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Abstract. In this paper, we study languages of finite and infinite bi-rooted words. We show how the embedding of free ω-semigroups of finite and infinite words into the monoid of birooted words can be generalized to the embedding of two-sorted ω-semigroups into (some notion of) one-sorted ordered ω-monoids. This leads to an algebraic characterization of regular languages of finite and infinite birooted words that generalizes and unifies the known algebraic characterizations of regular languages of finite and infinite words[1].

1 Introduction

Infinite strings naturally arise in Software Engineering as models of (potentially) non-terminating system behaviors. From an abstract point of view, infinite strings are defined as infinite concatenations of non-empty finite strings, with an infinite associativity law ensuring that this infinite product is compatible with the standard concatenation of finite strings. Such an infinite string is depicted Figure 1. The resulting algebraic structures are the (free) ω-semigroups [30],
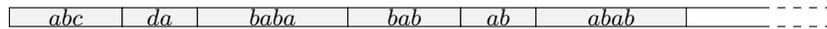
Fig. 1. An infinite product for $abcd(ab)^\omega$.

that is, the free semigroups of finite strings with string concatenation as product extended with the infinite product to produce infinite strings and a related mixed product to concatenate finite strings in front of infinite strings. This leads to the notions of ω-semigroups and morphisms that provide an algebraic characterization of regular languages of infinite words (see [30]). These results also offer quite a deep insight of the mathematical properties of Büchi's automata on infinite strings [3] and the associated algorithmic tools that have been developed for model-checking and program verification (see e.g. [9] for an overview).

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[1] This report is the long version of [5]

Programming languages such as Haskell [12] allow for effectively defining infinite streams of values by means of lazy evaluation mechanisms. In view of application to temporal media programming, that is, finite and infinite sequences of media type values such as images, sounds or control events, the abstract data type implicitly induced by ω-semigroups is enriched with the parallel product of finite or infinite strings. This leads to effective tools for handling temporal media types [10]. The notion of parallel product is depicted Figure 2.

Fig. 2. A parallel product of two words.

It has recently been advocated that there are benefits in embedding finite strings as well as infinite streams into (some notion of) *tiled temporal media* [13]. Typical temporal media synchronization constructs such as musical pickups (anacrusis) lead to distinguishing the effective starts of temporal media – the first note of a melody, from their logical starts – the first strong beat of that melody [1]. Then, the notion of tiled temporal media allows for a fairly simple modeling of these multi-level synchronization constructs. This comes from the fact that they are equipped with a *tiled product* that is neither a sequential nor a parallel product but both [21, 13].

More precisely, distinguishing string values from strings synchronization usage can be modeled by enriching the string data type by means of two synchronization marks: the *pre* synchronization mark or *input* root, and, respectively, the *post* synchronization mark or or *output* root (see Figure 3). These synchro-
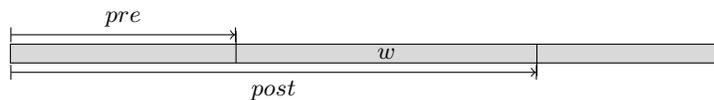
Fig. 3. A tiled temporal media (or birooted word).

nization marks tell how such enriched words are to be positioned with respect to the temporal media that comes before, or, respectively, after, the current temporal media. The resulting objects, here called birooted words, are then equipped with a single product, the *tiled product*, that is neither a sequential product as string concatenation, not a parallel product as string "zip", but both (see Figure 4 and [13] for a fairly simple implementation of these ideas on top of the musical library Euterpea [11] in Haskell).

In the general setting of higher-dimensional strings and tiling semigroups [22, 23, 20], the tiled temporal media and the related tiled product lay in the mathematical framework of inverse semigroup theory [25] and semigroups with local

Fig. 4. A tiled product instance

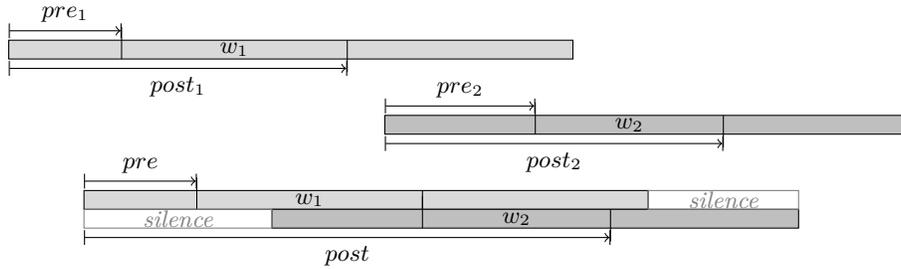units [24, 7, 8]. Projecting tiled temporal media onto semantical tags, we obtain sorts of tiled words. In the finite case, the induced algebra is the inverse semigroup of McAlister whose elements are birooted words [27, 26].

Aiming at providing a robust mathematical framework for handling languages of tagged tiled temporal media, our purpose here is to extend the language-theoretical tools available for languages of finite birooted words (see [17, 15, 4, 18]) to the case of infinite birooted words.

For such a purpose, we show that the obvious embedding of the free ω-semigroups of finite and infinite words into the monoid of birooted words can be generalized to the embedding of *two-sorted* ω-semigroups into *one-sorted* ordered ω-monoids.

This leads to an algebraic characterization of regular languages of finite and infinite birooted words that generalizes and unifies the algebraic characterizations of regular languages of finite and infinite words.

For the sake of simplicity, we restrict our presentation to one-sided birooted words. The resulting monoid turned out to be the filter completion (with respect to the natural order [25]) of the polycyclic monoid of Perrot and Nivat [29]. However, all proposed concepts and constructions can clearly be extended to two-sided birooted words, that is, within the filter completion of the submonoid of McAlister [25] of positive tiles. It follows that our results also apply to bi-infinite words with the resulting ordered ω-monoids equipped with both a left and a right infinite product.

Technically, our results capitalize on the quasi-inverse expansion of semigroups presented in [15, 16]. However, in the settings of infinite bi-rooted words, a *congruence property*, that appears in the notion of Ehresmann semigroups [24] is essential and was totally left unnoticed (and unused) in the finite case. The resulting algebraic framework is thus a refinement of the one proposed in [15, 14].

2   From finite or infinite words to birooted words

In this section we define the monoid of positive finite or infinite birooted words. From a mathematical point of view, that monoid is a submonoid of the filter

completion of the monoid of McAlister [26]. The language theory of the McAlister monoid has been developed in [17, 15, 18, 4] for finite tiles.

Let A be a finite alphabet. Let $(A^*, \cdot)$ be the free monoid of finite strings on the alphabet A, and let $(A^+, A^\omega, \cdot, *, \pi)$ be the associated free ω-semigroup (see [30]) with finite product $\cdot$, mixed product $*$ and infinite product π.

In the sequel, both the finite product $u_1 \cdot u_2$ when $u_1, u_2 \in A^*$ or the mixed product $u_1 * u_2$ when $u_1 \in A^*$ and $u_2 \in A^\omega$ may simply be denoted by $u_1 u_2$.

The set $A^\infty = A^* \cup A^\omega$ of finite and infinite strings is ordered by the *prefix order* $\leq_p$, defined, for every u and $v \in A^\infty$, by $u \leq_p v$ when either $u = v$, or u is finite and there is $w \in A^\infty$ such that $v = uw$. Extended with a maximum element denoted by 0, the set $A^\infty + 0$ ordered by the prefix order $\leq_p$ is a complete lattice. The *prefix join* $u \vee_p v$ of two words u and $v \in A^\infty$ is then the least word $w \in A^\infty$, if it exists, such that we have both $u \leq_p w$ and $v \leq_p w$, or 0 otherwise. Then, for every $u \in A^*$ and $v \in A^\infty$, the *right residual* $u^{-1}(v)$ of v by u, is defined as the word $w \in A^\infty$, unique if it exists, such that $v = uw$. We take $u^{-1}(v) = 0$ otherwise. By definition, $u^{-1}(v) \neq 0$ if and only if $u \leq_p v$.

Definition 1. A *positive (right) birooted word* u is a pair $u = (u_1, u_2)$ where $u_1 \in A^*$ and $u_2 \in A^\infty$. The word $u_1 u_2 \in A^\infty$ is called the *domain* of the birooted word u, and the word u_1, its *root path*. The birooted word (u_1, u_2) is finite when u_2 is finite. The set of positive birooted words on the alphabet A is denoted by $T^\infty(A)$. The set of finite positive birooted words on A is denoted by $T^+(A)$.

The product $u \cdot v$ of two birooted words $u = (u_1, u_2)$ and $v = (v_1, v_2)$ is defined as the birooted word $w = (w_1, w_2)$ with $w_1 = u_1 v_1$ and $w_2 = v_1^{-1}(u_2) \vee_p v_2$ when $v_1^{-1}(u_2) \vee_p v_2 \neq 0$. Otherwise, we take $u \cdot v = 0$ for some new birooted word 0, with $0 \cdot u = u \cdot 0 = 0$ for every $u \in T^\infty(A)$, and $0 \cdot 0 = 0$.

Examples. The following examples illustrate the definition of the product:

$$(ab, ab) \cdot (a, bc) = (aba, bc) \qquad (ab, (ab)^\omega) \cdot (a, bc) = 0$$
$$(ab, (ab)^\omega) \cdot (a, ba) = (aba, (ba)^\omega) \qquad (ab, ab) \cdot (a, (ba)^\omega) = (aba, (ba)^\omega)$$
$$(ab, ab) \cdot (a, (bc)^\omega) = (aba, (bc)^\omega) \qquad (ab, ac) \cdot (a, (ba)^\omega) = 0$$
$$(1, ab) \cdot (1, abc) = (1, abc) \qquad (1, ab) \cdot (1, ac) = 0$$

Let $T_0^\infty(A)$ (resp. $T_0^+(A)$) be the set of positive birooted words (resp. finite positive birooted words) extended with 0.

Theorem 2. *The set $T_0^\infty(A)$ equipped with the above product is a partially ordered monoid with unit $1 = (1, 1)$.*

Proof. Easily proved by routine arguments. $\square$

Let $u \mapsto u^R$ and, resp., $u \mapsto u^L$ be the right projection (or reset) and, resp., the left projection (or co-reset) defined on the monoid $T_0^\infty(A)$ by

$$0^R = 0^L = 0, \quad u^R = (1, u_1 u_2) \quad \text{and} \quad u^L = (1, u_2)$$

for every $u = (u_1, u_2) \in T_0^\infty(0)$. Then the *natural order relation* (see [24, 14]) is defined by

$$u \leq v \quad \text{when} \quad u = u^R \cdot v \cdot u^L, \quad \text{or, equivalently,} \quad u = u^R \cdot v$$

for every u and $v \in T_0^\infty(A)$. Elements u such that $u \leq 1$ are called *subunits* and the set of subunits is denoted by $U(T_0^\infty(A))$.

Theorem 3. *The natural order relation on $T_0^\infty(A)$ is a partial order relation stable under product, that is if $u \leq v$ then $uw \leq vw$ and $wu \leq wv$ for all $u, v, w \in T_0^\infty(A)$. Subunits and idempotents coincide, that is, we have $u \leq 1$ if and only if $u = uu$ for all $u \in T_0^\infty(A)$. Moreover, the set $U(T_0^\infty(A))$ ordered by the natural order is a complete lattice with product as meet.*

Proof. All statements can directly be proved by routine arguments. More interestingly, on can check that the set $T_0^\infty(A)$ of (right) birooted words is indeed isomorphic to the *filter completion* of the set $T_0^+(A)$ of *one-sided* finite birooted words ordered by the natural order, or, equivalently, the polycyclic monoid of Nivat and Perrot[29].

More precisely, an (order) filter is a *non empty* subset X of $T_0^+(A)$ that is *upward closed*, i.e. for every u and $v \in T_0^+(A)$, if $u \leq v$ and $u \in X$ then $v \in X$, and *downward directed*, i.e. for every u and $v \in X$ there exists $w \in X$ such that $w \leq u$ and $w \leq v$. The set $\mathcal{F}(T^+(A))$ of filters is equipped with the product defined, for every filters X and Y, by $X \cdot Y = \{w \in T_0^+(A) : \exists u \in X, v \in Y, u \cdot v \leq w\}$ that is well defined since the natural order is stable under product.

Then, one can show that the mapping $\varphi : T^\infty(A) \to \mathcal{F}(T^+(A))$ that maps every bi-rooted word $u \in T^\infty(A)$ to the set of finite birooted words $\varphi(u) = \{v \in T^+(A) : u \leq v\}$ is an isomorphism from $T^\infty(A)$ to the monoid of filters of $T^+(A)$. □

Remark. In the finite case [17, 15], we consider positive *two-sided* birooted words. Here, in a tile $(u_1, u_2) \in T_0^\infty(A)$, we always have empty domain on the left of the root path: the birooted words defined here are *one-sided*. A two-sided positive birooted word would be a triple (u_1, u_2, u_3) with a finite or infinite left word $u_1 \in {}^\infty A$, a finite root path $u_2 \in A^*$, and a finite or infinite right word $u_3 \in A^\infty$. All definitions and results presented here are easily extended to the two-sided case.

An immediate interest of the notion of finite and infinite birooted words is that the (two sorted) free omega monoid (A^+, A^ω) can be embedded into the (one sorted) monoid $T_0^\infty(A)$, that is, the pair $(T_0^\infty(A), U(T_0^\infty(A)))$ can be equipped with a mixed and an infinite product in such a way that it is an ω-semigroup into which the free ω-semigroup (A^+, A^ω) can be embedded. More precisely, let $u * v$ be the *mixed product* defined by $u * v = (u \cdot v)^R$ for all $u \in T_0^\infty(A)$ and $v \in U(T_0^\infty(A))$, and let $\pi(u_i)_{i \in \omega}$ be the *infinite product* defined by $\pi(u_i)_{i \in \omega} = \bigwedge_{n \in \omega}(u_0 \cdot u_1 \cdot \cdots \cdot u_{n-1})^R$ for all infinite sequences of birooted words $(u_i)_{i \in \omega}$. Then we have:

Theorem 4. *The pair* $(T_0^+(A), U(T_0^\infty(A)))$ *equipped with the finite product, the mixed product and the infinite product is a well-defined ω-semigroup.*

Moreover, let $\theta_f : A^* \to T_0^+(A)$ be the mapping defined by $\theta_f(v) = (v, 1)$ for every finite word $v \in A^*$ and let $\theta_\omega : A^\omega \to U(T_0^\infty(A))$ be the mapping defined by $\theta_\omega(w) = (1, w)$ for every infinite word $w \in A^\omega$.

Then, the pair of mappings $(\theta_f, \theta_\omega) : (A^+, A^\omega) \to (T_0^+(A), U(T_0^\infty(A)))$ is an ω-semigroup embedding, that is, a one-to-one mapping that preserves the finite, the mixed and the infinite products.

Proof. Clearly, the infinite product is well-defined since $U(T_0\infty)$ ordered by the natural order is a complete lattice with product as meet. Then, it is routine to check that the pair $(T_0^+(A), U(T_0^\infty(A))$ with product $\cdot$, mixed product $*$ and infinite product π is an ω-semigroup. Moreover, θ_f is clearly a one-to-one monoid morphism and θ_ω is one-to-one and, moreover, for every $v \in A^*$ and $w \in A^\omega$, we easily check that we have $\theta_\omega(v * w) = (\theta_f(v) \cdot \theta_\omega(w))^R$. Then it is routine to check that $\theta_\omega(v * w) = \theta_f(v) * \theta_\omega(w)$ for all $v \in A^*$ and $w \in A^\omega$ and that

$$\theta_\omega(\pi(u_i)_{i \in \omega}) = \pi_{i \in \omega} \theta_f(u_i)$$

for every $(u_i)_{i \in \omega} \in (A^+)^\omega$. In other words, the pair of mappings $(\theta_f, \theta_\omega)$ is an ω-semigroup morphism. $\square$

In the particular case where the infinite sequence $(u_i)_{i \in \omega}$ is *constant*, that is, when there is some $v \in T_0^\infty(A)$ such that $u_i = v$ for every $i \in \omega$, then we write v^ω for $\pi(u_i)_{i \in \omega}$. One can easily check that for every idempotent (or subunit) $u \leq 1$ we have $u^\omega = u$.

3   Embedding ω-semigroups into ordered ω-monoids

We show here that the above embedding of the free ω-semigroup (A^+, A^ω) into the monoid of birooted words $T^\infty(A)$ can be generalized to an embedding of any ω-semigroup S into (some notion of) ordered ω-monoid $M(S)$.

Let M be a monoid partially ordered by a relation $\leq$. We assume that the order relation $\leq$ is stable under product, i.e. if $x \leq y$ then $xz \leq yz$ and $zx \leq zy$ for every x, y and $z \in M$. The set $U(M)$ of *subunits* of the partially ordered monoid M is defined by $U(M) = \{y \in M : y \leq 1\}$.

The following definition is adapted from [18] and then, following [24], refined with the congruence property [14, 6].

Definition 5 (Adequately ordered and E-ordered monoids). A (stable) partially ordered monoid M is an *adequately ordered monoid* when:

(A1) idempotent subunits: for every $x \in M$, if $x \leq 1$ then $xx = x$,
(A2) left and right projection: for every $x \in M$, both the left projection defined by $x^L = \min\{y \in U(M) : xy = x\}$ and the right projection defined by $x^R = \min\{y \in U(M) : yx = x\}$ exist in $U(M)$,

It is an Ehresmann-ordered monoid (or E-ordered monoid) when, moreover:

(A3) congruence property: for every $x, y, z \in M$, if $x^L = y^L$ then $(xz)^L = (yz)^L$ and if $x^R = y^R$ then $(zx)^R = (zy)^R$,

Examples. Every monoid trivially ordered is an adequately ordered monoid. Every inverse monoid ordered by the natural order [25] is also an adequately ordered monoid with left and right projections defined by $x^R = x \cdot x^{-1}$ and $x^L = x^{-1} \cdot x$ for every element x.

The notion of adequately ordered monoid is extended here with an infinite (right) product as follows.

Definition 6 (E-ordered ω-monoid). An *E-ordered ω-monoid* if an E-ordered monoid M equipped with an infinite product operator $\pi : M^\omega \to U(M)$ satisfying the following properties:

(I1) *subunit preservation:* for every $(x_i)_{i \in \omega}$ such that for every $i \in \omega$ we have $x_i = x$ for some $x \in U(M)$, then $\pi(x_i)_{i \in \omega} = x$,

(I2) *monotonicity:* for every infinite sequences $(x_i)_{i \in \omega} \in M^\omega$ and $(y_i)_{i\omega}$, if $x_i \leq y_i$ for every $i \in \omega$ then $\pi(x_i)_{i \in \omega} \leq \pi(y_i)_{i \in \omega}$,

(I3) *mixed associativity:* for every infinite sequence $(x_i)_{i \in \omega} \in M^\omega$, for every $x \in M$, given $x'_i = x$ when $i = 0$ and $x'_i = x_{i-1}$ when $i > 0$, we have $(x \cdot (\pi(x_i)_{i \in \omega}))^R = \pi(x'_i)_{i \in \omega}$

(I4) *infinite associativity:* for every infinite sequence $(x_i)_{i \in \omega} \in M^\omega$, for every strictly increasing sequence $(k_i)_{i \in \omega}$ of positive integers with $k_0 = 0$, given $y_i = x_{k_i} \cdot x_{k_i + 1} \cdots x_{k_{i+1} - 1}$ defined for every $i \in \omega$, we have $\pi(y_i)_{i \in \omega} = \pi(x_i)_{i \in \omega}$.

Definition 7 (Monoid completion). Let $S = (S_f, S_\omega)$ be an ω-semigroup with finite product $\cdot : S_f \times S_f \to S_f$, mixed product $* : S_f \times S_\omega \to S_\omega$ and infinite product $\pi : (S_f)^\omega \to S_\omega$. By definition (see [30]) the finite, mixed and infinite product are related by mixed and infinite associativity laws. Let S^1_f be the semigroup S_f extended with a unit and let $\mathcal{P}^*(S_\omega)$ be the set of non-empty subsets of S_ω.

The *monoid completion $M(S)$* of the ω-semigroup S is defined to be the $M(S) = S^1_f \times \mathcal{P}^*(S_\omega) + 0$ equipped with the product $\cdot$ defined, for every non zero element (x, X) and $(y, Y) \in M(S)$ by $(x, X) \cdot (y, Y) = (xy, y^{-1}(X) \cap Y)$ when $y^{-1}(X) \cap Y \neq 0$ with $y^{-1}(X) = \{z \in S_\omega : y * z \in X\}$ and is defined to be 0 all other cases, with $y^{-1}(X) = \{z \in S_\omega : y * z \in X\}$. Element of $M(S)$ are also ordered by the relation $\leq$ defined over $M(S)$ by taking 0 to be the smallest element and by $(x, X) \leq (y, Y)$ when $x = y$ and $X \subseteq Y$ for every (x, X) and $(y, Y) \in M(S)$.

Theorem 8. *The set $M(S)$ with the above product $\cdot$ and the natural order $\leq$ is an E-ordered monoid with unit $1 = (1, S_\omega)$.*

Proof. The proof follows from the following Lemmas.

Lemma 9 (Soundness). *The set $M(S)$ with the above product $\cdot$ and the natural order $\leq$ is a partially ordered monoid with unit $1 = (1, S_\omega)$.*

Proof. Associativity and stability of the order relation are easily proved with essentially the same argument as in [14] though with subsets of S_ω instead of right ideals of S_f. The fact that $1 = (1, S_\omega)$ makes no difficulty either. $\qquad\square$

Let $U(M(S)) = \{x \in M(S) : x \leq 1\}$ be the set of *subunits* of $M(S)$.

Lemma 10 (Adequacy). *The set $M(S)$ is an adequately ordered monoid.*

Proof. By definition, a subunit of $M(S)$ is either 0 or an element of the form $(1, X)$ for some $X \subseteq S_\omega$. It follows that (A1) holds: subunits are indeed idempotent elements.

Let us prove (A2). We easily check that $0^L = 0^R = 0$. Let $x \in M(S)$ be some non zero element of $M(S)$, that is an element of the form $x = (s, X)$. We prove that we have $x^R = (1, s * X)$ and its left projection $x^L = (1, X)$.

For the right projection, let $z = (1, s * X)$. We have $z \cdot x = (x, X \cap s^{-1}(s * X))$ but $X \subseteq s^{-1}(s * X)$ hence $z \cdot x = x$. Now, if $z' \cdot x = x$ for some $z' = (1, Y)$ then we have $X = X \cap s^{-1}(Y)$ hence $X \subseteq s^{-1}(Y)$ hence $s * X \subseteq Y$ that is, following the order definition, $z \leq z'$.

For the left projection, let $z = (1, X)$. We have $x \cdot z = (s, X \cap 1^{-1}(X)) = (s, X) = z$. If $x \cdot z' = x$ for some $z = (1, Y)$ we have $x \cdot z' = (s, Y \cap 1^{-1}(X)) = (s, X)$ hence $X \subseteq Y$ that is $z \leq z'$. $\qquad\square$

Lemma 11 (Congruence property). *The monoid $M(S)$ is even an E-ordered monoid.*

Proof. We prove that $M(S)$ satisfies (A3). Let $x, y, z \in S$. If any of x, y or z is zero then the claim is satisfied. Assume $x = (x_1, X)$, $y = (y_1, Y)$ and $z = (z_1, Z)$.

For the left projection, assume that $x^L = y^L$. This means that $X = Y$. We have $xz = (x_1 z_1, z_1^{-1}(X) \cap Z)$ hence $(xz)^L = (1, z_1^{-1}(X) \cap Z)$. A similar computation shows that $(yz)^L = (1, z_1^{-1}(Y) \cap Z)$ hence the claim since $X = Y$.

For the right projection, assume that $x^R = y^R$. This means that $x_1 * X = y_1 * Y$. We have $zx = (z_1 x_1, x_1^{-1}(Z) \cap X)$ hence $(zx)^R = (1, z_1 * x_1 * (x_1^{-1}(Z) \cap X))$. A similar computation shows that $(zy)^R = (1, z_1 y_1 (y_1^{-1}(Z) \cap Y))$. Now let $z' \in z_1 * x_1 * (x_1^{-1}(Z) \cap X)$. By definition, and mixed associativity in S hence we do not bother about parenthesis, this means there is $x' \in X$ such that $z' = z_1 * x_1 * x'$ with the additional constraint that $x_1 * x' \in Z$ since $x' \in x_1^{-1}(Z)$. But since $x_1 * X = y_1 * Y$, this means there exists $y' \in Y$ such that $x_1 * x' = y_1 * y' \in Z$ hence $y' \in y_1^{-1}(Z) \cap Y$ and thus $z' = z_1 * x_1 * x' = z_1 * y_1 * y' \in z_1 * y_1 * (y_1^{-1}(Z) \cap Y)$. This proves that $(zy)^R \leq (zx)^R$. The reverse inequality is proved by applying a symmetrical argument. $\qquad\square$

This concludes the proof of Theorem 8. $\qquad\square$

Remark. The notion of E-ordered monoid, extending Ehresmann semigroups [24], appears in [6]. It is examined quite in detail in [14]. There, a general construction, based on left and right ideals, is provided to embed every monoid S into a non trivial adequately ordered monoid $\mathcal{Q}(S)$: the quasi-inverse expansion of S that

turned out to be E-ordered as well. One can observe that the construction of $M(S)$ is quite similar with right ideals replaced by non empty subsets of S_ω.

We define the infinite product π almost by iteration although, as well known in ω-language theory, the infinite product itself cannot be defined as a limit since many regula languages are not closed in prefix topology.

Among subunits of $M(S)$, the meet $\wedge$ in the order correspond to the product. Indeed, a subunit in $M(S)$ is either zero (with just behave like $(1, \emptyset)$) or of the form $(1, X)$ for some non empty $X \subseteq S_\omega$ with $(1, X) \cdot (1, Y) = (1, X \cap Y)$. The monoid of subunits $U(S)$ is thus isomorphic to the power set $\mathcal{P}(S_\omega)$ with intersection as product. In the next definition, when appropriate, we thus may use at will the meet operator $\wedge$ in place of the product.

Definition 12 (Infinite product). Let $(x_i)_{i \in \omega} \in (M(S))^\omega$. Let $\pi^0() = 1$ and for every $n \in \omega$, let $\pi^{n+1}(x_i)_{i \leq n} = (x_0 \cdot \pi^n(x_{i+1})_{i<n})^R$. Then, the infinite product $\pi(x_i)_{i \in \omega}$ is defined by

$$\pi(x_i)_{i \in \omega} = (1, X_\omega) \wedge \bigwedge_{n \in \omega} \pi^n(x_i)_{i \leq n}$$

with X_ω defined by $X_\omega = S_\omega$ when $J = \{i \in \omega : x_i \leq 1\}$ is finite and X_ω defined by $X_\omega = \{x_\omega\}$ when J is infinite, with $x_\omega = \pi(s_{j_i})_{i<\omega}$ where $(j_i)_{i \in \omega}$ is the increasing enumeration of the elements of J and, for very $j \in J$, the (non zero) element x_j is of the form $x_j = (s_j, X_j)$.

Theorem 13. *The E-ordered monoid $M(S)$ equipped with the above infinite product is an adequately ordered ω-monoid.*

Proof. Subunit preservation (I1) immediately follows from the definition. Monotonicity (I2) is also easily check. Indeed, if $(x_i)_{i \leq \omega} \leq (y_i)_{i \in \omega}$ then, by definition of the order relation, this means that for every $i \in \omega$ if $x_i = (s_i, X_i)$ for some $s_i \in S_f$ and $X_i \subseteq S_\omega$ then we have $y_i = (s_i, Y_i)$ for some $Y_i \subseteq S_\omega$ with $X_i \subseteq Y_i$. It is then routine to check that we indeed have $\pi(x_i)_{i \leq \omega} \leq \pi(y_i)_{i \in \omega}$.

Mixed associativity (I3) quite immediately follows from Lemma 11 and the mixed associativity in S.

It remains to check that the product and the infinite product also satisfies the infinite associativity law (I4). Let $(x_i)_{i \in \omega} \in (S_f \times \{S_\omega\})^\omega$, let $(j_i)_{i \in \omega}$ a strictly increasing sequence of integers with $j_0 = 0$, and, for every $i \in \omega$, let $y_i = x_{j_i} \cdot x_{j_i+1} \cdot \cdots \cdot x_{j_{i+1}-1}$. We have to prove that $\pi(x_i)_{i \in \omega} = \pi(y_i)_{i \in \omega}$. By definition, for every $i \in \omega$, we have

$$\pi^{j_{i+1}}(x_j)_{j<j_{i+1}} \leq \pi^i(y_j)_{j<i} \leq \pi^{j_i}(x_j)_{j<j_i}$$

hence $\bigwedge_{i \in \omega} \pi^i(x_j)_{j<i} = \bigwedge_{i \in \omega} \pi^i(y_j)_{j<i}$. In the case any of these meet is zero then we are done. Otherwise, assuming that, for every $i < \omega$ we have $x_i = (s_i, X_i)$ and $y_i = (t_i, Y_i)$ with $t_i = s_{j_i} \cdot s_{j_i+1} \cdot \cdots \cdot s_{j_{i+1}-1}$ we have to check that $X_\omega = Y_\omega$ as defined above. But this immediately follows from the definition and the fact S is an ω-semigroup. Indeed, either we have $X_\omega = Y_\omega = S_\omega$, or we have $X_\omega = \{x_\omega\}$

and $Y_\omega = \{y_\omega\}$ with $x_\omega = y_\omega$ that follows from the infinite associativity property in the ω-semigroup S. $\qquad\square$

Last, we aim at showing that the ω-monoid $S = (S_f, S_\omega)$ can be embedded as an ω-monoid into $M(S)$. This means defining over $M(S)$ an ω-monoid structure $(M_f(M), M_\omega(S))$, which can be done by taking $M_f(S) = M(S)$ for the "finitary" part and $M_\omega(S) = U(M(S))$ for the "infinitary" part. The infinite product is the one already defined, and the *mixed product* of two elements $x \in M(S)$ and $y \in U(M(S))$ is defined to be $x * y = (x \cdot y)^R$.

Theorem 14. *The mapping $\theta = (\theta_f, \theta_\omega) : (S_f, S_\omega) \to (M(S), U(M(S)))$, defined by $\theta_f(x) = (x, S_\omega)$ for every $x \in S_f$ and by $\theta_\omega(y) = (1, \{y\})$ for every $y \in S_\omega$, is a one-to-one ω-monoid morphism.*

Proof. We first check that the pair of sets $(M(S), U(M(S)))$ equipped with the product $\cdot$, the mixed product $*$ and the infinite product π is an ω-semigroup.

Lemma 9 ensures that $M(S)$ equipped with the product $\cdot$ is a semigroup. The mixed associativity law follows from Lemma 11. The infinite associativity law follows from Theorem 13.

The fact θ_f is a one-to-one monoid morphism follows from the fact that for every $x \in S_f$ we have $x^{-1}(S_\omega) = \{y \in S_\omega : x * y \in S_\omega\} = S_\omega$ hence we indeed have $\theta_f(x \cdot y) = \theta_f(x) \cdot \theta_f(y)$ for every $x, y \in S_f$.

Let us show it preserves the mixed product. Let $x \in S_f$ and $y \in S_\omega$. We have $\theta(x * y) = (1, \{x * y\})$ and $\theta(x) * \theta(y) = ((x, S_\omega) \cdot (1, \{y\}))^R = ((x, \{y\}))^R = (1, \{x * y\})$ hence we indeed have equality. Observe that this property is also a particular case of the congruence property proved in Lemma 11.

It remains to prove that θ also preserves infinite products. Let $(x_i)_{i<\omega} \in (S_f)^\omega$. By definition of θ_ω, we have $\theta_\omega(\pi(x_i)_{i\in\omega}) = (1, \{\pi(x_i)_{i\in\omega}\})$. By definition of the infinite product, since $\theta_f(x_i) \not\leq 1$ for every $i \in \omega$, we have $\pi(\theta_f(x_i))_{i\in\omega} = \theta_\omega(\pi(x_i)_{i\in\omega}) \wedge \bigwedge_{i\in\omega} \pi^i(\theta(x_j))_{j<i}$. Now, by definition of θ_f, for every $i \in \omega$, we have $\theta_f(x_i) = (x_i, S_\omega)_{i\in\omega}$ hence, by definition of π^i, $\pi^{i+1}\theta(x_j)_{j\leq i} = (1, \bigcap_{j\leq i} x_0 * x_1 * \cdots * x_j * S_\omega)$. Now, by mixed associativity in S, we have we have $\pi(x_i)_{i\in\omega} \in x_0 * x_1 * \cdots * x_i * S_\omega$ for every $i \in \omega$, hence $\theta_\omega(\pi(x_i)_{i\in\omega}) \leq \pi^i(\theta(x_j))_{j<i}$ henceforth $\pi(\theta_f(x_i))_{i\in\omega} = \theta_\omega(\pi(x_i)_{i\in\omega})$. $\qquad\square$

4 Application to language theory

We provide here a complete language theory for birooted words. Since every non-zero birooted word can just be seen as a pair composed of a finite and a finite or infinite word, the logical definability in Monadic Second-Order (MSO) logic of languages of birooted words boils down to the classical finite and infinite word language theory. More interestingly, we also provide a notion of finite-state birooted word automata and algebras that *essentially* captures definability in MSO logic.

A language of birooted words is a set $X \subseteq T^\infty(A)$ of *non-zero* birooted words. The class of such languages is equipped with the boolean operators union

(also called sum), intersection and complement, plus operators derived from the structure of $T_0^\infty(A)$: for all X and $Y \subseteq T^\infty(A)$, the product $X \cdot X$, the star X^*, as well as X^+ and the omega X^ω, are defined by extending the corresponding operators on $T_0^\infty(A)$ in a point-wise manner, *always omitting* the birooted word zero possibly resulting from these products. Additionally, we define the left and right projections X^L and X^R of the language X as the sets $X^L = \{x^L \in T^\infty(A) : x \in X\}$ and $X^R = \{x^R \in T^\infty(A) : x \in X\}$.

Theorem 15. *The class of languages of (positive) birooted words definable in MSO is closed under all the operators defined above. Moreover, it is finitely generated from the finite languages of (positive) birooted words, sum, product, star, omega and left and right projections.*

Proof. Easily follows from standard relativization techniques of mathematical logic. □

Combining the notions of Muller ω-word automata [28] (see also [30]) and of tile automata [18], we define birooted words ω-automata as follows.

Definition 16. A (finite) *birooted word ω-automaton* is a tuple $\mathcal{A} = \langle Q, \delta, K, W \rangle$ with a (finite) *set of states* Q, a *transition function* $\delta : A \to \mathcal{P}(Q \times Q)$, a *finitary acceptance condition* $K \subseteq Q \times Q$ and an *infinitary acceptance condition* $W \subseteq \mathcal{P}(Q)$. For technical reasons, we always assume that $\emptyset \in W$.

A *run* of the automaton $\mathcal{A}$ on a birooted word $u = (u_1, u_2)$ is a labeling mapping $\rho : [0, |u_1 * u_2| + 1[\to Q$ that satisfies the *local consistency* property : $(\rho(k), \rho(k+1)) \in \delta((u_1 * u_2)[k])$ for every $0 \leq k \leq |u_1| + |u_2|$.

The run ρ of the automaton $\mathcal{A}$ on the birooted word $u = (u_1, u_2)$ is *locally accepting* when the pair of states $(\rho(0), \rho(|u_1|))$ that marks the input and output roots belongs to K.

The run ρ is *globally accepting* when the set of states that occur infinitely often belongs to W, i.e. $\{q \in Q : |\varphi^{-1}(q)| = \infty\} \in W\}$. Observe that when (u_1, u_2) is finite then the global acceptance constraint is always satisfied since we assume that $\emptyset \in W$.

The language $L(\mathcal{A})$ is then defined as the set of birooted words $(u_1, u_2) \in T^\infty(A)$ for which there exists a locally and globally accepting run of $\mathcal{A}$ on (u_1, u_2).

Theorem 17. *A language $L \subseteq T^\infty(A)$ of non nul birooted words is recognized by a finite-state birooted word ω-automaton if and only if L is definable in MSO and upward closed.*

Proof. Assume that L is recognizable by a finite-state automaton. Since the existence of an accepting run on a birooted word amounts to checking the existence of some labeling of the vertices of (the graph representation of) the birooted word, this property is classically definable in MSO. Then, by definition of birooted word automata, the language L is upward closed w.r.t. in the birooted word order.

Conversely, assume that L is definable in MSO. By applying classical automata theoretic techniques, there exists an finite-state *word* automaton $\mathcal{A}'$ recognizing the language of finite and infinite words $L' = \{u_1 \# u_2 \in (A + \#)^\infty : (u_1, u_2) \in L\}$. It is not difficult to derive from the automaton $\mathcal{A}'$ a birooted word automaton $\mathcal{A}$ recognizing, under upward closure hypothesis, the language L. □

Corollary 18. *The class of languages recognized by a finite-state birooted word ω-automaton is closed under the sum, intersection, product, star and omega operations. Moreover, it is finitely generated from finite languages of (positive) birooted words, sum, product, star and omega, and the* upward closure *of left and right projections.*

The next definitions and theorem extend a similar result proved in [18] for languages of finite birooted words.

A mapping $\varphi : T_0^\infty(A) \to T$ from birooted words to an adequately ordered monoid T is a *premorphism* when $\varphi(1) = 1$ and $\varphi(xy) \leq \varphi(x)\varphi(y)$ and if $x \leq y$ then $\varphi(x) \leq \varphi(y)$ for every x and $y \in T_0^\infty(A)$.

The premorphism φ is *adequate* when, moreover, $\varphi(x^L) = (\varphi(x))^L$ and $\varphi(x^R) = (\varphi(x))^R$ for every $x \in S$, and if $x^L y^R \neq 0$ and $x^L \vee y^R = 1$ then $\varphi(xy) = \varphi(x)\varphi(y)$ for every x and $y \in S$. In the latter case we say that the product xy is disjoint.

The adequate premorphism φ is *ω-adequate* when T is an adequately ordered ω-monoid and, for every infinite sequence $(u_i)_{0 \leq i} \in (T_\infty^0(A))^\omega$, we have $\varphi(\pi(u_i)_{0 \leq i}) \leq \pi(\varphi(u_i))_{0 \leq i}$ and, moreover, if the product $\pi(u_i)_{0 \leq i}$ is disjoint (i.e. for every $0 < k$, given $x = u_0 \cdot u_1 \cdots u_{k-1}$ and $y = \pi(u_{i+k})_{0 \leq i}$ we have $x^L y^R \neq 0$ and $x^L \vee y^R = 1$), then we also have $\varphi(\pi(u_i)_{0 \leq i}) = \pi(\varphi(u_i))_{0 \leq i}$.

A language $L \subseteq T^\infty(A)$ is *quasi-recognizable* when there exists a finite adequately ordered ω-monoid S and an ω-adequate premorphism $\varphi : T_0^\infty(A) \to S$ such that $L = \varphi^{-1}(\varphi(T))$.

Theorem 19. *Let $L \subseteq T_0^\infty(A)$ be a language of non-zero finite or infinite birooted words. The language L is quasi-recognizable if and only if it is a finite boolean combination of upward-closed (for the natural order) languages definable in monadic second order logic (MSO).*

Proof. Assume that there is an ω-adequate premorphism $\varphi : T_0^\infty(A) \to S$ that recognizes L. Without loss of generality, possibly by adding a new zero to S, we may assume that $\varphi^{-1}(0) = 0$. Then, for every non-zero $s \in S$, $\varphi^{-1}(s)$ is definable in MSO.

Indeed, the mapping that maps every finite word $u \in A^*$ to $\varphi((u, 1))$ is a monoid morphism since, for every $a \in A$, every $u \in A^*$, the product $(a, 1) \cdot (u, 1)$ is disjoint. Then, given $(u_1, u_2) \in T^\infty(A)$, since $(u_1, u_2) = (u_1, 1) \cdot (1, u_2)$ with a disjoint product, we have $\varphi((u_1, u_2)) = \varphi((u_1, 1)) \cdot (\varphi(1, u_2))$. In the case u_2 is also finite, we are done since $\varphi(1, u_2) = \varphi((u_2, 1)^R) = \varphi((u_2, 1))^R$. In the case when u_2 is infinite, we can apply Ramsey's theorem and infinite associativity, following the technique used for infinite words [30], to show that $(1, u_2) = \pi((v_i, 1))_{0 \leq i}$ with $\varphi((v_i, 1)) = \varphi((v_1, 1))$ henceforth $\varphi((1, u_2)) = (\varphi((v_0, 1)) \cdot (\varphi((v_1, 1)))^\omega)^R$.

Then, by applying classical arguments of algebraic language theory, the fact that L is a finite boolean combination of upward closed languages is a consequence of the monotonicity of φ and the finiteness of S since, for every $s \in \varphi(L)$, we have $\varphi^{-1}(s) = \varphi^{-1}(s^\uparrow) \cap \varphi^{-1}(s^\downarrow)$ with $s^\uparrow$ (resp. $s^\downarrow$) is the upward closure (downward closure) of the singleton $\{s\}$.

Conversely, assume that L is a finite boolean combination of upward closed languages $\{L_i\}_{0 \leq i \leq n}$ definable in MSO. For every $0 \leq i \leq n$, by applying Theorem 17, the language L_i is recognized by a finite-state automaton $\mathcal{A}_i$. Now, by applying Lemma 20 below, for every $i \in I$, there exists an ω-adequate premorphisms $\varphi_i : T_0^\infty(A) \to S_i$ with finite S_i such that $L_i = \varphi_i^{-1}(\varphi_i(L_i))$ hence we conclude by applying classical algebraic techniques that the product premorphism

$$\varphi : T_0^\infty(A) \to S = S_0 \times S_1 \times \cdots \times S_n$$

defined for every $x \in T_0^\infty(A)$ by

$$\varphi(x) = (\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x))$$

quasi-recognized L. Proving that the product monoid S is a (finite) adequately ordered ω-monoid raises no particular difficulties. □

Lemma 20. *Let $\mathcal{A}$ be a finite-state birooted word ω-automaton. There exists a (finite) adequately ordered ω-monoid $S_\mathcal{A}$ and an ω-adequate premorphism $\varphi_\mathcal{A} : T_0^\infty(A) \to S_\mathcal{A}$ such that $L(\mathcal{A}) = \varphi_\mathcal{A}^{-1}(\varphi_\mathcal{A}(L(\mathcal{A}))$.*

Proof. Let $\mathcal{A} = \langle Q, \delta, K, W \rangle$ be the automaton. Let $T_Q \subseteq Q \times \mathcal{P}(Q) \times Q \times \mathcal{P}(Q)$ be the set of automaton traces, that is, quadruples of the from $(p, P, q, R) \in Q \times \mathcal{P}(Q) \times Q \times \mathcal{P}(Q)$ with $p \in P$ when $p \neq q$, with, intendedly, a input root state p, a set of states P occurring between the input and the output root, an output root state q and a set of states R occurring infinitely often.

The set T_Q is equipped with the partial product $\cdot$ defined, for every (p, P, q, R) and $(p', P', q', R') \in T_Q$, when $q = p'$ and either $R = \emptyset$ or $R' = \emptyset$ or $R = R'$, by

$$(p, P, q, R) \cdot (p', P', q', R') = (p, P \cup P', q', R \cup R')$$

One can check that, although partial, it is nevertheless associative in the sense that for every traces t, t', t", we have the product $(t \cdot t') \cdot t$" is defined if and only if the product $t \cdot (t' \cdot t$") is defined and, in that case, they are equal.

The set T_Q is also equipped with an infinite product π defined, for every sequence $(p_i, P_i, q_i, R_i)_{0 \leq i}$ of traces of T_Q, when $q_i = p_{i+1}$ and $R_i = \emptyset$ for every $0 \leq i$, by

$$\pi(p_i, P_i, q_i, \emptyset)_{0 \leq i} = (p_0, \{p_0\}, p_0, R)$$

with $R = \{q \in Q : |\{i \in \omega : q \in P_i\}| = \infty\}$.

Then we define $S_\mathcal{A}$ as the power set of T_Q, with product and infinite product defined as the point-wise extensions of the product and infinite product of T_Q. That is, for every X and $Y \subseteq T_Q$, we define $X \cdot Y$ as the set of all defined products $x \cdot y$ in T_Q with $x \in X$ and $y \in Y$, and, for every sequence $(X_i)_{0 \leq i}$

of subsets of traces, we define $\pi(X_i)_{0 \leq i}$ as the set of all defined products of the form $\pi(x_i)_{0 \leq i}$ with $x_i \in X_i$ for every $0 \leq i$.

It is routine to check that the monoid $S_{\mathcal{A}}$ obtained, ordered by inclusion, is an adequately ordered ω-monoid (following the technicalities used in [18, 16] for finite birooted words or trees).

We define $\varphi_{\mathcal{A}} : T_0^{\infty}(A) \rightarrow S_{\mathcal{A}}$ by $\varphi(0) = \emptyset$ and, for all birooted words (u_1, u_2), $\varphi((u_1, u_2))$ is the set of traces of the form (p, P, q, R) such that there exists a run ρ of the automaton $\mathcal{A}$ on (u_1, u_2) with $\rho(0) = p$, $P = \rho([0, |u_1|])$, $q = \rho(|u_1|)$ and $R = \{r \in |Q| : |\rho^{-1}(q)| = \infty\}$. Again, it is routine to check that $\varphi_{\mathcal{A}}$ is an ω-adequate premorphism.

We finally check that, by construction, a birooted word $x \in T_0^{\infty}(A)$ belongs to L if and only if there exist $(p, P, q, R) \in \varphi_{\mathcal{A}}(x)$ such that $(p, q) \in K$ and $R \in W$. $\qquad \square$

5 Conclusion

We have shown that the embedding of strings and streams into tiled streams, as done for programming purposes [21], can be lifted to arbitrary ω-semigroups, thus unifying the finite and infinite word language theory into the emerging language theory of birooted words.

Since regular languages of finite and infinite tiled streams can be used as programming language subtypes, this result is also a new step in developing the idea that, in the long term, inverse semigroups [25] may be tuned towards robust engineering practice.

Following [16, 19], we could also aim at generalizing the present approach to trees, possibly leading to further developments in the very subtle and difficult emerging algebraic theory of languages of infinite trees [2]. However, there is no evidence yet that such a generalization could lead to a successful algebraic characterization of infinite tree languages.

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