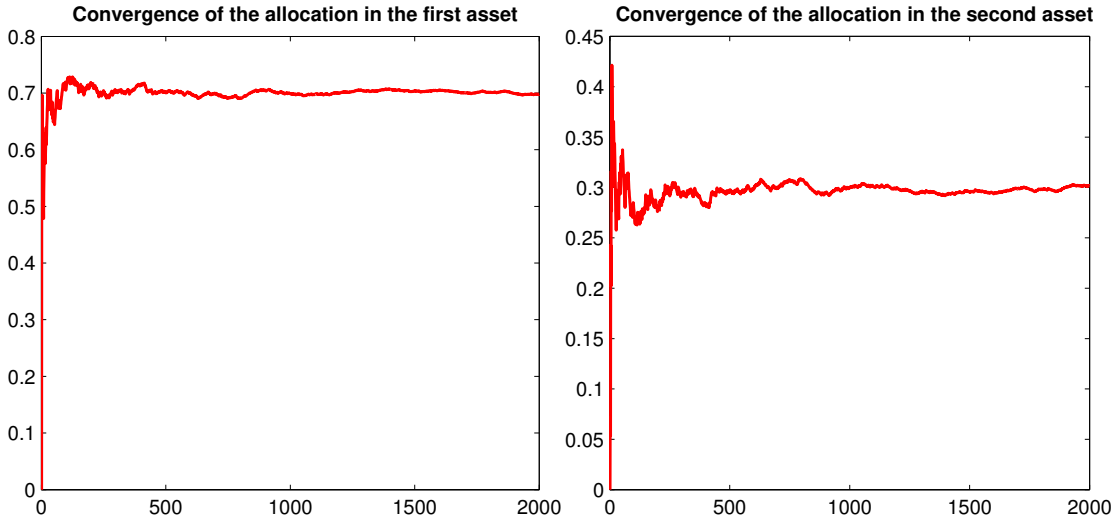


Figure 6.6: Convergence of $\widetilde{Y}_n$ toward y^* for $f(y) = y^4$ with $p_1 = 0.7$ and $p_2 = 0.75$.

asymmetric, because the trader chooses to invest two times more in one asset than in the other.

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Appendix

A Basic tools from Stochastic Approximation

Consider the following recursive procedure defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$ having values in a convex set $C \subset \mathbb{R}^d$,

$$\forall\, n \geq 0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1}\left(\Delta M_{n+1} + r_{n+1}\right), \tag{A.42}$$

where $(\gamma_n)_{n\geq 1}$ is a $(0, \bar{\gamma}]$-valued step sequence for some $\bar{\gamma} > 0$, $h : C \to \mathbb{R}^d$ is a continuous function with linear growth (the *mean field* of the algorithm) such that

$$(I_d - \gamma h)(C) \subset C \ \text{ for every } \quad \gamma \in (0, \bar{\gamma}], \tag{A.43}$$

and θ_0 is an $\mathcal{F}_0$-measurable finite random vector and, for every $n \geq 1$, ΔM_n is an $\mathcal{F}_n$-martingale increment and r_n is an $\mathcal{F}_n$-adapted remainder term.

Note that the assumptions of the theorems recalled below are possibly not minimal, but adapted to the problems we want to solve.

▷ *A.s.* **Convergence.** Let us introduce a few additional notions on differential systems. We consider the differential system $ODE_h \equiv \dot{x} = -h(x)$ associated to the (continuous) *mean field* $h : C \to \mathbb{R}^d$. We assume that this system has a C-valued *flow* ([1]) $\Phi(t,\xi)_{t\in\mathbb{R}_+,\xi\in C}$: For every $\xi \in C$, $(\Phi(t,\xi))_{t\geq 0}$ is the unique solution to ODE_h defined on the whole positive real line. This flow exists as soon as h is locally Lipschitz with linear growth.

Let K be a compact connected, flow invariant subset of C, *i.e.* such that $\Phi(t,K) \subset K$ for every $t \in \mathbb{R}_+$.

A non-empty subset $A \subset K$ is an *internal attractor* of K for ODE_h if

(*i*) $A \subsetneq K$,

(*ii*) $\exists\, \varepsilon_0 > 0$ such that $\sup_{x\in K, \mathrm{dist}(x,A)\leq\varepsilon_0} \mathrm{dist}\big(\Phi(t,x), A\big) \to 0$ as $t \to +\infty$.

[1] this is satisfied under the above stability condition (A.43).

A compact connected flow invariant set K is a *minimal attractor* for ODE_h if it contains no internal attractor. This terminology coming from dynamical systems may be misleading: Thus any equilibrium point of ODE_h (zero of h) is a minimal attractor by this definition, regardless of its stability (see Claim (b) in Theorem A.1 below, see also Definition 2.2).

Remark. When the flow does not exist, the above definition should be understood as follows: One replaces the flow $\Phi(x,\cdot)$ by the family of all solutions of ODE_h starting from x at time 0 (whose existence follow from Peano's Theorem). For more details on this natural extension, we refer to [20] (see Appendix "the ODE method without flow"). Up to this extension, the theorem below remains true even when uniqueness of solutions of ODE_h fails.

Theorem A.1. *(A.s. convergence with ODE method, see e.g. [9, 18, 27, 19, 6]). Assume that $h : C \to \mathbb{R}^d$ satisfies (A.43) and that ODE_h has a C-valued flow (e.g. because h is a locally Lipschitz function with linear growth). Assume furthermore that*

$$r_n \xrightarrow[n\to+\infty]{a.s.} 0 \quad \text{and} \quad \sup_{n\geq 0} \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n\right] < +\infty \quad a.s.,$$

and that $(\gamma_n)_{n\geq 1}$ is a positive sequence satisfying ($\gamma_n \in (0,\bar{\gamma}]$, $n\geq 1$) and

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

On the event $A_\infty = \big\{\omega : (h(\theta_n(\omega)))_{n\geq 0} \text{ is bounded}\big\}$, $\mathbb{P}(d\omega)$-a.s., the set $\Theta^\infty(\omega)$ of the limiting values of $(\theta_n(\omega)_{n\geq 0})$ as $n\to+\infty$ is a compact connected flow invariant minimal attractor for ODE_h (see Proposition 5.3 in Section 5.1 in [6]).

Furthermore:

(a) Equilibrium point(s) as limiting value(s). *If $\operatorname{dist}\big(\Phi(\theta_0,t),\{h=0\}\big)\to 0$ as $t\to+\infty$, for every $\theta_0\in\mathbb{R}^d$, then $\Theta^\infty(\omega)\cap\{h=0\}\neq\varnothing$.*

(b) Single stable equilibrium point. *If $\{h=0\}=\{\theta^*\}$ and $\Phi(\theta_0,t)\to\theta^*$ as $t\to+\infty$ locally uniformly in θ_0, then $\Theta^\infty(\omega)=\{\theta^*\}$ i.e. $\theta_n \xrightarrow{a.s.} \theta^*$ as $n\to+\infty$.*

(c) 1-dimensional setting. *If $d=1$ and $\{h=0\}$ is locally finite, then $\Theta^\infty(\omega)=\{\theta_\infty\}\subset\{h=0\}$ i.e. $\theta_n \xrightarrow{a.s.} \theta_\infty\in\{h=0\}$.*

A stochastic algorithm may *a.s.* converge under the existence of multiple equilibrium points, typically stochastic gradient or pseudo-descents, but we do not need such results to solve the urn problems under consideration in this paper. We refer to [18, 27, 19, 6], among others. Note also that examples of situation (a) where the algorithm *a.s. does not converge* are developed in [19], [20] or [6] (necessarily with $d\geq 2$ owing to Claim (c)).

▷ **Traps (Unstable equilibrium point).** This second theorem deals with "traps" *i.e.* repulsive zeros of the mean function h. It shows that, provided such a trap is noisy enough, such a trap cannot be a limiting point of the algorithm.

Theorem A.2. *(A.s. non-convergence toward a noisy trap, see e.g. [12, 17]). Assume that $z^*\in\mathbb{R}^d$ is a trap for the stochastic algorithm (A.42), i.e.*
(i) $h(z^*)=0$,
(ii) there exists a neighborhood $V(z^*)$ of z^* in which h is differentiable with a Lipschitz differential,
(iii) the eigenvalue of $J_h(z^*)$ with the lowest real part, denoted by $\lambda_{\min}$, satisfies $\Re e(\lambda_{\min})<0$.
Assume furthermore that a.s. on $\Gamma(z^*)=\{\theta_n \xrightarrow[n\to+\infty]{} z^*\}$,

$$\sum_{n\geq 1}\|r_n\|^2 < +\infty \quad \text{and} \quad \limsup_n \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n\right] < +\infty.$$

Let K_+ the subspace of $\mathbb{R}^d$ spanned by the eigenvectors whose associated eigenvalues have a non-negative real part and K_- the subset of $\mathbb{R}^d$ spanned by the eigenvectors whose associated eigenvalues have a negative real part (then $\mathbb{R}^d = K_+ \oplus K_-$). By setting $\Delta M_{n+1}^{(r)}$ the projection of ΔM_{n+1} on K_- alongside K_+, assume that a.s. on $\Gamma(z^*)$,

$$\liminf_n \mathbb{E}\left[\left\|\Delta M_{n+1}^{(r)}\right\| \middle| \mathcal{F}_n\right] > 0. \tag{A.44}$$

Moreover, if the positive sequence $(\gamma_n)_{n\geq 1}$ satisfies

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \textit{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty,$$

then $\mathbb{P}(\Gamma(z^*)) = 0$.

▷ **Rate(s) of convergence.** We will say that h is ϵ-differentiable ($\epsilon > 0$) at θ^* if

$$h(\theta) = h(\theta^*) + J_h(\theta^*)(\theta - \theta^*) + o(\|\theta - \theta^*\|^{1+\epsilon}) \quad \text{as} \quad \theta \to \theta^*.$$

Theorem A.3. *(Rate of convergence see [18] Theorem 3.III.14 p.131 (for the CLT see also e.g. [9, 27])). Let θ^* be an equilibrium point of $\{h = 0\}$ and $\{\theta_n \to \theta^*\}$ the convergence event associated to θ^* (supposed to have a positive probability). Set the gain parameter sequence $(\gamma_n)_{n\geq 1}$ as follows*

$$\forall n \geq 1, \quad \gamma_n = \frac{1}{n}. \tag{A.45}$$

Assume that the function h is differentiable at θ^* and all the eigenvalues of $J_h(\theta^*)$ have positive real parts. Assume that, for a real number $\delta > 0$,

$$\sup_{n\geq 0} \mathbb{E}\left[\|\Delta M_{n+1}\|^{2+\delta} \,|\, \mathcal{F}_n\right] < +\infty \, a.s., \quad \mathbb{E}\left[\Delta M_{n+1} \Delta M_{n+1}^t \,|\, \mathcal{F}_n\right] \underset{n\to+\infty}{\overset{a.s.}{\longrightarrow}} \Gamma^* \quad \textit{on} \quad \{\theta_n \to \theta^*\}, \tag{A.46}$$

where $\Gamma^* \in \mathcal{S}^+(d, \mathbb{R})$ (deterministic symmetric positive matrix) and for an $\varepsilon > 0$ and a positive sequence $(v_n)_{n\geq 1}$ (specified below),

$$n\, v_n \mathbb{E}\left[\|r_{n+1}\|^2 \mathbb{1}_{\{\|\theta_n - \theta^*\| \leq \varepsilon\}}\right] \underset{n\to+\infty}{\longrightarrow} 0. \tag{A.47}$$

Let $\lambda_{\min}$ denote the eigenvalue of $J_h(\theta^*)$ with the lowest real part and set $\Lambda := \Re e(\lambda_{\min})$.

(a) If $\Lambda > \frac{1}{2}$ and $v_n = 1$, $n \geq 1$, then, the weak convergence rate is ruled on the convergence event $\{\theta_n \overset{a.s.}{\longrightarrow} \theta^*\}$ by the following Central Limit Theorem

$$\sqrt{n}\,(\theta_n - \theta^*) \underset{n\to+\infty}{\overset{\mathcal{L}_{stably}}{\longrightarrow}} \mathcal{N}(0, \Sigma^*) \quad \textit{with} \quad \Sigma^* := \int_0^{+\infty} e^{-u\left(J_h(\theta^*)^t - \frac{I_d}{2}\right)} \Gamma^* e^{-u\left(J_h(\theta^*) - \frac{I_d}{2}\right)} du.$$

(b) If $\Lambda = \frac{1}{2}$, $v_n = \log n$, $n \geq 1$, and h is ϵ-differentiable at θ^*, then

$$\sqrt{\frac{n}{\log n}}\,(\theta_n - \theta^*) \underset{n\to+\infty}{\overset{\mathcal{L}_{stably}}{\longrightarrow}} \mathcal{N}(0, \Sigma^*) \quad \textit{on} \quad \{\theta_n \to \theta^*\},$$

where $\Sigma^* = \lim_n \frac{1}{n} \int_0^n e^{-u\left(J_h(\theta^*)^t - \frac{I_d}{2}\right)} \Gamma e^{-u\left(J_h(\theta^*) - \frac{I_d}{2}\right)} du$.

(c) If $\Lambda \in \left(0, \frac{1}{2}\right)$, $v_n = n^{2\Lambda - 1 + \eta}$, $n \geq 1$, for some $\eta > 0$, and h is ϵ-differentiable at θ^*, for some $\epsilon > 0$, then $n^\Lambda (\theta_n - \theta^*)$ is a.s. bounded on $\{\theta_n \to \theta^*\}$ as $n \to +\infty$.

If, moreover, $\Lambda = \lambda_{\min}$ ($\lambda_{\min}$ is real), then $n^\Lambda (\theta_n - \theta^*)$ a.s. converges as $n \to +\infty$ toward a finite random variable.