Nonlinear Randomized Urn Models: a Stochastic Approximation Viewpoint

Sophie Laruelle, Gilles Pagès

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SOPHIE LARUELLE *   GILLES PAGÈS †

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Abstract

This paper extends the link between stochastic approximation (SA) theory and randomized urn models developed in [26], and its applications to clinical trials introduced in [2, 3, 4]. The idea is that the drawing rule is no longer uniform among the balls of the urn (which contains d colors), but can be reinforced by a function f modeling in some sense aversion to risk. Firstly, by considering that f is concave or convex and by reformulating the dynamics of the urn composition as a standard stochastic approximation (SA) algorithm (with remainder), we derive the $a.s.$ convergence and the asymptotic normality (Central Limit Theorem, CLT) of the normalized procedure by calling upon the so-called ODE and SDE methods. An in-depth analysis of the case $d = 2$ exhibits two different behaviours: a single equilibrium point when f is concave, and when f is convex, a transition phase from a single to a system with two attracting equilibrium points and a repulsive one. The last setting is solved using results on noiseless traps in order to remove the repulsive point and to deduce the $a.s.$ towards one of the attractive point. Secondly, the special case of a Pólya urn ($i.e.$ when the addition rule the I_d matrix) is analyzed, still using result form (SA) about traps. Finally, these results are applied to functions with regular variation and optimal asset allocation in Finance.

Keywords *Stochastic approximation, extended Pólya urn models, reinforcement, non-homogeneous generating matrix, strong consistency, asymptotic normality, bandit algorithm.*

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1 Introduction

In this paper, we introduce and study in-depth a class of generalized Pólya urns (with d colors) characterized by their nonlinear drawing rules. These models appear as a generalization of randomized urn models originally devised for clinical trials which takes into account the risk aversion attitude of the agent. Randomized urn models have been extensively investigated by various authors (see [2, 3, 4]) for years based on *ad hoc* martingale arguments to solve $a.s.$ convergence as well as its rate of convergence. In a recent paper [26], we revisited, unified and often extended these results by showing that

*Université Paris-Est, Laboratoire d'Analyse et de Mathématiques Appliquées (UMR 8050), UPEMLV, UPEC, CNRS, F-94010, Créteil, France E-mail: `sophie.laruelle@u-pec.fr`. A part of this paper was designed in Laboratoire de Mathématiques Appliquées aux Systèmes, École Centrale de Paris, Grande voie des vignes, 92295 Châtenay-Malabry Cedex, France.

†Laboratoire de Probabilités et Modèles aléatoires, UMR 7599, Université Paris 6. E-mail: `gilles.pages@upmc.fr`

they can be proved by relying on the main results of Stochastic Approximation (SA) theory, especially $a.s.$ convergence and weak convergence rate (Central Limit Theorem (CLT)). We will again analysis these more demanding models by the mean of SA which will turn out to be very efficient for that purpose (see also [8] and the references therein). SA deals with the asymptotic behaviour of zero search stochastic recursive procedure and goes back to the seminal paper by Robbins & Monro in the 1950'. This theory has been developed extensively by many authors (see [22, 23, 9, 16, 17] and the references therein an overview and historical notes) and has been applied in various directions (Automatic Control, Mathematical Psychology, Artificial Neural Networks, Statistics, Stochastic Control, Numerical Probability).

Considering nonlinear drawing rules leads to dynamics for the (normalized) urn composition having several local attractors but also "parasitic" equilibrium points (where equilibrium point means zero of the mean function associated to the stochastic algorithm). This is a major difference with the linear case investigated in [26] since we will need this time to call upon the whole machinery of SA, namely "second order" results about noisy or noiseless non-attractive equilibrium (repellers, saddle points) also known as "traps" in the SA literature (see [12, 16, 25], see also [27, 29, 5]). Taking advantage of these trap results, we will establish the $a.s.$ convergence (strong consistency) even in the presence of multiple attractors prior to analysis the convergence rate of the normalized urn composition.

In this paper we both randomize a single urn in the sense that the addition rule matrix itself can be random (as introduced in [3, 4] and studied with SA theory in [26]) and investigate wide (non-parametric) classes of "convex" and "concave" drawing rules. $A.s.$ convergence of the normalized urn and of the drawing rule is proved, as well as the rate of convergence, either weak or strong depending on the structure of the updating rules. In the particular case of standard Pólya's urn (the addition rule matrix is equal to identity) but implemented with a "convex" drawing rule (for generalized Pólya urn, see for examples [20, 31, 15]), "noiseless traps" may appear ($i.e.$ unstable equilibrium vectors located at the boundary of the state space hence noiseless). To spruce whether they are parasitic, we developed a dedicated approach, close in spirit to that developed in [25] and [24] where the authors introduce a martingale and a kind of "oracle" inequality presented in Lemma 4.1. These results highlight the efficiency of SA theory, even in presence of noiseless repulsive equilibrium points.

Recently, a system of Pólya's urns with graph based interactions and a "power drawing" rule has also been investigated using SA techniques in [6, 14]. $A.s.$ convergence of the normalized urn composition is established.

Generalized Pólya Urn models (GPU) have been widely studied in the literature with different points of view: martingale method (see for example [19]), algebraic approach (see for example [30]), reinforcement process (see $e.g.$ [30]), branching process (see $e.g.$ [21]), stochastic approximation (see for example [8]). These models also have applications to many areas: biology, random walks, statistics, computer science, clinical trials, psychology, economics or finance for instance (see [31]).

In these adaptive models, the key point is the equation which governs the urn composition updating after each drawing. Basically, we will show that (a normalized version of) this urn composition can be formulated as a classical recursive stochastic algorithm with step $\gamma_n = \frac{1}{n+\mathrm{Tr}(Y_0)}$ where $\mathrm{Tr}(Y_0)$ denotes the number of balls in the urn at time 0. Doing so, we will be in position to first establish the $a.s.$ convergence of the procedure by calling upon the so-called Ordinary Differential Equation Method (ODE method) toward a finite set of equilibrium points (but usually not reduced to a single point). As a second step, we will rely on non-convergence results toward traps for SA (see [12, 16]) and on multiple targets (see [7, 16, 18]). As a third step, we entirely elucidate the rate of convergence (namely a CLT or an $a.s.$ rate) by using the Stochastic Differential Equation Method (SDE method, see $e.g.$ [17, 9]). The three main theoretical results from SA are recalled in a self-contained form in the

Appendix. Proofs of such results can be found in classical textbooks on SA ([9, 16, 17, 23]). As for the CLT, they go back to [22] and [11]. We will check again to what extent these general theorems are extremely powerful to solve these questions and spare tedious lengthy computations and somewhat repetitive proofs.

Let us be more precise on the urn model under consideration in this paper. We consider an urn containing balls of (at most) d different types. All random variables involved in the model are supposed to be defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Denote by $Y_0 = (Y_0^i)_{i=1,\dots,d} \in \mathbb{R}_+^d \setminus \{0\}$ the initial composition of the urn, where Y_0^i is the number of balls of type $i \in \{1, \dots, d\}$ (of course a more realistic though not mandatory assumption would be $Y_0 \in \mathbb{N}^d \setminus \{0\}$). The urn composition at draw n is denoted by $Y_n = (Y_n^i)_{i=1,\dots,d}$. At the n^{th} stage, one draws randomly (according to a law defined further on) a ball from the urn with instant replacement. If the drawing ball is of type j, then the urn composition is updated by adding D_n^{ij} balls of type $i \in \{1, \dots, d\}$. The procedure is then iterated. The urn composition at stage n, modeled by an $\mathbb{R}^d$-valued vector Y_n, satisfies the following recursive procedure between times $n-1$ and n:

$$Y_n = Y_{n-1} + D_n X_n, \quad n \geq 1, \quad Y_0 \in \mathbb{R}_+^d \setminus \{0\} \tag{1.1}$$

where $D_n = (D_n^{ij})_{1 \leq i,j \leq d}$ is the addition rule matrix and $X_n : (\Omega, \mathcal{A}, \mathbb{P}) \to \{e^1, \cdots, e^d\}$ models the color of the ball drawn at time n with $\{e^1, \cdots, e^d\}$ the canonical basis of $\mathbb{R}^d$ (e^j stands for ball of color j). We assume that there is no extinction *i.e.* $Y_n \in \mathbb{R}_+^d \setminus \{0\}$ *a.s.* for every $n \geq 1$: so is the case if all the entries D_n^{ij} are *a.s.* nonnegative (see [26]). The filtration of the model is defined by $\mathcal{F}_n = \sigma(Y_0, X_k, D_k, 1 \leq k \leq n)$, $n \geq 0$. Two types of drawing rules are considered and will be analyzed:

- *Normalized skewed empirical frequency (convex/concave function)*

$$\forall i \in \{1, \dots, d\}, \quad \mathbb{P}(X_{n+1} = e^i \mid \mathcal{F}_n) = \frac{f(Y_n^i/(n + \mathrm{Tr}(Y_0)))}{\sum_{j=1}^d f(Y_n^j/(n + \mathrm{Tr}(Y_0)))}, \quad n \geq 0, \tag{1.2}$$

  where $f : \mathbb{R}_+ \to \mathbb{R}_+$ will usually satisfy some convexity property.

- *Normalized skewed distribution (f with regular variations)*

$$\forall i \in \{1, \dots, d\}, \quad \mathbb{P}(X_{n+1} = e^i \mid \mathcal{F}_n) = \frac{f(Y_n^i)}{\sum_{j=1}^d f(Y_n^j)}, \quad n \geq 0, \tag{1.3}$$

  where $f : \mathbb{R}_+ \to \mathbb{R}_+$ will be assumed to have α-regular variations for some $\alpha > 0$ *i.e.* for every $t > 0$, $\frac{f(tx)}{f(x)} \underset{x \to +\infty}{\longrightarrow} t^\alpha$.

Moreover we will make the assumption that D_n and X_n are independent given $\mathcal{F}_{n-1}$ (see (**A2**) further on). Such a drawing procedure can be performed by using an i.i.d. sequence $(U_n)_{n\geq}$ of random variables with uniform distribution on the unit interval, independent of the sequence $(D_n)_{n\geq 1}$, to simulate the above conditional probabilities.

Let us remark that when $f = \mathrm{Id}_{\mathbb{R}_+}$ both updating rules coincide. Moreover, the regular variation case can be deduced from the convex one by noticing that, if f has regular variation and is bounded on each interval $(0, M]$, then $\frac{f(tx)}{f(x)} \underset{x\to\infty}{\longrightarrow} t^\alpha$ uniformly in t on each $(0, b]$ $(0 < b < \infty)$ if $\alpha > 0$ (see Theorem 1.5.2 p.22 in [10]), thus, if $\frac{Y_n}{n+\sum_{i=1}^d Y_0^i}$ lies in a compact, then

$$\max_{1\le i\le d}\left|\frac{f(Y_n^i)}{f(n+\sum_{i=1}^d Y_0^i)} - \left(\frac{Y_n}{n+\sum_{i=1}^d Y_0^i}\right)^\alpha\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

Then we will be able to apply the convex framework to the special function $x \mapsto x^\alpha$ (see Section 5.1).

The *generating matrices* are defined as the $\mathcal{F}_n$-compensator of the additions rule sequence *i.e.*

$$H_n = \left(\mathbb{E}\left[D_n^{ij} \,|\, \mathcal{F}_{n-1}\right]\right)_{1\le i,j\le d},\ n \ge 1. \tag{1.4}$$

We will also assume that the sequence of generating matrices *a.s.* converges toward a limit generating matrix denoted by H.

Other fields of application can be considered for such procedures like the adaptive asset allocation by an asset manager or a trader. One may also consider this type of procedure as a strategy to update the composition of a portfolio or even a whole fund, based on the (recent) past performances of the assets (see Section 5 for more details).

We will study randomized urn models (inspired from those introduced in [3]) but governed by the nonlinear drawing rules (1.2) and (1.3). We will consider random additional rule matrices but the limit generating matrix is a deterministic and co-stochastic (with a stochastic transpose). We will mostly analyze two opposite settings: when H is irreducible (in Section 3) and the case where $D_n = H_n = H = I_d$ (one adds a ball of the same color as the drawn one) which corresponds to the classical framework of Pólya urn (except for the reinforced drawing conditional distribution). This last setting, investigated in Section 4 leads to a "winner takes all" behaviour closely connected to bandit algorithm (as investigated in [25]) where the aim is to find the best arm. One difficulty is that the equilibria which lie on the boundary of the state space, hence being noiseless. When H is irreducible, the aim of the procedure applied to clinical tests is more to classify according to their performances the different treatments without excluding one of them: this is more related to cooperative/competitive systems.

As for the non-linear drawing rules, we will consider either that f satisfies a convexity/concavity property with the empirical frequencies as an argument (see (1.2)), or that f has regular variations with the composition of the urn as an argument, renormalized *a posteriori*. In the first setting, an in-depth analysis of the existence of equilibrium of the system and their attractiveness in full generality (d colors) spending on the convexity of f. Thus, if f is concave and H is bi-stochastic, $\frac{1}{d}\mathbf{1}$ is the only equilibrium and uniformly attracting so that the urn composition converges toward it. Otherwise, the situation is more involved and only partial result are established, including the *a.s.* non-convergence toward traps (repulsive or saddle equilibrium points).

When considering the case $d = 2$ we fully elucidate the dynamics of the urn: namely, when f is concave, we always have an attracting unique target, and when f is convex, we observe a phase transition from a setting with one attracting equilibrium to a setting with two attracting equilibrium points and a repulsive one depending on the exponent α when $f(y) = y^\alpha$, $\alpha > 0$. In both cases, we obtain *a.s.* convergence of the urn compositions toward an attractor of the system as well as convergence rate. It should be noted that three different convergence rate can occur.

Finally, we show that when f has regular variations with index $\alpha \in \mathbb{R}_+$, then the procedure behaves like $f_\alpha : y \mapsto y^\alpha$.

The paper is organized as follows. Section 2 presents the framework of skewed randomized urn models with the required assumptions on both the addition rule matrices and the generating matrices. After rewriting the dynamics of the urn composition as an SA procedure in Section 2.2, we analyze in Section 2.4 the equilibrium points and their stability for the associated ODE when the reinforced drawing rule is convex/concave. An in-depth analysis of the bi-dimensional case is proposed in Section 3. We exhibit two kinds of behaviours: either a unique stable equilibrium point when f is concave, or a single, two or three ones with two attractive and one repulsive points when f is convex. By calling upon result on traps in SA, we prove the $a.s.$ convergence towards one of the attractive equilibrium points; then we derive from SDE method all the possible rates of convergence. In Section 4, we study the case of a classical Pólya urn, *i.e.* the addition rule matrix equals to identity. We use results derive from the bandit algorithm to prove the convergence towards the target and the non-convergence towards traps. Finally, Section 5 introduces examples of application of such recursive procedures to drawing rule with regular variation and to portfolio allocation.

Notations. For $u = (u^i)_{i=1,\dots,d} \in \mathbb{R}^d$, $\|u\|$ denotes the canonical Euclidean norm of the column vector u on $\mathbb{R}^d$, $\mathrm{Tr}(u) = \sum_{k=1}^d u^k$ denotes its "weight", u^t denotes its transpose; $|||A|||$ denotes the operator norm of the matrix $A \in \mathcal{M}_{d,q}(\mathbb{R})$ with d rows and q columns with respect to the canonical Euclidean norm. When $d=q$, $\mathrm{Sp}(A)$ denotes the set of eigenvalues of A. $\mathbf{1} = (1 \cdots 1)^t$ denotes the unit column vector in $\mathbb{R}^d$, I_d denotes the $d \times d$ identity matrix, $\mathrm{diag}(u) = [\delta_{ij} u_i]_{1\le i,j\le d}$, where δ_{ij} is the Kronecker symbol and $\mathcal{S}_d = \left\{ u \in \mathbb{R}^d_+ : \sum_{i=1}^d u^i = 1 \right\}$ denotes the canonical simplex. $\mathcal{E}_d = \{y \in \mathcal{S}_d : h(y) = 0\}$ denotes the set of zeros of h.

2 Skewed randomized urn models

With the notations and definitions described in the introduction, we are in position to formulate the main assumptions needed to establish the $a.s.$ convergence of the urn composition.

$$(\mathbf{A1}) \equiv \begin{cases} (i) & \textit{Addition rule matrix:} \text{ For every } n \ge 1, \text{ the matrix } D_n \ a.s. \text{ has nonnegative entries.} \\ (ii) & \textit{Generating matrix:} \text{ For every } n \ge 1, \text{ the generating matrices } H_n = (H_n^{ij})_{1\le i,j\le d} \ a.s. \text{ satisfies} \\ & \qquad \forall\, j \in \{1,\dots,d\}, \quad \displaystyle\sum_{i=1}^d H_n^{ij} = c > 0. \\ (iii) & \textit{Starting value:} \text{ The starting urn composition vector } Y_0 \in \mathbb{R}^d_+ \setminus \{0\}. \end{cases}$$

The constant c is known as the balance of the urn. In fact, we may assume without loss of generality, up to a renormalization of Y_n, that $c = 1$. If we set for every $n \ge 0$, $\widehat{Y}_n = \frac{Y_n}{c}$ and $\widehat{D}_{n+1} = \frac{D_{n+1}}{c}$, then the couple $(\widehat{Y}_n, \widehat{D}_{n+1})_{n\ge 0}$, formally satisfies the dynamics (1.1), namely

$$\widehat{Y}_n = \widehat{Y}_{n-1} + \widehat{D}_n X_n, \quad n \ge 1, \quad \widehat{Y}_0 \in \mathbb{R}^d_+ \setminus \{0\},$$

whereas $\widehat{H}_n = \mathbb{E}[\widehat{D}_n | \mathcal{F}_{n-1}]$ satisfies now $(\mathbf{A1})$-(iii) with $c = 1$ *i.e.* $\widehat{H}_n$ is *co-stochastic* *i.e.* the transpose of a stochastic matrix.

From now on, throughout the paper, we will consider this normalized balanced version still denoted by Y_n and D_n for convenience.

(**A2**) The addition rule D_n and the drawing procedure X_n are conditionally independent given $\mathcal{F}_{n-1}$ and satisfy

$$\begin{aligned} &\forall j \in \{1,\ldots,d\}, \quad \sup_{n\geq 1} \mathbb{E}\left[\left\|D_n^{\cdot j}\right\|^2 \mid \mathcal{F}_{n-1}\right] < +\infty \quad a.s. \\ \Longleftrightarrow\ &\forall i,j \in \{1,\ldots,d\}, \quad \sup_{n\geq 1} \mathbb{E}\left[(D_n^{ij})^2 \mid \mathcal{F}_{n-1}\right] < +\infty \quad a.s. \end{aligned} \tag{2.5}$$

where $D_n^{\cdot j} = (D_n^{ij})_{i=1,\ldots,d}$ (column vector).

(**A3**) Assume that there exists an irreducible $d \times d$ matrix H (with nonnegative entries) such that

$$H_n \xrightarrow[n\to\infty]{a.s.} H \quad \text{and} \quad \sum_{n\geq 1} |||H_n - H|||^2 < +\infty \quad a.s. \tag{2.6}$$

H is called the *limit generating matrix*.

2.1 Convex or concave skewed drawing rule functions

We introduce a skewed drawing rule as follows

$$\forall i \in \{1,\ldots,d\}, \quad \mathbb{P}(X_{n+1} = e^i \mid \mathcal{F}_n) = \frac{f(\widetilde{Y}_n^i)}{\sum_{j=1}^d f(\widetilde{Y}_n^j)}, \quad n \geq 1, \tag{2.7}$$

where $\widetilde{Y}_n = \dfrac{Y_n}{n + \mathrm{Tr}(Y_0)}$, $n \geq 0$, and f is a non-decreasing *convex (or concave) function* satisfying $f(0) = 0$ and $f(1) = 1$. The drawing rule is called *skewed* as soon as $f \not\equiv Id_{[0,1]}$. We use this renormalization since

$$\forall n \geq 0, \quad \mathbb{E}\left[\mathrm{Tr}(Y_n)\right] = n + \mathrm{Tr}(Y_0)$$

so that $\mathbb{E}[\widetilde{Y}_n]$ lies in the simplex $\mathcal{S}_d$. Therefore this is a natural way to normalize the urn composition vector.

2.2 Representation as a stochastic algorithm

The starting point of an approach like in [26] is to reformulate the dynamics (1.1)-(1.2) as a recursive stochastic algorithm in order to take advantage of results from Stochastic Approximation Theory to elucidate the asymptotic properties ($a.s.$ convergence) of both the urn composition and the ball allocation. To do so, we start from (1.1) with $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$. For every $n \geq 0$, we note that

$$Y_{n+1} = Y_n + D_{n+1}X_{n+1} = Y_n + \mathbb{E}\left[D_{n+1}X_{n+1} \mid \mathcal{F}_n\right] + \Delta M_{n+1}, \tag{2.8}$$

where

$$\Delta M_{n+1} := D_{n+1}X_{n+1} - \mathbb{E}\left[D_{n+1}X_{n+1} \mid \mathcal{F}_n\right] \tag{2.9}$$

is an $\mathcal{F}_n$-local martingale increment (integrability follows from (**A2**)). By the definition of the generating matrix H_n, we have, owing to assumption (**A2**),

$$\begin{aligned} \mathbb{E}\left[D_{n+1}X_{n+1} \mid \mathcal{F}_n\right] &= \sum_{i=1}^d \mathbb{E}\left[D_{n+1}\mathbb{1}_{\{X_{n+1}=e^i\}}, \mid \mathcal{F}_n\right] = \sum_{i=1}^d \mathbb{E}\left[D_{n+1} \mid \mathcal{F}_n\right]\mathbb{P}\left(X_{n+1} = e^i \mid \mathcal{F}_n\right) e^i \\ &= H_{n+1}\sum_{i=1}^d \frac{f(\widetilde{Y}_n^i)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} e^i = H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} \end{aligned}$$

where

$$\widetilde{f}((y^1,\ldots,y^d)^t) = \big(f(y^i)\big)_{1\le i\le d} \in [0,1]^d$$

is a column vector, so that

$$Y_{n+1} = Y_n + H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} + \Delta M_{n+1}. \tag{2.10}$$

Now we can derive a stochastic approximation for the normalized urn composition Y_n. First we have for every $n \ge 0$,

$$\frac{Y_{n+1}}{n+1+\mathrm{Tr}(Y_0)} = \frac{Y_n}{n+\mathrm{Tr}(Y_0)} + \frac{1}{n+1+\mathrm{Tr}(Y_0)}\left(H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} - \frac{Y_n}{n+\mathrm{Tr}(Y_0)}\right) + \frac{\Delta M_{n+1}}{n+1+\mathrm{Tr}(Y_0)}. \tag{2.11}$$

Consequently, the $\mathcal{S}_d$-valued sequence $\widetilde{Y}_n = \dfrac{Y_n}{n+\mathrm{Tr}(Y_0)}$, $n \ge 0$, satisfies the canonical recursive stochastic approximation procedure

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n + \frac{1}{n+1+\mathrm{Tr}(Y_0)}\left(H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} - \widetilde{Y}_n\right) + \frac{1}{n+1+\mathrm{Tr}(Y_0)}\Delta M_{n+1} \tag{2.12}$$

or equivalently

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \gamma_{n+1}h(\widetilde{Y}_n) + \gamma_{n+1}\left(\Delta M_{n+1} + r_{n+1}\right) \tag{2.13}$$

where $h : \mathbb{R}^d \setminus \{0\} \mapsto \mathbb{R}^d$ is the mean field function of the procedure defined by

$$h(y) = \mathbb{E}\left[\widetilde{Y}_n - H\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} \mid \widetilde{Y}_n = y\right] = y - H\frac{\widetilde{f}(y)}{\mathrm{Tr}(\widetilde{f}(y))}, \tag{2.14}$$

$\gamma_n = \frac{1}{n+\mathrm{Tr}(Y_0)}$ is its step parameter and

$$r_{n+1} := (H_{n+1} - H)\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} \tag{2.15}$$

is its ($\mathcal{F}_n$-adapted) remainder term. Note that h can be canonically extended to $\mathbb{R}^d\setminus\{0\}$.

2.3 Composition boundedness

Our first task is to establish the boundedness off the sequence $(\widetilde{Y}_n)_{n\ge 0}$. By summing up the components of $\widetilde{Y}_n$ in (2.10) we obtain

$$\mathrm{Tr}(Y_{n+1}) = \mathrm{Tr}(Y_n) + \frac{\mathrm{Tr}(H_{n+1}\widetilde{f}(\widetilde{Y}_n))}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} + \mathrm{Tr}(\Delta M_{n+1}).$$

Using that the transpose of the generating matrix H_{n+1} is a stochastic matrix by (**A1**)-(i) (with $c = 1$), we obtain

$$\mathrm{Tr}(H_{n+1}\widetilde{f}(\widetilde{Y}_n)) = \sum_{i=1}^{d}(H_{n+1}\widetilde{f}(\widetilde{Y}_n))_i = \sum_{i=1}^{d}\sum_{j=1}^{d}H_{n+1}^{ij}f(\widetilde{Y}_n^j) = \sum_{j=1}^{d}\left(\sum_{i=1}^{d}H_{n+1}^{ij}\right)f(\widetilde{Y}_n^j) = \mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n)).$$

Consequently, for every $n \geq 0$,

$$\mathrm{Tr}(Y_{n+1}) = \mathrm{Tr}(Y_n) + 1 + \mathrm{Tr}(\Delta M_{n+1}). \tag{2.16}$$

Set $N_n := \sum_{k=1}^n X_k$, $n \geq 1$, $N_0 = 0$. Then we have, for every $n \geq 0$,

$$\begin{aligned} N_{n+1} &= N_n + X_{n+1} = N_n + \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} + \Delta \widetilde{M}_{n+1}, \\ \text{where } \Delta \widetilde{M}_{n+1} &:= X_{n+1} - \mathbb{E}\left[X_{n+1} \mid \mathcal{F}_n\right] = X_{n+1} - \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} \end{aligned}$$

is a true $\mathcal{F}_n$-martingale increment. Thus, for $\widetilde{N}_n := \dfrac{N_n}{n}$ we have, still for every $n \geq 0$,

$$\widetilde{N}_{n+1} = \widetilde{N}_n - \frac{1}{n+1}\left(\widetilde{N}_n - \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right) + \frac{1}{n+1}\Delta \widetilde{M}_{n+1}.$$

Proposition 2.1. *Let $(Y_n)_{n\geq 0}$ be the urn composition sequence defined by (1.1)-(1.2).*
(a) Under the assumptions **(A1)** *and* **(A2)**,

$$\frac{\mathrm{Tr}(Y_n)}{n + \mathrm{Tr}(Y_0)} \xrightarrow[n\to\infty]{a.s.} 1.$$

(b) If the addition rule matrices D_n themselves satisfy **(A1)**-*(ii) (with $c = 1$), then* $\mathrm{Tr}(Y_n) = n + \mathrm{Tr}(Y_0)$, *therefore the sequence $(\widetilde{Y}_n)_{n\geq 0}$ lies in the simplex $\mathcal{S}_d$.*

Proof. (a) We derive from the identity

$$D_{n+1}X_{n+1} = \sum_{j=1}^{d} D_{n+1}^{\cdot j} \mathbb{1}_{\{X_{n+1}=e^j\}}, \quad n \geq 0,$$

that

$$\|D_{n+1}X_{n+1}\|^2 = \sum_{j=1}^{d} \left\|D_{n+1}^{\cdot j}\right\|^2 \mathbb{1}_{\{X_{n+1}=e^j\}}.$$

Consequently, owing to (**A2**),

$$\begin{aligned} \mathbb{E}\left[\|D_{n+1}X_{n+1}\|^2 \mid \mathcal{F}_n\right] &= \sum_{j=1}^{d} \mathbb{E}\left[\left\|D_{n+1}^{\cdot j}\right\|^2 \mid \mathcal{F}_n\right] \mathbb{P}\left(X_{n+1} = e^j \mid \mathcal{F}_n\right) \\ &\leq \sup_{n\geq 0}\sup_{1\leq j\leq d} \mathbb{E}\left[\left\|D_{n+1}^{\cdot j}\right\|^2 \mid \mathcal{F}_n\right] < +\infty \quad a.s. \end{aligned}$$

Consequently $\sup_{n\geq 1} \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n\right] < +\infty$ $a.s.$ Therefore, thanks to the strong law of large numbers for conditionally L^2-bounded local martingale increments, we have $\frac{M_n}{n} \underset{n\to\infty}{\longrightarrow} 0$ $a.s.$. Consequently it follows from (2.16) that

$$\frac{\mathrm{Tr}(Y_n)}{n + \mathrm{Tr}(Y_0)} = 1 + \frac{\mathrm{Tr}(M_n)}{n + \mathrm{Tr}(Y_0)} \xrightarrow[n\to+\infty]{a.s.} 1.$$

(b) In this case $\mathrm{Tr}(M_n) = 0$, consequently for every $n \geq 0$, $\mathrm{Tr}(\widetilde{Y}_n) = 1$. □

2.4 Existence of equilibrium points

The procedure (2.13) appears in its canonical form as recursive is a recursive zero search algorithm of the mean field function $h : \mathcal{S}_d \mapsto \mathbb{R}^d$. In what follows we will use the following obvious property

$$\mathcal{S}_d = \mathrm{Tr}^{-1}\{1\} = \Big\{y \in \mathbb{R}_+^d \,|\, \mathrm{Tr}(y) = 1\Big\}.$$

Since the components of $\widetilde{Y}_n = \frac{Y_n}{n+\mathrm{Tr}(Y_0)}$ are nonnegative by construction and $\mathrm{Tr}(\widetilde{Y}_n) = \frac{\mathrm{Tr}(Y_n)}{n+\mathrm{Tr}(Y_0)} \underset{n\to\infty}{\longrightarrow} 1$ *a.s.*, it is clear that $\mathbb{P}(d\omega)$-*a.s.*, the sequence $(\widetilde{Y}_n)_{n\geq 0}$ is bounded and that the set $\mathcal{Y}_\infty(\omega)$ of all its limiting value is contained in the simplex. Consequently, we search for equilibrium points $y \in \mathcal{S}_d$ such that $h(y) = 0$.

Proposition 2.2. (a) *Assume they H is co-stochastic. The function $\varphi_H : y \mapsto H\frac{\widetilde{f}(y)}{\mathrm{Tr}(\widetilde{f}(y))}$ is continuous and $\mathcal{S}_d$-valued. Therefore, by Brouwer's theorem, it has at least one fixed point or, equivalently, h has at least one zero $y^* \in \mathcal{S}_d$.*
(b) *If, furthermore, for every $i, j \in \{1, \ldots, d\}$, $H^{ij} > 0$, then for every $i \in \{1, \ldots, d\}$, $\min_j H^{ij} \leq y^{*i} \leq \max_j H^{ij}$.*

Proof. (a) The functions $y \mapsto \widetilde{f}(y)$ and $y \mapsto \mathrm{Tr}(\widetilde{f}(y))$ are continuous on $\mathcal{S}_d$ and $\mathrm{Tr}(\widetilde{f}(y)) \geq f\left(\frac{1}{d}\right) > 0$ because if $y = (y^1, \ldots, y^d)^t \in \mathcal{S}_d$, there exists $i_0 \in \{1, \ldots, d\}$ such that $y^{i_0} \geq \frac{1}{d}$ so $\mathrm{Tr}(\widetilde{f}(y)) \geq f(y^{i_0}) \geq f\left(\frac{1}{d}\right) > 0$. Therefore, $y \mapsto \frac{\widetilde{f}(y)}{\mathrm{Tr}(\widetilde{f}(y))}$ is continuous on $\mathcal{S}_d$ so that h is continuous on $\mathcal{S}_d$. Then, by Brouwer's theorem, h has at least one zero.
(b) It follows from $\displaystyle\sum_{j=1}^d H^{ij}\frac{f(y^{*,j})}{\mathrm{Tr}(\widetilde{f}(y^*))} = y^{*,i}$, $i\in \{1, \ldots, d\}$, that $\displaystyle\min_j H^{ij} \leq y^{*i} \leq \max_j H^{ij}$. □

Proposition 2.3. *Let $\mathcal{E}_d = \big\{y \in \mathcal{S}_d : h(y) = 0\big\}$ denote the set of zeros of h.*
(a) *If H is bi-stochastic, then $\frac{1}{d}\mathbf{1} \in \mathcal{E}_d$.*
(b) *If $H = I_d$, then $\big\{\widetilde{e}_I,\, I \subset \{1, \ldots, d\}, I \neq \emptyset\big\} \subset \mathcal{E}_d$, where $\widetilde{e}_I = \frac{1}{|I|}\sum_{i\in I} e^i$, $(e^i)_{1\leq i\leq d}$ is the canonical basis of $\mathbb{R}^d$.*
(c) *If $H = I_d$ and f is strictly convex (or concave), then*

$$\mathcal{E}_d = \big\{\widetilde{e}_I, I \subset \{1, \ldots, d\}, I \neq \emptyset\big\}.$$

(d) *If H is bi-stochastic and irreducible, then $\mathcal{E}_d \subset \mathring{\mathcal{S}}_d = \Big\{y \in (0,1)^d : \sum_{i=1}^d y^i = 1\Big\}$.*
(e) *If furthermore, f is strictly concave, then $\mathcal{E}_d = \big\{\frac{1}{d}\mathbf{1}\big\}$.*

Proof. (a) Set $y(d) = \frac{1}{d}\mathbf{1} \in \mathcal{S}_d$. Thus $\frac{f(\frac{1}{d})}{d\frac{1}{d}} = \frac{1}{d}$ since $f\left(\frac{1}{d}\right) > 0$ and consequently $\displaystyle\sum_{j=1}^d H^{ij}\frac{1}{d} = 1 \times \frac{1}{d}$ for every $i\in \{1, \ldots, d\}$, therefore $h\big(y(d)\big) = y(d)$.
(b) Let $I \neq \emptyset$. Then

$$f(\widetilde{e}_I^i) = \begin{cases} 0 & \text{if} \quad i \notin I \\ f\left(\frac{1}{|I|}\right) & \text{if} \quad i \in I \end{cases}.$$

Thus $\mathrm{Tr}(\widetilde{f}(\widetilde{e}_I)) = |I| f\left(\frac{1}{|I|}\right)$ as well so that $h(\widetilde{e}_I) = 0$.

(c) Let $y^* \in \mathcal{E}_d$. Assume that there exists y^{*,i_0}, y^{*,i_1} such that $0 < y^{*,i_0} < y^{*,i_1} \le 1$. Then $y^{*,i_0} = \frac{f(y^{*,i_0})}{\mathrm{Tr}(\widetilde{f}(y^*))}$ and $y^{*,i_1} = \frac{f(y^{*,i_1})}{\mathrm{Tr}(\widetilde{f}(y^*))}$, so that $\frac{f(y^{*,i_0})}{y^{*,i_0}} = \frac{f(y^{*,i_1})}{y^{*,i_1}} = \mathrm{Tr}(\widetilde{f}(y^*))$. If $\pm f$ is strictly convex, $\xi \mapsto \frac{f(\xi)}{\xi}$ is strictly monotonic which yields to a contradiction.

As a consequence, $y^{*,i} \in \{0, \xi_0\}$, $\xi_0 \in (0,1]$ so that if $I_{\xi_0} = \{i : y^{*,i} = \xi_0\}$, $\mathrm{Tr}(\widetilde{f}(y^*)) = |I_{\xi_0}| f(\xi_0)$ and, for every $i \in I_{\xi_0}$, $y^{*,i} = \xi_0$ and $\frac{f(y^{*,i})}{\mathrm{Tr}(\widetilde{f}(y^*))} = \frac{f(\xi_0)}{|I_{\xi_0}| f(\xi_0)} = \frac{1}{|I_{\xi_0}|}$, *i.e.* $h(y^*) = \frac{1}{|I_{\xi_0}|}\sum_{i=1}^d e^i$.

(d) Let i_0 be such that $y^{*,i_0} = \min_i y^{*,i}$. If $y^{*,i_0} > 0$, then $y^* \in \overset{\circ}{\mathcal{S}}_d$. Assume there exists $i \in \{1, \ldots, d\}$ such that $y^{*,i} = 0$. Then

$$\sum_{j=1}^d H^{ij} \frac{f(y^{*,j})}{\mathrm{Tr}(\widetilde{f}(y^*))} = 0.$$

Hence, for every i such that $y^{*,i} = 0$, for every j such that $y^{*,j} \neq 0$, $H^{ij} = 0$ since $f(y^{*,j}) > 0$. This contradicts the irreducibility of H since $\{1, \ldots, d\}$ can be decomposed into two absorbing classes.

(e) We know that $\frac{1}{d}\mathbf{1} \in \mathcal{E}_d$ and that for any $y^* \in \mathcal{E}_d$, $\min_i y^{*,i} > 0$. Let i_0 such that $y^{*,i_0} = \min_i y^{*,i}$. Then

$$\sum_{j=1}^d H^{i_0 j} f(y^{*,j}) \ge \sum_{j=1}^d H^{i_0 j} f(y^{*,i_0}) = f(y^{*,i_0}).$$

On the other hand, using the concavity of f, we derive that

$$\forall y \in \mathcal{S}_d, \quad \mathrm{Tr}(\widetilde{f}(y)) = d\sum_{j=1}^d \frac{1}{d} f(y_j) \le d\, f\left(\frac{1}{d}\sum_{j=1}^d y_j\right) = d\, f\left(\frac{1}{d}\right)$$

with equality if and only if $y = y(d) = \frac{1}{d}\mathbf{1}$ owing to the strict concavity of f.

As a consequence

$$y^{*,i_0} = \frac{\sum_{j=1}^d H^{i_0 j} f(y^{*,j})}{\mathrm{Tr}(\widetilde{f}(y^*))} \ge \frac{\sum_{j=1}^d H^{i_0 j} f(y^{*,j})}{df\left(\frac{1}{d}\right)} = \frac{f(y^{*,i_0})}{df\left(\frac{1}{d}\right)},$$

which can be rewritten as $\frac{f(y^{*,i_0})}{y^{*,i_0}} \le \frac{f(1/d)}{1/d}$. As $\xi \mapsto \frac{f(\xi)}{\xi}$ is (strictly) decreasing, it implies that $y^{*,i_0} \ge \frac{1}{d}$ which in turn implies that $y^* = \frac{1}{d}\mathbf{1}$ since since $y^* \in \mathcal{S}_d$. ☐

Remark. When H is not bi-stochastic but simply co-stochastic, we have no closed form for an $\overset{\circ}{\mathcal{S}}_d$-valued equilibrium and we could not manage to establish uniqueness even if f is (strictly) concave.

Proposition 2.4. *If f is differentiable in the neighbourhood of $\frac{1}{d}$, then by an obvious extension of $\widetilde{f}$ and* Tr*, the function $\varphi(y) = \frac{\widetilde{f}(y)}{\mathrm{Tr}(\widetilde{f}(y))}$ can be extended on $[0,1]^d \setminus \{0\}$ into a differentiable function on $(0,1)^d$. Its Jacobian at $y(d) := \frac{1}{d}\mathbf{1}$ is given by*

$$J_\varphi(y(d)) = \begin{pmatrix} a & b & \ldots & \ldots & b \\ b & a & b & \ldots & b \\ \vdots & b & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & b \\ b & b & \ldots & b & a \end{pmatrix}, \quad a = \frac{f'(1/d)(d-1)}{d^2 f(1/d)} \quad \textit{and} \quad b = -\frac{f'(1/d)}{d^2 f(1/d)} = -\frac{a}{d-1}.$$

As a symmetric matrix, $J_\varphi(y(d))$ is diagonalizable in the orthogonal group with eigenvalues 0 (associated to the eigenvector $\mathbf{1}$) and $\frac{f'(1/d)}{df(1/d)}$ on the hyperplane $\mathbf{1}^\perp = \left\{u \in \mathbb{R}^d : \sum_{i=1}^d u^i = 0\right\}$.

Proof. If $y \in (0,1)^d$ and f is differentiable in the neighbourhood of $y^1, \ldots, y^d$, then φ is differentiable at y and, for every real numbers a, b,

$$\frac{\partial \varphi^i}{\partial y^j}(y) = -\frac{f(y^i)f'(y^j)}{\mathrm{Tr}(\widetilde{f}(y))^2} + \delta_{ij}\frac{f'(y^i)}{\mathrm{Tr}(\widetilde{f}(y))}.$$

The form of $J_\varphi(y(d))$ follows. Elementary and classical computations show that

$$\begin{vmatrix} a & & b \\ & \ddots & \\ b & & a \end{vmatrix} = (a-b)^{d-1}(a+b(d-1)),$$

which in turn implies that the characteristic polynomial of $J_\varphi(y(d))$ is given by $(a-b-\lambda)^{d-1}(a+b(d-1)-\lambda)$ so that the eigenvalues are

$$\lambda_0 = a + b(d-1) = 0 \text{ (order 1)} \quad \text{ and } \quad \lambda_1 = a - b = \frac{f'(1/d)}{df(1/d)} \text{ (order } d-1).$$

The eigenvector associated to λ_0 is clearly $\mathbf{1}$ and the eigenspace associated to λ_1 is $\mathbf{1}^\perp = \left\{u \in \mathbb{R}^d : \sum_{i=1}^d u^i = 0\right\}$, so that $J_\varphi(y(d))_{|\mathbf{1}^\perp} = \lambda_1 I_{|\mathbf{1}^\perp}$. $\square$

Before carrying on with the next proposition, we recall that a co-stochastic matrix has 1 as an eigenvalue of order 1 with eigenspace E_1, then $\mathbf{1}^\perp$ is left stable by H so that $\mathbb{R}^d = E_1 \oplus \mathbf{1}^\perp$. Moreover, note that if H is irreducible then $1 \notin \mathrm{Sp}(H_{|\mathbf{1}^\perp})$ owing to the Perron-Frobenius Theorem.

Remark. By attractor (or uniformly stable equilibrium) we mean that the flow of $ODE_h \equiv \dot{y} = -h(y)$ on $\mathcal{S}_d$ converges to $y(d) = \frac{1}{d}\mathbf{1}$ as $t \to \infty$ uniformly with respect to y in a (compact) neighbourhood of $y(d)$. Since h is differentiable, this holds true if all eigenvalues of $J_h(y(d))_{\mathbf{1}^\perp} = J_\varphi(y(d))_{\mathbf{1}^\perp} - I\mathbf{1}^\perp$ have (strictly) positive real parts, see [5]. If one of these eigenvalues has a negative real part then the equilibrium is unstable and if all eigenvalues have negative real parts the equilibrium is called a *repeller*.

Proposition 2.5. *Let H be a bi-stochastic matrix (so that $y(d) = \frac{1}{d}\mathbf{1}$ is a zero of h) such that $\mathrm{Sp}(H) \subset B_{\mathbb{C}}(0,1)$ (note that this second assumption is always satisfied if H is irreducible or if $H = I_d$). Let $\lambda_1 = \frac{f'(1/d)}{df(1/d)}$ and let $\mu_{\max}$ be the eigenvalue of $H_{|\mathbf{1}^\perp}$ with the highest real part.*

(a) If $\Re e(\mu_{\max}) < \frac{1}{\lambda_1}$ then $y(d)$ is an attractor of $ODE_h \equiv \dot{y} = -h(y)$ and if $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$ then $y(d)$ is unstable (even a repeller when $H = I_d$).

(b) Thus, if

$$\Big(\lambda_1 \le 1 \text{ and } 1 \notin \mathrm{Sp}(H_{|\mathbf{1}^\perp})\Big) \text{ or } (\lambda_1 < 1)$$

then $y(d)$ is always an attractor of the $ODE_h \equiv \dot{y} = -h(y)$.

Note that the first assumption is satisfied when H is irreducible and f concave whereas the second is satisfied as soon as f is strictly concave.

Remark • If $\lambda_1 > 1$ (*e.g.* since f is strictly convex), then $\frac{1}{\lambda_1} < 1$ and the two opposite situations $\Re e(\mu_{\max}) < \frac{1}{\lambda_1}$ and $\Re e(\mu_{\max}) > \frac{1}{\lambda_1}$ may occur *i.e.* $y(d)$ may switch from uniform stability to instability.
• Note that if f is strictly convex and $1\in \mathrm{Sp}(H_{|\mathbf{1}^\perp})$, then $y(d)$ is always unstable.

Proof. (a) Having in mind the above background, we derive from Proposition 2.4 that

$$\begin{aligned} Dh(y(d))_{|\mathbf{1}^\perp} &= D(I_d - H\varphi)(y(d))_{|\mathbf{1}^\perp} = I_{d|\mathbf{1}^\perp} - \big(HD\varphi(y(d))\big)_{|\mathbf{1}^\perp} \\ &= I_{d|\mathbf{1}^\perp} - H\lambda_1 I_{d|\mathbf{1}^\perp} = (I_d - \lambda_1 H)_{|\mathbf{1}^\perp}, \end{aligned}$$

so that $\mathrm{Sp}(Dh(y(d))_{\mathbf{1}^\perp}) = \{1 - \lambda_1\mu, \mu \in \mathrm{Sp}(H_{\mathbf{1}^\perp})\} \subset \mathbb{C}$.

Every $\mu\in \mathrm{Sp}(H)$ satisfies $|\mu| \le 1$. If $\lambda_1 < 1$ then $|\lambda_1\mu_{\max}| < 1$ and consequently $\Re e(1-\lambda_1\mu_{\max}) > 0$ which ensures the attractiveness of $y(d)$. The other case follows likewise.
(b) is obvious. Note that if f is convex $f(1/d) \le f(0) + f'(1/d)/d$ so that $\lambda_1 \ge 1$. $\square$

Proposition 2.6. (a) *If H is bi-stochastic irreducible and f strictly concave, then the flow of $ODE_h \equiv \dot y = -h(y)$, denoted by $(y(\xi,t))_{\xi\in\mathcal{S}_d, t\ge 0}$ is $\mathcal{S}_d$-valued and converges toward $\frac{\mathbf{1}}{d}$ (here $\mathcal{E}_d = \{\frac{\mathbf{1}}{d}\}$).*
(b) *If H is simply bi-stochastic (and possibly not irreducible) the flow $y(\xi,t)$ converges toward $y(d)$ for every $\xi\in \overset{\circ}{\mathcal{S}}_d$.*

Proof. (a) Let $\xi \in \mathcal{S}_d\backslash\{y(d)\}$. First note (by adding up all the components of $y(\xi,t)$ and using that H is co-stochastic, that for very $t \ge 0$, $\sum_j y^i(\xi,t) = 1$. Now let $i(t)$ such that $t \mapsto i(t)$ is càd and $y^{i(t)}(\xi,t) = \min_j y^j(\xi,t) \in [0, \frac{1}{d}]$. We know that $y^{i(0)}(\xi,0) = \xi^{i(0)} < \frac{1}{d}$ (since $\xi \neq y(d)$). Note that $t \mapsto y^{i(t)}(\xi,t)$ is continuous and right differentiable. Then

$$\left(\frac{\partial}{\partial t}\right)_r y^{i(t)}(\xi,t) = \sum_{j=1}^d H^{i(t)j}\frac{f(y^j(\xi,t))}{\mathrm{Tr}(\widetilde{f}(y(\xi,t)))} - y^{i(t)}(\xi,t) \ge 1 \times \frac{f(y^{i(t)}(\xi,t))}{\mathrm{Tr}(\widetilde{f}(y(\xi,t)))} - y^{i(t)}(\xi,t).$$

As $\mathrm{Tr}(\widetilde{f}(y(\xi,t))) \le df\left(\frac{1}{d}\right)$, it follows that

$$\left(\frac{\partial}{\partial t}\right)_r y^{i(t)}(\xi,t) \ge \frac{f(y^{i(t)}(\xi,t))}{df\left(\frac{1}{d}\right)} - y^{i(t)}(\xi,t) \ge 0 \tag{2.17}$$

since $u \mapsto \frac{f(u)}{u}$ is decreasing by strict concavity and $y^{i(t)}(\xi,t) \le \frac{1}{d}$.

If $\xi^{i(0)} = 0$, then $y^{i(t)}(\xi,t) \ge 0$ for every $t \ge 0$. Assume there exists $\varepsilon_0 > 0$ such that $y^{i(t)}(\xi,t) = 0$ for every $t\in (0,\varepsilon_0]$, then, one derives from the integrated form of the ODE_h that

$$\int_0^{\varepsilon_0}\left(\sum_{j=1}^d H^{i(t)j}\frac{f(y^j(\xi,s))}{\mathrm{Tr}(\widetilde{f}(y(\xi,s)))}\right)ds = 0,$$

so that, as the above integrand is nonnegative and continuous, $s \mapsto \sum_{j=1}^d H^{i(s)j}\frac{f(y^j(\xi,s))}{\mathrm{Tr}(\widetilde{f}(y(\xi,s)))} \equiv 0$ on $[0,\varepsilon_0]$. Let $I_0 = \{i \text{ s.t. } y^{i(t)}(\xi,\varepsilon_0) = 0,\ 1 \le i \le d\}$ and $I_1 = I_0^c$. Then, it follows that $i(\varepsilon_0)\in I_0$ and $H^{i(\varepsilon_0)j} = 0$ for every $j\in I_1$ which is not empty since $\sum_j y^i(\xi,t) = 1$. Hence $i(\varepsilon_0)$ (and more generally I_0) are not H-connected which contradicts the irreducibility. Consequently, $y^{(i(t)}(\xi,t) > 0$ on $(0,\varepsilon_0]$ or equivalently that $y(\xi,t)$ lives in $\overset{\circ}{\mathcal{S}}_d$ and consequently for every $t \ge 0$ (by monotony of $y^{i(t)}(\xi,t)$).

If $\xi^{i(0)} > 0$, the same conclusion follows simply form the monotony of of $y^{i(t)}(\xi, t)$.

Finally for every $t > 0$, $y^{i(t)}(\xi, t)$ is positive and increasing. We can integrate (2.17) between a fixed $\varepsilon > 0$ and $t > \varepsilon$ as follows:

$$-\log\left(\frac{d}{\xi^{i(0)}(\varepsilon)}\right) \geq \log\left(\frac{y^{i(t)}(\xi,t)}{\xi^{i(0)}(\varepsilon)}\right) \geq \int_\varepsilon^t \left(\underbrace{\frac{f(y^{i(s)}(\xi,s))}{y^{i(s)}(\xi,s)} - \frac{f\,(1/d)}{1/d}}_{\geq 0}\right) ds \geq 0 \quad \text{is increasing in } t$$

as long as $y^{i(t)}(\xi, t) < \frac{1}{d}$. As the left hand side of the above string of inequalities is finite, it follows that $\int_0^{+\infty} \left(\frac{f(y^{i(s)}(\xi,s))}{y^{i(s)}(\xi,s)} - \frac{f(1/d)}{1/d}\right) ds < +\infty$. Now the function $t \mapsto \frac{f(y^{i(t)}(\xi,t))}{y^{i(t)}(\xi,s)} - \frac{f(1/d)}{1/d}$ is decreasing since $t \mapsto y^{i(t)}(\xi, t)$ being decreasing since $y(\xi, t)$ is increasing and $u \mapsto \frac{f(u)}{u}$ is decreasing (by strict concavity), one shows that $\dfrac{f\,(1/d)}{1/d} - \dfrac{f(y^{i(t)}(\xi,t))}{y^{i(t)}(\xi,t)} \to 0$ as $t \to \infty$ or, equivalently, that

$$y^{i(t)}(\xi,t) = \min_{1 \leq i \leq d} y^i(\xi,t) \to \frac{1}{d} \quad \text{as} \quad t \to +\infty.$$

The convergence of every component $y^i(\xi, t)$ follows since $y(\xi, t) \in \mathcal{S}_d$.

(b) is obvious given the above proof. ☐

2.5 Application to the algorithm convergence

Proposition 2.7. *If H is bi-stochastic, irreducible and f is strictly concave, then*

$$\widetilde{Y}_n \longrightarrow \frac{\mathbf{1}}{d} \; a.s.$$

Proof. By combing the results of Proposition 2.3(e), the convergence of the flow from the above Proposition 2.6(a) to the uniformly stable point $\frac{1}{d}\mathbf{1}$, we know that the flow of ODE_h uniformly converges towards $\frac{1}{d}\mathbf{1}$ on $\mathcal{S}_d$, hence on the set Θ^∞ of it the limiting values of ODE_h. Consequently, we can apply Theorem A.1 from the Appendix with $\Theta^* = \{\frac{1}{d}\mathbf{1}\}$. ☐

We now give a (partial) result in the convex setting (a more precise one is provided in section 4 devoted to randomized Pólya's urns).

Proposition 2.8 (When $y(d)$ is a trap)**.** *Assume H is a bi-stochastic matrix and f is convex, then $y(d)$ is unstable. Assume that* $\sup_n \mathbb{E}\big(|\!|\!| D_{n+1} |\!|\!| \,|\, \mathcal{F}_{n-1}\big) \leq L \in \mathbb{R}_+$.

(a) *If, for every $v \in \mathbb{R}^d$ with $\mathrm{Tr}(v) = 0$ and $\|v\| = 1$,*

$$\liminf_n \mathbb{E}\big(\|D_{n+1}v\|^2 \,|\, \mathcal{F}_{n-1}\big) > 0$$

then $\mathbb{P}\big(\widetilde{Y}_n \to y(d)\big) = 0$.

(b) *Assume H is also symmetric. Let v_μ be a unitary eigenvector attached to an eigenvalue μ of $H_{\mathbf{1}^\perp}$ such that $1 - \lambda_1 \Re e(\mu) < 0$. If*

$$\liminf_n \mathbb{E}\big(\|D_{n+1}v_\mu\|^2 \,|\, \mathcal{F}_{n-1}\big) > 0$$

then $\mathbb{P}\big(\widetilde{Y}_n \to y(d)\big) = 0$.

We want to apply Theorem A.2 from the Appendix. Elementary computations show that

$$\mathbb{E}\big((\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\big) = v_\mu^t \mathbb{E}\big(D_{n+1}\mathrm{diag}(\varphi_H(\widetilde{Y}_n))D_{n+1}^t \,|\, \mathcal{F}_n\big)v_\mu - 2\varphi_H(\widetilde{Y}_n) \otimes \widetilde{Y}_n + \widetilde{Y}_n^{\otimes 2}.$$

On the event $\{\widetilde{Y}_n \to y(d)\}$ we derive, owing to Assumption (**A2**), the continuity of φ_H and $\varphi_H(y(d)) = y(d)$ that

$$\mathbb{E}\big((\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\big) = v_\mu^t \mathbb{E}\big(D_{n+1}\mathrm{diag}\big(y(d)\big)D_{n+1}^t \,|\, \mathcal{F}_n\big)v_\mu - (v_\mu|y(d))^2 + o(1).$$

Now, as $v_\mu \in \mathbf{1}^\perp$, $(v_\mu|y(d))^2 = 0$ and $\mathrm{diag}(y(d)) = \frac{1}{d}I_d$ so that

$$\liminf_n \mathbb{E}\big((\Delta M_{n+1}|v_\mu)^2 \,|\, \mathcal{F}_n\big) = \frac{1}{d}\liminf_n \mathbb{E}\big(\|D_{n+1}v_\mu\|^2 \,|\, \mathcal{F}_n\big) > 0.$$

Now, we note that

$$\|D_{n+1}v_\mu\| \geq \frac{\|D_{n+1}v_\mu\|^2}{|||D_{n+1}|||} \geq \frac{1}{L}$$

so that $\liminf_n \mathbb{E}\big(|(\Delta M_{n+1}|v_\mu)| \,|\, \mathcal{F}_n\big) > 0$.

3 Bi-dimensional skewed randomized urn model

When f is convex the situation becomes much more involved: thus if H is bi-stochastic and irreducible, then $y(d)$ is still an equilibrium of h and $\mathcal{E}_d \subset \mathring{\mathcal{S}}_d$, but $\mathcal{E}_d$ is not reduced to $y(d)$ (which is a repeller of ODE_h when $H = I_d$). To start elucidating this case, we limit ourself in this paper to a two-type urn ($d = 2$) and an irreducible matrix H. We will see, as expected, that the asymptotic behaviour of the urns is much more sophisticate, since a phase transition appears.

3.1 Existence of equilibrium points

The co-stochastic generating matrix H can be written as follows

$$H = \begin{pmatrix} p_1 & 1-p_2 \\ 1-p_1 & p_2 \end{pmatrix}, \qquad 0 < p_i < 1, \quad i = 1, 2.$$

As $0 < p_i < 1$, $i = 1, 2$, then H is irreducible. The mean function associated to the model is still given by (2.14). Solving $h(z) = 0$, $z \in \mathcal{S}_2$ is therefore equivalent to solve $h((y, 1-y)^t) = 0$, $y \in [0, 1]$. Consequently, the equation can be reduced to a one-dimensional problem, namely solving

$$(p_1 - y)f(y) + (1 - p_2 - y)f(1 - y) = 0. \tag{3.18}$$

First we remark that if $p_1 = 1 - p_2$, then $y = p_1$ is the unique solution of (3.18) because $f > 0$ on $(0, 1)$. Let y^* be a solution of (3.18). From Proposition 2.2-(b), we have that $y^* \in I^* := (p_1 \wedge (1 - p_2), p_1 \vee (1 - p_2))$.

Proposition 3.1. *1. If $p_1 + p_2 - 1 \leq 0$, then (3.18) has a unique solution lying in I^*.*

2. If $p_1 + p_2 - 1 > 0$ and f is concave, then (3.18) has a unique solution lying in I^*.

3. *If $p_1+p_2-1>0$ and f is strictly convex, then (3.18) has one, two or three solutions lying in I^*.*

Proof. 1. We have, for $y \in [0,1]$,

$$h^1(y) = y - \frac{p_1 f(y) + (1-p_2) f(1-y)}{f(y)+f(1-y)} \quad \text{and} \quad (h^1)'(y) = 1-(p_1+p_2-1)\frac{f'(y)f(1-y)+f(y)f'(1-y)}{(f(y)+f(1-y))^2}.$$

As f is increasing and nonnegative, $(h^1)'>0$ when $p_1+p_2-1 \leq 0$, therefore h^1 is increasing with $h^1(0) = p_2-1 < 0$ and $h^1(1) = 1-p_1 > 0$, so h^1 has a unique zero lying in I^*.

2. Assume that $p_1+p_2-1>0$. Then it is obvious that p_1 and $1-p_2$ are not solutions of (3.18) which can be rewritten as

$$g_1(y) = g_2(y) \quad \text{on} \quad \mathcal{J} = [0,1] \setminus \{p_1, 1-p_2\},$$

where

$$g_1(y) = \frac{f(1-y)}{p_1-y}, \quad y \in [0,1]\setminus\{p_1\} \quad \text{and} \quad g_2(y) = \frac{f(y)}{y-1+p_2}, \quad y \in [0,1]\setminus\{1-p_2\}. \tag{3.19}$$

Let us compute the first derivative of these functions: we obtain for $y \in \mathcal{J}$,

$$g_1'(y) = \frac{f(1-y)-f'(1-y)(p_1-y)}{(p_1-y)^2} \quad \text{and} \quad g_2'(y) = \frac{f'(y)(y-1+p_2)-f(y)}{(y-1+p_2)^2}.$$

As f is concave and nonnegative, $f(x)-xf'(x) \geq 0$, $x \in [0,1]$. Then, as $0<p_1,p_2<1$, we have $g_1'>0$ and $g_2'<0$ on I^*. Hence (3.18) has a unique solution $y^* \in I^*$.

3. If $p_1+p_2-1>0$ and f is strictly convex, then we have three possibilities. Let us illustrate this behaviour by setting $f(y)=y^\alpha$, $\alpha>1$. Then (3.18) can be rewritten as

$$\varphi_1(y) = \varphi_2(y) \quad \text{on} \quad I^*,$$

where

$$\varphi_1(y) = (p_1-y)f(y) = (p_1-y)y^\alpha \quad \text{and} \quad \varphi_2(y) = (y-1+p_2)f(1-y) = (y-1+p_2)(1-y)^\alpha.$$

Then the first derivatives of these functions reads

$$\varphi_1'(y) = (\alpha p_1-(\alpha+1)y)y^{\alpha-1} \quad \text{and} \quad \varphi_2'(y) = ((\alpha+1)(1-y)-\alpha p_2)(1-y)^{\alpha-1}.$$

These two derivatives have only one zero lying in I^* which are the maximum of each function : $\frac{\alpha p_1}{\alpha+1}$ for φ_1' and $1-\frac{\alpha p_2}{\alpha+1}$ for φ_2'. As $p_1+p_2-1>0$, we have that $1-\frac{\alpha p_2}{\alpha+1} < \frac{\alpha p_1}{\alpha+1}$. Besides, we have $\varphi_1(1-p_2) = (p_1+p_2-1)(1-p_2)^\alpha > 0$, $\varphi_1(p_1)=0$, $\varphi_2(1-p_2)=0$ and $\varphi_2(p_1) = (p_1+p_2-1)(1-p_1)^\alpha > 0$. Therefore we have three possibilities

- If $\varphi_1\left(1-\frac{\alpha p_2}{\alpha+1}\right) > \varphi_2\left(1-\frac{\alpha p_2}{\alpha+1}\right)$, then (3.18) has a unique solution lying in I^*.
- If $\varphi_1\left(1-\frac{\alpha p_2}{\alpha+1}\right) = \varphi_2\left(1-\frac{\alpha p_2}{\alpha+1}\right)$, then (3.18) has two solutions lying in I^*.
- If $\varphi_1\left(1-\frac{\alpha p_2}{\alpha+1}\right) < \varphi_2\left(1-\frac{\alpha p_2}{\alpha+1}\right)$, then (3.18) has three solutions lying in I^*.

The general case is obtained in the same way : when f is strictly convex, the functions φ_1 and φ_2 have a maximum on I^*. Let us set $\xi_1, \xi_2 \in I^*$ such that $\varphi'1(\xi_1)=0$ and $\varphi_2'(\xi_2)=0$. As $p_1+p_2-1>0$ and f is strictly convex, $\xi_2<\xi_1$ and depending on the relative position of $\varphi_1(\xi_2)$ and $\varphi_2(\xi_2)$, we obtain the three possible results. $\square$

Example of phase transitions

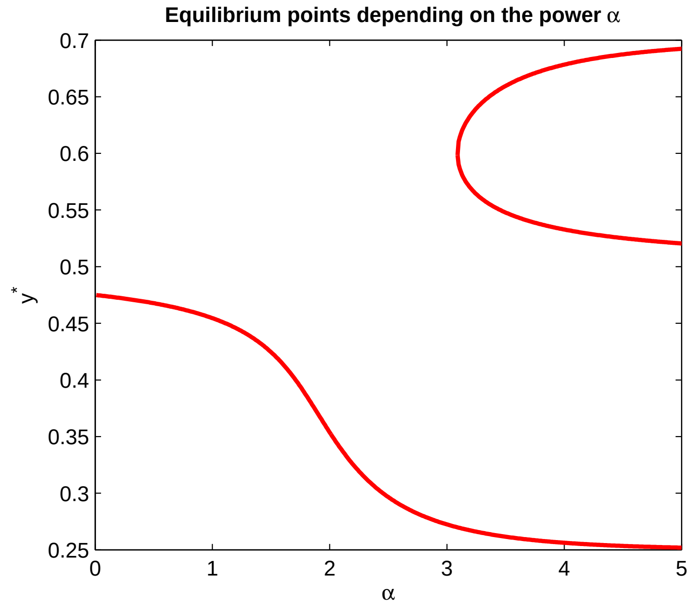

Figure 1: Equilibrium points for $f(y) = y^\alpha$ depending on α with $p_1 = 0.7$ and $p_2 = 0.75$.

3.2 Attractiveness : a simple phase transition model

After the study of the equilibrium points, we have to compute the eigenvalues of $Dh(y^*)$ at each equilibrium points y^* to deduce its attractiveness for ODE_h.

Indeed, this amounts to studying the zeros of h, the first component h^1 of h, namely $h^1(y) = y - \frac{p_1 f(y)+(1-p_2)f(1-y)}{f(y)+f(1-y)}$. By simple computations, we obtain that, for every zeros $y^* \in I^*$ of h^1,

$$(h^1)'(y^*) = 1 - \frac{f'(y^*)(p_1 - y^*) + f'(1-y^*)(y^* - 1 + p_2)}{f(y^*) + f(1-y^*)}, \quad y^* \in [0,1], \quad h^1(y^*) = 0.$$

Therefore, for each zero y^* of h^1, we have three possibilities:

- y^* is uniformly attractive ($i.e.$ $(h^1)'(y^*) > 0$),
- y^* is repulsive ($i.e.$ $(h^1)'(y^*) < 0$),
- y^* is *a priori* undetermined ($i.e.$ $(h^1)'(y^*) = 0$) ; in fact a higher order Taylor expansion can help to determine the nature of this equilibrium point).

Proposition 3.2. (i) *If h has a unique equilibrium point, then it is attractive.*

(ii) *If h has two equilibrium points, then the lowest equilibrium point for h^1 is attractive and the highest for h^1 is undetermined.*

(iii) *If h has three equilibrium points, then the lowest and the highest for h^1 are attractive and the one in the middle is repulsive.*

Proof. We have that $h^1(0) = -(1-p_2) < 0$ and $h^1(1) = 1 - p_1 > 0$. Then, if there exists a unique equilibrium point, the derivative at this point is positive, therefore the equilibrium point is attractive. Since there exists at most three equilibrium points, then:

- if h^1 has two zeros, then the first equilibrium point is attractive and the second is undetermined;
- if h^1 has three zeros, then we have two attractive equilibrium points (the first and the last) and one repulsive point (in the middle). □

Examples of function h^1

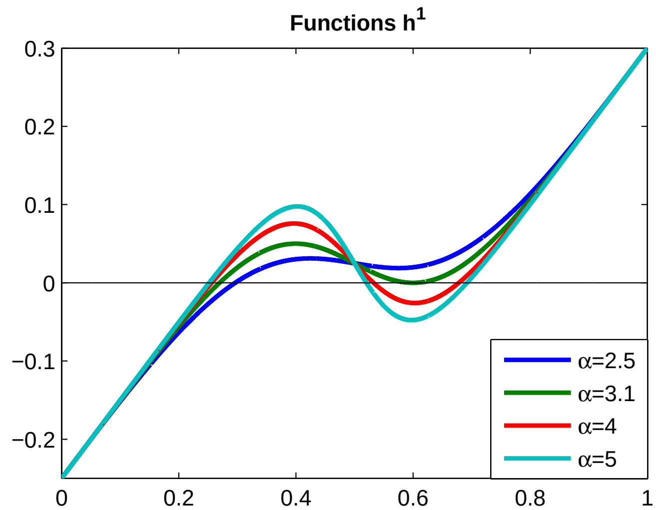

Figure 2: Functions h^1 for $f(y) = y^\alpha$ with $p_1 = 0.7$ and $p_2 = 0.75$.

Remark. If $f(y) = y^\alpha$, $\alpha > 1$, then for the case of two equilibrium points we have to look at the sign of the second derivative of h^1 to know if the undetermined zero is attractive or repulsive. We have, for every $y \in [0, 1]$,

$$(h^1)''(y) = (1-p_1-p_2)\alpha y^{\alpha-2}(1-y)^{\alpha-2}\frac{(\alpha-1)(1-2y)(y^\alpha+(1-y)^\alpha) - 2\alpha y(1-y)(y^{\alpha-1}-(1-y)^{\alpha-1})}{(y^\alpha+(1-y)^\alpha)^3}.$$

Let $y^* \in I^*$ such that $h^1(y^*) = 0$ and $(h^1)'(y^*) = 0$. Then $(h^1)''(y^*) > 0$ and a Taylor expansion gives $\dot{y} \approx -\frac{(h^1)''(y^*)}{2}(y - y^*)^2$ and the equilibrium point is then attractive.

Figure 2 shows that for $p_1 = 0.7$, $p_2 = 0.75$ and $\alpha = 3.09$, we have two equilibrium points $y_1^* \simeq 0.2699$ and $y_2^* \simeq 0.6002$ with $(h^1)'(y_2^*) = 0$. Figure 3 illustrates that $(h^1)''(y_2^*) > 0$.

Second Derivative of h^1 in the case of two equilibrium points

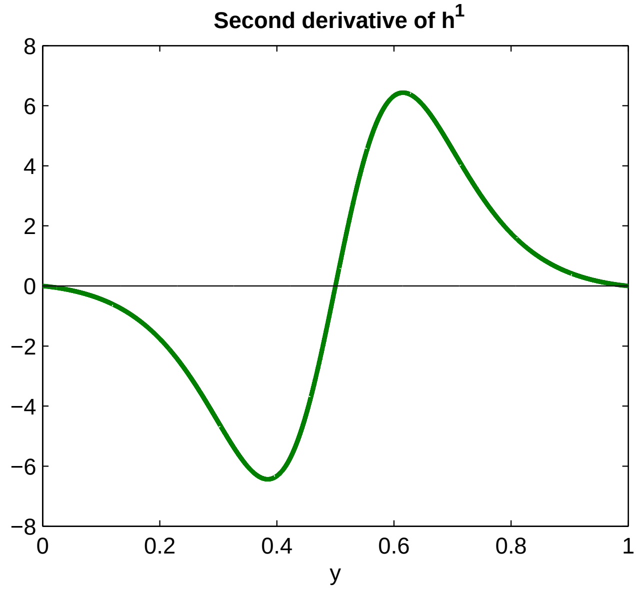

Figure 3: Second derivative of h^1 for $f(y) = y^{3.09}$ with $p_1 = 0.7$ and $p_2 = 0.75$.

It remains to show that the algorithm does not converge towards the repulsive equilibrium point denoted by $\hat{y}$. To show that there is an excitation in the repulsive direction, we have to prove that assumption (A.32) holds (see Theorem A.2 in the Appendix).

Proposition 3.3. *Let $\hat{y}$ be a repulsive equilibrium point for h^1 i.e. satisfying $(h^1)'(\hat{y}) < 0$. Then*

$$\mathbb{P}(\widetilde{Y}_n^1 \to \hat{y}) = 0.$$

Proof. As we consider the one-dimensional problem (namely the algorithm satisfied by the first component $\widetilde{Y}_n^1$), we have, using the notations of Theorem A.2 in the Appendix, that

$$\Delta M_{n+1}^{(r)} = \Delta M_{n+1}^1.$$

Using Assumption **(A1)**, we obtain

$$\Delta M_{n+1}^{(r)} = D_{n+1}^{11} X_{n+1}^1 + D_{n+1}^{12} X_{n+1}^2 - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}.$$

Therefore

$$\begin{aligned}
\mathbb{E}\left[\left\|\Delta M_{n+1}^{(r)}\right\| \mid \mathcal{F}_n\right] &= \mathbb{E}\left[\left|\Delta M_{n+1}^1\right| \mid \mathcal{F}_n\right] \\
&= \frac{f(\widetilde{Y}_n^1)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\mathbb{E}\left[\left|D_{n+1}^{11} - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right| \mid \mathcal{F}_n\right] \\
&\quad + \frac{f(\widetilde{Y}_n^2)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\mathbb{E}\left[\left|D_{n+1}^{12} - \frac{H_{n+1}^{11} f(\widetilde{Y}_n^1) + H_{n+1}^{12} f(\widetilde{Y}_n^2)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right| \mid \mathcal{F}_n\right].
\end{aligned}$$

By Jensen's inequality applied to both conditional expectation in the right hand side

$$\begin{aligned}\mathbb{E}\left[\left\|\Delta M^{(r)}_{n+1}\right\| \mid \mathcal{F}_n\right] \quad &\geq \quad \frac{f(\widetilde{Y}^1_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\left|H^{11}_{n+1} - \frac{H^{11}_{n+1} f(\widetilde{Y}^1_n) + H^{12}_{n+1} f(\widetilde{Y}^2_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right| \\ &\quad + \frac{f(\widetilde{Y}^2_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\left|H^{12}_{n+1} - \frac{H^{11}_{n+1} f(\widetilde{Y}^1_n) + H^{12}_{n+1} f(\widetilde{Y}^2_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right|.\end{aligned}$$

Owing to $\mathbf{(A5)}_v$, $H^{ij}_{n+1} \xrightarrow[n\to+\infty]{a.s.} H^{ij}$, where $H^{ii} = p_i$ and $H^{ij} = 1 - p_j$, $1 \leq i, j \leq 2$. Furthermore, on $\hat{\mathcal{Y}} = \left\{\omega : \widetilde{Y}^1_n(\omega) \to \hat{y}\right\}$,

$$\frac{H^{11}_{n+1} f(\widetilde{Y}^1_n) + H^{12}_{n+1} f(\widetilde{Y}^2_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))} \xrightarrow[n\to+\infty]{a.s.} \hat{y}.$$

Consequently,

$$\frac{f(\widetilde{Y}^1_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\left|H^{11}_{n+1} - \frac{H^{11}_{n+1} f(\widetilde{Y}^1_n) + H^{12}_{n+1} f(\widetilde{Y}^2_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right| \xrightarrow[n\to+\infty]{a.s.} \frac{f(\hat{y})}{\mathrm{Tr}(\widetilde{f}(\hat{y}))}\left|p_1 - \hat{y}\right| > 0$$

and

$$\frac{f(\widetilde{Y}^2_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\left|H^{12}_{n+1} - \frac{H^{11}_{n+1} f(\widetilde{Y}^1_n) + H^{12}_{n+1} f(\widetilde{Y}^2_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right| \xrightarrow[n\to+\infty]{a.s.} \frac{f(\hat{y})}{\mathrm{Tr}(\widetilde{f}(\hat{y}))}\left|1 - p_2 - \hat{y}\right| > 0$$

since $\hat{y} \in (1 - p_2, p_1)$. Thus (A.32) is satisfied. Then, by using (2.6) and by applying Theorem A.2 in the Appendix, $\mathbb{P}(\hat{\mathcal{Y}}) = 0$. ☐

3.3 *A.s.* convergence

Theorem 3.1. *Let $(Y_n)_{n\geq 0}$ be the urn composition sequence defined by (1.1)-(1.2). Under the assumptions* **(A1)**, **(A2)** *and* **(A3)**,

(a) $\dfrac{Y_n}{\mathrm{Tr}(Y_n)} \xrightarrow[n\to\infty]{a.s.} y^*$.

(b) $\widetilde{N}_n \xrightarrow[n\to\infty]{a.s.} \dfrac{\widetilde{f}(y^*)}{\mathrm{Tr}(\widetilde{f}(y^*))}$.

Proof. First, we will prove that $(a) \Rightarrow (b)$, then (a).

$(a) \Rightarrow (b)$. We have

$$\mathbb{E}\left[X_n \mid \mathcal{F}_{n-1}\right] = \sum_{i=1}^{d} \frac{f(\widetilde{Y}^i_{n-1})}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_{n-1}))} e^i = \frac{\widetilde{f}(\widetilde{Y}_{n-1})}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_{n-1}))}$$

and, by construction $\|X_n\|^2 = 1$ so that $\mathbb{E}\left[\|X_n\|^2 \mid \mathcal{F}_{n-1}\right] = 1$. Hence the martingale

$$\widetilde{M}_n = \sum_{k=1}^{n} \frac{X_k - \mathbb{E}\left[X_k \mid \mathcal{F}_{k-1}\right]}{k} \xrightarrow[n\to\infty]{a.s.\& L^2} \widetilde{M}_\infty \in L^2,$$

and by the Kronecker Lemma we obtain

$$\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} \frac{\widetilde{f}(\widetilde{Y}_{k-1})}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_{k-1}))} \xrightarrow[n\to\infty]{a.s.} 0.$$

This yields the announced implication owing to the Césaro Lemma.

(a) The algorithm is bounded by construction since it is $[0,1]$-valued. Assumption **(A2)** implies that $\sup_{n\geq n_0}\mathbb{E}\left[\|\Delta M_{n+1}\|^2\,|\,\mathcal{F}_n\right]<+\infty$ $a.s.$ and Assumption **(A3)** implies that $r_n \xrightarrow[n\to\infty]{a.s.} 0$. The set $\{h=0\}$ is finite hence $\widetilde{Y}_n$ $a.s.$ converges toward a zero of h (see Theorem A.1-(c) in the appendix). Moreover, it follows from Proposition 3.2 devoted to attractiveness that this zero cannot be a repulsive (a point at which h' is negative).

3.4 Weak rate of convergence

To establish a CLT for the sequence $(\widetilde{Y}_n)_{n\geq 0}$ we need to make the following additional assumptions:

(A4) The addition rules D_n $a.s.$ satisfy

$$\forall 1\leq j\leq d,\quad \begin{cases} \sup_{n\geq 1}\mathbb{E}\left[\|D_n^{\cdot j}\|^{2+\delta}\,|\,\mathcal{F}_{n-1}\right]\leq C<+\infty \quad \text{for a } \delta>0,\\ \mathbb{E}\left[D_n^{\cdot j}(D_n^{\cdot j})^t\,|\,\mathcal{F}_{n-1}\right]\xrightarrow[n\to\infty]{} C^j,\end{cases}$$

where $C^j=(C^j_{il})_{1\leq i,l\leq d}$, $j=1,\ldots,d$, are $d\times d$ positive definite matrices.

Note that **(A4)** $\Rightarrow$**(A2)** since $\mathbb{E}\left[\|D_n^{\cdot j}\|^2\,|\,\mathcal{F}_{n-1}\right]\leq\left(\mathbb{E}\left[\|D_n^{\cdot j}\|^{2+\delta}\,|\,\mathcal{F}_{n-1}\right]\right)^{\frac{2}{2+\delta}}$.

(A5)$_v$ There exists a sequence $(v_n)_{n\geq 1}$ such that matrices H_n and H satisfy

$$n\,v_n\mathbb{E}\left[|||H_n-H|||^2\right]\xrightarrow[n\to\infty]{} 0. \tag{3.20}$$

Theorem 3.2. *Assume* **(A1)**, **(A3)**, **(A4)** *and* **(A5)**$_v$. *We have*

$$Sp\left(Dh(y^*)_{|\mathcal{S}_2}\right)=\{1,1-\lambda\},$$

where

$$\lambda=\frac{f'(y^{*1})(p_1-y^{*1})+f'(1-y^{*1})(y^{*1}-1+p_2)}{f(y^{*1})+f(1-y^{*1})}.$$

(a) *If* $p_1+p_2-1\leq 0$ *and* $v_n=1$, $n\geq 1$, *then*

$$\sqrt{n}\left(\widetilde{Y}_n-y^*\right)\xrightarrow[n\to\infty]{\mathcal{L}}\mathcal{N}\left(0,\frac{1}{1-2\lambda}\Sigma\right)\qquad \text{with}\quad \Sigma=\int_0^{+\infty}e^{u(Dh(y^*)-\frac{I}{2})}\Gamma e^{u(Dh(y^*)-\frac{I}{2})^t}du$$

$$\text{and}\qquad \Gamma=\frac{f(y^{*1})C^1+f(1-y^{*1})C^2}{\mathrm{Tr}(\widetilde{f}(y^*))}-y^*(y^*)^t=a.s.\text{-}\lim_{n\to\infty}\mathbb{E}\left[\Delta M_n\Delta M_n^t\,|\,\mathcal{F}_{n-1}\right]. \tag{3.21}$$

(b) *If* $p_1+p_2-1>0$, *we have three possible rates of convergence depending on the second eigenvalue:*

(i) *If* $0<\lambda<\frac{1}{2}$ *and* $v_n=1$, $n\geq 1$, *then*

$$\sqrt{n}\left(\widetilde{Y}_n-y^*\right)\xrightarrow[n\to\infty]{\mathcal{L}}\mathcal{N}\left(0,\frac{1}{2\lambda-1}\Sigma\right).$$

(ii) *If* $\lambda=\frac{1}{2}$ *and* $v_n=\log n$, $n\geq 1$, *then*

$$\sqrt{\frac{n}{\log n}}\left(\widetilde{Y}_n-y^*\right)\xrightarrow[n\to\infty]{\mathcal{L}}\mathcal{N}\left(0,\Sigma\right).$$

(iii) *If* $\frac{1}{2}<\lambda<1$ *and* $v_n=n^{1-2\lambda+\eta}$, $\eta>0$, *then* $n^\lambda\left(\widetilde{Y}_n-y^*\right)$ *a.s. converges as* $n\to+\infty$ *towards a positive finite random variable* Υ.

Remarks. • Condition $0 < \lambda < \frac{1}{2}$ is satisfied as soon as

$$\begin{cases} \dfrac{(f'(1-p_1)+f'(1-p_2))(p_1+p_2-1)}{f(1-p_1)+f(1-p_2)} < \dfrac{1}{2} & \text{if } f \text{ is concave,} \\[2ex] \dfrac{(f'(p_1)+f'(p_2))(p_1+p_2-1)}{f(1-p_1)+f(1-p_2)} < \dfrac{1}{2} & \text{if } f \text{ is convex,} \end{cases}$$

by using the monotony of f and f' and that $y^* \in (1-p_2, p_1)$.

• If $f(y) = y$, then the above criteria reads

$$\frac{2(p_1+p_2-1)}{2-p_1-p_2} < \frac{1}{2} \quad i.e. \quad p_1+p_2 < \frac{6}{5},$$

which ensures a CLT for the unique equilibrium point. This case has already been investigated (see [3, 4, 26]) and we can compute the equilibrium point and the second eigenvalue, namely

$$y^{*1} = \frac{1-p_2}{2-p_1-p_2}, \quad y^{*2} = \frac{1-p_1}{2-p_1-p_2} \quad \text{and} \quad \lambda_2 = 2-p_1-p_2.$$

Thus, if $p_1+p_2 < \frac{3}{2}$, then the recursive procedure (2.13) satisfies a CLT; if $p_1+p_2 = \frac{3}{2}$, (2.13) satisfies Theorem 3.2-2.(ii); and if $p_1+p_2 > \frac{3}{2}$, (2.13) admits a $a.s.$-rate of convergence.

• In [13, 28], the properties of the random variable Υ are deeply investigated in the more standard framework of Pólya's urn with deterministic addition rule matrix. It is shown to be solution to a smoothing equation obtained by a smart decomposition of the urn into canonical components. Thus, it is proved that its distribution is characterized by its moments. It is clear that such results are out of reach of standard SA techniques although it would be challenging to check whether similar results about Υ in the randomized and nonlinear framework are true.

Proof. We will check the three assumptions of the CLT for SA algorithms recalled in the Appendix (Theorem A.3). Let us begin by computing the general differential matrix $Dh(y)$. We obtain

$$Dh(y) = \begin{pmatrix} 1+\frac{f'(y^1)}{f(y^1)+f(y^2)}\left(\frac{p_1 f(y^1)+(1-p_2)f(y^2)}{f(y^1)+f(y^2)} - p_1\right) & \frac{f'(y^2)}{f(y^1)+f(y^2)}\left(\frac{p_1 f(y^1)+(1-p_2)f(y^2)}{f(y^1)+f(y^2)} - (1-p_2)\right) \\ \frac{f'(y^1)}{f(y^1)+f(y^2)}\left(\frac{(1-p_1)f(y^1)+p_2 f(y^2)}{f(y^1)+f(y^2)} - (1-p_1)\right) & 1+\frac{f'(y^2)}{f(y^1)+f(y^2)}\left(\frac{(1-p_1)f(y^1)+p_2 f(y^2)}{f(y^1)+f(y^2)} - p_2\right) \end{pmatrix}.$$

As the equilibrium points y^* lie in the simplex $\mathcal{S}_2$, we have that $y^{*2} = 1-y^{*1}$. Furthermore, using that $h(y^*) = 0$, we obtain

$$Dh(y^*)_{|\mathcal{S}_2} = \begin{pmatrix} 1+\frac{f'(y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(y^{*1}-p_1\right) & \frac{f'(1-y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(y^{*1}-(1-p_2)\right) \\ \frac{f'(y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(p_1-y^{*1}\right) & 1+\frac{f'(1-y^{*1})}{f(y^{*1})+f(1-y^{*1})}\left(1-p_2-y^{*1}\right) \end{pmatrix}.$$

Thus

$$\mathrm{Sp}\left(Dh(y^*)_{|\mathcal{S}_2}\right) = \left\{1, 1-\frac{f'(y^{*1})(p_1-y^{*1})+f'(1-y^{*1})(y^{*1}-1+p_2)}{f(y^{*1})+f(1-y^{*1})}\right\}.$$

The condition (A.35) on the spectrum of $Dh(y^*)$ requested for algorithms with step $\frac{1}{n}$ in Theorem A.3 reads $\mathrm{Sp}(Dh(y^*)) > \frac{1}{2}$.
Secondly Assumption **(A4)** ensures that Condition (A.33) is satisfied since

$$\sup_{n\geq 1}\mathbb{E}\left[\|\Delta M_n\|^{2+\delta}\,|\,\mathcal{F}_{n-1}\right] < +\infty \quad a.s. \quad \text{and} \quad \mathbb{E}\left[\Delta M_n \Delta M_n^t\,|\,\mathcal{F}_{n-1}\right] \xrightarrow[n\to\infty]{a.s.} \Gamma \quad \text{as} \quad n\to\infty,$$

where Γ is the symmetric nonnegative matrix given by

$$\begin{aligned}
\mathbb{E}\left[\Delta M_{n+1}\Delta M_{n+1}^t\,|\,\mathcal{F}_n\right] &= \sum_{q=1}^{2}\mathbb{P}(X_{n+1}=e^q\,|\,\mathcal{F}_n)\big(\mathbb{E}\left[D_{n+1}^{\cdot q}(D_{n+1}^{\cdot q})^t\,|\,\mathcal{F}_n\right] \\
&\quad -\mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right]\mathbb{E}\left[D_{n+1}X_{n+1}\,|\,\mathcal{F}_n\right]^t\big) \\
&= \sum_{q=1}^{2}\frac{f(\widetilde{Y}_n^q)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\mathbb{E}\left(D_{n+1}^{\cdot q}(D_{n+1}^{\cdot q})^t\,|\,\mathcal{F}_n\right) - \left(H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right)\left(H_{n+1}\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}\right)^t \\
&\xrightarrow[n\to\infty]{a.s.} \Gamma = \frac{f(y^{*1})C^1 + f(1-y^{*1})C^2}{\mathrm{Tr}(\widetilde{f}(y^*))} - y^*(y^*)^t.
\end{aligned}$$

Finally, using $\mathbf{(A5)}_v$, the remainder sequence $(r_n)_{n\geq 1}$ satisfies (A.34) since $\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}$ owing to (2.15).
□

4 Pólya urn with reinforced drawing rule: a bandit approach

Assume that the drawing rule is given by (2.7), that $D_n = I_d$, $n \geq 1$, and that the initial urn composition vector $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$. In this framework, $H = H_n = I_d$, $n \geq 1$, therefore H is no more irreducible and we cannot use the results proved in Sections 2.1 and 3. We still normalize Y_n by setting $\widetilde{Y}_n := \frac{Y_n}{n+\mathrm{Tr}(Y_0)}$, $n \geq 0$.

The sequence $(\widetilde{Y}_n)_{n\geq 0}$ satisfies the following recursive stochastic algorithm (obvious consequence of (2.11))

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n+1+\mathrm{Tr}(Y_0)}\left(\widetilde{Y}_n - \frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}\left(\widetilde{f}(\widetilde{Y}_n)\right)}\right) + \frac{1}{n+1+\mathrm{Tr}(Y_0)}\Delta M_{n+1}, \quad n\geq 1, \tag{4.22}$$

where

$$\Delta M_{n+1} := X_{n+1} - \mathbb{E}\left[X_{n+1}\,|\,\mathcal{F}_n\right] \tag{4.23}$$

is an $(\mathcal{F}_n)_{n\geq 0}$-true martingale increment. Let us remark that in this case $\frac{\mathrm{Tr}(Y_n)}{n+\mathrm{Tr}(Y_0)} = \frac{n+\mathrm{Tr}(Y_0)}{n+\mathrm{Tr}(Y_0)} = 1$, so that the sequence $(\widetilde{Y}_n)_{n\geq 0}$ is bounded and lies in the canonical simplex $\mathcal{S}_d$ since it is a non-negative sequence.

4.1 Special case where $f \equiv Id$

The special case where $f(y) = y$ follows from the classical Athreya theorem that we recall below.

Theorem 4.1 (Athreya's Theorem, see [1]). *Let $(Y_n)_{n\geq 0}$ be the urn composition sequence defined by (1.1)-(1.2) with $D_n = I_d$, $n \geq 1$. Under the assumption (2.7) with $f(x) = x$, there exists a random vector $\widetilde{Y}_\infty$ having values in the simplex $\mathcal{S}_d$ such that*

$$\widetilde{Y}_n = \frac{Y_n}{\mathrm{Tr}(Y_n)} \xrightarrow[n\to\infty]{a.s.} \widetilde{Y}_\infty \quad a.s.$$

Furthermore,

(i) $\widetilde{Y}_\infty$ has a Dirichlet distribution with parameter Y_0.

(ii) In particular, if $d = 2$, $\widetilde{Y}^1_\infty$ has a beta distribution with parameter Y^1_0 and Y^2_0 (in particular, $\widetilde{Y}^1_\infty$ has a uniform distribution on $[0,1]$ if $Y^1_0 = Y^2_0 = 1$).

Proof (partial). Since the components of $\widetilde{Y}_n = \frac{Y_n}{n+\mathrm{Tr}(Y_0)}$ are nonnegative and $\mathrm{Tr}(\widetilde{Y}_n) = 1$, $n \geq 0$, it is clear that $(\widetilde{Y}_n)_{n\geq 0}$ is bounded and lies in the simplex $\mathcal{S}_d$ and that $\mathbb{P}(d\omega)$-$a.s.$, the set $\mathcal{Y}_\infty(\omega)$ of all its limiting values is contained in $\mathcal{S}_d$. We can rewrite the recursive procedure (4.22) for $f(x) = x$ in the following form

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n + \frac{\Delta M_{n+1}}{n+1+\mathrm{Tr}(Y_0)},$$

therefore $\widetilde{Y}_n$ is a non-negative bounded martingale. Consequently $\widetilde{Y}_n \xrightarrow[n\to\infty]{a.s.} \widetilde{Y}_\infty < +\infty$ $a.s.$

We only will prove claim (ii). We will use the moment method to prove that the law of $\widetilde{Y}^1_\infty$ is the beta law with parameter Y^1_0 and Y^2_0. Indeed, by Lebesgue's theorem $\lim_{n\to+\infty} \mathbb{E}[(\widetilde{Y}_n)^k] = \mathbb{E}[(\widetilde{Y}_\infty)^k]$.

Let us recall the moments of the beta distribution (which characterize it since it is compactly supported). Assume that a random variable X has the beta distribution with parameters α and β. Then for every $k \geq 1$,

$$\mathbb{E}[X^k] = \prod_{i=0}^{k-1} \frac{\alpha+i}{\alpha+\beta+i}.$$

We set, for $n \geq 0$,

$$\widetilde{M}_n = \frac{Y^1_n(Y^1_n+1)\cdots(Y^1_n+k-1)}{(n+\mathrm{Tr}(Y_0))(n+\mathrm{Tr}(Y_0)+1)\cdots(n+\mathrm{Tr}(Y_0)+k-1)}.$$

Let us show that $(\widetilde{M}_n)_{n\geq 0}$ is an $(\mathcal{F}_n)_{n\geq 0}$-martingale. We have $\mathbb{P}$-$a.s.$

$$\begin{aligned}
\mathbb{E}[\widetilde{M}_{n+1}|\mathcal{F}_n] &= \mathbb{E}[\widetilde{M}_{n+1}\mathbb{1}_{\{X^1_{n+1}=0\}}|\mathcal{F}_n] + \mathbb{E}[\widetilde{M}_{n+1}\mathbb{1}_{\{X^1_{n+1}=1\}}|\mathcal{F}_n] \\
&= \frac{n+\mathrm{Tr}(Y_0)-Y^1_n}{n+\mathrm{Tr}(Y_0)} \frac{Y^1_n(Y^1_n+1)\cdots(Y^1_n+k-1)}{(n+\mathrm{Tr}(Y_0)+1)(n+\mathrm{Tr}(Y_0)+2)\cdots(n+\mathrm{Tr}(Y_0)+k)} \\
&\quad + \frac{Y^1_n}{n+\mathrm{Tr}(Y_0)} \frac{(Y^1_n+1)(Y^1_n+2)\cdots(Y^1_n+k)}{(n+\mathrm{Tr}(Y_0)+1)(n+\mathrm{Tr}(Y_0)+2)\cdots(n+\mathrm{Tr}(Y_0)+k)} \\
&= \frac{Y^1_n(Y^1_n+1)\cdots(Y^1_n+k-1)[(n+\mathrm{Tr}(Y_0)-Y^1_n)+(Y_n+k)]}{(n+\mathrm{Tr}(Y_0))(n+\mathrm{Tr}(Y_0)+1)\cdots(n+\mathrm{Tr}(Y_0)+k)} \\
&= \frac{Y^1_n(Y^1_n+1)\cdots(Y^1_n+k-1)}{(n+\mathrm{Tr}(Y_0))(n+\mathrm{Tr}(Y_0)+1)\cdots(n+\mathrm{Tr}(Y_0)+k-1)} = \widetilde{M}_n.
\end{aligned}$$

Since $\widetilde{Y}^1_n \underset{n\to+\infty}{\longrightarrow} \widetilde{Y}^1_\infty$ $a.s.$ then also $\frac{Y^1_n+r}{n+\mathrm{Tr}(Y_0)+r} \underset{n\to+\infty}{\longrightarrow} \widetilde{Y}^1_\infty$ $a.s.$ for every fixed r, so that $\widetilde{M}_n \underset{n\to+\infty}{\longrightarrow} (\widetilde{Y}^1_\infty)^k$ $a.s.$ and, as $0 \le \widetilde{M}_n \le 1$, $\lim_{n\to+\infty} \mathbb{E}[\widetilde{M}_n] = \mathbb{E}[(\widetilde{Y}^1_\infty)^k]$. $(\widetilde{M}_n)_{n\ge1}$ being a martingale

$$\mathbb{E}[\widetilde{M}_n] = \mathbb{E}[\widetilde{M}_0] = \prod_{r=0}^{k-1} \frac{Y^1_0 + r}{\mathrm{Tr}(Y_0)+r}. \quad \square$$

4.2 Multi-dimensional bandit model with convex (or concave) skewed drawing rule

In this section we investigate the typically no-irreducible case, namely $H = I_d$ we will see that it requires new tools, especially a method to avoid the so-called "traps" in the SA literature.

Theorem 4.2. *(a) Let $I \subsetneq \{1,\dots,d\}$ be non-empty. If f satisfies either $f'_r(0) > |I| f\big(\frac{1}{|I|}\big)$. Then for every deterministic initial value such that $Y^j_0 > 0$ for some $j \notin I$,*

$$\mathbb{P}\left(\widetilde{Y}_\infty = \widetilde{e}_I\right) = 0.$$

(b) If $d = 2$, the above conclusion still holds if $f'_r(0) = 1$ and $f'_l(1) + \frac{f''_r(0)}{2} > 1$.
(c) If f is strictly concave then $\mathcal{E}_d = \{\widetilde{e}_I, I \subset \{1,\dots,d\}, I \neq \emptyset\}$ by Proposition 2.3(c). Then for every starting value $Y_0 \in (0,+\infty)^d$,

$$\widetilde{Y}_n \overset{a.s.}{\longrightarrow} \widetilde{e}_{\{1,\dots,d\}} = y(d) \quad as \quad n \to +\infty$$

Remarks. • Notice that in the previous theorem, no convexity property is requested on the function f. This result is indeed more general than the convex/concave framework that we usually study in this section.

• In claim (b), if $f(0) = 0$, $f(1) = 1$, $f'(0) = 1$ and f is convex or concave, then $f = \mathrm{Id}$ so this case is out of the convex/concave framework.

• If $Y^i_0 = 0$ for some $i \in \{1,\dots,d\}$, then, as proved below, $Y^i_n = 0$ for every $n \ge 0$. So, as soon as $Y_0 \in \mathbb{R}^d_+ \setminus \{0\}$, one may apply the above result (c) to the urn restricted to $I' = \{i \in I,\ Y^i_0 > 0\}$ to prove that $\widetilde{Y}_n \to \widetilde{e}_{I'}$ as $n \to +\infty$.

Proof. (a)-(b) STEP 1: It follows from (1.1) and the fact that $D_n \equiv I_d$ that, if $Y^i_0 = 0$ then, for every instant $n \ge 0$, $Y^i_n = 0$. So, up to a reduction of the dimension d, we may always assume that all $Y^i > 0$.

As a consequence we may assume that, for every $n \ge 0$, $\min_i \widetilde{Y}^i_n > 0$. Now we will prove that $\mathbb{P}\left(\widetilde{Y}^j_\infty = 0\right) = 0$ for every $j \notin I$. Without loss of generality we may assume that $1 \notin I$ and $j = 1$. Define the function $\widetilde{h}$ by $\widetilde{h}(y) = 1 - \frac{f(y^1)}{y^1 \mathrm{Tr}(\widetilde{f}(y))} \mathbf{1}_{\{y^1=0\}}$ which satisfies $\widetilde{h}(y) < 1$ for $y \in \mathcal{S}_d \setminus \{y^1 = 0\}$.

Starting from the dynamics of $\widetilde{Y}^1_n$ given by (4.22), we have, for $n \ge 0$,

$$\begin{aligned}
\widetilde{Y}^1_{n+1} &= \widetilde{Y}^1_n - \frac{1}{n+1+\mathrm{Tr}(Y_0)}\left(\widetilde{Y}^1_n - \frac{f(\widetilde{Y}^1_n)}{\mathrm{Tr}\big(\widetilde{f}(\widetilde{Y}_n)\big)}\right) + \frac{1}{n+1+\mathrm{Tr}(Y_0)} \Delta M^1_{n+1} \\
&= \widetilde{Y}^1_n \left(1 - \frac{1}{n+1+\mathrm{Tr}(Y_0)} \widetilde{h}(\widetilde{Y}^1_n)\right) + \frac{1}{n+1+\mathrm{Tr}(Y_0)} \Delta M^1_{n+1}.
\end{aligned}$$

We derive that the sequence

$$\widetilde{L}_n := \frac{\widetilde{Y}^1_n}{\prod_{k=1}^n \left(1 - \frac{1}{k+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right)}, \quad n \geq 0 \tag{4.24}$$

is a non-negative martingale satisfying

$$\widetilde{L}_{n+1} = \widetilde{L}_n + \frac{1}{n+1+\mathrm{Tr}(Y_0)} \frac{\Delta M^1_{n+1}}{\prod_{k=1}^{n+1} \left(1 - \frac{1}{k+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right)}, \quad n \geq 0.$$

• If $f'_r(0) > |I| f\left(\frac{1}{|I|}\right)$, then $\widetilde{h}(y) \longrightarrow \kappa := 1 - \frac{f'_d(0)}{|I|\left(\frac{1}{|I|}\right)} < 0$ as $y \to \widetilde{e}_{_I}$. Therefore, on the event $\{Y^1_n \to \widetilde{e}_{_I}\}$, $\widetilde{h}\left(Y^1_{n-1}\right) \overset{a.s.}{\sim} \kappa < 0$ so that $\prod_{k=1}^n \left(1 - \frac{1}{k+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right) \xrightarrow{a.s.} +\infty$ *a.s.*. It follows from its definition in (4.24) that $\widetilde{L_n} \xrightarrow{a.s.} 0$ on $\left\{\widetilde{Y}^1_n \to \widetilde{e}_{_I}\right\}$ since $\widetilde{Y}^1_n \leq \frac{\mathrm{Tr}(Y_n)}{n+\mathrm{Tr}(Y_0)} \xrightarrow{a.s.} 1$. Consequently

$$\left\{\widetilde{Y}^1_n \to \widetilde{e}_{_I}\right\} \subset \left\{\widetilde{L}_n \to 0\right\}.$$

• (Case $d = 2$) If $f'_r(0) = 1$ and $f'_l(1) + \frac{f''_r(0)}{2} > 1$, then $\widetilde{h}(y) < 0$ for y in the neighborhood of 0. So we still have $\left\{\widetilde{Y}_n \to e_2\right\} = \left\{\widetilde{Y}^1_n \to 0\right\} \subset \left\{\widetilde{L}_n \to 0\right\}$.
Consequently

$$\mathbb{P}(\widetilde{Y}^1_\infty = 0) \leq \mathbb{P}(\widetilde{L}_\infty = 0).$$

Step 2: The end of the proof is based on the following (short) key lemma (see [25]) reproduced here for the reader's convenience.

Lemma 4.1. *Let $(M_n)_{n\geq 0}$ be a non-negative martingale. Then*

$$\forall n \geq 0, \quad \mathbb{P}(M_\infty = 0 \,|\, \mathcal{F}_n) \leq \frac{\mathbb{E}\left[\Delta \langle M \rangle^\infty_{n+1} \,|\, \mathcal{F}_n\right]}{M_n^2}.$$

Proof of Lemma 4.1. It is sufficient to observe that, for every $n \geq 0$,

$$\begin{aligned}
\mathbb{P}(M_\infty = 0 \,|\, \mathcal{F}_n) &= \frac{\mathbb{E}\left[\mathbb{1}_{\{M_\infty = 0\}} M_n^2 \,|\, \mathcal{F}_n\right]}{M_n^2} \leq \frac{\mathbb{E}\left[(M_\infty - M_n)^2 \,|\, \mathcal{F}_n\right]}{M_n^2} \\
&= \frac{\mathbb{E}\left[\Delta \langle M \rangle^\infty_{n+1} \,|\, \mathcal{F}_n\right]}{M_n^2}.
\end{aligned}$$

□

First we note that

$$\mathbb{E}\left[(\Delta\widetilde{M}_{n+1})^2 \,|\, \mathcal{F}_n\right] = \left(\frac{1}{n+1+\mathrm{Tr}(Y_0)}\right)^2 \frac{\mathbb{E}\left[(\Delta M^1_{n+1})^2 \,|\, \mathcal{F}_n\right]}{\left(\prod_{k=1}^{n+1}\left(1 - \frac{1}{k+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right)\right)^2}$$

and

$$\mathbb{E}\left((\Delta M^1_{n+1})^2 \,|\, \mathcal{F}_n\right) = \frac{f(\widetilde{Y}^1_n)\left(\mathrm{Tr}\left(\widetilde{f}(\widetilde{Y}_n)\right) - f(\widetilde{Y}^1_n)\right)}{\mathrm{Tr}\left(\widetilde{f}(\widetilde{Y}_n)\right)^2} \leq \frac{d-1}{f(1/d)^2}$$

since $\mathrm{Tr}(f(y)) \geq f(\max_i Y_i) \geq f(1/d)$. As a consequence

$$\mathbb{E}\left[(\Delta\widetilde{M}_{n+1})^2 \mid \mathcal{F}_n\right] = \frac{1}{(n+1+\mathrm{Tr}(Y_0))^2\left(\prod_{k=1}^{n+1}\left(1-\frac{1}{k+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right)\right)^2}\frac{f(\widetilde{Y}^1_n)\big(\mathrm{Tr}\big(\widetilde{f}(\widetilde{Y}_n)\big)-f(\widetilde{Y}^1_n)\big)}{\left(\mathrm{Tr}\big(\widetilde{f}(\widetilde{Y}_n)\big)\right)^2}.$$

Then, applying Lemma 4.1 to the non-negative martingale $(\widetilde{L}_n)_{n\geq 1}$ yields

$$\begin{aligned}\mathbb{P}(\widetilde{M}_\infty = 0 \mid \mathcal{F}_n) &\leq \frac{\mathbb{E}\left[\Delta\left\langle \widetilde{M}\right\rangle_{n+1}^{\infty} \mid \mathcal{F}_n\right]}{\widetilde{M}_n^2}\\ &= \frac{1}{\widetilde{M}_n^2}\mathbb{E}\left[\sum_{k=n+1}^{\infty}\frac{1}{(k+\mathrm{Tr}(Y_0))^2\left(\prod_{\ell=1}^{k}\left(1-\frac{1}{\ell+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{\ell-1})\right)\right)^2}F(\widetilde{Y}^1_{k-1}) \mid \mathcal{F}_n\right]\end{aligned}$$

where the function F is defined by $F(y) := \frac{f(y^1)\left(\mathrm{Tr}(\widetilde{f}(y))-f(y^1)\right)}{\left(\mathrm{Tr}(\widetilde{f}(y))\right)^2}$, $y \in (0,1]\times[0,1]^{d-1}$ is clearly non-negative and bounded by $\kappa_d := (d-1)/f(1/d)^2$. Consequently,

$$\begin{aligned}&\mathbb{P}(\widetilde{M}_\infty = 0 \mid \mathcal{F}_n)\\ &\leq \frac{\kappa_d}{\widetilde{M}_n^2}\sum_{k=n+1}^{\infty}\frac{1}{(k+\mathrm{Tr}(Y_0))^2}\mathbb{E}\left[\underbrace{\frac{\widetilde{Y}^1_{k-1}}{\prod_{\ell=1}^{k-1}\left(1-\frac{1}{\ell+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{\ell-1})\right)}}_{=\widetilde{L}_{k-1}}\frac{\left(1-\frac{1}{k+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right)^{-1}}{\prod_{\ell=1}^{k}\left(1-\frac{1}{\ell+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{\ell-1})\right)} \mid \mathcal{F}_n\right].\end{aligned}$$

Let $\widetilde{h}^+ := \max(\widetilde{h},0)$ so that $\widetilde{h} \leq \widetilde{h}^+ \leq \left\|\widetilde{h}^+\right\|_\infty$. Now $\widetilde{h}^+(y) < 1$, $y \in \mathcal{S}_d \setminus \{y^1 = 0\}$, since $f(y^1) > 0$ on $(0,1]$ and $\limsup_{y^1\to 0}\widetilde{h}(y) \leq 1 - f_r'(0)\inf_{\mathcal{S}_d}\frac{1}{\mathrm{Tr}(\widetilde{f}(y))} \leq 1 - f_r'(0)/d < 1$ since $f_r'(0) > 0$ by assumption.

In 2-dimension, under the additional assumption in the critical care ($f_r'(0) = 1$), the extension of the function h over $[0,1]$ is negative so that $h^+ \equiv 0$.

Finally, we obtain

$$\left(1-\frac{1}{k+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{k-1})\right)^{-1} \leq \left(1-\frac{1}{k+\mathrm{Tr}(Y_0)}\left\|\widetilde{h}^+\right\|_\infty\right)^{-1}, \quad k \geq 1.$$

Then, as $\mathbb{E}\left[\widetilde{M}_{k-1}\,|\,\mathcal{F}_n\right] = \widetilde{M}_n$ since $\widetilde{M}$ is a $(\mathbb{P},\mathcal{F}_n)$-martingale,

$$\begin{aligned}
&\mathbb{P}(\widetilde{M}_\infty = 0\,|\,\mathcal{F}_n) \\
\le\ & \frac{\kappa_d}{\widetilde{M}_n^2}\sum_{k=n+1}^{\infty}\frac{1}{(k+\mathrm{Tr}(Y_0))^2}\frac{\widetilde{M}_n}{\left(1-\frac{1}{k+\mathrm{Tr}(Y_0)}\left\|\widetilde{h}^+\right\|_\infty\right)\prod_{\ell=1}^{n}\left(1-\frac{1}{\ell+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{\ell-1})\right)\prod_{\ell=n+1}^{k}\left(1-\frac{1}{\ell+\mathrm{Tr}(Y_0)}\left\|\widetilde{h}^+\right\|_\infty\right)} \\
=\ & \frac{\kappa_d}{\widetilde{M}_n\prod_{\ell=1}^{n}\left(1-\frac{1}{\ell+\mathrm{Tr}(Y_0)}\widetilde{h}(\widetilde{Y}^1_{\ell-1})\right)}\sum_{k=n+1}^{\infty}\frac{1}{\left(1-\frac{1}{k+\mathrm{Tr}(Y_0)}\left\|\widetilde{h}^+\right\|_\infty\right)(k+\mathrm{Tr}(Y_0))^2\prod_{\ell=n+1}^{k}\left(1-\frac{1}{\ell+\mathrm{Tr}(Y_0)}\left\|\widetilde{h}^+\right\|_\infty\right)} \\
=\ & \frac{\kappa_d C_{n_0}}{\widetilde{Y}^1_n}\sum_{k=n+1}^{\infty}\frac{1}{(k+\mathrm{Tr}(Y_0))^2}\exp\left(-\sum_{\ell=n+1}^{k}\ln\left(1-\frac{\left\|\widetilde{h}^+\right\|_\infty}{\ell+\mathrm{Tr}(Y_0)}\right)\right) \\
\le\ & \frac{\kappa_d C_{n_0}C_{\widetilde{h}}(n+\mathrm{Tr}(Y_0))}{Y^1_n}\sum_{k=n+1}^{\infty}\frac{e^{\left\|\widetilde{h}^+\right\|_\infty\ln\frac{k+\mathrm{Tr}(Y_0)}{n+\mathrm{Tr}(Y_0)}}}{(k+\mathrm{Tr}(Y_0))^2}
\end{aligned}$$

since $\widetilde{Y}^1_n = \dfrac{Y^1_n}{n+\mathrm{Tr}(Y_0)}$. Hence

$$\begin{aligned}
\mathbb{P}(\widetilde{M}_\infty = 0\,|\,\mathcal{F}_n) &\le \widetilde{Y}^1_n = \frac{Y^1_n}{n+\mathrm{Tr}(Y_0)} \\
&= \frac{\kappa_d C_{n_0}C_{\widetilde{h}}(n+\mathrm{Tr}(Y_0))}{Y^1_n}\sum_{k=n+1}^{\infty}\frac{(n+\mathrm{Tr}(Y_0))^{-\left\|\widetilde{h}^+\right\|_\infty}}{(k+\mathrm{Tr}(Y_0))^{2-\left\|\widetilde{h}^+\right\|_\infty}} \\
&= \frac{\kappa_d C_{n_0}C_{\widetilde{h}}(n+\mathrm{Tr}(Y_0))^{1-\left\|\widetilde{h}^+\right\|_\infty}}{Y^1_n}\sum_{k=n+1}^{\infty}\frac{1}{(k+\mathrm{Tr}(Y_0))^{2-\left\|\widetilde{h}^+\right\|_\infty}} \\
&\le \frac{\kappa_d C_{n_0}C_{\widetilde{h}}}{Y^1_n}.
\end{aligned}$$

Now, it remains to prove that $Y^1_n \xrightarrow{a.s.} +\infty$. One checks that

$$\{Y^1_\infty < +\infty\} = \bigcup_{n\ge 0}\bigcap_{k>n}\left\{U_k > \frac{f(\frac{Y^1_n}{k-1+\mathrm{Tr}(Y_0)})}{f(\frac{Y^1_n}{k-1+\mathrm{Tr}(Y_0)}) + f(1-\frac{Y^1_n}{k-1+\mathrm{Tr}(Y_0)})}\right\},$$

then

$$\forall n\in\mathbb{N},\quad \mathbb{P}\left(Y^1_\infty < +\infty\,|\,Y^1_n = y\right) = \prod_{k>n}\left(1-\frac{f(y/(k-1+\mathrm{Tr}(Y_0)))}{f(y/(k-1+\mathrm{Tr}(Y_0))) + f(1-y/(k-1+\mathrm{Tr}(Y_0)))}\right) = 0$$

since $\sum_k \frac{f(y/k+\mathrm{Tr}(Y_0))}{f(y/k+\mathrm{Tr}(Y_0))+f(1-y/k+\mathrm{Tr}(Y_0))} = +\infty$ because $f'_r(0) > 0$. Therefore $Y^1_\infty = \lim_n Y^1_n = +\infty$ *a.s.*. Consequently,

$$\mathbb{P}\left(\widetilde{M}_\infty = 0\right) = 0$$

which in turn implies

$$\mathbb{P}\left(\widetilde{Y}^1_\infty = 0\right) = \lim_n \mathbb{E}\left(\mathbb{P}(\widetilde{Y}^1_\infty = 0\,|\,\mathcal{F}_n)\right) \le \lim_n \mathbb{E}\left(\mathbb{P}(\widetilde{M}_\infty = 0\,|\,\mathcal{F}_n)\right) = 0$$

(where we used that bounded martingales converge in L^1).
(c) First assume that all $Y_i > 0$, $i \in \{1, \dots, d\}$. From what precedes, we derive that $\mathbb{P}\big(\widetilde{Y}_n \to \partial\mathcal{S}_d)\big) = 0$. As a consequence, following Theorem A.1, $\mathbb{P}(d\omega)$-$a.s.$, the set $\Theta^\infty(\omega)$ of limiting values of $(\widetilde{Y}_n(\omega))_{n\geq 0}$ is a connected compact set of $\overset{\circ}{\mathcal{S}}_d$. We know that $\frac{1}{d}\mathbf{1}$ is a uniformly attracting point for ODE_h and that the flow of ODE_h $(y(y_0,t))_{t\geq 0, y_0 \in \overset{\circ}{\mathcal{S}}_d}$ converges toward $\frac{1}{d}\mathbf{1}$, so it converges uniformly with respect to $y_0 \in \Theta^\infty(\omega)$. The conclusion of Theorem A.1 then provides the expected $a.s.$ convergence. □

5 Applications

5.1 Function with regular variation for the drawing rule

Let define the law of the drawings as follows

$$\forall 1 \leq i \leq d, \quad \mathbb{P}(X_{n+1} = e^i \,|\, \mathcal{F}_n) = \frac{f(Y_n^i)}{\sum_{j=1}^d f(Y_n^j)}, \quad n \geq 0, \tag{5.25}$$

where f has regular variation with index $\alpha > 0$, namely $\forall t > 0$, $\dfrac{f(tx)}{f(x)} \underset{x\to\infty}{\longrightarrow} t^\alpha$ and f is bounded on each interval $(0, M]$. Then, by applying Theorem 1.5.2 p.22 in [10], $\dfrac{f(tx)}{f(x)} \underset{x\to\infty}{\longrightarrow} t^\alpha$ uniformly in t on each $(0, b]$, $0 < b < \infty$.

We can reformulate the dynamics (1.1)-(1.2) into a recursive stochastic algorithm like in the Section 2.1, and we obtain the following recursive procedure satisfied by the sequence $(\widetilde{Y}_n)_{n\geq 0}$, namely

$$\widetilde{Y}_{n+1} = \widetilde{Y}_n - \frac{1}{n + \mathrm{Tr}(Y_0) + 1}\left(\widetilde{Y}_n - H\frac{\widetilde{Y}_n^\alpha}{\mathrm{Tr}(\widetilde{Y}_n^\alpha)}\right) + \frac{1}{n + \mathrm{Tr}(Y_0) + 1}\left(\Delta M_{n+1} + \widehat{r}_{n+1}\right) \tag{5.26}$$

where $\widetilde{Y}_n^\alpha = \left((\widetilde{Y}_n^i)^\alpha\right)_{1\leq i\leq d}$ with step $\gamma_n = \frac{1}{n+\mathrm{Tr}(Y_0)}$ and a remainder term given by

$$\widehat{r}_{n+1} := H_{n+1}\frac{\widetilde{f}(Y_n)}{\mathrm{Tr}(\widetilde{f}(Y_n))} - H\frac{\widetilde{Y}_n^\alpha}{\mathrm{Tr}(\widetilde{Y}_n^\alpha)} \in \mathcal{F}_n. \tag{5.27}$$

Notice that, in the convex case, the remainder term was $r_{n+1} = (H_{n+1} - H)\frac{\widetilde{f}(\widetilde{Y}_n)}{\mathrm{Tr}(\widetilde{f}(\widetilde{Y}_n))}$, therefore assumption **(A3)** implied directly that $r_n \xrightarrow[n\to+\infty]{a.s.} 0$. Here we have to use the uniform convergence of the regular variation to prove the required assumption on $\widehat{r}_{n+1}$.

By the same arguments like in Section 2.1, $\mathrm{Tr}(Y_n)$ satisfies (2.16). Moreover, for the quantity $\widetilde{N}_n := \frac{1}{n}\sum_{k=1}^n X_k$, we also devise a stochastic recursive procedure in the same way as before, namely

$$\widetilde{N}_{n+1} = \widetilde{N}_n - \frac{1}{n+1}\left(\widetilde{N}_n - \frac{\widetilde{Y}_n^\alpha}{\mathrm{Tr}(\widetilde{Y}_n^\alpha)}\right) + \frac{1}{n+1}\left(\Delta\widetilde{M}_{n+1} + \widetilde{r}_{n+1}\right),$$

where $\widetilde{r}_{n+1} = \dfrac{\widetilde{f}(Y_n)}{\mathrm{Tr}(\widetilde{f}(Y_n))} - \dfrac{\widetilde{Y}_n^\alpha}{\mathrm{Tr}(\widetilde{Y}_n^\alpha)}$, thus $\widetilde{r}_{n+1} \in \mathcal{F}_n$.

Theorem 5.1. *Assume that* **(A1)**, **(A2)** *and* **(A3)** *hold and* $d = 2$.
If $0 < \alpha \le 1$, *then* h *has a unique zero* $y^* \in I^*$ *and*

$$\frac{\mathrm{Tr}(Y_n)}{n + \mathrm{Tr}(Y_0)} \xrightarrow[n\to\infty]{a.s.} 1, \quad \frac{Y_n}{\mathrm{Tr}(Y_n)} \xrightarrow[n\to\infty]{a.s.} y^* \quad \text{and} \quad \widetilde{N}_n \xrightarrow[n\to\infty]{a.s.} \frac{(y^*)^\alpha}{\mathrm{Tr}((y^*)^\alpha)}.$$

If $\alpha > 1$, *then* h *has a unique zero* $y^* \in I^*$ *or* ODE_h *has two attractive equilibrium points in* I^* *(as we have established in Section 2.3). Thus, the stochastic recursive procedure a.s. converges to one of the possible limit values.*

Proof. By the same arguments like in Section 2.3, $\mathrm{Tr}(Y_n)$ satisfies (2.16), therefore Proposition 2.1 holds. Consequently, $\widetilde{Y}_n$ lies in a compact of $\mathbb{R}_+$, thus

$$\max_{1\le i\le d}\left|\frac{f(Y_n^i)}{f(n+\mathrm{Tr}(Y_0))} - \left(\frac{Y_n}{n+\mathrm{Tr}(Y_0)}\right)^\alpha\right| \xrightarrow[n\to+\infty]{} 0.$$

Set $a_n^i = \frac{f(Y_n^i)}{f(n+\mathrm{Tr}(Y_0))}$ and $b_n^i = (\widetilde{Y}_n^i)^\alpha$, $i \in \{1,\dots,d\}$. Then, for every $i \in \{1,\dots,d\}$,

$$\frac{a_n^i}{\mathrm{Tr}(a_n)} - \frac{b_n^i}{\mathrm{Tr}(b_n)} = \frac{a_n^i - b_n^i}{\mathrm{Tr}(b_n)} + \frac{a_n^i}{\mathrm{Tr}(a_n)}\left(1 - \frac{\mathrm{Tr}(a_n)}{\mathrm{Tr}(b_n)}\right).$$

But

$$\mathrm{Tr}(b_n) = \sum_{i=1}^d (\widetilde{Y}_n^i)^\alpha \ge \begin{cases} \left(\sum_{i=1}^d \widetilde{Y}_n^i\right)^\alpha = \mathrm{Tr}(\widetilde{Y}_n)^\alpha & \text{if} \quad \alpha \in [0,1] \\ d^{1-\alpha}\mathrm{Tr}(\widetilde{Y}_n)^\alpha & \text{if} \quad \alpha > 1, \end{cases}$$

therefore

$$\mathrm{Tr}(b_n) \ge \frac{\mathrm{Tr}(\widetilde{Y}_n)^\alpha}{d^{(\alpha-1)_+}} \underset{a.s.}{\sim} \frac{(n+\mathrm{Tr}(\widetilde{Y}_0))^\alpha}{d^{(\alpha-1)_+}}.$$

Consequently, for every $i \in \{1,\dots,d\}$,

$$\frac{a_n^i}{\mathrm{Tr}(a_n)} - \frac{b_n^i}{\mathrm{Tr}(b_n)} \le \frac{\max_{1\le i\le d}|a_n^i - b_n^i| + \sum_{j=1}^d |a_n^j - b_n^j|}{\mathrm{Tr}(b_n)}$$

i.e.

$$\max_{1\le i\le d}\left|\frac{a_n^i}{\mathrm{Tr}(a_n)} - \frac{b_n^i}{\mathrm{Tr}(b_n)}\right| \le \frac{d+1}{\mathrm{Tr}(b_n)} \max_{1\le i\le d}|a_n^i - b_n^i| \xrightarrow[n\to+\infty]{a.s.} 0.$$

Thus

$$|\widehat{r}_{n+1}| \le |||H||| \max_{1\le i\le d}\left|\frac{a_n^i}{\mathrm{Tr}(a_n)} - \frac{b_n^i}{\mathrm{Tr}(b_n)}\right| + |||H_{n+1} - H||| \xrightarrow[n\to+\infty]{a.s.} 0$$

and in the same way $\widetilde{r}_{n+1} \xrightarrow[n\to+\infty]{a.s.} 0$. Consequently item *1.* follows from Proposition 3.1-*1.* and Theorem 3.1.
2. We have to check the assumption on the remainder term to apply result on traps for SA. We have that

$$\max_{1\le i\le d}\left|\frac{a_n^i}{\mathrm{Tr}(a_n)} - \frac{b_n^i}{\mathrm{Tr}(b_n)}\right| \lesssim \frac{(d+1)d^{(\alpha-1)_+}}{(n+\mathrm{Tr}(Y_0))^\alpha} \max_{1\le i\le d}|a_n^i - b_n^i| = o(n^{-\alpha}). \tag{5.28}$$

So, for $\alpha > 1$, under assumption **(A3)** on the generating matrices,

$$\sum_{n\ge 0} \|r_{n+1}\|^2 < +\infty.$$

The end of the proof follows from Proposition 3.1-*2.*-*3.* and Theorem 3.1. $\square$

To establish a CLT for the sequence $(\widetilde{Y}_n)_{n\geq 0}$ we need that the remainder term $(r_n)_{n\geq 1}$ satisfies (A.34). Then we will assume that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy **(A1)**-(ii) to ensure that $(\widetilde{Y}_n)_{n\geq 0}$ lies in the simplex (which implies that the rate in (5.28) is no more *a.s.*) and we assume also that $\alpha > 1/2$.

Theorem 5.2. *Assume that the index of regular variation $\alpha > 1/2$, that the addition rule matrices $(D_n)_{n\geq 1}$ satisfy* **(A1)**-(ii), **(A3)**, **(A4)** *and* $\mathbf{(A5)}_v$. *We have*

$$Sp\left(Dh(y^*)_{|\mathcal{S}_2}\right) = \{1, 1-\lambda\},$$

where

$$\lambda = \frac{f'(y^{*1})(p_1 - y^{*1}) + f'(1-y^{*1})(y^{*1} - 1 + p_2)}{f(y^{*1}) + f(1-y^{*1})}.$$

(a) If $p_1 + p_2 - 1 \leq 0$ and $v_n = 1$, $n \geq 1$, then

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{1-2\lambda}\Sigma\right) \quad \text{with} \quad \Sigma = \int_0^{+\infty} e^{u(Dh(y^*)-\frac{I}{2})}\Gamma e^{u(Dh(y^*)-\frac{I}{2})^t} du$$

$$\text{and} \quad \Gamma = \frac{(y^{*1})^\alpha C^1 + (1-y^{*1})^\alpha C^2}{\mathrm{Tr}((y^*)^\alpha)} - y^*(y^*)^t = a.s.\text{-}\lim_{n\to\infty} \mathbb{E}\left[\Delta M_n \Delta M_n^t \,|\, \mathcal{F}_{n-1}\right]. \tag{5.29}$$

If $p_1 + p_2 - 1 > 0$, we have three possible rate of convergence depending on the second eigenvalue:

(i) If $0 < \lambda < \frac{1}{2}$ and $v_n = 1$, $n \geq 1$, then

$$\sqrt{n}\left(\widetilde{Y}_n - y^*\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{1}{2\lambda - 1}\Sigma\right).$$

(ii) If $\lambda = \frac{1}{2}$ and $v_n = \log n$, $n \geq 1$, then

$$\sqrt{\frac{n}{\log n}}\left(\widetilde{Y}_n - y^*\right) \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma).$$

(iii) If $\frac{1}{2} < \lambda < 1$ and $v_n = n^{1-2\lambda+\eta}$, $\eta > 0$, then $n^\lambda\left(\widetilde{Y}_n - y^*\right)$ a.s. converges as $n \to +\infty$ towards a positive finite random variable Υ.

This result follows from Theorem A.3 and Theorem 3.2.

5.2 An application to Finance: adaptive asset allocation

Such urn based recursive procedures can be applied to adaptive portfolio allocation by an asset manager or a trader or to optimal split across liquidity pools. Indeed the first setting has already been done in [25] and successfully implemented with multi-armed bandit procedure. We develop in this section the adaptive portfolio allocation, but the optimal split across liquidity pools can be implemented in the same way, by considering that the different colors represent the different liquidity pools, and the trader want to optimally split a large volume of a single asset among the different possible destinations.

Imagine an asset manager who deals with a portfolio of d tradable assets. To optimize the yield of her portfolio, she can modify the proportions invested in each asset. She starts with the initial allocation vector Y_0. At stage n, she chooses a tradable asset according to the distribution (1.2) or (1.3) of X_n, then evaluates its performance over one time step and modifies the portfolio composition accordingly (most likely virtually) and proceeds. Thus the normalized urn composition $\widetilde{Y}_n$ represents the allocation vector among the assets and the addition rule matrices D_n model the successive reallocations depending on the past performances of the different assets. The evaluation of the asset performances can be carried out recursively with an estimator like with multi-arm clinical trials (see [4, 26]). In practice, it can be used to design the addition rule matrices D_n. For example, we may consider sequences of d independent $[0,1]$-valued random variables $(T_n^i)_{n\geq 1}$, $i \in \{1,\dots,d\}$, independent of the drawing X_n, such that

$$\mathbb{E}[T_n^i] = p_i, \quad 0 < p^i < 1, \quad i \in \{1,\dots,d\}.$$

If $(T_n^i)_{n\geq 1}$, $i \in \{1,\dots,d\}$, is simply a *success indicator*, namely d independent sequences of i.i.d. $\{0,1\}$-valued Bernoulli trials with respective parameter p_i, then the convention is to set $T_n^i = 1$ if the return of the i^{th} asset in the n^{th} reallocation is positive and $T_n^i = 0$ otherwise.

Let $N_n^i := \sum_{k=1}^n X_k^i$ be the number of times the i^{th} asset is selected among the first n stages with $N_0^i = 1$, $i \in \{1,\dots,d\}$, and let S_n be the d dimensional vector defined by

$$S_n^i = S_{n-1}^i + T_n^i X_n^i, \quad n \geq 1, \quad S_0^i = 1, \quad i \in \{1,\dots,d\},$$

denoting the number of successes of the i^{th} asset among these N_n^i reallocations. Define Π_n an estimator of the vector of success probabilities, namely $\Pi_n^i = \frac{S_n^i}{N_n^i}$, $i \in \{1,\dots,d\}$. We can prove that $\Pi_n \xrightarrow[n\to+\infty]{a.s.} p := (p^1,\dots,p^d)^t$ (see [4, 26]). Then we build the following addition rule matrices

$$D_{n+1} = \begin{pmatrix} T_{n+1}^1 & \frac{\Pi_n^1(1-T_{n+1}^2)}{\sum_{j\neq 2}\Pi_n^j} & \cdots & \frac{\Pi_n^1(1-T_{n+1}^d)}{\sum_{j\neq d}\Pi_n^j} \\ \frac{\Pi_n^2(1-T_{n+1}^1)}{\sum_{j\neq 1}\Pi_n^j} & T_{n+1}^2 & \cdots & \frac{\Pi_n^2(1-T_{n+1}^d)}{\sum_{j\neq d}\Pi_n^j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Pi_n^d(1-T_{n+1}^1)}{\sum_{j\neq 1}^d\Pi_n^j} & \frac{\Pi_n^d(1-T_{n+1}^2)}{\sum_{j\neq 2}^d\Pi_n^j} & \cdots & T_{n+1}^d \end{pmatrix}, \tag{5.30}$$

i.e. at stage $n+1$, if the return of the j^{th} asset is positive, then one ball of type j is added in the urn. Otherwise, $\frac{\Pi_n^i}{\sum_{k\neq j}\Pi_n^k}$ (virtual) balls of type i, $i \neq j$, are added. This addition rule matrix clearly satisfies **(A1)**-(i) and **(A2)**. Then, one easily checks that the generating matrices are given by

$$H_{n+1} = \mathbb{E}\left[D_{n+1}\,|\,\mathcal{F}_n\right] = \begin{pmatrix} p_1 & \frac{\Pi_n^1(1-p_2)}{\sum_{j\neq 2}\Pi_n^j} & \cdots & \frac{\Pi_n^1(1-p_d)}{\sum_{j\neq d}\Pi_n^j} \\ \frac{\Pi_n^2(1-p_1)}{\sum_{j\neq 1}\Pi_n^j} & p_2 & \cdots & \frac{\Pi_n^2(1-p_d)}{\sum_{j\neq d}\Pi_n^j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\Pi_n^d(1-p_1)}{\sum_{j\neq 1}\Pi_n^j} & \frac{\Pi_n^d(1-p_2)}{\sum_{j\neq 2}\Pi_n^j} & \cdots & p_d \end{pmatrix}$$

and satisfy **(A1)**-(ii). As soon as $Y_0 \in \mathbb{R}_+^d \setminus \{0\}$, $H_n \xrightarrow{a.s.} H$ (see [4, 26]) where

$$H = \begin{pmatrix} p^1 & \frac{p^1(1-p^2)}{\sum_{j\neq 2} p^j} & \cdots & \frac{p^1(1-p^d)}{\sum_{j\neq d} p^j} \\ \frac{p^2(1-p^1)}{\sum_{j\neq 1} p^j} & p^2 & \cdots & \frac{p^2(1-p^d)}{\sum_{j\neq d} p^j} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p^d(1-p^1)}{\sum_{j\neq 1} p^j} & \frac{p^d(1-p^2)}{\sum_{j\neq 2} p^j} & \cdots & p^d \end{pmatrix}.$$

Let remark that H is $\mathbb{R}$-diagonalizable since it is symmetric with respect to its invariant measure. Therefore, the number of each asset in the portfolio Y_n follows the dynamics (1.1) and the repartition of the portfolio in each asset follows the dynamics (2.13) or (5.26) depending on the drawing rule.

Here the components of the limit generating matrix H can be interpreted as constraints on the composition of the portfolio. Indeed, in presence of two assets (or colors), we prove that the first component of the allocation vector y^{*1} lies in I^* (see Proposition 3.1), therefore the portfolio will contain at least $p_1 \vee (1-p_2)\%$ and no more than $p_1 \wedge (1-p_2)\%$ of the first asset. Such rules may be prescribed by the regulation, the bank policy or the bank customer, and our approach is a natural way to have them satisfied (at least asymptotically).

The idea of reinforcing the drawing rule (instead of considering the uniform drawing) like in (1.2) or (1.3) can be interpreted as a way to take into account the risk aversion of the trader or the customer. Indeed, if f is concave the equilibrium point will be in the middle of the simplex (see Theorem 3.2 and Theorem 3.3), so the trader prefers to have diversification in her portfolio. On the contrary, if f is convex, the equilibrium points will lie on the boundary of the set of constraints induced by the limit generating matrix H, so she prefers to take advantage of the most money-making asset (like in a "winner take all" or a "0-1" strategy).

Numerical experiments. We present some numerical experiment for the drawing rule defined by (1.2), firstly with a concave function $f : y \mapsto \sqrt{y}$ and secondly with a convex function $f : y \mapsto y^4$. Therefore we have a unique equilibrium point in the first setting and two attractive targets in the second framework. We consider an asset manager who deal with a portfolio of 2 tradable assets. We model the addition rule matrices like in the multi-arm clinical trials, namely D_n is defined by (5.30). We use the same success probabilities, namely $p_1 = 0.7$ and $p_2 = 0.75$, and the initial urn composition is chosen randomly in the simplex $\mathcal{S}_2$.

$\triangleright$ *Convergence of the portfolio allocation with concave drawing rule.* We have that $y^{*1} \in (0.25, 0.7)$ and y^{*1} and y^{*2} are close to $\frac{1}{2}$, so the portfolio is diversified because in this case the investor is risk adverse.

$\triangleright$ *Convergence of the portfolio allocation with convex drawing rule.*

In the convex framework, we have two possible strategies and they are close to the boundaries defined by regulation. Moreover the repartition of the portfolio between the two assets is more asymmetric, because the trader chooses to invest two times more in one asset than in the other.

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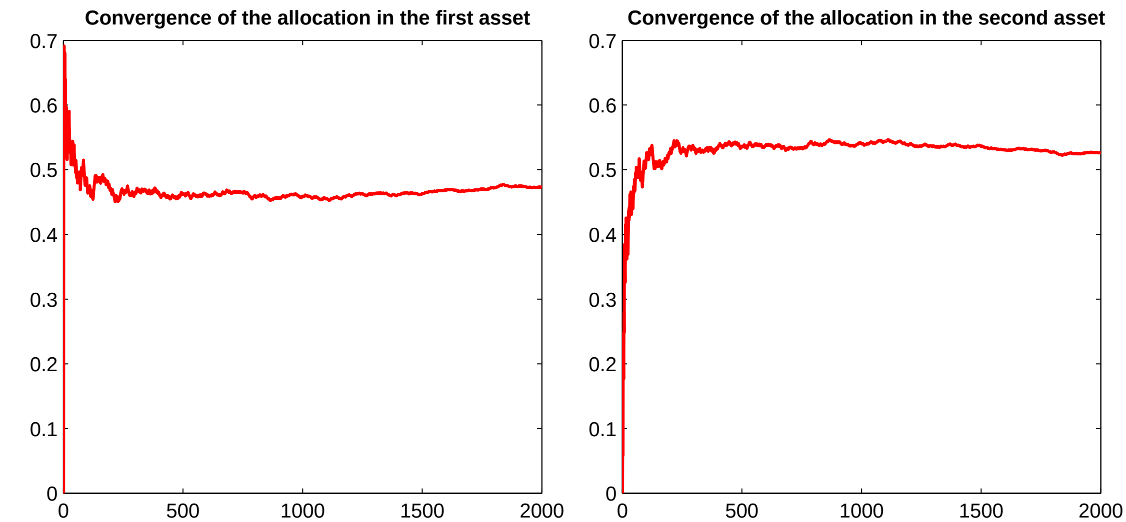

Figure 4: Convergence of $\widetilde{Y}_n$ toward y^* for $f(y) = \sqrt{y}$ with $p_1 = 0.7$ and $p_2 = 0.75$.

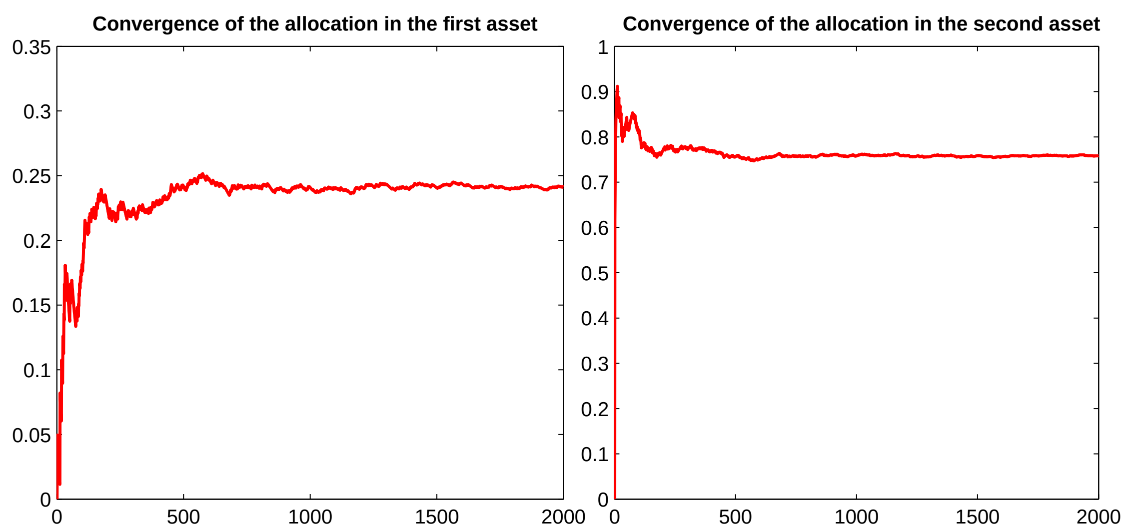

Figure 5: Convergence of $\widetilde{Y}_n$ toward y^* for $f(y) = y^4$ with $p_1 = 0.7$ and $p_2 = 0.75$.

Appendix

A Basic tools from Stochastic Approximation

Consider the following recursive procedure defined on a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_n)_{n\geq 0}, \mathbb{P})$

$$\forall\, n \geq n_0, \quad \theta_{n+1} = \theta_n - \gamma_{n+1} h(\theta_n) + \gamma_{n+1} \left(\Delta M_{n+1} + r_{n+1}\right), \tag{A.31}$$

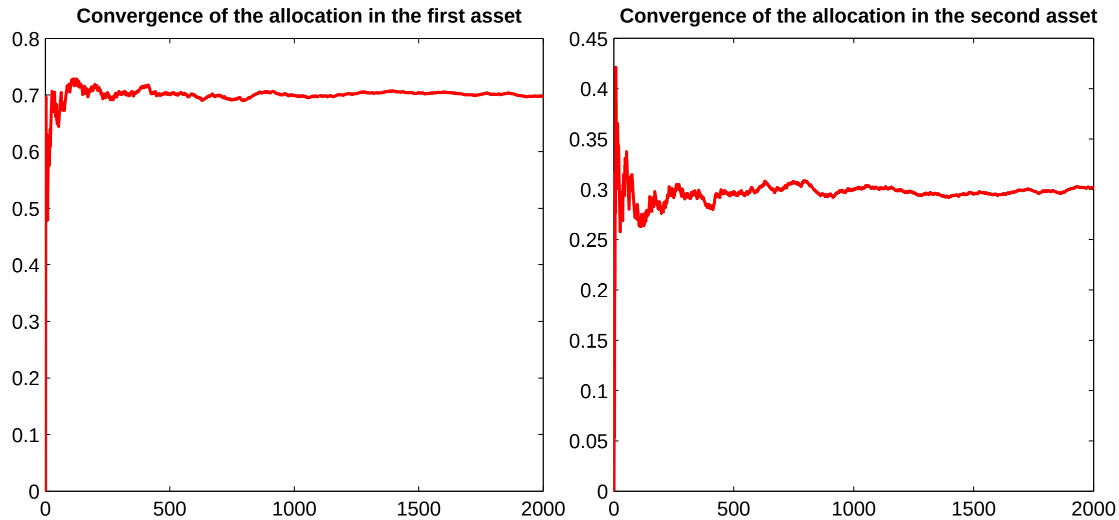

Figure 6: Convergence of $\widetilde{Y}_n$ toward y^* for $f(y) = y^4$ with $p_1 = 0.7$ and $p_2 = 0.75$.

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous function with linear growth, θ_{n_0} an $\mathcal{F}_{n_0}$-measurable finite random vector and, for every $n \geq n_0$, ΔM_{n+1} is an $\mathcal{F}_n$-martingale increment and r_n is an $\mathcal{F}_n$-adapted remainder term.

Note that the assumptions of the theorems recalled below are possibly not minimal, but adapted to the problems we want to solve.

$\triangleright$ *A.s.* **Convergence.** Let us introduce a few additional notions on differential systems. We consider the differential system $ODE_h \equiv \dot{x} = -h(x)$ where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous vector field. We assume that this system has a *flow* $\Phi(t,x)_{t\in\mathbb{R}_+, x\in\mathbb{R}^d}$: for every $x \in \mathbb{R}^d$, $(\Phi(t,x))_{t\geq 0}$ is the unique solution to ODE_h defined on the whole positive real line. This flow exists as soon as h is locally Lipschitz with linear growth.

Let K be a compact connected, flow invariant subset of $\mathbb{R}^d$ *i.e.* such that $\Phi(t, K) \subset K$ for every $t \in \mathbb{R}_+$.

A subset $A \subset K$ is an internal attractor of K for ODE_h if

(i) $A \subsetneq K$,

(ii) $\exists\, \varepsilon_0 > 0$ such that $\displaystyle\sup_{x\in K, \mathrm{dist}(x,A)\leq\varepsilon_0} \mathrm{dist}\big(\Phi(t,x), A\big) \to 0$ as $t \to +\infty$.

A compact connected flow invariant set K is a *minimal attractor* for ODE_h if it contains no internal attractor.

Theorem A.1. *(A.s. convergence with ODE method, see e.g. [9, 17, 23, 18, 5]). Assume that $h : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz function with linear growth,*

$$r_n \xrightarrow[n\to\infty]{a.s.} 0 \quad \text{and} \quad \sup_{n\geq n_0} \mathbb{E}\left[\|\Delta M_{n+1}\|^2 \mid \mathcal{F}_n\right] < +\infty \quad a.s.,$$

and that $(\gamma_n)_{n\geq 1}$ *is a positive sequence satisfying*

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty.$$

On the event $A_\infty = \big\{\omega \,|\, (h(\theta_n(\omega)))_{n\geq 0} \text{ is bounded}\big\}$, $\mathbb{P}(d\omega)$*-a.s., the set* $\Theta^\infty(\omega)$ *of the limiting values of* $(\theta_n(\omega)_{n\geq 0})$ *as* $n \to +\infty$ *is a compact connected flow invariant minimal attractor for* ODE_h *(see Proposition 5.3 in Section 5.1 in [5]).*

Furthermore

(a) *If* $\operatorname{dist}\big(\Phi(\theta_0, t), \{h = 0\}\big) \to 0$ *as* $t \to +\infty$*, for every* $\theta_0 \in \mathbb{R}^d$*, then* $\Theta^\infty(\omega) \cap \{h = 0\} \neq \emptyset$.

(b) *If* $\{h = 0\} = \{\theta^*\}$ *and* $\Phi(\theta_0, t) \to \theta^*$ *as* $t \to +\infty$ *locally uniformly in* θ_0*, then* $\Theta^\infty(\omega) = \{\theta^*\}$ *i.e.* $\theta_n \xrightarrow{a.s.} \theta^*$ *as* $n \to +\infty$.

(c) *If* $d = 1$ *and* $\{h = 0\}$ *is locally finite, then* $\Theta^\infty(\omega) = \{\theta_\infty\} \subset \{h = 0\}$ *i.e.* $\theta_n \xrightarrow{a.s.} \theta_\infty \in \{h = 0\}$.

$\triangleright$ **Traps (unstable equilibrium point).** This second theorem deals with "traps" *i.e.* repulsive zeros of the mean function h. It shows that, provided such a trap is noisy enough, such a trap cannot be a limiting point of the algorithm.

Theorem A.2. *(Non-a.s. convergence toward a trap, see e.g. [12, 16]). Assume that* $z^* \in \mathbb{R}^d$ *is a trap for the stochastic algorithm (A.31), i.e. (i)* $h(z^*) = 0$*, (ii) there exists a neighborhood* $V(z^*)$ *of* z^* *in which* h *is differentiable with a Lipschitz continuous differential, (iii) the eigenvalue of* $Dh(z^*)$ *with the lowest real part, denoted by* $\lambda_{\min}$*, satisfies* $\Re e(\lambda_{\min}) < 0$*. Assume furthermore that a.s. on* $\Gamma(z^*) = \{\theta_n \xrightarrow[n\to\infty]{a.s.} z^*\}$,

$$\sum_{n\geq 1} \|r_n\|^2 < +\infty \quad \text{and} \quad \limsup_n \mathbb{E}\Big[\|\Delta M_{n+1}\|^2 \,|\, \mathcal{F}_n\Big] < +\infty.$$

Let K_+ *the subspace of* $\mathbb{R}^d$ *spanned by the eigenvectors whose associated eigenvalues have a non-negative real part and* K_- *the subset of* $\mathbb{R}^d$ *spanned by the eigenvectors whose associated eigenvalues have a negative real part (then* $\mathbb{R}^d = K_+ \oplus K_-$*). By setting* $\Delta M^{(r)}_{n+1}$ *the projection of* ΔM_{n+1} *on* K_- *alongside* K_+*, assume that a.s. on* $\Gamma(z^*)$

$$\liminf_n \mathbb{E}\Big[\Big\|\Delta M^{(r)}_{n+1}\Big\| \,|\, \mathcal{F}_n\Big] > 0. \tag{A.32}$$

Moreover, if the positive sequence $(\gamma_n)_{n\geq 1}$ *satisfies*

$$\sum_{n\geq 1} \gamma_n = +\infty \quad \text{and} \quad \sum_{n\geq 1} \gamma_n^2 < +\infty,$$

then the event $\Gamma(z^*)$ *is negligible.*

$\triangleright$ **Rate(s) of convergence.** We will say that h is ϵ-differentiable ($\epsilon > 0$) at θ^* if

$$h(\theta) = h(\theta^*) + Dh(\theta^*)(\theta - \theta^*) + o(\|\theta - \theta^*\|^{1+\epsilon}) \quad \text{as} \quad \theta \to \theta^*.$$

Theorem A.3. *(Rate of convergence see [17] Theorem 3.III.14 p.131 (for the CLT see also e.g. [9, 23])). Let θ^* be an equilibrium point of $\{h = 0\}$. Assume that the function h is differentiable at θ^* and all the eigenvalues of $Dh(\theta^*)$ have positive real parts. Assume that for some $\delta > 0$,*

$$\sup_{n\geq n_0} \mathbb{E}\left[\|\Delta M_{n+1}\|^{2+\delta} \,|\, \mathcal{F}_n\right] < +\infty \; a.s., \quad \mathbb{E}\left[\Delta M_{n+1}\Delta M_{n+1}^t \,|\, \mathcal{F}_n\right] \xrightarrow[n\to+\infty]{a.s.} \Gamma, \tag{A.33}$$

where $\Gamma \in \mathcal{S}^+(d,\mathbb{R})$ (deterministic symmetric positive matrix) and for an $\varepsilon > 0$ where

$$n\, v_n \mathbb{E}\left[\|r_{n+1}\|^2 \mathbb{1}_{\{\|\theta_n-\theta^*\|\leq\varepsilon\}}\right] \xrightarrow[n\to+\infty]{} 0. \tag{A.34}$$

Specify the gain parameter sequence as follows

$$\forall n \geq 1, \quad \gamma_n = \frac{1}{n}. \tag{A.35}$$

Let $\lambda_{\min}$ denote the eigenvalue of $Dh(\theta^*)$ with the lowest real part and set $\Lambda := \Re(\lambda_{\min})$.
(a) If $\Lambda > \frac{1}{2}$ and $v_n = 1$, $n \geq 1$, then, the above a.s. convergence is ruled on the convergence event $\{\theta_n \longrightarrow \theta^*\}$ by the following Central Limit Theorem

$$\sqrt{n}\,(\theta_n - \theta^*) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}\left(0, \frac{\Sigma}{2\Lambda - 1}\right) \quad \text{with} \quad \Sigma := \int_0^{+\infty} \left(e^{-\left(Dh(\theta^*)-\frac{I_d}{2}\right)u}\right)^t \Gamma e^{-\left(Dh(\theta^*)-\frac{I_d}{2}\right)u} du.$$

(b) If $\Lambda = \frac{1}{2}$, $v_n = \log n$, $n \geq 1$, and h is ϵ-differentiable at θ^*, then

$$\sqrt{\frac{n}{\log n}}\,(\theta_n - \theta^*) \xrightarrow[n\to+\infty]{\mathcal{L}} \mathcal{N}(0, \Sigma).$$

(c) If $\Lambda \in \left(0, \frac{1}{2}\right)$, $v_n = n^{2\Lambda-1+\eta}$, $n \geq 1$ for some $\eta > 0$ and h is ϵ-differentiable at θ^* for some $\epsilon > 0$, then $n^\Lambda(\theta_n - \theta^*)$ is a.s. bounded as $n \to +\infty$.

If, moreover, $\Lambda = \lambda_{\min}$ ($\lambda_{\min}$ is real), then $n^\Lambda(\theta_n - \theta^*)$ a.s. converges as $n \to +\infty$ toward a finite random variable.

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