Large deviation exponential inequalities for supermartingales
Xiequan Fan, Ion Grama, Quansheng Liu

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Xiequan Fan† Ion Grama‡ Quansheng Liu§

Abstract

Let \((X_i, F_i)_{i \geq 1}\) be a sequence of supermartingale differences and let \(S_k = \sum_{i=1}^k X_i\). We give an exponential moment condition under which

\[
P(\max_{1 \leq k \leq n} S_k \geq n) = O(\exp\{-C_1 n^{\alpha}\}), \quad n \to \infty,
\]

where \(\alpha \in (0, 1)\) is given and \(C_1 > 0\) is a constant. We also show that the power \(\alpha\) is optimal under the given moment condition.

Keywords: Large deviation; martingales; exponential inequality; Bernstein type inequality.

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1 Introduction

Let \((X_i, F_i)_{i \geq 1}\) be a sequence of martingale differences and let \(S_k = \sum_{i=1}^k X_i, k \geq 1\). Under the Cramér condition \(\sup_i E|X_i| < \infty\), Lesigne and Volný [9] proved that

\[
P(S_n \geq n) = O(\exp\{-C_1 n^{\frac{1}{3}}\}), \quad n \to \infty,
\]

for some constant \(C_1 > 0\). Here and throughout the paper, for two functions \(f\) and \(g\), we write \(f(n) = O(g(n))\) if there exists a constant \(C > 0\) such that \(|f(n)| \leq C|g(n)|\) for all \(n \geq 1\). Lesigne and Volný [9] also showed that the power \(\frac{1}{3}\) in (1.1) is optimal even for stationary and ergodic sequence of martingale differences, in the sense that there exists a stationary and ergodic sequence of martingale differences \((X_i, F_i)_{i \geq 1}\) such that \(E|X_1| < \infty\) and \(P(S_n \geq n) \geq \exp\{-C_2 n^{\frac{1}{3}}\}\) for some constant \(C_2 > 0\) and infinitely many \(n\)'s. Liu and Watbled [10] proved that the power \(\frac{1}{3}\) in (1.1) can be improved to 1 under the conditional Cramér condition \(\sup_i E(e^{\int_{F_{i-1}}^{X_i}}) \leq C_3\), for some constant \(C_3\). It is natural to ask under what condition

\[
P(S_n \geq n) = O(\exp\{-C_1 n^{\alpha}\}), \quad n \to \infty,
\]

where \(\alpha \in (0, 1)\) is given and \(C_1 > 0\) is a constant. In this paper, we give some sufficient conditions in order that (1.2) holds for supermartingales \((S_k, F_k)_{k \geq 1}\).

The paper is organized as follows. In Section 2, we present the main results. In Sections 3-5, we give the proofs of the main results.

* Université de Bretagne-Sud, LMBA, UMR CNRS 6205, Campus de Tohannic, 56017 Vannes, France.
† E-mail: fanxiequan@hotmail.com
‡ E-mail: ion.grama@univ-ubs.fr
§ E-mail: quansheng.liu@univ-ubs.fr
2 Main Results

Our first result is an extension of the bound (1.1) of Lesigne and Volný [9].

**Theorem 2.1.** Let $\alpha \in (0, 1)$. Assume that $(X_i, F_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i E \exp\{X_i^{2\alpha} / 2\alpha \} \leq C_1$ for some constant $C_1 \in (0, \infty)$. Then, for all $x > 0$,

$$
P\left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq C(\alpha, x) \exp\left\{ -\left(\frac{x}{4}\right)^{2\alpha} n^\alpha \right\}, 
$$

where

$$
C(\alpha, x) = 2 + 35C_1 \left( \frac{1}{x^{2\alpha} 16^{1-\alpha}} + \frac{1}{x^2} \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right)
$$

does not depend on $n$. In particular, with $x = 1$, it holds

$$
P\left( \max_{1 \leq k \leq n} S_k \geq n \right) = O\left( \exp\left\{ -\frac{1}{16} n^\alpha \right\} \right), \quad n \to \infty.
$$

Moreover, the power $\alpha$ in (2.2) is optimal in the class of martingale differences: for each $\alpha \in (0, 1)$, there exists a sequence of martingale differences $(X_i, F_i)_{i \geq 1}$ satisfying $\sup_i E \exp\{X_i^{2\alpha} / 2\alpha \} < \infty$ and

$$
P\left( \max_{1 \leq k \leq n} S_k \geq n \right) \geq \exp\{ -3n^\alpha \},
$$

for all $n$ large enough.

In fact, we shall prove that the power $\alpha$ in (2.2) is optimal even for stationary martingale difference sequences.

It is clear that when $\alpha = \frac{1}{3}$, the bound (2.2) implies the bound (1.1) of Lesigne and Volný.

Our second result shows that the moment condition $\sup_i E \exp\{X_i^{2\alpha} / 2\alpha \} < \infty$ in Theorem 2.1 can be relaxed to $\sup_i E \exp\{(X'_i)^{2\alpha} / 2\alpha \} < \infty$, where $X'_i = \max\{X_i, 0\}$, if we add a constraint on the sum of conditional variances

$$
\langle S \rangle_k = \sum_{i=1}^k E(X_i^2 | F_{i-1}).
$$

**Theorem 2.2.** Let $\alpha \in (0, 1)$. Assume that $(X_i, F_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i E \exp\{(X'_i)^{2\alpha} / 2\alpha \} \leq C_1$ for some constant $C_1 \in (0, \infty)$. Then, for all $x, v > 0$,

$$
P\left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \leq \exp\left\{ -\frac{x^2}{2(v^2 + \frac{1}{5} x^{2-\alpha})} \right\} + nC_1 \exp\{ -x^\alpha \}. \quad (2.4)
$$

For bounded random variables, some inequalities closely related to (2.4) can be found in Freedman [5], Dedecker [11], Dzhaparidze and van Zanten [3], Merlevède, Peligrad and Rio [11] and Delyon [2].

Adding a hypothesis on $\langle S \rangle_n$ to Theorem 2.2, we can easily obtain the following Bernstein type inequality which is similar to an inequality of Merlevède, Peligrad and Rio [12] for weakly dependent sequences.
**Corollary 2.3.** Let $\alpha \in (0, 1)$. Assume that $(X_i, F_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i E \exp\left(\frac{(X_i^+)^{\alpha}}{C}\right) \leq C_1$ and $E \exp\left(\frac{(S_i)^{\alpha}}{n}\right) \leq C_2$ for some constants $C_1, C_2 \in (0, \infty)$. Then, for all $x > 0$,

$$
\Pr\left(\max_{1 \leq k \leq n} S_k \geq n x\right) \leq \exp\left(-\frac{x^{1+\alpha}}{2(1 + \frac{1}{2}x)} n^\alpha\right) + (nC_1 + C_2) \exp\{-x^\alpha n^\alpha\}. \tag{2.5}
$$

In particular, with $x = 1$, it holds

$$
\Pr\left(\max_{1 \leq k \leq n} S_k \geq 1\right) = O(\exp\{-C n^\alpha\}), \quad n \to \infty, \tag{2.6}
$$

where $C > 0$ is an absolute constant. Moreover, the power $\alpha$ in (2.6) is optimal for the class of martingale differences: for each $\alpha \in (0, 1)$, there exists a sequence of martingale differences $(X_i, F_i)_{i \geq 1}$ satisfying $\sup_i E \exp\left(\frac{(X_i^+)^{\alpha}}{C}\right) < \infty$, $\sup_n E \exp\left(\frac{(S_i)^{\alpha}}{n}\right) < \infty$ and

$$
\Pr\left(\max_{1 \leq k \leq n} S_k \geq n\right) \geq \exp\{-3n^\alpha\} \tag{2.7}
$$

for all $n$ large enough.

Actually, just as (2.2), the power $\alpha$ in (2.6) is optimal even for stationary martingale difference sequences.

In the i.i.d. case, the conditions of Corollary 2.3 can be weakened considerably, see Lanzinger and Stadmüller [8] where it is shown that if $E \exp\left(\frac{(X_i^+)^{\alpha}}{C}\right) < \infty$ with $\alpha \in (0, 1)$, then

$$
\Pr\left(\max_{1 \leq k \leq n} S_k \geq 1\right) = O(\exp\{-C n^\alpha\}), \quad n \to \infty. \tag{2.8}
$$

## 3 Proof of Theorem 2.1

We shall need the following refined version of the Azuma-Hoeffding inequality.

**Lemma 3.1.** Assume that $(X_i, F_i)_{i \geq 1}$ is a sequence of martingale differences satisfying $|X_i| \leq 1$ for all $i \geq 1$. Then, for all $x \geq 0$,

$$
\Pr\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \exp\left(-\frac{x^2}{2n}\right). \tag{3.1}
$$

A proof can be found in Laib [7].

For the proof of Theorem 2.1 we use a truncating argument as in Lesigne and Volný [9]. Let $(X_i, F_i)_{i \geq 1}$ be a sequence of supermartingale differences. Given $u > 0$, define

$$
X_i' = X_i 1_{\{|X_i| \leq u\}} - E(X_i 1_{\{|X_i| \leq u\}} | F_{i-1}),
$$

$$
X_i'' = X_i 1_{\{|X_i| > u\}} - E(X_i 1_{\{|X_i| > u\}} | F_{i-1}),
$$

$$
S_k' = \sum_{i=1}^k X_i', \quad S_k'' = \sum_{i=1}^k X_i'', \quad S_k''' = \sum_{i=1}^k E(X_i | F_{i-1}).
$$

Then $(X_i', F_i)_{i \geq 1}$ and $(X_i'', F_i)_{i \geq 1}$ are two martingale difference sequences and $S_k = S_k' + S_k'' + S_k'''$. Let $t \in (0, 1)$. Since $S_k''' \leq 0$, for any $x > 0$,

$$
\Pr\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq \Pr\left(\max_{1 \leq k \leq n} S_k' + S_k'' \geq x t\right) + \Pr\left(\max_{1 \leq k \leq n} S_k' \geq x(1- t)\right) \tag{3.2}
$$

$$
\leq \Pr\left(\max_{1 \leq k \leq n} S_k' \geq x t\right) + \Pr\left(\max_{1 \leq k \leq n} S_k'' \geq x(1- t)\right).
$$

Using Lemma 3.1 and the fact that $|X_i| \leq 2u$, we have
\[
P \left( \max_{1 \leq k \leq n} S_k' \geq x \right) \leq \exp \left\{ -\frac{x^2t^2}{8nu^2} \right\}. \tag{3.3}
\]

Let $F_i(x) = P(|X_i| \geq x), x \geq 0$. Since $E \exp \{ |X_i|^{2\alpha/\beta} \} \leq C_1$, we obtain, for all $x \geq 0$,
\[
F_i(x) \leq \exp \{ -x^{2\alpha/\beta} \} E \exp \{ |X_i|^{2\alpha/\beta} \} \leq C_1 \exp \{ -x^{2\alpha/\beta} \}.
\]

Using the martingale maximal inequality (cf. e.g. p. 14 in [6]), we get
\[
P \left( \max_{1 \leq k \leq n} S_k'' \geq x(1-t) \right) \leq \frac{1}{x^2(1-t)^2} \sum_{i=1}^{n} \mathbb{E}X_i''^2. \tag{3.4}
\]

It is easy to see that
\[
\mathbb{E}X_i''^2 = -\int_u^\infty t^2 dF_i(t)
= u^2 F_i(u) + \int_u^\infty 2t F_i(t) dt
\leq C_1 u^2 \exp \{ -u^{2\alpha/\beta} \} + 2C_1 \int_u^\infty t \exp \{ -t^{2\alpha/\beta} \} dt. \tag{3.5}
\]

Notice that the function $g(t) = t^3 \exp \{ -t^{2\alpha/\beta} \}$ is decreasing in $[\beta, +\infty)$ and is increasing in $[0, \beta]$, where $\beta = \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1}{2\alpha}}$. If $0 < u < \beta$, we have
\[
\int_u^\infty t \exp \{ -t^{2\alpha/\beta} \} dt \leq \int_u^\beta t \exp \{ -t^{2\alpha/\beta} \} dt + \int_\beta^\infty t - t^3 \exp \{ -t^{2\alpha/\beta} \} dt
\leq \int_u^\beta t \exp \{ -u^{2\alpha/\beta} \} dt + \int_\beta^\infty t - 2t^3 \exp \{ -\beta^{2\alpha/\beta} \} dt
\leq \frac{3}{2} \beta^2 \exp \{ -u^{2\alpha/\beta} \}. \tag{3.6}
\]

If $\beta \leq u$, we have
\[
\int_u^\infty t \exp \{ -t^{2\alpha/\beta} \} dt = \int_u^\infty t - t^3 \exp \{ -t^{2\alpha/\beta} \} dt
\leq \int_u^\infty t - 2t^3 \exp \{ -u^{2\alpha/\beta} \} dt
= u^2 \exp \{ -u^{2\alpha/\beta} \}. \tag{3.7}
\]

By (3.5), (3.6) and (3.7), we get
\[
\mathbb{E}X_i''^2 \leq 3C_1(u^2 + \beta^2) \exp \{ -u^{2\alpha/\beta} \}. \tag{3.8}
\]

From (3.4), it follows that
\[
P \left( \max_{1 \leq k \leq n} S_k'' \geq x(1-t) \right) \leq \frac{3nC_1}{x^2(1-t)^2} (u^2 + \beta^2) \exp \{ -u^{2\alpha/\beta} \}. \tag{3.9}
\]

Combining (3.2), (3.3) and (3.9), we obtain
\[
P \left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq 2 \exp \left\{ -\frac{x^2t^2}{8nu^2} \right\} + \frac{3nC_1}{(1-t)^2} \left( \frac{u^2}{x^2} + \frac{\beta^2}{x^2} \right) \exp \{ -u^{2\alpha/\beta} \}. 
\]
Taking $t = \frac{1}{\sqrt{2}}$ and $u = \left(\frac{x}{x \sqrt{\pi}}\right)^{1-\alpha}$, we get, for all $x > 0$,

$$
P\left(\max_{1 \leq k \leq n} S_k \geq x\right) \leq C_n(\alpha, x) \exp\left\{ -\left(\frac{x^2}{16\alpha}\right)^{\alpha}\right\},
$$

where

$$
C_n(\alpha, x) = 2 + 35C_1 \left(\frac{1}{x^{2\alpha} (16\alpha)^{1-\alpha}} + \frac{\beta^2}{x^2}\right).
$$

Hence, for all $x > 0$,

$$
P\left(\max_{1 \leq k \leq n} S_k \geq nx\right) \leq C(\alpha, x) \exp\left\{ -\left(\frac{x}{4}\right)^{2\alpha} n^\alpha\right\},
$$

where

$$
C(\alpha, x) = 2 + 35C_1 \left(\frac{1}{x^{2\alpha} (16\alpha)^{1-\alpha}} + \left(\frac{3(1-\alpha)}{2\alpha}\right)^{\frac{1-\alpha}{\alpha}}\right).
$$

This completes the proof of the first assertion of Theorem 2.1.

Next, we prove that the power $\alpha$ in (2.2) is optimal by giving a stationary sequence of martingale differences satisfying (2.3). We proceed as in Lesigne and Volný ([9], p. 150). Take a positive random variable $X$ such that

$$
P(X > x) = \frac{2e}{1 + x^{\frac{1}{1-\alpha}}} \exp\left\{ -x^{\frac{2\alpha}{1-\alpha}}\right\} \tag{3.10}
$$

for all $x > 1$. Using the formula $E f(X) = f(1) + \int_1^{\infty} f'(t)P(X > t)dt$ for $f(t) = \exp\{t^{\frac{2\alpha}{1-\alpha}}\}, t \geq 1$, we obtain

$$
E \exp\{X^{\frac{2\alpha}{1-\alpha}}\} = e + \frac{4\alpha}{1-\alpha} \int_1^{\infty} \frac{t^{\frac{2\alpha}{1-\alpha}-1}}{1 + t^{\frac{1}{1-\alpha}}} dt < \infty.
$$

Assume that $(\xi_i)_{i \geq 1}$ are Rademacher random variables independent of $X$, i.e. $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$. Set $X_i = X\xi_i$, $\mathcal{F}_0 = \sigma(X)$ and $\mathcal{F}_i = \sigma(\xi_i, \mathcal{F}_0, \ldots, \mathcal{F}_{i-1})$. Then, $(X_i, \mathcal{F}_i)_{i \geq 1}$ is a stationary sequence of martingale differences satisfying

$$
\sup_{i} E \exp\{|X_i|^{\frac{2\alpha}{1-\alpha}}\} = E \exp\{X^{\frac{2\alpha}{1-\alpha}}\} < \infty.
$$

For $\beta \in (0, 1)$, it is easy to see that

$$
P\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq P(S_n \geq n) \geq P\left(\sum_{i=1}^{n} \xi_i \geq n^\beta\right)P\left(X \geq n^{1-\beta}\right).
$$

Since, for $n$ large enough,

$$
P\left(\sum_{i=1}^{n} \xi_i \geq n^\beta\right) \geq \exp\{-n^{2\beta-1}\},
$$

(cf. Corollary 3.5 in Lesigne and Volný [9]), we get, for $n$ large enough,

$$
P\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq \frac{2e}{1 + (n^{1-\beta})^{\frac{2\alpha}{1-\alpha}}} \exp\left\{ -n^{2\beta-1} - (n^{1-\beta})^{\frac{2\alpha}{1-\alpha}}\right\}. \tag{3.11}
$$

Setting $2\beta - 1 = \alpha$, we obtain, for $n$ large enough,

$$
P\left(\max_{1 \leq k \leq n} S_i \geq n\right) \geq \frac{2e}{1 + n^{\beta-1}} \exp\{-2n^\alpha\} \geq \exp\{-3n^\alpha\},
$$

which proves (2.3). This ends the proof of Theorem 2.1.
4 Proof of Theorem 2.2

To prove Theorem 2.2, we need the following inequality.

**Lemma 4.1** ([3], Remark 2.1). Assume that \((X_i, \mathcal{F}_i)_{i \geq 1}\) are supermartingale differences satisfying \(X_i \leq 1\) for all \(i \geq 1\). Then, for all \(x, v > 0\),

\[
P \left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x)} \right\}.
\] (4.1)

Assume that \((X_i, \mathcal{F}_i)_{i \geq 1}\) are supermartingale differences. Given \(u > 0\), set

\[
X'_i = X_i 1_{\{X_i \leq u\}}, \quad X''_i = X_i 1_{\{X_i > u\}}, \quad S'_k = \sum_{i=1}^{k} X'_i \quad \text{and} \quad S''_k = \sum_{i=1}^{k} X''_i.
\]

Then, \((X'_i, \mathcal{F}_i)_{i \geq 1}\) is also a sequence of supermartingale differences and \(S_k = S'_k + S''_k\). Since \(\langle S' \rangle_k \leq \langle S \rangle_k\), we deduce, for all \(x, u, v > 0\),

\[
P \left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \leq P \left( S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) + P \left( S''_k \geq 0 \text{ and } \langle S'' \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \leq P \left( S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) + P \left( \max_{1 \leq k \leq n} S''_k \geq 0 \right).
\] (4.2)

Applying Lemma 4.1 to the supermartingale differences \((X'_i/u, \mathcal{F}_i)_{i \geq 1}\), we have, for all \(x, u, v > 0\),

\[
P( S'_k \geq x \text{ and } \langle S' \rangle_k \leq v^2 \text{ for some } k \in [1, n] ) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\}.
\] (4.3)

Using the exponential Markov’s inequality and the condition \(E \exp\{ (X^+_i)^{\frac{1}{1-\alpha}} \} \leq C_1\), we get

\[
P \left( \max_{1 \leq k \leq n} S''_k \geq 0 \right) \leq \sum_{i=1}^{n} P(X_i > u) \leq \sum_{i=1}^{n} E \exp\{ (X^+_i)^{\frac{1}{1-\alpha}} - u \} \leq n C_1 \exp\{-u \}. \] (4.4)

Combining the inequalities (4.2), (4.3) and (4.4) together, we obtain, for all \(x, u, v > 0\),

\[
P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\} + n C_1 \exp\{-u \}. \] (4.5)

Taking \(u = x^{1-\alpha}\), we get, for all \(x, v > 0\),

\[
P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})} \right\} + n C_1 \exp\{-x^{\alpha} \}. \] (4.6)

This completes the proof of Theorem 2.2.
5 Proof of Corollary 2.3.

To prove Corollary 2.3 we make use of Theorem 2.2. It is easy to see that

\[
P\left(\max_{1 \leq k \leq n} S_k \geq nx, \langle S \rangle_n \leq n v^2 \right) \leq P(S_k \geq nx \text{ and } \langle S \rangle_k \leq n v^2 \text{ for some } k \in [1, n]) + P\left(\langle S \rangle_n > n v^2 \right).
\]

(5.1)

By Theorem 2.2 it follows that, for all \( x, v > 0 \),

\[
P\left(\max_{1 \leq k \leq n} S_k \geq nx \right) \leq \exp\left\{-\frac{x^2}{2 \left( n^{\alpha-1} v^2 + \frac{1}{4} x^2 - \alpha \right)} n^{\alpha}\right\} + n C_1 \exp\{-x^\alpha n^\alpha\} + P\left(\langle S \rangle_n > n v^2 \right).
\]

Using the exponential Markov’s inequality and the condition \( E \exp\{\left( \langle S \rangle_n \right)^{\alpha 1 - \alpha} \} \leq C_2 \), we get, for all \( v > 0 \),

\[
P\left(\langle S \rangle_n > n v^2 \right) \leq E \exp\left\{\left( \frac{\langle S \rangle_n}{n} \right)^{\alpha 1 - \alpha} - v^2 \frac{\alpha 1 - \alpha}{2} \right\} \leq C_2 \exp\{-v^2 \frac{\alpha 1 - \alpha}{2}\}.
\]

Taking \( v = (nx)^{\frac{1}{1-\alpha}} \), we obtain, for all \( x > 0 \),

\[
P\left(\max_{1 \leq k \leq n} X_k \geq nx \right) \leq \exp\left\{-\frac{x^{1+\alpha}}{2 \left( 1 + \frac{1}{4} x \right)} n^\alpha\right\} + (n C_1 + C_2) \exp\{-x^\alpha n^\alpha\},
\]

which gives inequality (2.5).

Next, we prove that the power \( \alpha \) in (2.6) is optimal. Let \((X_i, \mathcal{F}_i)_{i \geq 1}\) be the sequence of martingale differences constructed in the proof of the second assertion of Theorem 2.1. Then \( \langle S \rangle_n = X^2 \),

\[
\sup_i E \exp\left\{\left( X^+_i \right)^{\frac{\alpha}{1-\alpha}} \right\} = \frac{1}{2} E \exp\{X^{\frac{\alpha}{1-\alpha}}\} < \infty
\]

and

\[
\sup_n E \exp\left\{\left( \frac{\langle S \rangle_n}{n} \right)^{\frac{\alpha}{1-\alpha}} \right\} = E \exp\{X^{\frac{\alpha}{1-\alpha}}\} < \infty.
\]

Using the same argument as in the proof of Theorem 2.1 we obtain, for \( n \) large enough,

\[
P\left(\max_{1 \leq k \leq n} S_k \geq n \right) \geq \exp\{-3 n^\alpha\}.
\]

This ends the proof of Corollary 2.3.

References


Exponential inequalities for supermartingales


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