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Static and dynamic consistent perturbation analysis for nonlinear inextensible planar frames

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ABSTRACT
An asymptotically exact method for static and dynamic analysis of geometrically nonlinear planar frames is illustrated. The method is based on the integration of the nonlinear equations for the beam, carried out via a perturbation method, aiming to express the forces at the ends as series expansion of the displacements at the ends and of the distributed loads. Since the beams are assumed to be inextensible and unshearable, also reactive stresses appear among the unknowns, while compatibility conditions must be appended to the equilibrium equations. The element state relations are assembled for the frame, and discrete, nonlinear perturbation equations are derived. Examples are worked out and relevant results compared with purely numerical solutions.

1. Introduction
Perturbation methods consist in series solutions, able to extrapolate information from linear systems to nonlinear systems, provided nonlinearities are sufficiently weak (typically, when the amplitude of the response is comparatively small). Perturbation methods, of course, cannot compete in accuracy with purely numerical methods, but they offer the strong advantage to supply analytical solutions, in one or more parameters, with a little computational effort, which are particularly useful when the response of a family of systems is sought for, instead of that of a specific system.

Perturbation methods have been widely used to study free and forced nonlinear oscillatory phenomena [1], as well as bifurcation problems, both in static [2] and dynamical fields [3]. While straightforward expansions or the strained coordinate method [4] are sufficient to analyze static or buckling problems, respectively, a richer variety of approaches is likely to be followed in dynamics. Among them, the Multiple Scale Method [4] seems to be the most performing (see e.g. [5] for a discussion); moreover, it appears as the natural counterpart of the 'static perturbation method' [6], as discussed in [7]. The method calls for solving a chain of ode/pde linear equations, each requiring proper solvability conditions, whose combination furnishes 'amplitude modulation equations' which govern the slow flow. The method works in lowering the dimension of the original system, thus reducing the original multidimensional dynamic system into a smaller equivalent problem, similarly to what performed by the Center Manifold reduction [8] and Invariant Manifolds theories [9].

Although the algorithm is tailored to an analytical approach (so that innumerable examples have been worked out in literature), numerical analytical (or semi analytical) versions have also been proposed. A first attempt was made in [10], where it was stated that "Purely analytical techniques are capable of determining the response of structural elements having simple geometries [...], but they are not applicable to elements with complicated structure and boundaries. Numerical techniques are effective in determining the linear response of complicated structures, but they are not optimal for determining the nonlinear response of even simple elements [...]. Therefore, the optimum is a combined numerical and perturbation technique".

Indeed, numerical perturbation approaches have extensively been applied in literature (see, e.g. [11–14]). They usually consist in two steps, which combine (a) the FEM, which is used to formulate a discrete model, starting from a continuous one; and (b) the perturbation method, which is applied to solve the finite dimensional problem. The solution, therefore, is affected by two types of errors, one of local type, related to discretization adopted in each subdomain constituting the structure, the other of global type, related to solution by series of the final problem. Indeed, step (a) could be replaced itself by a local perturbation solution, able to express (although only in an asymptotic way) the solution of the field equation. Such an approach would avoid using 'a priori' selected interpolating functions, thus overcoming locking problems, and by making the two kinds of approximations mutually consistent. For this reason, such an approach will be referred here as a 'consistent perturbation analysis'. Moreover, it is important to stress, that
the adjective ‘numerical’, used here to partially connote the algorithm, should not be understood in the usual meaning of ‘approximation’, but rather as an ‘automatic procedure’ to numerically evaluate the coefficients of some series expansions, that would be impossible to perform manually. For these reasons, the process should be considered as ‘asymptotically exact’.

This approach, however, does not seem to be thoroughly investigated; yet, an attempt along this line was performed in [15], where, by dealing with continuous beams on sliding supports, an algorithm transforming a nonlinear continuous problem into a nonlinear discrete map was illustrated. Of course the consistent perturbation method can only be applied to systems made of elements whose field equations can be asymptotically integrated for general non homogeneous boundary conditions, typically one dimensional elements. However, it is expected that it can be applied also for bi or three dimensional elements, in the framework of a semi variational (Kantorovich) approach, as those used, e.g. in [16,17].

In this paper, the algorithm is detailed for a geometrically non linear planar frame, made of inextensible and shear indeformable beams, whose masses are lumped at their ends. The inextensible model appears physically reasonable and analytically suitable, since it reduces the number of field equations; in contrast, however, it calls for a proper treatment of the axial reactive stress, in the framework of a mixed displacement stress approach. Nevertheless, a constrained elastic problem for a single beam element basis, with \( \{a_x,a_y,a_z\} \) spanning the plane in which the beam bends, and \( a_z \) aligned with the beam axis in the reference configuration, it follows that (Fig. 1):

\[
\begin{align*}
\mathbf{M}(s) + S(s) \mathbf{R} \mathbf{R}(s) \sin \varphi(s) &= 0, \\
\mathbf{R}(s) &= B(s), \\
\mathbf{S}(s) &= B(s).
\end{align*}
\]

(2)

So that the equilibrium conditions (1) lead to three scalar equations:

\[
\begin{align*}
\mathbf{M}(s) + S(s) \cos \varphi(s) &\mathbf{R}(s) \sin \varphi(s) = 0, \\
\mathbf{R}(s) &= B(s), \\
\mathbf{S}(s) &= B(s).
\end{align*}
\]

(3)

while kinematic compatibility calls for internal geometrical con strains (Fig. 1):

\[
\begin{align*}
\cos \varphi(s) &\mathbf{R}(s) \sin \varphi(s) = 0, \\
\sin \varphi(s) &\mathbf{R}(s) \sin \varphi(s).
\end{align*}
\]

(4)

The material is assumed to behave as linearly elastic, so that the bending moment and the curvature are related by:

\[
\begin{align*}
\mathbf{M}(s) &= EI \mathbf{K}(s) \mathbf{R}(s) \sin \varphi(s),
\end{align*}
\]

(5)

with \( EI \) the (uniform) flexural stiffness.

The elastic problem of the internally constrained beam there fore consists of seven scalar Eqs. (2) (5), in which \( \mathbf{M}(s), \mathbf{R}(s), \mathbf{S}(s), u(s), \varphi(s), \kappa(s) \) are the scalar unknowns. After combination and partial integration, the problem is recast in the following form:

\[
\begin{align*}
EI \varphi(s) &\mathbf{R}(s) \sin \varphi(s) \\
R(s) &= B(s), \\
S(s) &= B(s).
\end{align*}
\]

(6)

where \( u_x = u(0), \varphi_x = \varphi(0), R_B = R(l), S_B = S(l) \) are integration con stants and \( l \) is the beam length. Eq. (6) would supply the solution of the problem once \( u_x, \varphi_x, R_B, S_B \) were assigned, together with prescribed rotations \( \varphi_x, \varphi_B \) at the ends:

\[
\begin{align*}
\varphi(0) &= \varphi_x, \\
\varphi(l) &= \varphi_B.
\end{align*}
\]

(7)

However, when the beam is considered as a frame element, the translations \( u_x, \varphi_x \) (equal to those of the attached joint) should be considered as assigned, instead of the reactive internal forces \( R_B, S_B \). In this perspective, it is convenient to consider the reactive formulation

2. Continuous formulation

A straight beam is considered, as an element of a planar frame. The beam is assumed internally constrained and massless. The continuous problem for the single element is formulated, and asymptotically solved.

2.1. Constrained elastic problem for a single beam

The (static) nonlinear elastic problem for a rectilinear beam, is formulated here. The beam is considered to be axially inextensible and shear indeformable, and modeled as an elastic, polar, one dimensional, internally constrained continuum. The field equilibrium equations, in vector form, turn out to be (Fig. 1):

\[
\begin{align*}
\mathbf{t}(s) + \mathbf{b}(s) &= 0, \\
\mathbf{m}(s) &= \mathbf{a}(s) \times \mathbf{t}(s) - 0.
\end{align*}
\]

(1)

where \( \mathbf{t}(s) \) and \( \mathbf{m}(s) \) are the internal force and couple, of reactive and active nature, respectively; \( \mathbf{b}(s) \) is the linear density of the external body forces; \( \mathbf{a}(s) \) is the unit vector tangent at the actual configuration; \( s \) is the curvilinear abscissa (both in the reference and actual configuration); finally, a prime denotes differentiation with respect to \( s \).

By introducing the components with respect to the \( \{a_x,a_y,a_z\} \) element basis, with \( \{a_x,a_y,a_z\} \) spanning the plane in which the beam bends, and \( a_z \) aligned with the beam axis in the reference configuration, it follows that (Fig. 1):

\[
\begin{align*}
\mathbf{M}(s) + S(s) \cos \varphi(s) &\mathbf{R}(s) \sin \varphi(s) = 0, \\
\mathbf{R}(s) &= B(s), \\
\mathbf{S}(s) &= B(s).
\end{align*}
\]

(2)

The beam undergoes a planar displacement field, where \( u(s) = u(s) a_x + v(s) a_y \), \( u(s) \) is the beam axis translation field and \( \varphi(s) \) the rotation field of the sections. The strain displacement relation expresses the link between the unique strain component, the curvature \( \kappa(s) \), and the rotation:

\[
\begin{align*}
\kappa(s) &= \varphi(s) \quad (3)
\end{align*}
\]

while kinematic compatibility calls for internal geometrical con straints (Fig. 1):

\[
\begin{align*}
u(s) \cos \varphi(s) &= 1, \\
v(s) \sin \varphi(s).
\end{align*}
\]

(4)

The material is assumed to behave as linearly elastic, so that the bending moment and the curvature are related by:

\[
\begin{align*}
\mathbf{M}(s) &= EI \mathbf{K}(s) \mathbf{R}(s) \sin \varphi(s),
\end{align*}
\]

(5)

with \( EI \) the (uniform) flexural stiffness.

The elastic problem of the internally constrained beam there fore consists of seven scalar Eqs. (2) (5), in which \( \mathbf{M}(s), \mathbf{R}(s), \mathbf{S}(s), u(s), \varphi(s), \kappa(s) \) are the scalar unknowns. After combination and partial integration, the problem is recast in the following form:

\[
\begin{align*}
EI \varphi(s) &\mathbf{R}(s) \sin \varphi(s) \\
R(s) &= B(s), \\
S(s) &= B(s).
\end{align*}
\]

(6)

where \( u_x = u(0), \varphi_x = \varphi(0), R_B = R(l), S_B = S(l) \) are integration con stants and \( l \) is the beam length. Eq. (6) would supply the solution of the problem once \( u_x, \varphi_x, R_B, S_B \) were assigned, together with precribed rotations \( \varphi_x, \varphi_B \) at the ends:

\[
\begin{align*}
\varphi(0) &= \varphi_x, \\
\varphi(l) &= \varphi_B.
\end{align*}
\]

(7)

However, when the beam is considered as a frame element, the translations \( u_x, \varphi_x \) (equal to those of the attached joint) should be considered as assigned, instead of the reactive internal forces \( R_B, S_B \). In this perspective, it is convenient to consider the reactive
Forces as Lagrange multipliers associated with kinematic constraints deduced by Eqs. (6a,5); namely:

\[
\begin{align*}
\mathbf{u}_b &= \mathbf{u}_A \int_0^l (\cos \phi(s)) \, ds, \\
\mathbf{v}_b &= \mathbf{v}_A \int_0^l \sin \phi(s) \, ds
\end{align*}
\]

in which the ‘known’ quantities \(\mathbf{u}_A, \mathbf{v}_A, \mathbf{u}_b, \mathbf{v}_b\) and the unknown function \(\phi(s)\) appear.

By summarizing, the problem is governed by a mixed differential algebraic problem in the unknowns \(\phi(s), R_b, S_b\), constituted by:

(i) a second order differential equation, obtained combining the first three Eqs. (6); (ii) the relevant boundary conditions Eqs. (7); (iii) the side algebraic conditions (8). By introducing the nondimensional quantities:

\[
\tilde{s} = \frac{s}{L}, \quad \tilde{u}_l = \frac{u_l}{L}, \quad \tilde{u}_h = \frac{u_h}{L}, \quad \tilde{v}_h = \frac{v_h}{L}, \quad \tilde{b}_x = \frac{b_x L^2}{E I}, \quad \tilde{R}_b = \frac{R_b L^2}{E I}, \quad \tilde{M}(s) = \frac{M(s) L}{E I}
\]

with \(H = A, B; \alpha = x, y\) and \(L\) a characteristic length, the problem reads:

\[
\begin{align*}
\tilde{s}''(s) + \left( \tilde{S}_B + \int_0^l \tilde{b}_y(s) \, ds \right) \cos \phi(s) + \left( \tilde{R}_B + \int_0^l \tilde{b}_y(s) \, ds \right) \sin \phi(s) &= 0, \\
\phi(0) &= \phi_A, \quad \phi(l) = \phi_B
\end{align*}
\]

where the prime denotes now differentiation with respect to \(s\) and the tilde has been dropped for notational convenience. As a solution strategy, Eqs. (10) should be solved to obtain \(\phi(s; R_b, S_b)\), in which \(R_b, S_b\) are parameters; after substitution into Eqs (10a), two nonlinear algebraic equations follow for these latter quantities. Finally, the nondimensional bending moment follows (tilde suppressed):

\[
\tilde{M}(s) \quad \phi(s)
\]

2.2. Perturbation solution

Since the solution to Eqs. (7) (10) cannot be pursued in closed form, a perturbation approach is followed. It is based on the hypothesis that all the quantities involved, loads, displacements, active and reactive stresses, are first order quantities, whose order of magnitude is \(O(\varepsilon)\), where \(0 < \varepsilon \ll 1\), is a small perturbation parameter, acting as bookkeeping, to be eliminated at the end of the analysis. Accordingly, displacements and reactions are expanded in truncated \(n\) term series as:

\[
\begin{align*}
\begin{pmatrix}
\phi(s) \\
R_B \\
S_B \\
\mathbf{u}_A \\
\mathbf{v}_A \\
\mathbf{u}_b \\
\mathbf{v}_b
\end{pmatrix}
&= 
\begin{pmatrix}
\phi_0(s) \\
R_{B0} \\
S_{B0} \\
\mathbf{u}_{A0} \\
\mathbf{v}_{A0} \\
\mathbf{u}_{b0} \\
\mathbf{v}_{b0}
\end{pmatrix} + 
\varepsilon \begin{pmatrix}
\phi_1(s) \\
\mathbf{R}_1 \\
\mathbf{S}_1 \\
\mathbf{u}_1 \\
\mathbf{v}_1 \\
\mathbf{u}_b \\
\mathbf{v}_b
\end{pmatrix} + 
\varepsilon^2 \begin{pmatrix}
\phi_2(s) \\
\mathbf{R}_2 \\
\mathbf{S}_2 \\
\mathbf{u}_2 \\
\mathbf{v}_2 \\
\mathbf{u}_b \\
\mathbf{v}_b
\end{pmatrix} + 
\varepsilon^3 \begin{pmatrix}
\phi_3(s) \\
\mathbf{R}_3 \\
\mathbf{S}_3 \\
\mathbf{u}_3 \\
\mathbf{v}_3 \\
\mathbf{u}_b \\
\mathbf{v}_b
\end{pmatrix}
\end{align*}
\]

Eqs. (14) (16) are sets of linear differential algebraic equations linking the \(k\)th order part of the unknowns, \(\phi_k(s), R_k, S_k\), to the \(k\)th order part of the assigned end displacements \(u_{A_k}, v_{A_k}, u_{B_k}, v_{B_k}\). Each linear problem is governed by the same linear operator (the tangent operator at the reference configuration of the beam), while the known term contains low order quantities, already determined at the previous steps.

The perturbation equations highlight the different role of the two reactive forces. While \(S_b\) enters the linear differential algebraic operator, \(R_b\), in contrast, does not appear in it. Consequently, \(R_{B0}\) is undetermined at the \(e\) order in the single beam problem,
and, similarly, \( R_2, R_3, \ldots \) are undetermined at \( \varepsilon^2, \varepsilon^3, \ldots \) orders. However, as it will be clearer ahead, they will be determined by the joint equilibrium conditions when elements will be assembled.

To limit algebra, the loads \( b_1, b_2 \) will be assumed ahead to be constant along the beam axis. By performing integrations and accounting for boundary conditions, the \( \varepsilon^3 \) order Eqs. (14) lead to:

\[
\begin{align*}
\phi_1(s) &= \left( 1 - \frac{3}{2} \varepsilon^2 \right) \phi_{A1} + \frac{1}{2} \phi_{B1} + \left( \frac{3}{2} \varepsilon^2 \right) \frac{d^2}{ds^2} S_{B1} + \frac{1}{6} \left( \frac{d^3}{ds^3} \right) b_y, \\
\phi_{B1} &= 0, \\
\phi_{B1} &= \frac{1}{2} \phi_{A1} + \frac{1}{2} \phi_{B1} + \frac{1}{12} \varepsilon^2 (3b_1^2 + 2s^3) b_y, \\
S_{B1} &= \frac{1}{2} \phi_{A1} + \frac{1}{2} \phi_{B1} + \frac{1}{2} (\phi_{B1} + \phi_{B1}) + \frac{\phi_{B1}}{2} \\
\phi_{B1} &= 0.
\end{align*}
\]

(17)

Eq. (17) can be solved with respect to \( S_{B1} \) and this, in turn, substituted in Eq. (17), to obtain \( \phi_1(s) \) expressed as a function of the nodal displacements and load; it results:

\[
\begin{align*}
\phi_1(s) &= \mathcal{F}_1(v_1, b_1) \\
S_{B1} &= \mathcal{S}_1(v_1, b_1) \\
\phi_{B1} &= 0
\end{align*}
\]

(19)

or, in symbolic form:

\[
\begin{align*}
\phi_1(s) &= \mathcal{F}_1(v_1, b_1, s) \\
S_{B1} &= \mathcal{S}_1(v_1, b_1, s) \\
\phi_{B1} &= 0
\end{align*}
\]

(20)

3.1. Element relation

To formulate the problem in a discrete form, it needs to link the external forces acting at the ends of the beam, namely \( \textbf{f} : (x_A, x_B, y_A, y_B, m_A, m_B)^T \), to the loads \( \textbf{b} \) and to the displacements at the same points, \( \textbf{u} : (u_A, u_B, v_A, v_B, \phi_A, \phi_B)^T \), where all the components are expressed in the element basis (Fig. 2). However, due to the inextensibility condition, forces depend on the transversal displacements \( \textbf{v} : (v_A, v_B, \phi_A, \phi_B)^T \) and on the reactive stress \( R_0 \) not on the longitudinal displacements \( u_A, u_B \); on the other hand, the kinematic constraint (10a) must be appended to this relation as a scalar condition involving all the displacements and \( R_0 \). Consequently, the relations sought for are of the following type:

\[
\begin{align*}
\textbf{f} &= \mathbf{f}(\textbf{v}, R_0, \textbf{b}) \\
0 &= \mathbf{g}(\textbf{v}, R_0, \textbf{b})
\end{align*}
\]

(22)

They will be named the state relations for the element. Their form suggests to introduce the following vectors:

\[
\begin{align*}
\textbf{Q} : (x_A, x_B | y_A, y_B, m_A, m_B) \\
\textbf{q} : (u_A, u_B, v_A, v_B, \phi_A, \phi_B)
\end{align*}
\]

(23)

said of the dependent and the independent state variables, respectively, which permit to write Eqs. (22) in the compact form:

\[
\textbf{Q} = \mathbf{Q}(\textbf{q}, \textbf{b})
\]

(24)

with \( \mathbf{Q} : (u_A, u_B, v_A, v_B, \phi_A, \phi_B) \). Of course, since the continuous problem has been solved in asymptotic form, also the state relation (23) will be asymptotic, namely \( \mathbf{Q}(\varepsilon) = \mathbf{Q}(\varepsilon, \textbf{b}) \) (with load rescaled). By letting \( \mathbf{Q} = \sum_\varepsilon \mathbf{Q}_\varepsilon \mathbf{q} \), expanding Eq. (24), it follows that:

\[
\varepsilon : \mathbf{Q}_1 \mathbf{L}_1 \mathbf{q}_1 \\
\varepsilon^2 : \mathbf{Q}_2 \mathbf{L}_2 \mathbf{q}_2 + \mathbf{N}_1(\mathbf{q}_1, \mathbf{b}) \\
\varepsilon^3 : \mathbf{Q}_3 \mathbf{L}_3 + \mathbf{N}_1(\mathbf{q}_1, \mathbf{b}), (\mathbf{q}_2, 0)) + \mathbf{N}_1(\mathbf{q}_1, \mathbf{b})
\]

(25)

where:

\[
\begin{align*}
\mathbf{L}_1 &= \begin{pmatrix} \frac{d\mathbf{Q}}{d\mathbf{q}} \end{pmatrix}_0, \\
\mathbf{p}_1 &= \begin{pmatrix} \frac{d\mathbf{Q}}{d\mathbf{b}} \end{pmatrix}_0 \\
\mathbf{b}
\end{align*}
\]

(26)

are the tangent matrix and the load vector, respectively; moreover:

\[
\begin{align*}
\mathbf{N}_1(\mathbf{q}_1, \mathbf{b}) &= \begin{pmatrix} \frac{1}{2} \left( \frac{d^2 \mathbf{Q}}{d\mathbf{q}^2} \right)_0 \mathbf{q}_1^2 + \frac{1}{2} \left( \frac{d \mathbf{Q}}{d\mathbf{b}} \right)_0 \mathbf{b}^2 \\
\mathbf{N}_1(\mathbf{q}_1, \mathbf{b}) &= \begin{pmatrix} \frac{1}{6} \left( \frac{d^3 \mathbf{Q}}{d\mathbf{q}^3} \right)_0 \mathbf{q}_1^3 + \frac{1}{2} \left( \frac{d^2 \mathbf{Q}}{d\mathbf{b}^2} \right)_0 \mathbf{b}^3 + \frac{\mathbf{Q}}{2} \left( \frac{d \mathbf{Q}}{d\mathbf{b}} \right)_0 \mathbf{b}
\end{align*}
\]

(27)

are quadratic, bilinear and cubic vector valued functions, related to second and third derivatives of \( \mathbf{Q} \). Eqs. (25a,b,c) are the \( \varepsilon \), \( \varepsilon^2 \) and \( \varepsilon^3 \) order parts, respectively, of the state relation (24).

Now, an explicit form for Eqs. (25) is derived. Forces at the ends are expressed by the boundary equilibrium conditions as:

\[
\begin{align*}
X_A &= R(0), R_B \\
X_B &= R(l), R_B \\
Y_A &= S(0), S_B \\
Y_B &= S(l), S_B \\
m_A &= M(0), \phi'(0) \\
m_B &= M(l), \phi'(l)
\end{align*}
\]

(28)
where \( L = L^T, \ b, n \) is a collocation vector, \( K \) is the familiar stiffness matrix of the (shear undeformable) rod and \( p \) is the relevant load vector. Explicit expressions for the operators \( N_j(q_i, b), N_{i,1}((q_i, b), (q_0, 0)) \) are reported in the Appendix B.

3.2. The static problem

When a planar frame is considered, made of \( M \) elements and \( N \) joints, the independent state variables are a set of \( 3 N \) displacements and \( M \) reactive stresses, i.e. \( q : (u_1, v_1, \varphi_1), \ldots, u_N, v_N, \varphi_N, R_{1b}, R_{2b}, \ldots, R_{Mb})^T \); here and in the following, an underbar denotes a vector (or a tensor) whose components are evaluated in an extrinsic (global) basis \((a, a, b)\), with \((a, a)\) spanning the plane of the frame. The relevant elastic problem is governed by a set of \((3N + M)\) equations, made of \( 3 N \) joint equilibrium equations, plus \( M \) constraint equations. In order to write these equations, the elements state vectors \( q^e \left( u^e, R_{eb}^e \right)^T \) and \( Q^e \left( f^e, 0 \right)^T \) appearing in the state relation (24) (the apex \( e \) being introduced here for convenience), must be expressed in the global basis. However, while displacements \( u^e \) and forces \( f^e \) obey the usual rule for basis rotations, the reactive force \( R_{eb}^e \) must be left unaltered in such a change. Therefore, the state vectors for the element transform as:

\[
q^e \rightarrow T^e q^e, \quad Q^e \rightarrow T^{eT} Q^e
\]

(32)

where:

\[
T^e = \begin{bmatrix} R_{eb}^e & 0 \\ 0 & 1 \end{bmatrix}
\]

is a \( 7 \times 7 \) transformation matrix, including the well known \( 6 \times 6 \) rotation matrix \( R_{eb}^e \). By using the coordinate transformations (32), in the state relation (24), namely \( Q^e = q^e(q_i, b) \), this latter becomes:

\[
Q^e \rightarrow T^{eT} Q^e \left( T^e q^e, b^e \right)
\]

(34)

or, in asymptotic form:

\[
\varepsilon : Q_1^e, \quad L' q_1^e, \quad p_1^e
\]

\[
\varepsilon^2 : L_2^e q_2^e + N_1^e \left( T^{eT} q_1^e, b^e \right)
\]

\[
\varepsilon^3 : L_3^e q_3^e + N_2^e \left( \left( T^{eT} q_1^e, \left( T^{eT} q_2^e, T^{eT} f^e \right), 0 \right) + N_3^e \left( T^{eT} q_1^e, b^e \right) \right)
\]

(35)

where:

\[
L_1^e \rightarrow T^{eT} L_1^e, \quad p_1^e \rightarrow T^{eT} p_1^e, \quad N_1^e \rightarrow T^{eT} N_2^e
\]

(36)

Finally, Eqs. (34) are assembled, to express the nodal equilibrium equations \( \sum_\varepsilon \Omega^e q^e + \sum_\varepsilon p^e = 0 \) and the compatibility equations \( \Omega^e = \Omega^e q^e \), where \( \Omega^e \) are collocation matrices and \( p^e \) nodal active forces. The state relation for the frame is therefore:

\[
\sum_\varepsilon \Omega^e q^e + \sum_\varepsilon p^e = 0
\]

(37)

in which a load parameter \( \mu \), affecting all the forces, has been introduced for later convenience. Finally, by using Eqs. (35) in Eq. (37), the following perturbation equations are generated:

\[
\varepsilon : L q_1^e + \mu q_1^e
\]

\[
\varepsilon^2 : L_2^e q_2^e + N_1^e (q_1^e, 0) - N_1^e (q_1^e, \mu b)
\]

\[
\varepsilon^3 : L_3^e q_3^e + N_2^e \left( \left( T^{eT} q_1^e, \left( T^{eT} q_2^e, T^{eT} f^e \right), 0 \right) + N_3^e \left( T^{eT} q_1^e, b^e \right) \right)
\]

(38)

where \( L^e, \sum_\varepsilon^e L^e T^e \) is the global system matrix; \( p^e, \sum_\varepsilon^e p^e, \sum_\varepsilon^e p^e \) is the global load vector; \( N_1(q_i), b) \) and \( (f_i, 0) \) (similar) are the global vectors of nonlinearities. Eqs. (38) asymptotically govern the static problem for the frame.
3.3. The dynamic problem and the multiple scale method

To tackle the dynamic problem, one has to model the inertia forces. Now, the mass density of the beam, \( m_e \), entails inertia forces that should be included in the field equations (1). These call for solving partial differential equations in space and time. A much simpler approach, however, consists in modelling the beams as massless and in lumping their total mass at the ends, although this requires a finer discretization. Lumped mass at the joints introduce inertia forces in the load vector \( P \), which therefore splits in an active and an inertial part, \( P_a + P_i \). To use quantities consistent with those in Eqs. (9), a nondimensional time \( \tau \) and a nondimensional linear mass density \( \tilde{m_e} \) are defined, as:

\[
\tau = \frac{\omega t}{\omega_f}, \quad \tilde{m_e} = \frac{m_e \omega^2 L^4}{EI}
\]  

where \( t \) is the true time and \( \omega_f \) is a characteristic frequency. By lumping the total mass of the element, \( m_e \tilde{r} \) at its ends and neglecting rotatory effects, it follows that forces \( \bar{P}_r(t) \) act at the ends of the element, where the dot denotes differentiation with respect to \( t \), and:

\[
\bar{M}' = \frac{1}{2} \tilde{m_e} \omega^2 \tilde{r} \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \end{bmatrix}
\]  

Fig. 4. Load-paths for the CS and HH frames of Fig. 3, displaying displacements \( u_B \) and \( v_D \) (in meters) vs the load multiplier \( \mu \) for the configurations of Fig. 3a and Fig. 3b respectively; curves: I linear, II quadratic, III cubic perturbation solution; \( \ast \) FEM solution.

Fig. 5. Deformed equilibrium configurations of the CS and HH frames, when \( \mu = 1 \): (a) and (c) I linear, II quadratic, III cubic solutions; (b) and (d) incremental contributions at different orders (Ic, IIc, IIIc).

Fig. 6. Hinged–Hinged and Clamped-Sliding frames and relevant nonlinear fundamental modes.
is the $7 \times 7$ element mass matrix; accordingly, the global inertia force vector is $\mathbf{p}(t) := \sum_l M_l q_l(t)$, with $M := \sum_l \Omega^2 \mathbf{M} \Omega^2 \mathbf{T}$ the global mass matrix. The state relation for the structure, Eq. (37), consequently modifies in:

$$M \ddot{q}(t) + \sum_l \Omega^2 \mathbf{T}^T Q^e \mathbf{T} \dot{\mathbf{q}}(t) + \mu \mathbf{p}(t) \mu \mathbf{p}(t) \quad (41)$$

where tilde has been dropped.

Eqs. (41) are rewritten in asymptotic form according to the expressions (35) for $Q^e$; moreover the Multiple Scale Method is used to express time dependence of the solution [1]. By introducing independent time scales $\tau_1 := \varepsilon \tau_2 := \varepsilon^2 \tau$ and by applying the chain rule for the second derivative, i.e. $d^2/d\tau^2 = (d/d\tau_0 + \varepsilon d/d\tau_1 + \varepsilon^2 d/d\tau_2)^2 = d^2/d\tau^2_0 + 2\varepsilon d^2/d\tau_1^2 + \varepsilon^2 (2\varepsilon d/d\tau_0 + \varepsilon d/d\tau_0 + \varepsilon^2 d/d\tau_1)^2 + \ldots$ where $d/d\tau_0$, $d/d\tau_1$, the following perturbation equations are derived:

$$\varepsilon : \mathbf{M}_s q_1 + \mathbf{L} q_1 + \mu \mathbf{p}(t) \mu \mathbf{p}(t)$$

$$\varepsilon^2 : \mathbf{M}_s q_2 + \mathbf{L} q_2 + 2\mathbf{M}_d d_1 q_1 + \mathbf{N}_2(q_1, q_2, \mu)$$

$$\varepsilon^3 : \mathbf{M}_s q_3 + \mathbf{L} q_3 + 2\mathbf{M}_d d_2 q_2 + 2\mathbf{M}_d d_2 q_3 + \mathbf{N}_3(q_1, q_2, q_3, \mu) \mu \mathbf{p}(t_0) \sigma$$

$$+ \mathbf{N}_4(\mathbf{q}_1, \mu \mathbf{b}(t_0), (\mathbf{q}_2, 0)) + \mathbf{N}_5(\mathbf{q}_3, \mu \mathbf{b}(t_0)) \quad (42)$$

where, $\mathbf{p}(t_0) := \sum \Omega^3 \mathbf{p}(t_0)$ and $\mathbf{q}_k := q_k(t_0 \tau_1, \tau_2)$. Eqs. (42) asymptotically govern the dynamic problem for the frame.

4. Solution to discrete equations

Eqs. (38) and (42), respectively governing the static and the dynamic problems for the frame, are solved.

4.1. The static response

When Eqs. (38) are solved in sequence, they furnish (underbar omitted):

$$\varepsilon : q_1 := \mu \mathbf{L}^1 \mathbf{p}$$

$$\varepsilon^2 : q_2 := \mathbf{L}^1 \mathbf{N}_2(\mu \mathbf{L}^1 \mathbf{p}, \mu \mathbf{b})$$

$$\varepsilon^3 : q_3 := \mathbf{L}^1 \mathbf{N}_3(\mu \mathbf{L}^1 \mathbf{p}, \mu \mathbf{b}), (\mathbf{L}^1 \mathbf{N}_2(\mu \mathbf{L}^1 \mathbf{p}, \mu \mathbf{b}), 0)$$

$$\mathbf{L}^1 \mathbf{N}_4(\mu \mathbf{L}^1 \mathbf{p}, \mu \mathbf{b}) \quad 0$$

The series $q := \sum \varepsilon^k q_k$, accordingly, reads:

$$q := \mu \mathbf{L}^1 \mathbf{p} + \varepsilon^3 \left\{ \mathbf{L}^1 \mathbf{N}_2(\mathbf{L}^1 \mathbf{p}, \mathbf{b}) + \mu \mathbf{L}^1 \mathbf{N}_4(\mathbf{L}^1 \mathbf{p}, \mathbf{b}) \right\} \left\{ \mathbf{L}^1 \mathbf{N}_2(\mathbf{L}^1 \mathbf{p}, \mathbf{b}), 0 \right\} \left( \mathbf{L}^1 \mathbf{N}_3(\mathbf{L}^1 \mathbf{p}, \mathbf{b}) \right) \quad (43)$$

where homogeneity of the $N$ functions has been accounted for and the perturbation parameter $\varepsilon$ has been reabsorbed through the inverse load rescaling $\varepsilon \mu \to \mu$. Eq. (44) provides a third order extrapolation from the origin of the relationship linking nodal displacements and reactions with the load multiplier.

4.2. The dynamic response to external harmonic excitation

The (undamped) harmonically forced response of the frame is now analyzed.

All loads, acting at the nodes, $P_0(t)$, and distributed on the beams, $b(t)$, are assumed, to be synchronous, obeying to the law:

$$
P_0(t) := P_0 \cos \Omega t \quad b(t) := b_0 \cos \Omega t$$

$$\Omega = \omega_0$$

$$\Omega = \omega_1$$

Fig. 8. Frequency response curves for the HH and CS frames, for different excitation amplitudes $2\rho_0 \Delta \omega = 0.001, 0.005, 0.01$; curves: third-order perturbation solution; FEM solution: ◦ true axial stiffness, • magnified axial stiffness.
with the excitation frequency. Accordingly, terms in Eqs. (42) that explicitly depend on time read:

\[ p(t_0) = p_0 \cos \omega_0 t_0 \]

\[ N_1(q_1, \mu b_0(t_0)) = N^{(0)} (q_1) + \mu N^{(1)} (q_1, b_0) \cos \omega_0 t_0 + \mu^2 N^{(2)} (q_1, b_0) \cos^2 \omega_0 t_0 \]

\[ N_1(q_1, \mu b_0(t_0)) = N^{(0)} (q_1) + \mu N^{(1)} (q_1, b_0) \cos \omega_0 t_0 + \mu^2 N^{(2)} (q_1, b_0) \cos^2 \omega_0 t_0 + \mu^3 N^{(3)} (b_0) \cos^3 \omega_0 t_0 \]  

(46)

where underbars have been omitted. It should be noticed that, since the distributed loads enter quadratic and cubic nonlinearities, parametric excitation terms of frequency \( \omega \) or \( 2 \omega \) and external excitation terms of frequency \( 3 \omega \) are generated. If, in contrast, only nodal forces are applied to the frame (i.e. \( b_0 = 0 \)), then, just the external excitation (46) exists, while nonlinearities (46) are of autonomous type.

The excitation frequency is here assumed to be nearly resonant with the \( k \)th natural frequency \( \omega_k \), i.e.:

\[ \Omega = \omega_k + \epsilon \sigma \]

(47)

where \( \sigma \) is a detuning parameter. Moreover, any internal resonances occurring among the natural frequencies, as well as any combination resonances occurring among the external and natural frequencies, are excluded. Hence, the response of the system is mono modal (at \( \epsilon \) order, but all modes contribute at higher orders, as forced by the leading one). Since the external excitation is resonant, it is assumed of soft type and therefore shifted at the \( \epsilon^3 \) order in the perturbation scheme [1]; this is formally achieved by taking \( \mu = O(\epsilon^3) \) instead of \( O(\epsilon) \), as so far assumed. Consequently the perturbation Eqs. (42) become:
\[
\varepsilon : M_d \ddot{\mathbf{q}}_1 + L_{\mathbf{q}}_1 = 0
\]
\[
\ddot{\mathbf{q}}_2 : M_d \ddot{\mathbf{q}}_2 + L_{\mathbf{q}}_2 = 2M_d \ddot{d} \mathbf{d}_1, \quad N_2^{(0)}(\mathbf{q}_1)
\]
\[
\ddot{\mathbf{q}}_3 : M_d \ddot{\mathbf{q}}_3 + L_{\mathbf{q}}_3 = M(2d_2 \ddot{d}_1 + 2d_2 d_1 \ddot{q}_1 + d_2^2 \ddot{\mathbf{q}}_1), \quad N_3^{(0)}(\mathbf{q}_1, \mathbf{q}_2)
\]

\[
N_3^{(0)}(\mathbf{q}_1, \mathbf{q}_2) + \mu \mathbf{p}_0 \cos \Omega_0 \tag{48}
\]

while the nonlinear effects of the distributed loads shift at orders higher than \( n \) of Eq. \((48)\), admits the generating solution:

\[
\mathbf{q}_1 \quad A_i(t_1, t_2) \mathbf{u}_i e^{i \omega t} + c.c.
\]

where \( \mathbf{u}_i \) is the \( i \)th natural mode, \( \omega_n \) the associated linear natural frequency, \( A_i(t_1, t_2) \) a slow time depending complex amplitude, and c.c. denotes complex conjugate terms. With Eq. \((49)\), Eq. \((48)\) reads:

\[
M_d \ddot{\mathbf{q}}_2 + L_{\mathbf{q}}_2 = 2i \omega_n d \mathbf{d}_1 A_i \mathbf{u}_i e^{i \omega t}
\]

\[
\left( A_i^2 e^{i \omega t} + A_i \bar{A}_i \right) N_2(\mathbf{u}_i) + c.c. \tag{50}
\]

where an overbar indicates the complex conjugate. By removing secular terms, \( d \mathbf{d}_1 = 0 \) is followed, and the Eq. \((50)\) furnishes:

\[
\mathbf{q}_2 \quad A_i^2(t_1, t_2) z_{11} e^{i \omega t} + A_i \bar{A}_i z_{11} + c.c.
\]

where \( z_{11} \) and \( z_{11} \) are solution of:

\[
(LK) \bar{M} z_{11} = N_2(\mathbf{u}_i),
\]

\[
L z_{11} = N_3(\mathbf{u}_i). \tag{52}
\]

Substitution in Eq. \((48_b)\) and imposition of solvability condition lead to:

\[
2i \omega_n \mathbf{u}_3^T M_d A_i \bar{A}_i (2N_2(\mathbf{u}_i, z_{11}) + 3N_3(\mathbf{u}_i, \mathbf{u}_i))
\]

\[
+ \mu \mathbf{p}_0 e^{i \omega t} \tag{53}
\]

Coming back to the true time \( t \), reabsorbing the parameter \( \varepsilon \), introducing the polar form \( A = a e^{i \varphi/2} \) and letting \( \gamma : \sigma t \ \varphi \) be the phase difference, it follows:

\[
\dot{\varphi} = \frac{c_2}{4c_1} a^2 \pm 2 \mu \frac{c_2}{c_1} \cos(\gamma)
\]

(54)

where

\[
\sigma a \quad \frac{c_2}{4c_1} a^2 + 2 \mu \frac{c_2}{c_1} = 0 \tag{55}
\]

Eq. \((55)\) supplies, in implicit form, the detuning vs the amplitude, \( \sigma = \sigma(a; \mu) \), with the load as a parameter, from which the forcing frequency amplitude response, \( \Omega = \omega + \sigma(a; \mu) \) is derived from Eq. \((47)\). This, also furnishes the natural frequency amplitude response (also known as backbone curve), when the excitation amplitude \( \mu = 0 \) is zero, namely:

\[
\Omega = \omega_k + \frac{c_2}{4c_1} a^2 \tag{56}
\]

Finally, the response to the harmonic excitation is evaluated by Eqs. \((49)\) and \((51)\), namely \( \mathbf{q} = \mathbf{q}_1 + \mathbf{q}_3 \) and making use of Eq. \((47)\) and definition of \( \gamma \). In real form it reads:

\[
\mathbf{q} \quad \mathbf{u}_i \cos(\Omega t - \gamma) + \frac{1}{2} a^2 (z_{11} + z_{11} \cos 2(\Omega t - \gamma)) \tag{57}
\]

with \( \gamma \) given by Eq. \((55)\).
with the commercial code SAP2000. The FEM solutions have been obtained for few selected values of the load intensity, and the relevant displacements reported in Fig. 4 by dots. Both an extensible and an inextensible model were considered, the first by accounting for the real axial stiffness of the beams, and the second by artificially magnifying the cross section area. However, it was found that relevant results were practically indistinguishable. Overall, the approximation offered by the third order asymptotic solutions is found to be excellent, although the magnitude of the displacements is very large. It should be stressed that while the perturbation approach requires solving three linear problems to build up the whole paths, the FEM solution requires solving a nonlinear problem for each value of the load multiplier.

The deformed shapes of the frame, relevant to \( \mu = 1 \), are illustrated in Fig. 5: again, linear (I), quadratic (II), and cubic (III) solutions are shown (Fig. 5a-c), while the separated (incremental) contributions are illustrated in Fig. 5a,b. In the first frame, consistently with the hardening effect, nonlinearities reduce the horizontal displacement of the joint B; in the second one, which instead exhibits a softening behavior, nonlinearities increase the vertical displacement of the joint D.

5.2. Dynamics of simple frames

The dynamic behavior of the simple frame of Fig. 6 has been analyzed. The two beams have equal length \( l = 30 \) m, axial stiffness \( EA = 1.53 \times 10^6 \) N, bending stiffness \( EI = 34.17 \times 10^6 \) Nm², and inertia radius \( \rho = 0.149 \) m, so that slenderness is \( \lambda = 200 \). Moreover, the mass per unit length is \( m = 5.84 \) kg/m. The two structures differ in the boundary conditions, namely hinged hinged (HH) and clamped supported (CS). Due to the inertia forces acting along the beam, it is necessary to divide each beam in several elements, in order that the assumed lumped mass model can adequately approximate the continuous model. A preliminary error analysis was therefore carried out, by evaluating the nonlinear fundamental frequency \( \Omega \) of both structures, as given by Eq. (56), for a given value of the nondimensional amplitude to length ratio, taken as \( a/l = 0.15 \) (see Fig. 6 for the meaning of the amplitude, and for the shape of the 1st nonlinear normal mode). Each beam has been divided in \( M/2 \) elements (hence \( N = M + 1 \) joints) and results reported in Table 1. It is seen that the frequency converges when \( M = 16 \), so that the analysis has been carried out with this discretization.

For the two frames, the relative correction \( \Omega/\omega_1 \) of the fundamental natural frequency \( \Omega \) with respect to the linear one \( \omega_1 \), is shown in Fig. 7 vs the nondimensional amplitude \( a/l \). While the HH frame possesses a strong softening behavior, the CS frame exhibits an almost linear, very weak hardening behaviour, similar to the fixed free beam, as well known in literature (see, e.g., [22]).

The response of the HH and CS frames to a harmonic load is now addressed. A horizontal force \( P(t) = P_0 \cos(\Omega t) \) is applied at the joint between the two beams, and the frequency response curve given by Eqs. (55) is plotted in Fig. 8 for different values of the nondimensional excitation amplitude \( 2P_0/C_6 \omega_1 = 0,0.001,0.005,0.01 \) and true or magnified cross section area.

The previous semi analytical results were validated via comparison with a FEM analysis carried out by the code SAP 2000. An har monically varying force of given \( \Omega \) frequency was assigned and the nonlinear response recorded for approximately 500 s. After that, a FFT was performed to extract from the response the amplitude of the \( \Omega \) frequency content. Both a script in Mathematica and the pro gram QtiPlot were used to check errors in FFT. Especially when a magnified value of the cross section area was considered, this procedure required extremely long computations on a PC with a multicore CPU (six cores, 12 threads, clock 3.2 GHz, 32 Gb RAM), of the order of 20 h to obtain a single point of the diagram! The relevant results, obtained for few points, both adopting the true or magnified axial stiffness are reported in Fig. 8. Again, an excellent agreement with the consistent perturbation method is found. This latter, however, requires few minutes to evaluate the whole curve.

During the forced motion, and according to Eq. (57), the structure experiences a linear \( \Omega \) harmonic response and a second order response, consisting in a drift and a \( 2\Omega \) harmonic motion, so that the resulting response is no more harmonic. Therefore, if some shots are taken during a period of oscillation, the shape of structure modifies itself. These are displayed in Fig. 9 when \( 2P_0/C_6 \omega_1 = 0.01, (\Omega \omega_1)/\omega_1 = 0.025,0.0002 \) and \( t = (0,0.3,0.6,0.9)\pi/\Omega \).

5.3. Dynamics of a Vierendeel beam

The dynamics of a more complex frame (see Fig. 10), also known as Vierendeel beam, has finally been analyzed. Such a problem, that would be intractable by a traditional manual approach, is indeed useful to show the capability of the proposed numerical perturbation method. The frame is made of 5 equal square meshes of side \( l/5 \), with \( l = 30 \) m the total length. All the 14 beams are equal, of axial stiffness \( EA = 1.53 \times 10^6 \) N, bending stiffness \( EI = 34.17 \times 10^6 \) Nm², slenderness \( \lambda = 40 \) and mass per unit length \( m = 5.84 \) kg/m. Both upper and bottom longerons are clamped on the left and supported on the right. Each beam has been divided in 4 elements, so that the whole structure is made of 56 elements and 54 joints. If symmetry of the structure with respect the horizontal axis is taken into account, then the number of joints reduces to 29.

The analysis has been focused on several modes; here, only the results relevant to the 1st and 2nd mode are reported. For each considered mode the corrections of the natural frequency versus the amplitude of free oscillation have been evaluated by Eq. (56) and plotted in Fig. 11. In the figure, the amplitude \( \alpha \) represents the maximum vertical displacement of the linear mode, and \( n = 1,2 \) is the number of the mode. It appears that the frame posses a softening behavior, stronger for higher modes.

When a harmonic vertical force is applied in the middle of the upper beam, having nondimensional amplitude \( 2P_0/C_6 \omega_1 = 0,0.001,0.005,0.01 \), the frequency response curves of Fig. 12 follow from Eq. (56).

Fig. 13 reports some shots of the evolution of the deformed configuration at \( t = (0.3,0.6,0.9)\pi/\Omega \), when the beam is forced closely to the 1st and 2nd mode, respectively (\( 2P_0/C_6 \omega_1 = 0.01, (\Omega \omega_1)/\omega_1 = 0.0423,0.1139 \)).

6. Conclusions

In this paper, a numerical perturbation method for analyzing statics and dynamics of general geometrically nonlinear planar frames, made of inextensible and shear undeformable beams, has been illustrated. The method is based on a ‘consistent perturbation analysis’, in which the displacement fields are not interpolated between the nodes, as in the FEM approach, but rather obtained by asymptotic integration of the field equations, under the simplifying assumption that masses are lumped at the ends. The numerical aspect of the algorithm does not consist in any approximation, but only in the fact that calculations, that would be manually impossible for their complexity, are instead rearranged in a way they can be made by a computer. Goal of the analysis is to evaluate the numerical values of the coefficients of the series expansions, which describe the response in terms of a control parameter (e.g. the load multiplier in a static problem, or the forcing frequency, in a dynamic problem). To achieve this goal, few linear problems (typically three) must be solved to obtain the whole curve, instead of a large number of nonlinear problems, as instead required by a continuation incremental iterative method.
The paper addressed all the aspect of the problem, namely: (a) the formulation of the model, (b) the solution algorithm, (c) the discussion of the results.

Formulations was aimed to find a third order relation between end forces and end displacements. Due to the internal constraints, the longitudinal component of the reactive internal contact forces also appeared in the set of end forces, counterbalanced by an additional kinematic condition. Modeling was therefore carried out in the context of the mixed formulation of elasticity. Once the relations have been obtained for the element, they were assembled, to furnish global equation for the frame, leading to perturbation equations to be solved in cascade.

The straightforward perturbation expansion method was used for solving static problems, and the multiple scale method for dynamic problems, both furnishing solutions which are analytical in the control parameter. Attention was focused on evaluating the frequency amplitude relationships, for nonlinear free vibrations as well as harmonic excitation.

A number of sample systems were studied, consisting in simple planar frames and a larger Vierendeel beam. Several aspects were addressed as: convergence of results, comparison between different order asymptotic solutions, mechanical interpretation of the structural behaviour. Remarkably, the semi analytical results were found to be in excellent agreement with computationally expansive FEM analyses.

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Appendix A. Perturbation solution for the element problem

The solution to the perturbation equations for the element, Eqs. (12) (14), is given, in symbolic form, by Eqs. (17) (19). There, linear (index 1), quadratic (index 2), bilinear (index 1,1) and cubic (index 3) operators are involved, whose explicit form is reported here. The \( F_i \) operators, appearing in the expression of \( \phi(s) \), are:

\[
F_1(v_1, b, s) = f_1(s)\phi_{A1} + f_2(s)\phi_{B1} + f_3(s)\phi_{A1}^{2} + f_4(s)\phi_{B1}^{2}
\]

\[
F_2(v_1, b, s) = f_3(s)\phi_{A2} + f_4(s)\phi_{B2} + f_5(s)(v_{A1} + v_{A1})\phi_{A3} + f_6(s)(v_{B1} + v_{B1})\phi_{B3}
\]

\[
F_3(v_1, b, s) = f_1(s)(v_{A1} + v_{A1})\phi_{A3} + f_2(s)(v_{B1} + v_{B1})\phi_{B3}
\]

\[
F_4(v_1, b, s) = f_3(s)(v_{A1} + v_{A1})\phi_{A3} + f_4(s)(v_{B1} + v_{B1})\phi_{B3}
\]

\[
F_5(v_1, b, s) = f_5(s)(v_{A1} + v_{A1})\phi_{A3} + f_6(s)(v_{B1} + v_{B1})\phi_{B3}
\]

\[
F_6(v_1, b, s) = f_7(s)(v_{A1} + v_{A1})\phi_{A3} + f_8(s)(v_{B1} + v_{B1})\phi_{B3}
\]

\[\text{where } f_i(s) \text{ are polynomials, given ahead. The } S_i \text{ operators, appearing in the expression of } S_n, \text{ are:}
\]

\[
S_1(v_1, b) = \frac{6}{f_1(\phi_{A1} + \phi_{B1})} + \frac{12}{f_1(\phi_{B1})}\phi_{A1} + \frac{1}{2}\phi_{B1}
\]

\[
S_2(v_1, b, s) = \frac{6}{f_1(\phi_{A1} + \phi_{B1})} + \frac{12}{f_1(\phi_{B1})}\phi_{A1} + \frac{1}{2}\phi_{B1}
\]

\[
S_3(v_1, b, s) = \frac{6}{f_1(\phi_{A1} + \phi_{B1})} + \frac{12}{f_1(\phi_{B1})}\phi_{A1} + \frac{1}{2}\phi_{B1}
\]

\[
S_4(v_1, b, s) = \frac{6}{f_1(\phi_{A1} + \phi_{B1})} + \frac{12}{f_1(\phi_{B1})}\phi_{A1} + \frac{1}{2}\phi_{B1}
\]

\[
S_5(v_1, b, s) = \frac{6}{f_1(\phi_{A1} + \phi_{B1})} + \frac{12}{f_1(\phi_{B1})}\phi_{A1} + \frac{1}{2}\phi_{B1}
\]
Finally, the polynomial \( f_k(s) \) is defined as follows:

\[ f_1(s) = 1, \quad f_2(s) = 6, \quad f_3(s) = 6 \]

\[ f_4(s) = 1^2, 3^2 + 2s^3, \quad f_5(s) = 6 \]

\[ f_6(s) = 2^2 + 4s^3, \quad f_7(s) = 8^2 + 3s^3 + 4s^4, \quad f_8(s) = 16, \quad f_9(s) = 16 \]

\[ f_{10}(s) = 2^2 + 3s^3 + 4s^4, \quad f_{11}(s) = 1^2 + 3, \quad f_{12}(s) = 16 \]

\[ f_{13}(s) = 16, \quad f_{14}(s) = 100, \quad f_{15}(s) = 100 \]

\[ f_{16}(s) = 100, \quad f_{17}(s) = 100 \]

\[ f_{18}(s) = 100, \quad f_{19}(s) = 100 \]

\[ f_{20}(s) = 100, \quad f_{21}(s) = 100 \]

\[ f_{22}(s) = 100, \quad f_{23}(s) = 100 \]

\[ f_{24}(s) = 100, \quad f_{25}(s) = 100 \]

\[ f_{26}(s) = 100, \quad f_{27}(s) = 100 \]

\[ f_{28}(s) = 100, \quad f_{29}(s) = 100 \]

\[ f_{30}(s) = 100, \quad f_{31}(s) = 100 \]

\[ f_{32}(s) = 100, \quad f_{33}(s) = 100 \]

\[ f_{34}(s) = 100, \quad f_{35}(s) = 100 \]
\[ f_{x}(s) = \frac{p_{s}}{s^{4} + 1400 + 1000 + 201 s + 1057 s^{3}} \]
\[ f_{y}(s) = \frac{p_{s}}{s^{4} + 1400 + 1000 + 201 s + 1057 s^{3}} \]
\[ f_{u}(s) = \frac{p_{s}}{s^{4} + 1400 + 1000 + 201 s + 1057 s^{3}} \]
\[ f_{v}(s) = \frac{p_{s}}{s^{4} + 1400 + 1000 + 201 s + 1057 s^{3}} \]

Appendix B. Nonlinear part of nodal forces

The nonlinear terms appearing in the state relation, Eq. (24), are defined as follows:

\[ N_{2}(q, b) = \begin{pmatrix} 0 & 0 & Y^{(2)}_{A} \phi^{(2)}_{B} m^{(2)}_{A} m^{(1)}_{B} \end{pmatrix} u_{t} \]
\[ N_{11}(q, b, q_{i}, 0) = \begin{pmatrix} 0 & 0 & Y^{(1)}_{A} \phi^{(2)}_{B} m^{(1)}_{A} m^{(1)}_{B} \end{pmatrix} u_{t1} \]
\[ N_{3}(q, b) = \begin{pmatrix} 0 & 0 & Y^{(3)}_{A} \phi^{(3)}_{B} m^{(3)}_{A} m^{(3)}_{B} \end{pmatrix} u_{t} \]

In these equations, \( u_{t}, u_{t1}, u_{t} \) are defined in the Appendix A. Concerning the end forces, the quadratic components are:

\[ Y^{(2)}_{A} = \begin{pmatrix} 6 \frac{\delta}{10} \nu_{A} R_{B1} + \frac{1}{10} \phi_{A} \nu_{R_{B1}} \end{pmatrix} \phi^{(1)}_{B} m^{(1)}_{B} \]
\[ Y^{(2)}_{B} = \begin{pmatrix} 6 \frac{\delta}{10} \nu_{A} R_{B1} + \frac{1}{10} \phi_{A} \nu_{R_{B1}} \end{pmatrix} \phi^{(1)}_{B} m^{(1)}_{B} \]
\[ m^{(2)}_{A} = \begin{pmatrix} 1 \frac{\delta}{10} \nu_{A} R_{B1} + \frac{1}{10} \phi_{A} \nu_{R_{B1}} \end{pmatrix} \phi^{(1)}_{B} m^{(1)}_{B} \]
\[ m^{(2)}_{B} = \begin{pmatrix} 1 \frac{\delta}{10} \nu_{A} R_{B1} + \frac{1}{10} \phi_{A} \nu_{R_{B1}} \end{pmatrix} \phi^{(1)}_{B} m^{(1)}_{B} \]

The bilinear components are:

\[ Y^{(1)}_{A} = \begin{pmatrix} 6 \frac{\delta}{10} \nu_{A} R_{B2} + \frac{1}{10} \phi_{A} \nu_{R_{B2}} \end{pmatrix} \phi^{(1)}_{B} m^{(1)}_{B} \]
\[ m^{(1)}_{A} = \begin{pmatrix} 1 \frac{\delta}{10} \nu_{A} R_{B2} + \frac{1}{10} \phi_{A} \nu_{R_{B2}} \end{pmatrix} \phi^{(1)}_{B} m^{(1)}_{B} \]

Finally, the cubic components are:

\[ Y^{(3)}_{A} = \begin{pmatrix} 1296 \frac{\delta}{35} \nu_{A} \nu_{B2} \end{pmatrix} \phi^{(3)}_{B} m^{(3)}_{B} \]
\[ m^{(3)}_{A} = \begin{pmatrix} 18 \frac{\delta}{144} \nu_{A} R_{B1} \end{pmatrix} \phi^{(3)}_{B} m^{(3)}_{B} \]
\begin{align*}
&+ \frac{l^5}{5040} \varphi_{A1} b_y^3 - \frac{l^5}{10080} \varphi_{B1} b_y^3 + \frac{l^5}{33600} R_{B1} b_x b_y \\
&- \frac{13l^3}{1814400} b_x^3 b_y + \frac{l^3}{725760} b_y^3 \\
&+ \frac{18}{7l^3} \left( v_{A1}^3 - v_{B1}^3 \right) + \frac{54}{7l} \left( - v_{A1}^2 v_{B1} + v_{A1} v_{B1}^2 \right) \\
&+ \frac{72}{35l^3} \left( v_{A1}^3 \varphi_{B1} + v_{B1}^3 \varphi_{A1} \right) + \frac{6}{7l} \left( v_{A1}^2 \varphi_{A1} + v_{B1}^2 \varphi_{B1} \right) \\
&+ \frac{144}{35l^3} \varphi_{B1} v_{A1} v_{B1} - \frac{12}{7l^3} \varphi_{B1} v_{A1} v_{B1} \\
&+ \frac{3}{7l^3} \left( \varphi_{A1} v_{A1} - \varphi_{B1} v_{B1} \right) + \frac{16}{35l^3} \left( \varphi_{A1} v_{B1} - \varphi_{B1} \varphi_{A1} \varphi_{B1} \right) \\
&+ \frac{8}{35l^3} \left( \varphi_{A1} v_{A1} - \varphi_{B1} v_{B1} \right) - \frac{3}{35l^3} \varphi_{B1} \varphi_{A1} + \frac{4}{35l^3} \varphi_{B1}^3 \\
&- \frac{1}{35l^3} \varphi_{A1}^3 - \frac{8}{35l^3} \varphi_{B1}^3 + \frac{l^3}{1400} \left( - v_{A1} R_1^2 + v_{B1} R_1^2 \right) \\
&- \frac{11l^3}{6500} \varphi_{B1} R_1^2 + \frac{13l^3}{12600} \varphi_{A1} R_1^2 - \frac{1}{28} \left( v_{A1} b_y + v_{B1} b_y \right) \\
&+ \frac{14}{70} \varphi_{A1} b_y + \frac{l^3}{700} \varphi_{A1} b_y + \frac{l^3}{1400} \varphi_{A1} b_y \\
&+ \frac{l^3}{30240} R_1 b_y + \frac{l^3}{4200} \left( - v_{A1} R_1 b_x + v_{B1} R_1 b_x \right) \\
&+ \frac{l^3}{11l^3} \varphi_{B1} R_1 b_x - \frac{13l^3}{12600} \varphi_{A1} R_1 b_x \\
&+ \frac{l^3}{2800} \left( v_{A1} b_x^2 - v_{B1} b_x^2 \right) - \frac{l^3}{1400} \varphi_{B1} b_x^2 + \frac{l^3}{3600} \varphi_{A1} b_x^2 \\
&+ \frac{l^3}{10080} \left( v_{A1} b_x^2 - v_{B1} b_x^2 \right) + \frac{l^3}{5040} \varphi_{B1} b_x^2 - \frac{l^3}{10080} \varphi_{A1} b_x^2 \\
&- \frac{11l^3}{302400} R_1 b_x b_y + \frac{19l^3}{1814400} b_x^3 - \frac{l^3}{725760} b_y^3
\end{align*}

(B.4)

References