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Central limit theorems for a supercritical branching process in a random environment

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Abstract
For a supercritical branching process $(Z_n)$ in a stationary and ergodic environment $\xi$, we study the rate of convergence of the normalized population $W_n = Z_n/E[Z_n|\xi]$ to its limit $W_\infty$: we show a central limit theorem for $W_\infty - W_n$ with suitable normalization and derive a Berry-Esseen bound for the rate of convergence in the central limit theorem when the environment is independent and identically distributed. Similar results are also shown for $W_{n+k} - W_n$ for each fixed $k \in \mathbb{N}^*$.

Keywords: Branching processes, random environment, central limit theorem, martingale, rate of convergence.
2000 MSC: 60J80, 60F05

1. Introduction

Galton-Watson processes have been studied by many authors, due to a wide range of applications. See for example the books by [Harris 1963] and [Athreya and Ney 1972]. In a Galton-Watson process \{Z_n, n = 0, 1, \ldots\}, particles behave independently, each gives birth to a random number of particles of the next generation with a fixed distribution \{p_k : k = 0, 1, \ldots\}.

A branching process in a random environment is a natural and important extension of the Galton-Watson process. It is a class of non-homogeneous Galton-Watson processes indexed by a time-environment $\xi = (\xi_0, \xi_1, \xi_2, \ldots)$, which is

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supposed to be stationary and ergodic; given the environment \( \xi \), the particles of \( n \)-th generation have offspring distribution \( \{ p_k(\xi_n) : k \in \mathbb{N} \} \) depending on \( \xi_n \).

For first important works on the subject, see Smith and Wilkinson (1969) and Athreya and Karlin (1971a,b).

For a Galton-Watson process with \( Z_0 = 1 \) and \( m = EZ_1 \in (0, \infty) \), it is well known that \( \{ W_n = Z_n / m^n : n = 0, 1, \ldots \} \) forms a non-negative martingale, and converges almost surely to a random variable \( W_\infty \). For the convergence rate of the martingale, Heyde (1971) and Bühler (1969) obtained respectively that if \( \text{Var}(Z_1) = \sigma^2 < \infty \), then conditioned on \( Z_n > 0 \), the conditional laws of \( (m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W_\infty - W_n) \) and

\[
(m^k / (m^k - 1))^\frac{1}{2} (m^2 - m)^{\frac{1}{2}} \sigma^{-1} Z_n^{-\frac{1}{2}} m^n (W_{n+k} - W_n) \quad k \in \mathbb{N}^*
\]

converge to the normal law \( \mathcal{N}(0, 1) \); Heyde and Brown (1971) gave an estimation of its convergence rate under a third moment condition.

The object of this paper is to extend the theorems of Bühler (1969), Heyde (1971) and Heyde and Brown (1971) to a branching process in a random environment. The main results are Theorems 2.1 and 2.2.

2. Main Results

As usual, we write \( \mathbb{N} = \{ 0, 1, 2, \ldots \} \), \( \mathbb{N}^* = \{ 1, 2, \ldots \} \) and \( \mathbb{R} \) for the set of real numbers.

Let us first recall the definition of a branching process in a random environment. For reference on the subject, see for example Athreya and Karlin (1971a,b), and Athreya and Ney (1972).

A random environment \( \xi = (\xi_n) \) is formulated as a stationary and ergodic sequence of random variables taking values in some measurable space \( (\Theta, \mathcal{F}) \). Each realization of \( \xi_n \) corresponds to a probability distribution \( p(\xi_n) = \{ p_i(\xi_n) : i \in \mathbb{N} \} \) where

\[
p_i(\xi_n) \geq 0, \quad \sum_{i=0}^{\infty} p_i(\xi_n) = 1, \quad 0 < \sum_{i=0}^{\infty} ip_i(\xi_n) < \infty. \quad (1)
\]

Without loss of generality, we can take \( \xi_n \) as coordinate functions defined on the product space \( (\Theta^\mathbb{N}, \mathcal{F}^{\otimes \mathbb{N}}) \), equipped with a probability law \( \tau \), which is invariant and ergodic under the usual shift transformation \( \theta \) on \( \Theta^\mathbb{N} \): \( \theta(\xi_0, \xi_1, \ldots) = (\xi_1, \xi_2, \ldots) \). A branching process \( (Z_n)_{n \geq 0} \) in the random environment \( \xi \) is a class of non-homogeneous branching processes indexed by \( \xi \). By definition,

\[
Z_0 = 1, \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i}, \quad n \geq 0, \quad (2)
\]
where given $\xi$, $\{X_{n,i} : n \geq 0, i \geq 1\}$ is a family of (conditionally) independent random variables, each $X_{n,i}$ has the common law $p(\xi_n)$. Notice that when all $\xi_n$ are the same constant, $(Z_n)$ reduces to the classical Galton-Watson process.

Let $(\Gamma, P_\xi)$ be the probability space under which the process is defined when the environment $\xi$ is given. As usual, $P_\xi$ is called quenched law. The total probability space can be formulated as the product space $(\Gamma \times \Theta^\mathbb{N}, P)$, where $P = P_\xi \otimes \tau$ in the sense that for all measurable and positive function $g$, we have

$$\int gdP = \int \int g(\xi, y)dP_\xi(y)d\tau(\xi),$$

(recall that $\tau$ is the law of the environment $\xi$). The total probability $P$ is usually called annealed law. The quenched law $P_\xi$ may be considered to be the conditional probability of the annealed law $P$ given $\xi$. The expectation with respect to $P_\xi$ (resp. $P$) will be denoted $E_\xi$ (resp. $E$).

For $n \geq 0$, define

$$m_n(a) = m(\xi_n, a) = \sum_{i=1}^{\infty} i^a p_i(\xi_n), \quad a \in \mathbb{R},$$

$$m_n = m_n(1), \quad \sigma_n^2 = m_n(2) - m_n^2,$$

$$\pi_0 = 1 \quad \text{and} \quad \pi_n = \pi_n(\xi) = m_0 \cdots m_{n-1} \quad \text{for} \quad n \geq 1.$$

Then $\pi_n = E_\xi Z_n$ for $n \geq 0$. It is well known that

$$W_n = Z_n/\pi_n$$

is a martingale with respect to the filtration

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_n = \sigma\{\xi, X_{j,i} : j \leq n-1, i \geq 1\} \quad (n \geq 1),$$

so that the limit

$$W_\infty = \lim_{n \to \infty} W_n$$

exists almost surely (a.s.) with $EW \leq 1$ by Fatou’s lemma.

Throughout the paper, we always assume that

$$E \ln m_0 > 0 \quad \text{and} \quad E \left(\frac{Z_1}{m_0} \ln^+ Z_1\right) < \infty.$$  

The first assumption ensures that the process is supercritical (cf. [Athreya and Karlin (1971a)]); the second one together with the first implies that $EW_\infty = 1$; moreover,

$$P_\xi(W_\infty > 0) = P_\xi(Z_n \to \infty) = \lim_{n \to \infty} P_\xi(Z_n > 0) = 1 - q(\xi) > 0 \quad \text{a.s.,}$$

where $q(\xi) = \lim_{n \to \infty} P_\xi(Z_n = 0)$ is the extinct probability.
In this paper, we search for central limit theorems on $W_{\infty} - W_n$ and $W_{n+k} - W_n$ for fixed $k \geq 1$ with an appropriate normalization. Assume that $m_0(2) < \infty$ a.s., and let

$$\Delta_k^2 = \Delta_k^2(\xi) = \sum_{0 \leq i < k} \frac{1}{\pi_i} \sigma_i^2 m_i^2$$

for $k \in \mathbb{N}^* \cup \{\infty\}.$

Then for $k \in \mathbb{N}^*$, $\Delta_k^2(\xi)$ is the variance of $W_k$ under $P_\xi$; $\Delta_\infty^2(\xi)$ is the variance of $W_{\infty}$ if the series converges (i.e. $\Delta_\infty^2(\xi) < \infty$): see Lemma 3.2.

We can now formulate our first main result.

**Theorem 2.1.** Suppose that (10) holds and that $m_0(2) < \infty$ a.s.. In the case where $k = \infty$, assume additionally that $E \ln^+ (\sigma_0^2/m_0^2) < \infty$. Write

$$U_{n,k} = \frac{\pi_n(W_{n+k} - W_n)}{\sqrt{Z_n \Delta_k(\theta^n \xi)}}$$

for $k \in \mathbb{N}^* \cup \{\infty\},$

where by convention $W_{n+k} = W_\infty$ if $k = \infty$. Then for each $k \in \mathbb{N}^* \cup \{\infty\}$, as $n \to \infty$,

$$\sup_{x \in \mathbb{R}} |P_\xi(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \to 0 \quad \text{in } L^1,$$

and

$$\sup_{x \in \mathbb{R}} |P(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \to 0.$$ (13)

We believe that for each $k \in \mathbb{N} \cup \{\infty\}$

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |P_\xi(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| = 0 \quad \text{a.s.}$$

We notice that in the classical Galton-Watson process, (13) reduces to the results of Bühler (1969) and Heyde (1971). Our second main result concerns the rate of convergence in the above central limit theorem for a branching process with an independent and identically distributed environment.

**Theorem 2.2.** Let the environment $\{\xi_n\}$ be independent and identically distributed. Assume that (10) holds and that $m_0(2) < \infty$ a.s.. In the case where $k = \infty$, assume additionally that $E \ln^+ (\sigma_0^2/m_0^2) < \infty$. For each $k \in \mathbb{N}^* \cup \{\infty\}$, if $E|\frac{W_{n,k}}{\Delta_k}|^{2+\delta} < \infty$ for some $\delta \in (0,1]$, then

$$\sup_{x \in \mathbb{R}} |P(U_{n,k} \leq x | Z_n > 0) - \Phi(x)| \leq C_\delta \left(Em_0(-\frac{\delta}{2})\right)^n E\left|\frac{W_{n,k}}{\Delta_k}\right|^{2+\delta},$$

where $U_{n,k}$ is defined in Theorem 2.1 and $C_\delta$ is the Berry-Esseen constant.

**Remark 2.3.** It maybe useful to notice that if

$$E(Z_1/m_0)^{2+\delta} < \infty, \quad Em_0^{-(1+\delta)} < 1 \quad \text{and} \quad m_0(2)/m_0^2 \geq A$$
for some constant $A > 1$, then $E\left|\frac{W_{n} - 1}{A}\right|^{2+\delta} < +\infty$. In fact by Theorem 3 of Guivarc’h and Liu (2001), the first two conditions imply that $E|W_{\infty} - 1|^{2+\delta} < \infty$, while the last one implies that $\Delta_{\infty}^{2} \geq A - 1 > 0$.

For the classical Galton-Watson process with $\delta = 1$, Theorem 2 reduces to Theorem 2 of (Heyde and Brown 1971, p.272).

3. Proof of Theorem 2.1

In this section, we consider a central limit theorem under a second moment condition in proving Theorem 2.1. We first give some lemmas.

**Lemma 3.1 (Grincevičius (1974)).** Let $\{(\alpha_{n}, \beta_{n}), n = 0, 1, 2, \cdots\}$ be a stationary and ergodic sequence of random variables with values in $\mathbb{R}^2$. If

\[ E \ln |\alpha_0| < 0 \quad \text{and} \quad E \ln^+ |\beta_0| < \infty, \]

then

\[ \sum_{n=0}^{\infty} |\alpha_0 \alpha_1 \cdots \alpha_{n-1} \beta_n| < \infty \quad \text{a.s.} \]

In fact, the result is a direct consequence of the ergodic theorem and Cauchy’s criterion for the convergence of series.

Using the above lemma, we can easily obtain the following result.

**Lemma 3.2.** Under the assumptions in Theorem 2.1, for each $k \in \mathbb{N}^* \cup \{\infty\}$,

\[ \text{Var}_{\xi}(W_k) = \Delta_k^2(\xi) = \sum_{0 \leq i < k} \frac{1}{\pi_i} \frac{\sigma_i^2}{m_i^2}. \]  

(15)

This has been known for branching processes in varying environment, see e.g. (Jagers, 1974, p.175) in a slightly different form. For reader’s convenience, we present a proof in the following.

**Proof of Lemma 3.2.** By (2) and the definition of $W_n$, we have

\[ W_{n+1} - W_n = \frac{1}{\pi_n} \sum_{j=1}^{Z_n} (X_{n,j} m_n - 1). \]

Recall that under $P_{\xi}$, the random variables $\{X_{n,j}\}$ are independent of each other and have the common distribution $p(\xi_n)$ with expectation $m_n$. Hence a direct calculation shows that

\[ E_{\xi}((W_{n+1} - W_n)^2) = E_{\xi} \left( E_{\xi}((W_{n+1} - W_n)^2 | F_n) \right) \]

\[ = E_{\xi} \left( \frac{Z_n}{\pi_n^2} \frac{\sigma_n^2}{m_n^2} \right) = \frac{1}{\pi_n^2} \frac{\sigma_n^2}{m_n^2}. \]
As \( \{W_n\} \) is a martingale, it follows that

\[
E_\xi W_k^2 = E_\xi W_0^2 + \sum_{i=0}^{k-1} E((W_{i+1} - W_i)^2) = 1 + \sum_{i=0}^{k-1} \frac{1}{\pi_i m_i^2}.
\]

Therefore for each fixed integer \( k \),

\[
\text{Var}_\xi(W_k) = E_\xi(W_k^2) - 1 = \sum_{i=0}^{k-1} \frac{1}{\pi_i m_i^2}.
\]

Now we turn to the calculation of \( \text{Var}_\xi(W_\infty) \). By Lemma 3.1 when \( E \ln m_0 > 0 \) and \( E \ln^+ \frac{\sigma_i^2}{m_i} < \infty \),

\[
\sup_n E_\xi(W_n^2) = 1 + \sum_{i=0}^{\infty} \frac{1}{\pi_i m_i^2} < \infty \quad \text{a.s.}
\]

So \( W_n \) converges to \( W_\infty \) in \( L^2 \) under \( P_\xi \) and

\[
E_\xi(W_\infty^2) = \lim_{k \to \infty} E_\xi(W_k^2) = 1 + \sum_{i=0}^{\infty} \frac{1}{\pi_i m_i^2}.
\]

It follows that

\[
\text{Var}_\xi(W_\infty) = E_\xi(W_\infty^2) - 1 = \Delta_\infty^2(\xi) = \sum_{i=0}^{\infty} \frac{1}{\pi_i m_i^2} < \infty \quad \text{a.s.}
\]

To give our next lemma, we will need some notations, which will also be used in the proof of the main theorems. By definition,

\[
Z_{n+k} = \sum_{j=1}^{Z_n} Z_k(n, j), \tag{16}
\]

where \( Z_k(n, j) \) denotes the number of descendants in the \((n+k)\)-th generation of the \( j \)-th particle among the \( Z_n \) particles in \( n \)-th generation.

Writing \( W_k(n, j) = \frac{Z_k(n, j)}{\pi_k(\theta^n \xi)} \) and using (16), we obtain the following decomposition:

\[
\pi_n(W_{n+k} - W_n) = \sum_{j=1}^{Z_n} (W_k(n, j) - 1). \tag{17}
\]

Letting \( k \to \infty \), it follows that

\[
\pi_n(W_\infty - W_n) = \sum_{j=1}^{Z_n} (W_\infty(n, j) - 1), \tag{18}
\]
where under $P_\xi$, the random variables $\{W_\infty(n,j)\}_j$ are independent of each other and have the common conditional distribution

$$P_\xi(W_\infty(n,j) \in \cdot) = P_{\theta^n,\xi}(W_\infty \in \cdot).$$

**Lemma 3.3.** Suppose that the assumptions of Theorem 2.1 hold. Let $r_n \in \mathbb{N}$ with $r_n \to \infty$. For $k \in \mathbb{N}^* \cup \{\infty\}$, define

$$Y_{k,n} = \frac{1}{\sqrt{r_n}} \sum_{j=1}^{r_n} \frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi)}.$$  

Fix $k \in \mathbb{N}^* \cup \{\infty\}$. Then for each subsequence $\{n'\}$ of $\mathbb{N}$ with $n' \to \infty$, there is a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that for a.e. $\xi$ and all $x \in \mathbb{R}$, as $n'' \to \infty$,

$$P_\xi(Y_{k,n''} \leq x) \to \Phi(x).$$

**Proof.** Fix $k \in \mathbb{N}^* \cup \{\infty\}$. In order to use Lindeberg’s theorem, for $n \in \mathbb{N}$ and $\epsilon > 0$, we consider the quantity

$$L_k(\xi, \epsilon, n) = \frac{1}{r_n} \sum_{j=1}^{r_n} E_\xi \left( \left( \frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi)} \right)^2 \left| \frac{W_k(n,j) - 1}{\Delta_k(\theta^n \xi) \sqrt{r_n}} > \epsilon \right. \right),$$

where for a set $A$, we write $E_\xi(x; A)$ for $E_\xi(X 1_A)$, $1_A$ denoting the indicator function of $A$. By the stationarity and ergodicity of the environment, for all $\epsilon > 0$, as $n \to \infty$,

$$EL_k(\xi, \epsilon, n) = E \left[ \left( \frac{W_k - 1}{\Delta_k} \right)^2 \left| \frac{W_k - 1}{\Delta_k} > \sqrt{r_n} \epsilon \right. \right] \to 0.$$  

(19)

Let $\{n'\}$ be a subsequence of $\mathbb{N}$. Notice that from (19), we can choose a subsequence $\{n''\}$ for which $L_k(\xi, \epsilon, n'') \to 0$ a.s., but this sequence may depend of $\epsilon$. We will use a diagonal argument to select a subsequence $\{n''\}$ of $\{n'\}$ such that a.s. $L_k(\xi, \epsilon, n'') \xrightarrow{n'' \to \infty} 0$ for all $\epsilon > 0$. Set

$$\epsilon_m = 1/m \quad \text{for} \quad m \geq 1.$$

Let $\{n_{0,i}\} = \{n'\}$. Because of (19), there is a subsequence $\{n_{1,i}\}$ of $\{n_{0,i}\}$ and a set $\Lambda_1$ with $\tau(\Lambda_1) = 1$ such that $\forall \xi \in \Lambda_1$,

$$\lim_{i \to \infty} L_k(\xi, \epsilon_1, n_{1,i}) = 0.$$

Inductively for $m \geq 1$, when $\Lambda_m$ and $\{n_{m,i}\}$ are defined such that $\tau(\Lambda_m) = 1$ and $\forall \xi \in \Lambda_m$, $L_k(\xi, \epsilon_m, n_{m,i}) \to 0$, there is a subsequence $\{n_{m+1,i}\} \subset \{n_{m,i}\}$ and a set $\Lambda_{m+1}$ with $\tau(\Lambda_{m+1}) = 1$ such that $\forall \xi \in \Lambda_{m+1}$,

$$\lim_{i \to \infty} L_k(\xi, \epsilon_{m+1}, n_{m+1,i}) = 0.$$
We now consider the diagonal sequence \( \{ n_{i,i} \}_{i \geq 1} \) and \( \Lambda = \bigcap_{j=1}^{\infty} \Lambda_j \). For each fixed \( \epsilon > 0 \), let \( m \geq \frac{1}{\epsilon} \). Then \( \epsilon_m \leq \epsilon \) and by the monotonicity of \( L_k(\xi, \epsilon, n) \) in \( \epsilon \), we see that \( \forall \xi \in \Lambda \),

\[
L_k(\xi, \epsilon, n_{m,i}) \leq L_k(\xi, \epsilon_m, n_{m,i}) \to 0 \quad \text{as} \quad i \to \infty.
\]

As \( \{ n_{i,i} \} \) is a subsequence of \( \{ n_{m,i} \} \) whenever \( i > m \), this implies that

\[
\lim_{i \to \infty} L_k(\xi, \epsilon, n_{i,i}) = 0. \tag{20}
\]

Since \( \tau(\Lambda) = 1 \), we have shown that for all \( \epsilon > 0 \), (20) holds a.s.. It follows that a.s. \( \tau(\Lambda) = 1 \) holds for all rational \( \epsilon > 0 \), and therefore for all real \( \epsilon > 0 \) by the monotonicity of \( L_k(\xi, \epsilon, n_{ii}) \) in \( \epsilon \). So by Lindeberg’s theorem, it is a.s. that for all \( x \in \mathbb{R} \), as \( i \to \infty \),

\[
P_\xi(\xi \leq x) \to \Phi(x).
\]

Thus the lemma has been proved with \( \{ n'' \} = \{ n_{i,i} \} \).

**Proof of Theorem 2.1.** We shall only deal with the case where \( k = \infty \), as the case where \( k \in \mathbb{N}^* \) can be treated similarly.

We first prove the following assertion: for each sequence \( \{ n' \} \) of \( \mathbb{N} \) with \( n' \to \infty \), there exist a subsequence \( \{ n'' \} \) of \( \{ n' \} \) with \( n'' \to \infty \) such that for a.e. \( \xi \) and all \( x \), as \( n'' \to \infty \),

\[
P_\xi(U_{n'',\infty} \leq x \mid Z_{n''} > 0) \to \Phi(x). \tag{21}
\]

By the definition of \( U_{n,\infty} \) and the relation (18), we get:

\[
U_{n,\infty} = \pi_n(W_\infty - W_n) / \sqrt{Z_n} \Delta_\infty(\theta^n \xi) = \frac{1}{\sqrt{Z_n}} \sum_{j=1}^{Z_n} \frac{W_\infty(n,j) - 1}{\Delta_\infty(\theta^n \xi)},
\]

where we recall that under \( P_\xi \), \( \{ W_\infty(n,j), j \geq 1 \} \) is a family of random variables independent of each other and independent of \( Z_n \), each has the same law as \( W_\infty \) under \( P_{\theta^n \xi} \). Set

\[
u_n(r, x) = P_\xi \left( \frac{1}{\sqrt{r}} \sum_{j=1}^{r} \frac{W_\infty(n,j) - 1}{\Delta_\infty(\theta^n \xi)} \leq x \right), \quad r \in \mathbb{N}^*, \quad x \in \mathbb{R}.
\]

Then

\[
P_\xi(U_{n,\infty} \leq x \mid Z_n > 0) = [P_\xi(Z_n > 0)]^{-1} \sum_{r=1}^{\infty} P_\xi(U_{n,\infty} \leq x, Z_n = r)
\]

\[
= \sum_{r=1}^{\infty} u_n(r, x) \frac{P_\xi(Z_n = r)}{P_\xi(Z_n > 0)}. \tag{22}
\]
To show the main idea, let us first consider the special case where $q(\xi) = 0$ a.s., i.e. for a.e. $\xi$,

$$Z_n \to \infty \quad P_\xi^*\text{-a.s.}.$$ 

In this case, the relation (22) becomes

$$P_\xi(U_{n,\infty} \leq x) = \sum_{r=1}^{\infty} u_n(r, x) P_\xi(Z_n = r) = E_\xi u_n(Z_n, x).$$

By Lemma 3.3, for each subsequence $\{n'\}$ of $\mathbb{N}$ with $n' \to \infty$, there exist a subsequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that for a.e. $\xi$ and all $x$, as $n'' \to \infty$,

$$u_{n''}(Z_{n''}, x) \to \Phi(x).$$

By the dominated convergence theorem, for a.e. $\xi$ and all $x$, as $n'' \to \infty$,

$$P_\xi(U_{n'',\infty} \leq x) = E_{\xi_u^n} u_n(Z_{n'',} x) \to \Phi(x).$$

So we have proved (21) when $q(\xi) = 0$ a.s..

We now consider the general case where $0 \leq q(\xi) < 1$ a.s..

For each $\xi \in \Theta^\mathbb{N}$, let $Z_n^*$ be random variables defined on some probability space $(\Gamma^*, \mathbb{P}_\xi^*)$ with law

$$P_\xi^*(Z_n = r) = \frac{P_\xi(Z_n = r)}{P_\xi(Z_n > 0)}, \quad r \in \mathbb{N}^*.$$ 

Then

$$P_\xi(U_{n,\infty} \leq x|Z_n > 0) = E_{\xi_u^n} u_n(Z_n^*, x),$$

where $E_{\xi_u^n}$ denotes the expectation with respect to $P_\xi^*$.

Let $\{n'\}$ be a sequence of $\mathbb{N}$ with $n' \to \infty$. If for a.e. $\xi$,

$$Z_{n'}^* \to \infty \quad P_\xi^*\text{-a.s.},$$

then as above we can use Lemma 3.3 and the dominated convergence theorem to show that there is a sequence $\{n''\}$ of $\{n'\}$ with $n'' \to \infty$ such that for all $x$, as $n'' \to \infty$,

$$E_{\xi_u^n} u_n(Z_{n''}, x) \to \Phi(x).$$

By the fact that $Z_{n''}^* \to \infty$ in probability under $P_\xi^*$, we can choose a subsequence for which $Z_{n''}^* \to \infty P_\xi^*\text{-a.s.}$ But to apply Lemma 3.3 we need that the sequence does not depend on $\xi$. We therefore pass to the probability $P^*$ to overcome this difficulty, where $P^* = P_\xi^* \otimes \tau$ is defined on the product space $\Gamma^* \times \Theta^\mathbb{N}$ just as $P$ was defined on $\Gamma \times \Theta^\mathbb{N}$.

Notice that for each $r \in \mathbb{N}^*$, as $n \to \infty$,

$$P_\xi^*(Z_n^* = r) = \frac{P_\xi(Z_n = r)}{P_\xi(Z_n > 0)} \to 0,$$
where the last step holds as $Z_n \to \infty$ a.s. on the survival event $S = \{Z_n > 0, \forall n \geq 1\}$ (see [11] or [Tanny 1977] for this fact). Then $Z^*_n \to +\infty$ in probability under $P^*_\xi$. By the dominated convergence theorem, this implies that $Z^*_n \to +\infty$ in probability under $P^*$. Therefore for each subsequence \{\tilde{n}'\} of $N$ with $n' \to \infty$, there is a subsequence \{\tilde{n}\} \subset \{n'\}$ with $\tilde{n} \to \infty$ such that $Z^*_n \to +\infty$ a.s. under $P^*$. This implies that for a.e. $\xi$, as $\tilde{n} \to \infty$,

$$Z^*_\tilde{n} \to +\infty\quad P^*_\xi \text{-a.s.}$$

Now by Lemma 3.3, there exists a subsequence \{\tilde{n}''\} of \{\tilde{n}\} such that for a.e. $\xi$ and all $x$, as $\tilde{n}'' \to \infty$, $u_{n''}(Z^*_n, x) \to \Phi(x).$

By the dominated convergence theorem, for almost every $\xi$ and each $x$, as $n'' \to \infty$,

$$P_\xi(U_{n''}, \infty \leq x|Z_{n''} > 0) = E_\xi u_{n''}(Z^*_n, x) \to \Phi(x). \quad (23)$$

So combining the above two cases, we have proved (21).

Since $P_\xi(U_{n''}, \infty \leq x|Z_{n''} > 0)$ are distribution functions and $\Phi(x)$ is a continuous distribution function, by Dini’s Theorem we see that for a.e. $\xi$, as $n'' \to \infty$,

$$\sup_x |P_\xi(U_{n''}, \infty \leq x|Z_{n''} > 0) - \Phi(x)| \to 0. \quad (24)$$

By the dominated convergence theorem, (24) implies that as $n'' \to \infty$,

$$E \sup_x |P_\xi(U_{n''}, \infty \leq x|Z_{n''} > 0) - \Phi(x)| \to 0. \quad (25)$$

Therefore we have proved that for each sequence \{n''\} of $N$ with $n'' \to \infty$, there is a subsequence \{n'''\} of \{n''\} with $n''' \to \infty$ such that (25) holds. Hence

$$E \sup_x |P_\xi(U_{n''}, \infty \leq x|Z_n > 0) - \Phi(x)| \to 0.$$

This gives (12) for $k = \infty$. The proof for $k \in \mathbb{N}^*$ is similar.

We now begin to prove (13).

As we have proved that for each subsequence \{n'\} of $N$, there is a subsequence \{n''\} so that (24) holds, which implies: for a.e. $\xi$ and all $x \in \mathbb{R}$, as $n'' \to \infty$,

$$|P_\xi(U_{n''}, \infty \leq x, Z_{n''} > 0) - \Phi(x)| \to 0.$$

It follows that for a.e. $\xi$ and all $x \in \mathbb{R}$,

$$|P_\xi(U_{n''}, \infty \leq x, Z_{n''} > 0) - P_\xi(Z_{n''} > 0)\Phi(x)| \to 0.$$

So by the dominated convergence theorem, we see that for each $x \in \mathbb{R}$, as $n'' \to \infty$,

$$|P(U_{n''}, \infty \leq x, Z_{n''} > 0) - P(Z_{n''} > 0)\Phi(x)| \to 0.$$
and hence
\[ P(U_{n',\infty} \leq x | Z_{n'} > 0) \rightarrow \Phi(x). \]
By Dini’s Theorem, it follows that
\[ \sup_x |P(U_{n',\infty} \leq x | Z_{n'} > 0) - \Phi(x)| \rightarrow 0. \] (26)
Therefore we have proved that for each sequence \( \{n'\} \) of \( \mathbb{N} \), there is a subsequence \( \{n''\} \) of \( \{n'\} \) with \( n'' \rightarrow \infty \) such that (26) holds. Hence
\[ \sup_x |P(U_{n,\infty} \leq x | Z_n > 0) - \Phi(x)| \rightarrow 0. \]
Thus the proof is completed. □

4. Proof of Theorem 2.2

In this section, we consider the rate of convergence in the central limit theorem under a moment condition of order \( 2 + \delta \), in proving Theorem 2.2.

Notice that by the definition (4) of \( m_n(a) \), we have
\[ m_n(a) = E_\xi X_{n,i}^a \text{ if } a > 0, \quad m_n(a) = E_\xi X_{n,i}^a \mathbf{1}_{\{X_{n,i} > 0\}} \text{ if } a \leq 0, \] (27)
where \( X_{n,i} \) is as in (2). For \( a > 0 \), define
\[ R_n = |m_0(-a) \cdots m_{n-1}(-a)|^{-1} Z_{n,a}^{-1} \mathbf{1}_{\{Z_n > 0\}}, \quad n \geq 0. \]

Lemma 4.1. \( (R_n, \mathcal{F}_n)_{n \geq 0} \) is a supermartingale, where \( \mathcal{F}_n \) were defined in (8).

Proof. Using the decomposition (2) of \( Z_{n+1} \), we have
\[ Z_{n+1}^{-a} \mathbf{1}_{\{Z_{n+1} > 0\}} = \left( \sum_{i=1}^{Z_n} X_{n,i} \right)^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}} \]
\[ = Z_n^{-a} \left[ \frac{1}{Z_n} \sum_{i=1}^{Z_n} X_{n,i} \mathbf{1}_{\{X_{n,i} > 0\}} \right]^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}} \]
\[ \leq Z_n^{-a} \frac{1}{Z_n} \sum_{i=1}^{Z_n} \left( X_{n,i} \mathbf{1}_{\{X_{n,i} > 0\}} \right)^{-a} \mathbf{1}_{\{Z_n > 0\}} \mathbf{1}_{\{Z_{n+1} > 0\}}, \]
where the last inequality is due to the convexity property of the function \( x^{-a}(a > 0) \).

Taking conditional expectation with respect to \( \mathcal{F}_n \) and \( P_\xi \) on both sides of the above inequality, we obtain that
\[ E_\xi(Z_{n+1}^{-a} \mathbf{1}_{\{Z_{n+1} > 0\}} | \mathcal{F}_n) \leq Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}} m_n(-a), \] (28)
which gives the desired result. □
Since $Z_0 = 1$, by (28), we immediately obtain the following

**Lemma 4.2.** For $a > 0$, we have

\[
E \xi Z_n^{-a} \mathbf{1}_{\{Z_n > 0\}} \leq m_0(-a) \cdots m_{n-1}(-a) \tag{29}
\]

If the environment sequence $\{\xi_n\}$ is independent and identically distributed, then

\[
EZ_n^{-a} \mathbf{1}_{\{Z_n > 0\}} \leq (Em_0(-a))^n. \tag{30}
\]

Now we give the proof of Theorem 2.2.

**Proof of Theorem 2.2.** We shall only deal with the case $k = \infty$, as the case where $k \in \mathbb{N}^*$ can be treated similarly.

Consider the probability space $(\Gamma^* \times \Theta^N, \mathbb{P}^*)$ and define random variables $Z^*_n$ as in the proof of Theorem 2.1. By definition,

\[
u_n(Z^*_n, x) = \mathbb{P} \left( \frac{Z^*_n}{\sqrt{2}} \sum_{j=1}^{Z^*_n} \frac{W_\infty(n, j)}{\Delta_\infty(\theta^n \xi)} \leq x \right).
\]

By our hypothesis and the Berry-Esseen theorem (see e.g. Theorem 6 of [Petrov, 1995, p.115]), we have

\[
|\nu_n(Z^*_n, x) - \Phi(x)| \leq C_\delta \left( \frac{Z^*_n}{(Z^*_n)^{1/2}} \sum_{j=1}^{Z^*_n} \mathbb{E} \left| W_\infty(n, j) - 1 \right| \Delta_\infty(\theta^n \xi) \right)^{2+\delta}
\]

\[
= C_\delta (Z^*_n)^{-\frac{1}{2}} \mathbb{E} \theta^n \xi \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta},
\]

where $C_\delta$ is the Berry-Esseen constant. Using this evaluation, we can derive that

\[
|\mu_n(U_{n, \infty} \leq x | Z_n > 0) - \Phi(x)| \leq C_\delta \mathbb{E} \theta^n \xi \left( \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}
\]

\[
= C_\delta \mathbb{E} \theta^n \xi \left( \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}
\]

By the definition of $Z^*_n$, this implies that

\[
|P \xi(U_{n, \infty} \leq x, Z_n > 0) - P \xi(Z_n > 0)\Phi(x)|
\]

\[
\leq C_\delta \mathbb{E} \xi \left( Z_n^{-\frac{1}{2}} \mathbb{I}_{\{Z_n > 0\}} \right) \mathbb{E} \theta^n \xi \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}. \tag{31}
\]

Using (31) and the fact that the sequence $\{\xi_n\}$ is independent and identically distributed, we get

\[
|P(U_{n, \infty} \leq x, Z_n > 0) - P(Z_n > 0)\Phi(x)|
\]

\[
\leq \mathbb{E}|P \xi(U_{n, \infty} \leq x, Z_n > 0) - P \xi(Z_n > 0)\Phi(x)|
\]

\[
\leq C_\delta \mathbb{E} \xi \left( Z_n^{-\frac{1}{2}} \mathbb{I}_{\{Z_n > 0\}} \right) \mathbb{E} \left| \frac{W_\infty - 1}{\Delta_\infty} \right|^{2+\delta}. \tag{31}
\]

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Together with (30), we obtain that

$$ |P(U_{n,\infty} \leq x|Z_n > 0) - \Phi(x)| \leq \frac{C\delta(Em_0(-\delta^2)^nE[W_{\infty}^{-1}]^{2+\delta}}{P(Z_n > 0)}. $$

Then the proof is completed.

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