Supplementary material to: Back-pressure traffic signal control with unknown routing rates

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***Abstract*—This is the supplementary material to the paper: back-pressure traffic signal control with unknown routing rates. It details some model details and proofs not included in the paper due to space limitations. The characterization of the capacity region, the optimality of BP* and the behaviour of the Lyapunov drift under BP control are proved.**

I. Routing process assumptions

When a quantity of vehicles arrives at node $N_a \in \mathcal{I}(J_i)$ during slot t, exogenously and endogenously, it is split and added into queues Q_{ab}, $b \in \mathcal{O}(J_i)$, according to an exogenous routing process $R_{ab}^{(t)}$, defined for all $a, b \in \mathcal{N}$. The arrival process and the routing process are independent, and for all t, $R_{ab}^{(t)}$ is independent from $\{Q(\tau)\}_{\tau \le t}$. $R_{ab}^{(t)}$ takes an integer, returns an integer, and for $X \in \mathbb{N}$, $\sum_b R_{ab}^{(t)}(X) \le X$. For all process $X(t)$ such that for all t, $R_{ab}^{(t)}$ is independent from $\{X(\tau)\}_{\tau \le t}$, there exists a rate $r_{ab} \ge 0$ for all $a, b \in \mathcal{N}$ such that $R_{ab}^{(t)}(X(t)) - r_{ab}X(t)$ is rate convergent with rate 0.

As a consequence of the above assumptions:

$$\sum_b r_{ab} \le 1 \tag{1}$$

II. Network dynamics

Consider the network under phase control $p(t)$. Let define the following flow variables:

$$f_{ab}(t) = \min\left[Q_{ab}(t), \mu_{ab}\left(p(t)\right)\right] \tag{2}$$

$$f_a^{\text{in}}(t) = \sum_b f_{ba}(t) \tag{3}$$

The network dynamics under control $p(t)$ is completely described as follows:

$$Q_{ab}(t+1) = Q_{ab}(t) + R_{ab}^{(t)}\left(A_a(t) + f_a^{\text{in}}(t)\right) - f_{ab}(t) \tag{4}$$

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III. Characterization of the capacity region

A. Stability definition

A key property of queuing systems is stability, defined below:

Definition 1 (Stability)**.** *The queuing network is stable if each individual queue U satisfies:*

$$\limsup_{T \to +\infty} \mathbb{E}\left\{\frac{1}{T}\sum_{t=0}^{T-1} 1_{U(t)>V}\right\} \to 0 \text{ as } V \to +\infty \tag{5}$$

This definition of stability is standard and is applicable to networks with arbitrary inputs and control laws [1].

B. The capacity region

It is possible to define a capacity region which describes the set of arrivals rates vectors that can be stably handled by the network.

Definition 2 (Capacity region [1])**.** *Given a routing matrix r, the capacity region Λ_r is the closure of the set of all arrival rate vectors λ that can be stabilized by some control.*

The following theorem provides a characterization of the capacity region in our particular model:

Theorem (Capacity region characterisation)**.** *Given a routing matrix r, the capacity region Λ_r is the set of arrival vectors λ such that there exists $g \in \Gamma$ satisfying:*

$$\forall a, b \in \mathcal{N}, r_{ab}(\lambda_a + g_a^{\text{in}}) \le g_{ab} \tag{6}$$

where Γ is the set of feasible long-term endogenous service rates, defined below:

$$\Gamma = \text{Convex\_Hull}\{\mu(p) : p \in \mathcal{P}\} \tag{7}$$

Moreover,

- *$\lambda \in \Lambda_r$ is a necessary condition for network stability, considering all possible controls (including those that have perfect knowledge of future events)*
- *$\lambda \in \text{int}(\Lambda_r)$ is a sufficient condition for the network to be stabilized by a control that does not have knowledge of future events.*

Proof. The proof is a slightly modified version of the characterization of the capacity region of [1]. Let $\tilde{\Lambda}_r$ denote the set of arrival rates vectors such that there exists $g \in \Gamma$ satisfying:

$$\forall a, b \in \mathcal{N}, r_{ab}(\lambda_a + g_a^{\text{in}}) \leq g_{ab} \tag{8}$$

Let prove that $\lambda \in \tilde{\Lambda}_r$ is a necessary condition for network stability, considering all possible controls (including those that have perfect knowledge of future events). Suppose that the network is empty at $t = 0$, using the equation of the dynamics of the network, we obtain:

$$Q_{ab}(T) = \sum_{t=0}^{T-1} R_{ab}^{(t)} \left(A_a(t) + f_a^{\text{in}}(t) \right) - \sum_{t=0}^{T-1} f_{ab}(t) \tag{9}$$

Suppose the network is stabilized and fix an arbitrary small value $\epsilon > 0$. By the network stability necessary condition of [1], there must exist some finite value V such that at arbitrary large times T, the queues lengths are simultaneously less than V with probability at least $1/2$. Hence, there exists a time T such that with probability at least $1/2$, the following inequalities hold for all $a, b \in \mathcal{N}$:

$$Q_{ab}(T) \leq V \tag{10}$$

$$\frac{V}{T} \leq \epsilon \tag{11}$$

$$\frac{\sum_{t=0}^{T-1} R_{ab}^{(t)} A_a(t)}{T} \geq r_{ab}\lambda_a - \epsilon \tag{12}$$

$$\frac{\sum_{t=0}^{T-1} R_{ab}^{(t)} f_a^{\text{in}}(t)}{T} \geq r_{ab} f_a^{\text{in}}(t) - \epsilon \tag{13}$$

$$\tag{14}$$

Define variables $g_{ab} = \sum_{t=0}^{T-1} f_{ab}(t)/T$. Using the above inequalities together with Equation 9 provides:

$$r_{ab} \left(\lambda_a + g_a^{\text{in}} \right) \leq g_{ab} + 3\epsilon \tag{15}$$

Hence, with probability greater that $1/2$, the flows g_{ab} come arbitrary close to satisfying Inequality 8. As a result, there must exist sample paths $f_{ab}(t)$ from which flow variables g_{ab} are defined that satisfy Inequality 15. This proves that λ is a limit point of $\tilde{\Lambda}_r$. As $\tilde{\Lambda}_r$ is a compact, it contains its limit points, and we finally obtain: $\Lambda_r \subset \tilde{\Lambda}_r$.

Now, assume that λ is interior to $\tilde{\Lambda}_r$. Then, as proved in Section IV, one can build a randomized control that stabilizes the network under arrival rates vector λ. Hence, $\lambda \in \text{int}(\tilde{\Lambda}_r)$ is a sufficient condition for the network to be stabilized by some control. Combining the two above results proves that $\tilde{\Lambda}_r = \Lambda_r$.

□

IV. Existence of a stabilizing randomized policy

The following theorem proves the existence of a stabilizing stationary randomized policy for all arrival rates vectors interior to the capacity region.

Theorem (Existence of a stabilizing randomized policy)**.** *Suppose there exists $\epsilon > 0$ such that $\lambda + \epsilon \in \text{int}(\Lambda_r)$, i.e. λ and $\lambda + \epsilon$ interior to the capacity region. Then, there exists a stationary randomized control $\tilde{p}(t)$ such that:*

$$\mathbb{E}\left\{ \tilde{\mu}_{ab}(p(t)) - r_{ab}\left(\tilde{\mu}_a^{\text{in}}(p(t)) + \lambda_a \right) \right\} \geq r_{ab}\epsilon_a \tag{16}$$

Proof. For the sake of simplicity, we assume in this proof that $\mathbb{E}\{A_a(t)\} = \lambda_a$ and $\mathbb{E}\{R_{ab}^{(t)}(X(t))\} = r_{ab}\mathbb{E}\{X(t)\}$ for all $X(t)$ independent from $R_{ab}^{(t)}$. It is not true in the general case (it is the particular case of i.i.d. arrivals and routing), and the reader can refer to [1] for the principle of an extension to the general case using a K-steps Lyapunov drift.

Suppose λ is interior to the capacity region, i.e. there exists a positive vector ϵ such that $\lambda + \epsilon \in \Lambda$. By Theorem III-B, there exists $g \in \Gamma$ such that:

$$\forall a, b \in \mathcal{N}, r_{ab}(\lambda_a + \epsilon_a + g_a^{\text{in}}) \leq g_{ab} \tag{17}$$

Since $g \in \Gamma$, it can be expressed as a weighted sum as follows:

$$g = \sum_{p \in \mathcal{P}} w_p \mu(p) \tag{18}$$

where weights w_p sum to 1. Let define the randomized policy $\tilde{p}$ that selects randomly the phase to apply at every time slot according to probabilities $(w_p)_{p \in \mathcal{P}}$.

It is direct that it will result in a randomized stationary service matrix $\mu(\tilde{p}(t))$ verifying:

$$\mathbb{E}\{\mu_{ab}(\tilde{p}(t))\} = g_{ab} \tag{19}$$

As a result,

$$\mathbb{E}\left\{ \mu_{ab}(\tilde{p}(t)) - r_{ab}\left(\tilde{\mu}_a^{\text{in}}(\tilde{p}(t)) + \lambda_a \right) \right\} \geq r_{ab}\epsilon_a \tag{20}$$

Now, assume that $\tilde{p}(t)$ is applied to the queuing network. Then, using the equation of the dynamics of the network:

$$\begin{aligned} Q_{ab}(t+1) \leq \max\left(Q_{ab}(t) - \mu_{ab}(\tilde{p}(t)), 0 \right) + \\ R_{ab}^{(t)} \left(A_a(t) + \mu_a^{\text{in}}(\tilde{p}(t)) \right) \end{aligned} \tag{21}$$

An inequality holds instead of an equality because the number of vehicles transferred is less or equal to the transmission rate offered by servers.

Squaring both sides and using $\max^2(x, 0) \leq x^2$, we obtain:

$$\begin{aligned} Q_{ab}(t+1)^2 - Q_{ab}(t)^2 \leq \left(R_{ab}^{(t)} \left(A_a(t) + \mu_a^{\text{in}}(\tilde{p}(t)) \right) \right)^2 \\ + \mu_{ab}(\tilde{p}(t))^2 \\ - 2Q_{ab}(t) \left(\mu_{ab}(\tilde{p}(t)) - R_{ab}^{(t)} \left(A_a(t) + \mu_a^{\text{in}}(\tilde{p}(t)) \right) \right) \end{aligned} \tag{22}$$

Define the Lyapunov function $\mathcal{V}(Q(t)) = V(t) = \sum_{a,b} Q_{ab}(t)^2$. Taking expectations, summing over all $a, b \in \mathcal{N}$, using independences and noting that $\mathbb{E}\{A\} \leq \sqrt{\mathbb{E}\{A^2\}}$, we obtain:

$$\begin{aligned} &\mathbb{E}\{V(t+1) - V(t) | Q(t)\} \leq B \\ &- 2\sum_{a,b} Q_{ab}(t)\mathbb{E}\{\mu_{ab}(\tilde{p}(t)) - r_{ab}\left(\lambda_a + \mu_a^{\text{in}}(\tilde{p}(t)) \right)\} \end{aligned} \tag{23}$$

Using Inequality 20, we obtain:

$$\mathbb{E}\{V(t+1)-V(t)|Q(t)\} \leq B - 2\sum_{a,b} r_{ab}\epsilon_a Q_{ab}(t) \quad (24)$$

Let define $\eta = 2\min_{a,b} r_{ab}\epsilon_a > 0$, we finally obtain:

$$\mathbb{E}\{V(t+1)-V(t)|Q(t)\} \leq B - \eta\sum_{a,b} Q_{ab}(t) \quad (25)$$

The sufficient condition using Lyapunov drift proved in [1] enables to conclude stability of the queuing network.

□

V. Optimality of BP*

Theorem (Back-pressure optimality)**.** *Assuming that pressure functions are linear with strictly positive slopes. Then, BP* is stability-optimal.*

Proof. Again, for the sake of simplicity, we assume in this proof that $\mathbb{E}\{A_a(t)\} = \lambda_a$ and $\mathbb{E}\{R_{ab}^{(t)}(X(t))\} = r_{ab}\mathbb{E}\{X(t)\}$ for all $X(t)$ independent from $R_{ab}^{(t)}$. The reader can refer to [1] for the principle of an extension to the general case using a K-steps Lyapunov drift.

Let $\theta_{ab} > 0$ denote the slope of linear pressure function P_{ab} and $\Pi_{ab}(t) = P_{ab}(Q_{ab}(t))$ the evolution of pressures over time. Define the Lyapunov function $\mathcal{V}(Q(t)) = V(t) = \sum_{a,b}\theta_{ab}Q_{ab}(t)^2$ and let $p(t)$ denote the control applied to the queuing network. With the same manipulations as for the proof of Theorem III-B, we obtain:

$$\begin{aligned}
V(t+1)-V(t) &= \sum_{a,b}\Pi_{ab}(t+1)^2 - \Pi_{ab}(t)^2 = \\
&\sum_{a,b}\theta_{ab}\left(Q_{ab}(t+1)^2 - Q_{ab}(t)^2\right) \leq \\
&\sum_{a,b}\theta_{ab}\left[\left(R_{ab}^{(t)}\left(A_a(t)+\mu_a^{\text{in}}(p(t))\right)\right)^2 + \mu_{ab}(p(t))^2\right] \\
&- 2\sum_{a,b}\theta_{ab}Q_{ab}(t)\left(\mu_{ab}(p(t)) - R_{ab}^{(t)}\left(A_a(t)+\mu_a^{\text{in}}(p(t))\right)\right) \leq \\
&B(t) - 2\sum_{a,b}\theta_{ab}Q_{ab}(t)\left(\mu_{ab}(p(t)) - R_{ab}^{(t)}\left(A_a(t)+\mu_a^{\text{in}}(p(t))\right)\right)
\end{aligned} \quad (26)$$

with the upper-bound $B(t)$ defined below:

$$B(t) = \sum_{a,b}\theta_{ab}\left[R_{ab}^{(t)}\left(A_a(t) + \sup_{p\in\mathcal{P}}\mu_a^{\text{in}}(p)\right)\right]^2 + \left(\sup_{p\in\mathcal{P}}\mu_{ab}(p)\right)^2 \quad (27)$$

Taking expectation and using independences, we get :

$$\begin{aligned}
&\mathbb{E}\{V(t+1)-V(t)|Q(t)\} \leq B \\
&\quad - 2\sum_{a,b}\theta_{ab}Q_{ab}(t)\mathbb{E}\left\{\mu_{ab}(p(t)) - r_{ab}\mu_a^{\text{in}}(p(t))|Q(t)\right\} \\
&\quad + 2\sum_{a,b}\theta_{ab}Q_{ab}(t)r_{ab}\lambda_a
\end{aligned} \quad (28)$$

The upper-bound B is obtained using $\mathbb{E}\{A_a\} \leq \sqrt{\mathbb{E}\{A_a^2\}}$:

$$B = \sum_{a,b}\theta_{ab}r_{ab}^2\left(A_a^{\max} + \sup_{p\in\mathcal{P}}\mu_a^{\text{in}}(p)\right)^2 + \theta_{ab}\left(\sup_{p\in\mathcal{P}}\mu_{ab}(p)\right)^2 \quad (29)$$

By simple sum manipulation, the following identity is obtained:

$$\sum_{a,b} M_{ab}(g_{ab} - r_{ab}g_a^{\text{in}}) = \sum_{a,b}(M_{ab} - \sum_c r_{bc}M_{bc})g_{ab} \quad (30)$$

Using identity 30, Equation 28 becomes:

$$\begin{aligned}
&\mathbb{E}\{V(t+1)-V(t)|Q(t)\} \leq B \\
&- 2\sum_{a,b}\left(\theta_{ab}Q_{ab}(t) - \sum_c r_{bc}\theta_{bc}Q_{bc}(t)\right)\mathbb{E}\left\{f_{ab}(t)|Q(t)\right\} \\
&\quad + 2\sum_{a,b}\theta_{ab}Q_{ab}(t)r_{ab}\lambda_a = B \\
&\quad - 2\sum_{a,b}\left(\Pi_{ab}(t) - \sum_c r_{bc}\Pi_{bc}(t)\right)\mathbb{E}\left\{f_{ab}(t)|Q(t)\right\} \\
&\quad + 2\sum_{a,b}\Pi_{ab}(t)r_{ab}\lambda_a
\end{aligned} \quad (31)$$

Now, assume that BP* control $p^\star(t)$ is applied and let $V^\star(t)$ denote the Lyapunov function under $p^\star(t)$. It is assumed that in case of equality when selecting the phase that maximizes the weighted sum, the selected phase $p^\star(t)$ satisfies $\mu_{ab}\left(p^\star(t)\right) = 0$ if $W_{ab}(t) = 0$.

As a result, we obtain:

$$\begin{aligned}
&\mathbb{E}\{V^\star(t+1)-V^\star(t)|Q(t)\} \leq B \\
&\quad - 2\sum_{a,b}W_{ab}(t)\mathbb{E}\left\{\mu_{ab}\left(p^\star(t)\right)|Q(t)\right\} \\
&\quad + 2\sum_{a,b}\Pi_{ab}(t)r_{ab}\lambda_a
\end{aligned} \quad (32)$$

By construction of back-pressure control $p^\star(t)$ (see the $\arg\max$ in the algorithm), $p^\star(t)$ maximizes $\sum_{a,b}W_{ab}(t)\mu_{ab}(p(t))$ over all possible alternative controls $p(t)$.

Now, suppose that the arrival rates vector is interior to the capacity region Λ_r, i.e. there exists $\epsilon > 0$ such that $\lambda + \epsilon \in \Lambda$. Then, as proved in the supplementary material not provided in this paper due to space limitations, there exists a stabilizing stationary randomized control $\tilde{p}(t)$ such that for all $a, b \in \mathcal{N}$:

$$\mathbb{E}\left\{\mu_{ab}(\tilde{p}(t)) - r_{ab}\left(\mu_a^{\text{in}}(\tilde{p}(t)) + \lambda_a\right)\right\} \geq r_{ab}\epsilon_a \tag{33}$$

Combining the two above statements, taking expectations, and noting that the control $\tilde{p}(t)$ is stationary result in:

$$\begin{aligned}\sum_{a,b} W_{ab}(t)\mathbb{E}\{\mu_{ab}(p^\star(t))|Q(t)\} \geq \\ \sum_{a,b} W_{ab}(t)\mathbb{E}\{\mu_{ab}(\tilde{p}(t))\} \geq \\ \sum_{a,b}\left(\Pi_{ab}(t) - \sum_c r_{bc}\Pi_{bc}(t)\right)\mathbb{E}\{\mu_{ab}(\tilde{p}(t)))\} = \\ \sum_{a,b}\Pi_{ab}(t)\mathbb{E}\left\{\mu_{ab}(\tilde{p}(t)) - r_{ab}\mu_a^{\text{in}}(\tilde{p}(t))\right\} \geq \\ \sum_{a,b}\Pi_{ab}(t)(r_{ab}\lambda_a + r_{ab}\epsilon_a)\end{aligned} \tag{34}$$

Injecting the above result in the Lyapunov drift inequality results in:

$$\mathbb{E}\{V^\star(t+1) - V^\star(t)|Q(t)\} \leq B - 2\sum_{a,b}\theta_{ab}r_{ab}\epsilon_a Q_{ab}(t) \tag{35}$$

Let $\eta = 2\min_{a,b}\theta_{ab}r_{ab}\epsilon_a > 0$. We finally obtain:

$$\mathbb{E}\{V^\star(t+1) - V^\star(t)|Q(t)\} \leq B - \eta\sum_{a,b}Q_{ab}(t) \tag{36}$$

The sufficient condition using Lyapunov drift proved in [1] enables to conclude stability of the queuing network. □

VI. Behaviour of the Lyapunov drift in heavy load conditions

Theorem (Lyapunov drift under heavy load conditions)**.** *Assume $\lambda + \epsilon \in \Lambda_r$, BP control is applied and the network is in heavy load conditions, then there exists $B, \eta > 0$ such that :*

$$\mathbb{E}\{V(t+1) - V(t) \mid Q(t)\} \leq B - \eta\sum_a Q_a(t) \tag{37}$$

for sufficiently large ϵ.

Proof. Again, for the sake of simplicity, we assume in this proof that $\mathbb{E}\{A_a(t)\} = \lambda_a$ and $\mathbb{E}\{R_{ab}^{(t)}(X)\} = r_{ab}X$ for all $X \in \mathbb{N}$.

Let $\Pi_a(t)$ denote the evolution of $P_a(Q_a(t))$ over time and $p(t)$ the control applied to the queuing network. By simple manipulations, we get:

$$\begin{aligned}V(t+1) - V(t) = \\ \sum_a \theta_a\left(Q_a(t+1) - Q_a(t)\right)^2 \\ + 2\sum_a \theta_a Q_a(t)\left(Q_a(t+1) - Q_a(t)\right)\end{aligned} \tag{38}$$

As a result, Inequality 38 becomes:

$$\begin{aligned}V(t+1) - V(t) \leq B(t) - 2\sum_a \theta_a Q_a(t) \\ \sum_b\left(f_{ab}(t) - R_{ab}^{(t)}\left(A_a(t) + f_a^{\text{in}}(t)\right)\right)\end{aligned} \tag{39}$$

with the upper-bound $B(t)$ defined below:

$$\begin{aligned}B(t) = \sum_a \theta_a \sum_b\left[R_{ab}^{(t)}\left(A_a(t) + \sup_{p\in\mathcal{P}}\mu_a^{\text{in}}(p)\right)\right]^2 \\ + \sum_a \theta_a \sum_b\left(\sup_{p\in\mathcal{P}}\mu_{ab}(p)\right)^2\end{aligned} \tag{40}$$

Taking expectations and using independences, we obtain :

$$\begin{aligned}\mathbb{E}\left\{V(t+1) - V(t) \mid Q(t)\right\} \leq B + 2\sum_a \theta_a Q_a(t)\lambda_a\sum_b r_{ab} \\ - 2\mathbb{E}\left\{\sum_{a,b}\theta_a Q_a(t)\left(f_{ab}(t) - r_{ab}f_a^{\text{in}}(t)\right) \mid Q(t)\right\}\end{aligned} \tag{41}$$

Moreover, by simple sum manipulations, we get the below identity:

$$\sum_{a,b} P_a\left(f_{ab} - r_{ab}f_a^{\text{in}}\right) = \sum_{a,b}\left(P_a - \sum_c r_{bc}P_b\right)f_{ab} \tag{42}$$

Using identity 42, Equation 41 becomes:

$$\begin{aligned}\mathbb{E}\left\{V(t+1) - V(t) \mid Q(t)\right\} \leq B + 2\sum_a \theta_a Q_a(t)\lambda_a \\ - 2\sum_{a,b}\left(\Pi_a(t) - \left(\sum_c r_{bc}\right)\Pi_b(t)\right)\mathbb{E}\left\{f_{ab}(t) \mid Q(t)\right\}\end{aligned} \tag{43}$$

Since $\sum_c r_{bc} \leq 1$, we obtain:

$$\begin{aligned}\mathbb{E}\left\{V(t+1) - V(t) \mid Q(t)\right\} \leq B + 2\sum_a \theta_a Q_a(t)\lambda_a \\ - 2\sum_{a,b}\left(\Pi_a(t) - \Pi_b(t)\right)\mathbb{E}\left\{f_{ab}(t) \mid Q(t)\right\}\end{aligned} \tag{44}$$

Now, assume that BP control $p^{\text{BP}}(t)$ is applied and let $V^{\text{BP}}(t)$ denote the Lyapunov function under $p^{\text{BP}}(t)$. It is assumed that in case of equality when selecting the phase that maximizes the weighted sum, the selected phase $p^\star(t)$ satisfies

$\mu_{ab}\left(p^{\star}(t)\right) = 0$ if $W_{ab}(t) = 0$. Moreover, by definition of $d_{ab}(t)$, under infinite capacities, $f_{ab}(t) = d_{ab}(t)\mu_{ab}\left(p(t)\right)$. Hence, we obtain:

$$\mathbb{E}\left\{V^{\mathrm{BP}}(t+1) - V^{\mathrm{BP}}(t) \mid Q(t)\right\} \leq B + 2\sum_a \theta_a Q_a(t)\lambda_a - 2\sum_{a,b} W_{ab}(t)\mathbb{E}\left\{\mu_{ab}\left(p^{\mathrm{BP}}(t)\right) \mid Q(t)\right\} \quad (45)$$

By construction of back-pressure control $p^{\mathrm{BP}}(t)$ (see $\arg\max$ in the algorithm), $p^{\mathrm{BP}}(t)$ maximizes $\sum_{a,b} W_{ab}(t)\mu_{ab}(p(t))$ over all possible alternative controls $p(t)$.

Now, assume $\lambda + \epsilon \in \Lambda_r$ where ϵ is a positive vector. Then, as proved in Section IV, there exists a stabilizing stationary randomized control $\tilde{p}(t)$ such that:

$$\exists g \in \Gamma : \forall a, b \in \mathcal{N},\ \mathbb{E}\{\mu_{ab}(\tilde{p}(t))\} = g_{ab} \quad (46)$$

Combining the two above statements and taking expectations results in:

$$\sum_{a,b} W_{ab}(t)\mathbb{E}\{\mu_{ab}(p^{\mathrm{BP}}(t)) \mid Q(t)\} \geq \sum_{a,b} W_{ab}(t)\mathbb{E}\{\mu_{ab}(\tilde{p}(t)) \mid Q(t)\} = \sum_{a,b} d_{ab}(t)\left(\Pi_a(t) - \Pi_b(t)\right) g_{ab} \quad (47)$$

Now, assume that the network in heavy load conditions, then $d_{ab}(t) = 1$ and we obtain:

$$\sum_{a,b} W_{ab}(t)\mathbb{E}\{\mu_{ab}(p^{\mathrm{BP}}(t)) \mid Q(t)\} \geq \sum_{a,b} \left(\Pi_a(t) - \Pi_b(t)\right) g_{ab} \quad (48)$$

By simple manipulation, we get the following identity:

$$\sum_{a,b} \left(\Pi_a - \Pi_b\right) g_{ab} = \sum_a \Pi_a\left(g_a^{\mathrm{out}} - g_a^{\mathrm{in}}\right) \quad (49)$$

Hence,

$$\sum_{a,b} W_{ab}(t)\mathbb{E}\{\mu_{ab}(p^{\mathrm{BP}}(t)) \mid Q(t)\} \geq \sum_a \Pi_a(t)\left(g_a^{\mathrm{out}} - g_a^{\mathrm{in}}\right) = \sum_a \theta_a Q_a(t)\left(g_a^{\mathrm{out}} - g_a^{\mathrm{in}}\right) \quad (50)$$

Moreover, by definition of the input/output flow,

$$g_a^{\mathrm{out}} - g_a^{\mathrm{in}} = \sum_b g_{ab} - g_a^{\mathrm{in}} = \sum_b \left(g_{ab} - r_{ab}g_a^{\mathrm{in}}\right) - (1 - \sum_b r_{ab})g_a^{\mathrm{in}} \quad (51)$$

As a result, using the inequalities verified by $g \in \Gamma$, we obtain:

$$\sum_{a,b} W_{ab}(t)\mathbb{E}\{\mu_{ab}(p^{\mathrm{BP}}(t)) \mid Q(t)\} \geq \sum_a \theta_a Q_a(t)\left[\sum_b \left(r_{ab}\lambda_a + r_{ab}\epsilon_a\right) - \left(1 - \sum_b r_{ab}\right) g_a^{\mathrm{in}}\right] \quad (52)$$

Injecting the latter result in Inequality 45 provides:

$$\mathbb{E}\left\{V^{\mathrm{BP}}(t+1) - V^{\mathrm{BP}}(t) \mid Q(t)\right\} \leq B - 2\sum_a \theta_a Q_a(t)\left[\sum_b r_{ab}\epsilon_a - \left(1 - \sum_b r_{ab}\right) g_a^{\mathrm{in}}\right] \quad (53)$$

Let define η as follows:

$$\eta = 2\min_a \theta_a\left(\sum_b r_{ab}\epsilon_a - \left(1 - \sum_b r_{ab}\right) g_a^{\mathrm{in}}\right) \quad (54)$$

We finally obtain:

$$\mathbb{E}\{V^{\mathrm{BP}}(t+1) - V^{\mathrm{BP}}(t) \mid Q(t)\} \leq B - \eta \sum_a Q_a(t) \quad (55)$$

If for all $a \in \mathcal{N}$,

$$\sum_b r_{ab}\epsilon_a > \left(1 - \sum_b r_{ab}\right) g_a^{\mathrm{in}} \quad (56)$$

which can be verified for sufficiently large ϵ then $\eta > 0$ and the inequality of the sufficient condition for network stability using Lyapunov drift of [1] is verified in heavy load conditions. However, it does not imply that the network is stable under BP control since the heavy load assumption is not necessarily verified at any time.

References

[1] M. J. Neely, *Dynamic power allocation and routing for satellite and wireless networks with time varying channels*. PhD thesis, LIDS, Massachusetts Institute of Technology, 2003.

□