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On the Use of the Multiple Scale Method in Solving 'Difficult' Bifurcation Problems

Abstract

Several algorithms consisting in 'non-standard' versions of the Multiple Scale Method are illustrated for 'difficult' bifurcation problems. Preliminary, the 'easy' case of bifurcation from a cluster of distinct eigenvalues is addressed, which requires using integer power expansions, and it leads to bifurcation equations all of the same order. Then, more complex problems are studied. The first class concerns bifurcation from a defective eigenvalue, which calls for using fractional power expansions and fractional time-scales, as well Jordan or Keldysh chains. The second class regards the interaction between defective and non-defective eigenvalues. This problem also requires fractional powers, but it leads to differential equations which are of different order for the involved amplitudes. Both autonomous and parametrically excited non-autonomous systems are studied. Moreover, the transition from a codimension-3 to a codimension-2 bifurcation is explained. As a third class of problems, singular systems possessing an evanescent mass, as Nonlinear Energy Sinks, are considered, and both autonomous systems undergoing Hopf bifurcation and non-autonomous systems under external resonant excitation, are studied. The algorithm calls for a suitable combination of the Multiple Scale Method and the Harmonic Balance Method, this latter to be applied exclusively to the singular equations. Several applications are shown, to test the effectiveness of the proposed methods. They include discrete and continuous systems, autonomous, parametrically and externally excited systems.

Keywords

Bifurcation Analysis — Perturbation Methods — Defective Eigenvalues — Parametric Stabilization — Relaxation Oscillations — Nonlinear Energy Sinks

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1. Introduction

Dynamical systems depend on parameters. When these are varied quasi-statically, the dynamics can, either, remain unchanged, or strongly modify. A *bifurcation* is said to occur at a point of the parameter space at which the dynamics abruptly changes. Here, new equilibria or cycles arise or disappear, and/or their stability changes. The bifurcation points occur on manifolds which separate the parameter space in regions in which the system possesses equivalent dynamics. Therefore bifurcations organize the scenario. The codimension of the manifold (equal to the number of independent algebraic conditions satisfied by parameters at bifurcation) is said the *codimension of the bifurcation*. In this paper, only bifurcations from equilibria are considered, and, although not strictly necessary, mechanical systems are addressed.

Bifurcation problems are almost universally tackled via the Center Manifold Method (CMM) and the Normal Form Theory (NFT) [1]. The CMM *reduces the dimension* of the dynamical system to that of the center subspace of the Jacobian (i.e. to the number of its eigenvalues having zero real part); the NFT *reduces the complexity* of the equations, via a quasi-identical change of coordinates, aimed to eliminate the possible largest number of nonlinear terms. The CMM is not only an algorithm, but reveals the true essence of the dynamics of a system close

to bifurcation. From a geometrical point of view, it states that an *attractive* surface (Center Manifold, CM) exists in the (finite- or infinite-dimensional) state-space. The CM is found to be tangent at the equilibrium point to the center subspace, so that it can be tough as originated by a *deformation* of this subspace, caused by nonlinearities. From a mechanical point of view, the CM establishes a *constraint* among the state variables. Few of them, describing the motion in the tangent plane, participate with their own natural frequency, and therefore are said *active coordinates*; the remaining ones, describing motions orthogonal to the plane, do not contribute with their own frequencies, but are driven from the active coordinates, and therefore are said *passive coordinates*. The active coordinates are first-order quantities, while the passive coordinates are second- or higher-order quantities. However, they cannot be neglected, since they contribute to determine stability and to organize the dynamics.

A completely different approach to derive the normal form of the bifurcation equations, is offered by the Multiple Scale Method (MSM). Although this algorithm is very popular in nonlinear dynamics [2], it is less known, or even looked at with suspicion, in the bifurcation community. This standpoint is probably due to the fact the, while the CMM is based on a theorem, the MSM is a heuristic approach, based on formal series expansions. On the other hand, it has been shown in liter-

ature, that the MSM is computationally more convenient, and furnishes the same results of the classical approach [3, 4, 5, 6]. Namely, it supplies reduced equations of motion *without describing the center manifold in advance*; moreover, it furnishes bifurcation equations *directly in normal form*. Thus, it performs simultaneously the two operations of the classical approach.

However, there exist 'difficult bifurcations' for which it has been erroneously believed in literature, that the MSM (or, more in general, *any* perturbation method) fails in determining the bifurcation equations. In spite of this opinion, the author and his co-workers showed that, to be successful, the MSM must only be *adapted* to the problem; once again the method turned out to be computationally competitive with other methods.

Here, we will discuss three classes of 'difficult' problems:

1. *Defective autonomous systems*, in which the Jacobian operator at bifurcation does not possess a complete system of eigenvectors [7, 8];
2. *Partially defective autonomous systems*, in which only some of the center eigenvalues are defective [9]; or, Hamiltonian systems close to divergence, *stabilized by resonant parametric excitation* [10];
3. *Singular, partially defective systems*, in which an evanescent mass coefficient exists, either autonomous [11] or externally excited [12].

All these problems require special treatments that will be sketched ahead; additional details can be found in the cited papers.

2. Bifurcation analysis from a cluster of distinct eigenvalues: the 'easy' case

We introduce the MSM referring to the simplest case in which the center eigenvalues are distinct, so that their associated eigenvectors form a complete basis for the center subspace (non-defective cases, see also [13, 14, 15]).

2.1 Finite-dimensional systems

A finite-dimensional, autonomous, dynamical system is considered, whose nondimensional equations of motion, in state-form and expanded around the trivial equilibrium point $\mathbf{x} = \mathbf{0}$, read:

$$\dot{\mathbf{x}} = \mathbf{J}(\boldsymbol{\mu})\mathbf{x} + \mathbf{n}_2(\mathbf{x}, \mathbf{x}) + \mathbf{n}_3(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \dots \quad (1)$$

Here \mathbf{x} is the n -dimensional state-vector, $\mathbf{J}(\boldsymbol{\mu}) := \mathbf{J}_0 + \boldsymbol{\mu}\mathbf{J}_1$ is the Jacobian matrix at the origin, depending on a vector of bifurcation parameters $\boldsymbol{\mu}$, and $\mathbf{n}_i(\cdot)$ are vector-valued homogeneous polynomials of degree $i = 2, 3, \dots$; moreover, the dot denotes differentiation with respect to the time t . It is assumed that $\boldsymbol{\mu} = \mathbf{0}$ is a bifurcation point, at which the Jacobian $\mathbf{J}(\mathbf{0}) = \mathbf{J}_0$ admits one or more *distinct* center eigenvalues $\lambda_c := (\lambda_{cj})$. To exemplify, in a codimension-3 zero-Hopf-Hopf bifurcation (ZHH, see [6]) there are five eigenvalues in the center spectrum, namely $\lambda_c = (0, \pm i\omega_1, \pm i\omega_2)$, while the remaining stable eigenvalues have negative real part. We want to analyze the dynamics of the system around the equilibrium point $\mathbf{x} = \mathbf{0}$, when $\boldsymbol{\mu}$ is varied in a ball of radius $0 < \varepsilon \ll 1$, centered at $\boldsymbol{\mu} = \mathbf{0}$.

First, in order a perturbation parameter appear in the equations, a rescaling is performed, namely $\mathbf{x} \rightarrow \varepsilon\mathbf{x}$, $\boldsymbol{\mu} \rightarrow \varepsilon\boldsymbol{\mu}$. Then,

according to the MSM, $\mathbf{x}(t) = \mathbf{x}(t_0(t), t_1(t), t_2(t), \dots)$ is taken, where $t_k := \varepsilon^k t$ are independent time-scales. Therefore, by the chain rule, $d/dt = \sum_{k=0}^{\infty} \varepsilon^k d_k$, where $d_k := \partial/\partial t_k$. Finally, the state variables are expanded as $\mathbf{x} = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{x}_k$, with \mathbf{x}_k unknowns. When all these rescaling/expansions are substituted into the Eq (1) and terms with the same power of ε separately equated to zero, a set of *linear* perturbation equations is drawn, namely:

$$\begin{aligned} (d_0 - \mathbf{J}_0)\mathbf{x}_0 &= \mathbf{0} \\ (d_0 - \mathbf{J}_0)\mathbf{x}_1 &= -d_1\mathbf{x}_0 + \boldsymbol{\mu}\mathbf{J}_1\mathbf{x}_0 + \mathbf{n}_2(\mathbf{x}_0, \mathbf{x}_0) \\ (d_0 - \mathbf{J}_0)\mathbf{x}_2 &= -(d_2\mathbf{x}_0 + d_1\mathbf{x}_1) + \boldsymbol{\mu}\mathbf{J}_1\mathbf{x}_1 + \mathbf{n}_3(\mathbf{x}_0, \mathbf{x}_0, \mathbf{x}_0) \\ &\quad + 2\mathbf{n}_2(\mathbf{x}_0, \mathbf{x}_1) \end{aligned} \quad (2)$$

The solution to Eq (2-a), when $t \rightarrow \infty$, reads (generating solution):

$$\begin{aligned} \mathbf{x}_0 &= A_0(t_1, t_2, \dots)\mathbf{u}_0 + A_1(t_1, t_2, \dots)\mathbf{u}_1 e^{i\omega_1 t_0} \\ &\quad + A_2(t_1, t_2, \dots)\mathbf{u}_2 e^{i\omega_2 t_0} + c.c. \end{aligned} \quad (3)$$

where \mathbf{u}_j are the (right) eigenvectors of \mathbf{J}_0 which span its center subspace, i.e. $(\mathbf{J}_0 - \lambda_{cj}\mathbf{I})\mathbf{u}_j = \mathbf{0}$; the real A_0 and the complex A_1, A_2 are amplitudes depending on the slower time-scales; finally *c.c.* means complex conjugates. Note that Eq (3) states that the leading part of the motion belongs to the center subspace, according to the CM theory.

When the generating solution is substituted in Eq (2-b), several combinations among the eigenvalues appear in the 'forcing' terms, namely $(0, \omega_1, \omega_2; 2\omega_1, 2\omega_2, \omega_1 \pm \omega_2)$. To avoid that \mathbf{x}_1 diverges on the t_0 -scale, thus making the series not uniformly valid, resonant terms must be removed on the right hand member, by *solvability conditions* requiring the known term is orthogonal to the *left* eigenvectors \mathbf{v}_j of \mathbf{J}_0 , having equal eigenvalue. In selecting the resonant terms, possible internal resonance must be accounted for; for example the 'forcing' frequency $\omega_1 - \omega_2$ is resonant with the natural frequency ω_2 if $\omega_1 = 2\omega_2$. The solvability conditions, since they always involve $d_1\mathbf{x}_0$, are of the type:

$$d_1 A_j = f_{1j}(A_0, A_1, A_2; \boldsymbol{\mu}), \quad j = 0, 1, 2 \quad (4)$$

i.e., they are first-order differential equations on the t_1 -scale. By solving Eq (2-b) \mathbf{x}_1 is determined, to within some arbitrary quantity, that can be chosen by requiring \mathbf{x}_1 is orthogonal to the center subspace. Thus, \mathbf{x}_1 describes the main contribution of the passive coordinates, in the CM view.

The procedure can be carried out to higher-orders, to evaluate $d_2 A_k, d_3 A_k, \dots$. At the end, all these partial differential equations can be recombined in first-order ordinary differential equations, according to $\dot{A}_j = \sum_{k=0}^{\infty} \varepsilon^k d_k A_j$, i.e.:

$$\dot{A}_j = f_j(A_0, A_1, A_2; \boldsymbol{\mu}), \quad j = 0, 1, 2 \quad (5)$$

where the perturbation parameter has been reabsorbed via a backward rescaling. These are the bifurcation equations sought for; they state that the (asymptotic) motion occurs on a (five-dimensional) manifold having the same dimension of the center

subspace. They are found to coincide with those furnished by the CMM, and already appear in their normal form.

The lower-codimension cases of zero-Hopf (ZH) and Hopf-Hopf (HH) bifurcations can easily be obtained from the previous analysis by zeroing the amplitude of the missing mode, i.e.:

$$\dot{A}_j = \begin{cases} f_j(A_0, A_1; \mu) & j = 0, 1 & \text{for ZH} \\ f_j(A_1, A_2; \mu) & j = 1, 2 & \text{for HH} \end{cases} \quad (6)$$

2.2 Infinite-dimensional systems

The method illustrated above can be straightforwardly extended to infinite-dimensional systems [16, 17, 18]. The relevant equations of motion read:

$$\dot{\mathbf{w}} = \mathbf{L}(\mu) \mathbf{w} + \mathbf{n}_2(\mathbf{w}, \mathbf{w}) + \mathbf{n}_3(\mathbf{w}, \mathbf{w}, \mathbf{w}) + \dots \quad (7)$$

where $\mathbf{w} = \mathbf{w}(\mathbf{x}, t)$ are state variables defined in a domain Ω , $\mathbf{L} := \mathbf{L}_0 + \mu \mathbf{L}_1$ is a linear differential operator, and $\mathbf{n}_i(\cdot)$ are vector-valued homogeneous polynomial of degree $i = 2, 3, \dots$. All the operators include field equations and boundary conditions.

By proceeding as for discrete systems, the following perturbation equations are drawn:

$$\begin{aligned} (d_0 - \mathbf{L}_0) \mathbf{w}_0 &= \mathbf{0} \\ (d_0 - \mathbf{L}_0) \mathbf{w}_1 &= -d_1 \mathbf{w}_0 + \mu \mathbf{L}_1 \mathbf{w}_0 + \mathbf{n}_2(\mathbf{w}_0, \mathbf{w}_0) \\ (d_0 - \mathbf{L}_0) \mathbf{w}_2 &= -(d_2 \mathbf{w}_0 + d_1 \mathbf{w}_1) + \mu \mathbf{L}_1 \mathbf{w}_1 \\ &\quad + \mathbf{n}_3(\mathbf{w}_0, \mathbf{w}_0, \mathbf{w}_0) + 2\mathbf{n}_2(\mathbf{w}_0, \mathbf{w}_1) \end{aligned} \quad (8)$$

The steady generating solution, in the Hopf-Hopf-divergence case, is:

$$\begin{aligned} \mathbf{x}_0 &= A_0(t_1, t_2, \dots) \phi_0(\mathbf{x}) + A_1(t_1, t_2, \dots) \phi_1(\mathbf{x}) e^{i\omega_1 t_0} \\ &\quad + A_2(t_1, t_2, \dots) \phi_2(\mathbf{x}) e^{i\omega_2 t_0} + c.c. \end{aligned} \quad (9)$$

where $\phi_j(\mathbf{x})$ are the (right) eigenvectors of \mathbf{L}_0 which span its center subspace, i.e. $(\mathbf{L}_0 - \lambda_{c_j} \mathbf{I}) \phi_j = \mathbf{0}$.

When this solution is substituted into the next perturbation equation, resonant terms appear. To avoid that \mathbf{w}_1 diverges on the t_0 -scale, each resonant terms must be made orthogonal to the left eigenvector $\psi_j(\mathbf{x})$ of equal eigenvalue. These latter are solutions of the *adjoint homogeneous problem* $(\mathbf{L}_0^* - \bar{\lambda}_{c_j} \mathbf{I}) \psi_j = \mathbf{0}$, where \mathbf{L}_0^* is the adjoint operator (including adjoint boundary conditions), supplied by the Green (or bilinear) Identity:

$$\int_{\Omega} \bar{\psi}^T \mathbf{L}_0 \phi d\Omega = \int_{\Omega} \bar{\phi}^T \mathbf{L}_0^* \psi d\Omega \quad (10)$$

Once the solvability condition is systematically enforced at any steps, partial time-differential equations for the amplitudes, $d_k A_j = f_{kj}(A_0, A_1, A_2; \mu)$, $k = 1, 2, \dots$, are drawn. These equations, when are recombined to come back to the true time, turn out to be formally equal to Eqs (5). Thus, an infinite-dimensional system is reduced to a 5-dimensional system.

We conclude, by the MSM, that the dimensions of the bifurcation equations do not depend on the dimensions of the original

system, but rather of those of its center sub-space, as stated by the CM theory.

As an example, an elastic cantilevered beam, grounded with a visco-elastic device applied at the free end, and there loaded by a follower force, was studied in [18]. The scenarios relevant to HH and ZH bifurcations are shown in Fig 1, in the plane of the control parameters $\mu = (\beta, \gamma)$, where β is a stiffness coefficient and γ a load parameter. The bifurcation charts display the qualitative dynamics occurring in the two-dimensional phase-space $\{a_j\} := \{|A_j|\}$, in each region of the parameter space bounded by bifurcation loci.

3. Defective bifurcations

3.1 Finite-dimensional systems

Let us assume that the Jacobian matrix is defective at the bifurcation, i.e. the eigenvectors associated with the center eigenvalues are less than their algebraic multiplicity, so that the basis is incomplete. For example, let $\lambda_c = (0, 0)$ be a double-zero eigenvalue, with just a (proper) eigenvector \mathbf{u}_0 associated. Then a Takens-Bogdanov (or double-zero, DZ) bifurcation, of codimension-2, is said to occur.

To complete the basis, a (index-2) generalized eigenvector must be associated, so that a *Jordan chain* $(\mathbf{u}_{01}, \mathbf{u}_{02})$, with $\mathbf{u}_{01} := \mathbf{u}_0$ must be built-up by solving the sequence of algebraic problems:

$$\begin{aligned} \mathbf{J}_0 \mathbf{u}_{01} &= \mathbf{0} \\ \mathbf{J}_0 \mathbf{u}_{02} &= \mathbf{u}_{01} \end{aligned} \quad (11)$$

under arbitrarily chosen normalization conditions. Now, it is known from algebra, that, said \mathbf{v}_0 the unique proper left eigenvector (satisfying $\mathbf{J}_0^T \mathbf{v}_0 = \mathbf{0}$), it results that $\mathbf{v}_0^T \mathbf{u}_{01} = 0$. From a geometrical point of view, since \mathbf{v}_0 spans the kernel of the adjoint operator \mathbf{J}_0^T , \mathbf{u}_{01} is in the range of the operator. Only the highest-index of the chain (in our case \mathbf{u}_{02}) has a component out the the range, namely $\mathbf{v}_0^T \mathbf{u}_{02} \neq 0$.

Due to these properties, *the integer power expansions used in the standard approach fail*, since, in removing resonant terms, $\mathbf{v}_0^T d_1 \mathbf{x}_0 = 0$ is found, so that the left hand member of Eq (4) disappears! To overcome the drawback, *fractional series expansions* must be used, in order to create intermediate perturbation equations able to generate the highest vector of the chain, when the resonant terms appear first [7]. Thus, if the eigenvalue has multiplicity m , fractional powers $\varepsilon^{1/m}$ must be used; here we discuss the simplest case $m = 2$.

With the same rescaling as before, we introduce *fractional time scales* $t_0, t_{1/2}, t_1, \dots$, so that $d/dt = \sum_{k=0}^{\infty} \varepsilon^{k/2} d_{k/2}$; moreover, we expand the state variables as $\mathbf{x} = \sum_{k=0}^{\infty} \varepsilon^{k/2} \mathbf{x}_{k/2}$. Thus we obtain the following perturbation equations:

$$\begin{aligned} (d_0 - \mathbf{J}_0) \mathbf{x}_0 &= \mathbf{0} \\ (d_0 - \mathbf{J}_0) \mathbf{x}_{1/2} &= -d_{1/2} \mathbf{x}_0 \\ (d_0 - \mathbf{J}_0) \mathbf{x}_1 &= -(d_1 \mathbf{x}_0 + d_{1/2} \mathbf{x}_{1/2}) + \mu \mathbf{J}_1 \mathbf{x}_0 + \mathbf{n}_2(\mathbf{x}_0, \mathbf{x}_0) \end{aligned} \quad (12)$$

and so on.

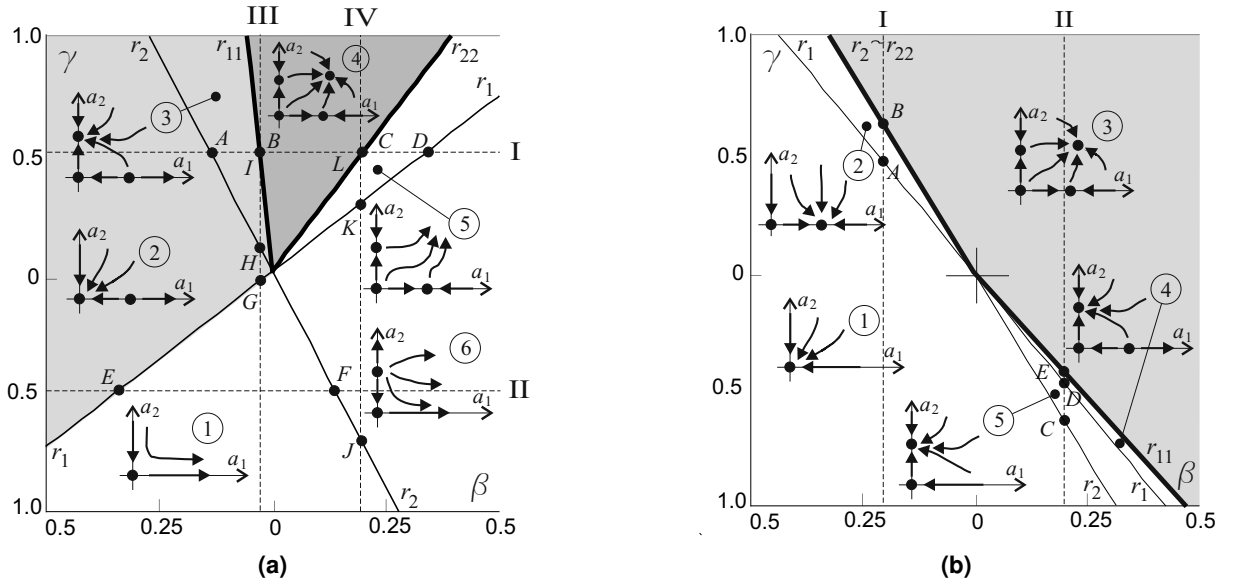


Figure 1. Bifurcation charts for an elastic beam under follower force [18]: Hopf-Hopf bifurcation, (b) zero-Hopf bifurcation.

The generating solution is:

$$\mathbf{x}_0 = A_0 (t_{1/2}, t_1, \dots) \mathbf{u}_{01} \quad (13)$$

where a real. It could be erroneously believed, at this level, that the leading motion takes place on a one-dimensional manifold, since the generalized eigenvector has not taken into account (as, instead, this is the case in the CMM); we will show soon, however, that this is not the case.

When the generating solution is substituted in Eq (12-b), since the 'known' term $-(d_{1/2}A_0) \mathbf{u}_{01}$ belongs to the range of the operator, no solvability is required and the solution reads $\mathbf{x}_{1/2} = -(d_{1/2}A_0) \mathbf{u}_{02}$. Note that the highest-index eigenvector of the chain appeared, and that, due to the lack of solvability conditions, $(d_{1/2}A_0)$ is still unknown.

With the previous results used in Eq (12-b), 'resonant' 0-frequency terms appear, generated by the last two terms in the equation. However, now $d_{1/2}\mathbf{x}_{1/2} = -(d_{1/2}^2A_0) \mathbf{u}_{02}$ is available to remove them, since it is out of the range of the operator. This solvability condition is of the type:

$$d_{1/2}^2A_0 = f_{1/2}(A_0; \mu) \quad (14)$$

When further steps are performed, and the solvability conditions reconstituted in a whole, the following bifurcation equation is recovered:

$$\ddot{A}_0 = f(A_0, \dot{A}_0; \mu) \quad (15)$$

Note that this is a *second-order* ordinary differential equation in the amplitude A_0 ; therefore, the leading motion occurs on a two-dimensional manifold, as predicted by the CMM.

3.2 Infinite-dimensional systems

When the system is infinite-dimensional, the perturbation analysis closely follows that for finite-dimensional systems [19, 20,

21, 22]. However, due to the occurrence of defective eigenvalues, we need to extend to continuous systems the concept of Jordan chain. This task has been accomplished by Keldysh [23, 24], who defined *Keldysh chains* for one-dimensional systems, similarly to the Jordan chains. Thus, if $\lambda_c = (0,0)$ is a double-zero eigenvalue, a Keldysh chain (ϕ_{01}, ϕ_{02}) is defined, satisfying the equations:

$$\begin{aligned} \mathbf{L}_0 \phi_{01} &= \mathbf{0} \\ \mathbf{L}_0 \phi_{02} &= \phi_{01} \end{aligned} \quad (16)$$

where ϕ_{01} is a proper eigenvector, and ϕ_{02} an index-2 generalized eigenvector. The same orthogonality conditions observed for the algebraic system holds, i.e. $\int_{\Omega} \bar{\psi}^T \phi_{01} d\Omega = 0$, $\int_{\Omega} \bar{\psi}^T \phi_{02} d\Omega \neq 0$, where ψ is the proper eigenvector which spans the kernel of \mathbf{L}_0^* .

By repeating the former analysis, we find $\mathbf{w} = \mathbf{w}_0 + \varepsilon^{1/2} \mathbf{w}_1 + \dots$, i.e., with ε reabsorbed:

$$\mathbf{w} = A_0 (t_{1/2}, t_1, \dots) \phi_{01} - \dot{A}_0 \phi_{02} + \dots \quad (17)$$

where the evolution of the real amplitude is governed by the bifurcation equation (15).

As an example, when a visco-elastic externally damped cantilevered beam is loaded at the tip by a follower μ and a gravitational v force, it undergoes a double-zero bifurcation for suitable combinations of the parameters [21, 22]. The relevant bifurcation chart is shown in Fig 2 for two different damping combinations, which display a non-catastrophic or catastrophic behavior (i.e. with or without an attractor in each region).

4. Interaction between defective and non-defective eigenvalues

4.1 Autonomous systems

We saw that distinct eigenvalues call for integer power expansions, while defective eigenvalues require fractional expansions,

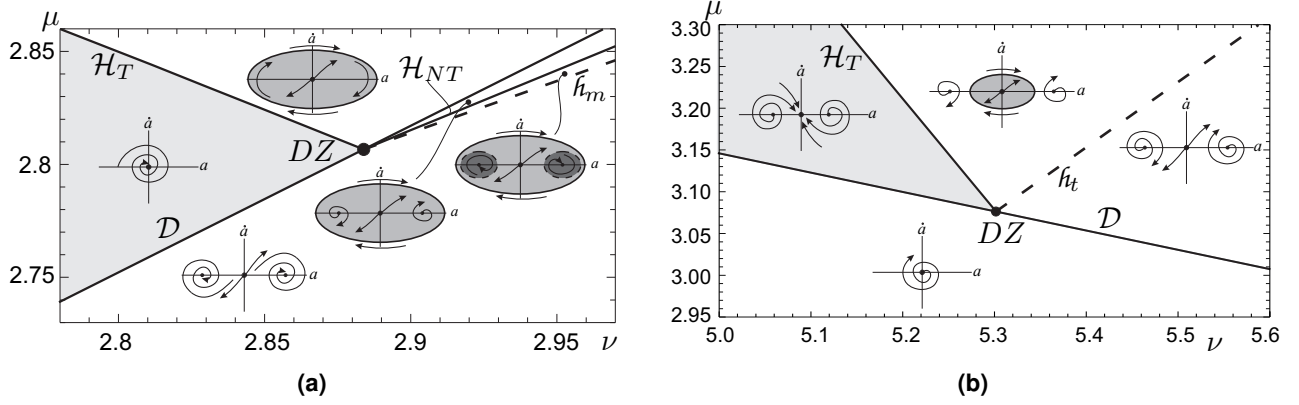


Figure 2. Bifurcation chart around a double-zero bifurcation point for a visco-elastic beam under gravitational and follower forces [21]: (a) non-catastrophic, (b) catastrophic behavior.

sions. What happens when eigenvalues of the two types interact? An example of such an occurrence is offered by the double-zero/Hopf bifurcation (DZH, see [9]). In this case, $\lambda_c = (0, 0, \pm i\omega_1)$, and the associated auto-space is spanned by $(\mathbf{u}_{01}, \mathbf{u}_{02}, \mathbf{u}_1, \bar{\mathbf{u}}_1)$, where $(\mathbf{u}_{01}, \mathbf{u}_{02})$ is a real Jordan chain, and $(\mathbf{u}_1, \bar{\mathbf{u}}_1)$ are complex conjugate complex eigenvalues. Since the defective eigenvalue $(0, 0)$ calls for power expansions of $\varepsilon^{1/2}$ and the non-defective eigenvalues $(\pm i\omega_1)$ for integer powers of ε , we chose the stronger condition, and expand in series of $\varepsilon^{1/2}$; consistently, we introduce time-scales multiple of the same power. As a result, we again obtain the perturbation equations (12).

The generating solution, in this case, reads:

$$\mathbf{x}_0 = A_0(t_{1/2}, t_1, \dots) \mathbf{u}_{01} + A_1(t_{1/2}, t_1, \dots) \mathbf{u}_1 e^{i\omega_1 t_0} + c.c. \quad (18)$$

where A_0 is real and A_1 is complex. When this is substituted in the Eq (12-b), this latter becomes:

$$(d_0 - \mathbf{J}_0) \mathbf{x}_{1/2} = -(d_{1/2} A_0) \mathbf{u}_{01} - (d_{1/2} A_1) \mathbf{u}_1 e^{i\omega_1 t_0} + c.c. \quad (19)$$

To remove secular terms, the 0-frequency 'known term' must be rendered orthogonal to the left proper eigenvector \mathbf{v}_0 , and the ω_1 -frequency term to the left proper eigenvector \mathbf{v}_1 . However, due to the properties of the Jordan chains, while $\bar{\mathbf{v}}_1^T \mathbf{u}_1 \neq 0$, it is $\bar{\mathbf{v}}_0^T \mathbf{u}_{01} = 0$; consequently, $d_{1/2} A_1 = 0$ but $d_{1/2} A_0$ is undetermined at this order. Therefore, as expected, A_1 follows the rule for non-defective amplitudes, while A_0 behaves as defective amplitudes. However, this is not a general rule, since, due to the interaction, this simple scheme does not hold at higher-orders. Moreover, it has been explained in [9] that, differently from the previous cases, in which the complementary solution of each perturbation equation can be ignored, here, in order to avoid inconsistencies, *some* of them must be taken into account. By skipping this discussion for the sake of brevity, we find $\mathbf{x}_{1/2} = -(d_{1/2} A_0) \mathbf{u}_{02}$. By going one step further, we obtain solvability conditions of the type:

$$\begin{aligned} d_{1/2}^2 A_0 &= f_{1/2,0}(A_0, A_1; \mu) \\ d_2 A_1 &= f_{1/2,1}(A_0, A_1; \mu) \end{aligned} \quad (20)$$

When the analysis is carried out to the higher-orders and the solvability conditions reconstituted, the following bifurcation equations are finally obtained:

$$\begin{aligned} \ddot{A}_0 &= f_0(A_0, \dot{A}_0, A_1; \mu) \\ \dot{A}_1 &= f_1(A_0, \dot{A}_0, A_1; \mu) \end{aligned} \quad (21)$$

They govern the dynamics of a 4-dimensional system, consistently with the dimension of the CM.

4.2 Transition from a higher- to a lower-codimension bifurcation

Let us consider a damped Hamiltonian systems at the divergence, under self-excitation causing Hopf bifurcation. There exist three non-hyperbolic eigenvalues $\lambda_c = (0, \pm i\omega)$ plus a fourth hyperbolic (stable) eigenvalue $\lambda_s = -\xi$ where $\xi > 0$ is a damping coefficient. If the damping is large, i.e. $\xi = O(1)$, we can analyze this bifurcation as a (easy) codimension-2 ZH bifurcation (see Fig 3-a); however, if the damping is small, i.e. $\xi = O(\varepsilon)$, the system is close to a (difficult) codimension-3 DZH bifurcation (see Fig 3-b). The question is: how large must be the damping in order for the fourth eigenvalue can be considered stable and not a perturbation of a zero eigenvalue? To answer the question, it needs to analyze the transition from the higher-codimension (partially-defective) to the lower-codimension (non-defective) bifurcation, by increasing ξ from zero.

A paradigmatic system introduced in [9] to analyze the problem, consists of two Duffing-Van der Pol coupled oscillators, of nondimensional equations:

$$\begin{aligned} \ddot{x} + x - \mu \dot{x} + g(x, y, \dot{x}, \dot{y}) &= 0 \\ \ddot{y} - \nu y - \xi \dot{y} + h(x, y, \dot{x}, \dot{y}) &= 0 \end{aligned} \quad (22)$$

where $\mu := (\mu, \nu, \xi) \rightarrow \mathbf{0}$. Here, the x -oscillator, of finite stiffness, undergoes a Hopf bifurcation at $\mu = 0$; the y -oscillator, of evanescent stiffness ν and evanescent damping ξ , undergoes a DZ bifurcation at $(\nu, \xi) = (0, 0)$. The nonlinear terms $g(\cdot), h(\cdot)$ couple the two oscillators. Therefore, a DZH occurs at $\mu = \mathbf{0}$; however, if ξ is allowed to become large of order 1, a ZH bifurcation occurs.

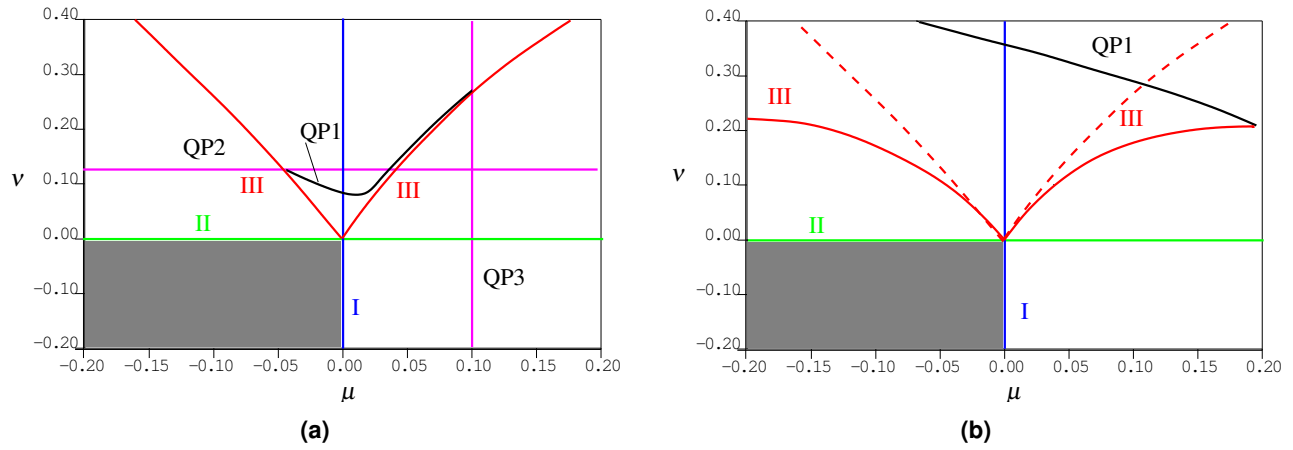


Figure 4. Bifurcation chart for a paradigmatic system, showing transition from double-zero/Hopf to zero-Hopf bifurcation [9]: (a) small damping, (b) large damping.

the dynamic mode of amplitude A_1 , and the static mode of amplitude A_0 , when $j = 1$ is taken. Therefore, solvability, leads to the following differential equations:

$$\begin{aligned} d_{1/2}^2 A_0 &= \mathcal{L}_{1/2,0}(A_1, \bar{A}_1; \delta) \\ d_2 A_1 &= \mathcal{L}_{2,1}(A_0; \delta) \end{aligned} \quad (27)$$

When the procedure is carried out up to the ε^2 -order, where the parameters μ have been ordered, and the solvability conditions reconstituted, the following bifurcation equations are obtained:

$$\begin{aligned} \ddot{A}_0 &= \mathcal{L}_0(A_0, A_1, \bar{A}_1; \delta, \mu) \\ \dot{A}_1 &= \mathcal{L}_1(A_0, \dot{A}_0, A_1, \bar{A}_1; \delta, \mu) \end{aligned} \quad (28)$$

which are a particular (linear) form of the more general Eqs (21).

These equations have been derived in [10] for a upright double pendulum, under gravitational forces triggering static instability, and harmonic vertical motion of amplitude δ , prescribed at the support and providing the parametric excitation. The analysis led to the results displayed in Fig 5-a, where the stability region of the equilibrium position $\mathbf{x} = \mathbf{0}$ is shown in the frequency-amplitude plane. It is seen that, when the system is unexcited ($\delta = 0$), the equilibrium is unstable; however, a frequency-dependent amplitude δ exists, which stabilizes the equilibrium. The phenomenon, however, only takes place on the right of the resonance value, while no stable equilibrium exist on the left. Results have also been compared with numerical solutions based on the Floquet theory (Fig 5-b). It is seen that, while the accordance is excellent for small δ , it is not good for large δ . In spite of this, the MSM captures the qualitative behavior, including the existence of a narrow stability zone at higher-amplitudes.

5. Bifurcations of singular systems: the Non-linear Energy Sinks

5.1 Autonomous systems undergoing Hopf bifurcation

The occurrence, at the bifurcation, of the center eigenvalues $\lambda_c = (0, 0, \pm i\omega)$ is not only an accidental (and detrimental) event, but even a desired circumstance adopted in passive control. As a matter of fact, when a mechanical system undergoes a Hopf bifurcation, e.g. triggered by wind flow forces, a control strategy recently studied in literature consists in attaching to it one (or more) essentially nonlinear oscillators (i.e. single d.o.f. devices with zero linear stiffness) having *evanescent mass*, said Nonlinear Energy Sinks (NES, [27, 28]). Thus, the augmented system also possesses a double zero eigenvalue. The NES works as a (well-known) Tuned Mass Damper (TMD), i.e. it is able to capture the input energy by preserving the main structure [29, 30]; however, being essentially nonlinear, it is able to adapt its own frequency, thus working in a frequency range wider than the mono-frequent TMD. Many semi-analytical or numerical methods were proposed in literature to solve this problem, and it was common opinion that 'standard perturbation methods' do not work [31]. This is, indeed, true, but a 'non-standard' version of the MSM, combined with the Harmonic Balance Method (HBM), and therefore denoted by MS-HBM, was indeed shown to work in [11, 12].

To illustrate the algorithm, we consider a symmetric (odd nonlinearities) finite dimensional system of state-variables \mathbf{x} , with a NES attached, whose displacement is y (Fig 6); the relevant equations of motion, in nondimensional form, read:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{J}(\mu) \mathbf{x} + \mathbf{n}_3(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \dots - (\xi \dot{z} + \kappa z^3) \hat{\mathbf{r}} \\ m(\ddot{z} - \mathbf{r}^T \ddot{\mathbf{x}}) + \xi \dot{z} + \kappa z^3 &= 0 \end{aligned} \quad (29)$$

where $z := \mathbf{r}^T \mathbf{x} - y$ is the elongation of the NES, and $\mathbf{r}, \hat{\mathbf{r}} := (\mathbf{0}, \mathbf{r})^T$ are influence coefficient vectors. Moreover, $m \ll 1$ is the NES mass, $\xi \ll 1$ the damping and $\kappa = \mathcal{O}(1)$ the nonlinear stiffness. Since m is small, but different from zero, the system possesses a

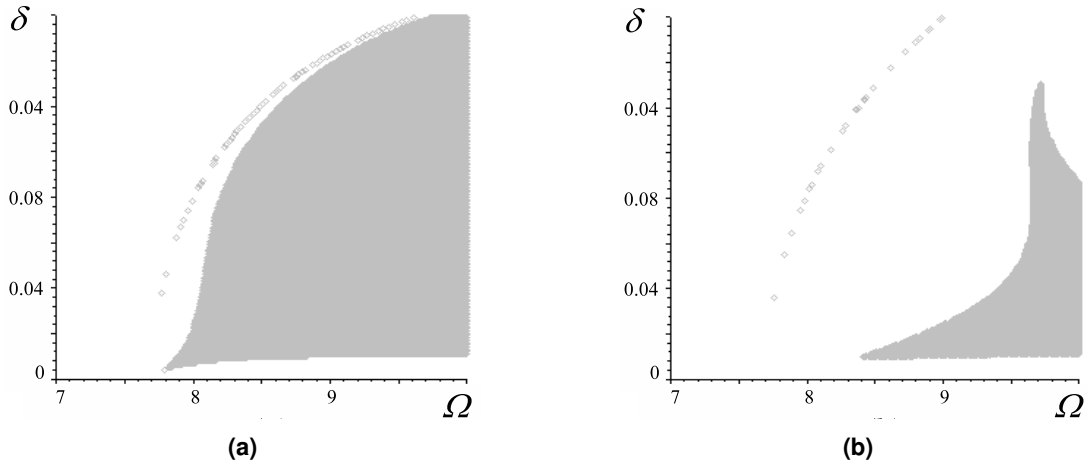


Figure 5. Stability region for the parametrically excited double pendulum close to divergence [10]: (a) MSM solution, (b) Numerical solution.

pair of (nearly) zero eigenvalues; however, since the mass is evanescent, the equations are of singular type.

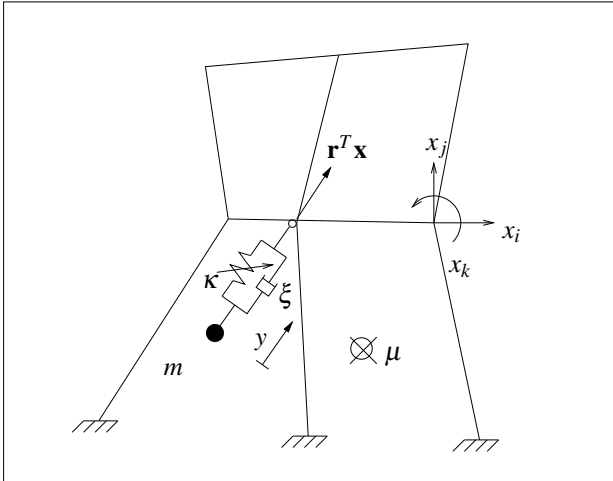


Figure 6. Sketch of a multi-d.o.f. system equipped with a NES, under normal wind flow.

To solve the problem, we first rescale the variables as $(\mathbf{x}, z) \rightarrow \varepsilon^{1/2}(\mathbf{x}, z)$ and the parameters as $(\mu, m, \xi) \rightarrow \varepsilon(\mu, m, \xi)$; then, we expand the variables in *integer* series, i.e. $\mathbf{x} = \sum_{k=0}^{\infty} \varepsilon^k \mathbf{x}_k$, $z = \sum_{k=0}^{\infty} \varepsilon^k z_k$ and use integer time-scales $t_k := \varepsilon^k t$. Thus we obtain the generating perturbation equation:

$$\varepsilon^0 : (d_0 - \mathbf{J}_0) \mathbf{x}_0 = \mathbf{0} \quad (30)$$

the ε^1 -order equations:

$$\begin{aligned} (d_0 - \mathbf{J}_0) \mathbf{x}_1 &= -d_1 \mathbf{x}_0 + \mu \mathbf{J}_1 \mathbf{x}_0 + \mathbf{n}_3(\mathbf{x}_0, \mathbf{x}_0, \mathbf{x}_0) \\ &\quad - (\xi d_0 z_0 + \kappa z_0^3) \hat{\mathbf{r}} \\ m d_0^2 z_0 + \xi d_0 z_0 + \kappa z_0^3 &= m \mathbf{r}^T d_0^2 \mathbf{x}_0 \end{aligned} \quad (31)$$

and so on. Note that, due to the fact m is evanescent, *the NES-equation shifts at a higher order*, this being the main difference with the case studied in the previous Section.

The generating solution, therefore, accounts for the uncontrolled main system. If, at $\mu = \mathbf{0}$, it undergoes a Hopf bifurcation, then:

$$\mathbf{x}_0 = A(t_1, t_2, \dots) \mathbf{u}_1 e^{i\omega_1 t_0} + c.c. \quad (32)$$

with A a complex amplitude. When this is substituted in the NES equation (31-b), a *strongly* nonlinear equation in z_0 , with harmonic excitation, is found. Since it cannot be solved by elementary functions, the Harmonic Balance Method (HBM) is used, by letting:

$$z_0 = \sum_{k=1}^{\infty} B_{0k}(t_1, t_2, \dots) e^{ik\omega_1 t_0} + c.c. \quad (33)$$

where B_{0k} are complex amplitudes. The HBM furnishes a set of cubic algebraic equations of the type:

$$\mathbf{g}_0(A, \mathbf{B}_0) = \mathbf{0} \quad (34)$$

where $\mathbf{B}_0 := (B_{01}, B_{02}, \dots)$. These provide an algebraic constraint between the amplitude of oscillation A of the main system and the harmonic components of the NES elongation. They describe a (parameter independent) manifold in the amplitude space on which the motion occurs. The Figure 7 shows a typical manifold in the space of the moduli (a, b_{01}, b_{03}) of the amplitudes (A, B_{01}, B_{03}) .

By substituting the z_0 solution in Eq (31-a) and removing secular terms, a differential equation governing the evolution of A on the t_1 -scale is found, having the form:

$$d_1 A = f_1(A, \mathbf{B}_0; \mu) \quad (35)$$

If the analysis is truncated at this order, the dynamics is ruled by this last equation, with \mathbf{B}_0 acting as a passive variable via the constraint (34), so that the motion is restrained to the manifold. However, if this latter exhibits turning point with respect to A (as it occurs in Fig. 7), the motion cannot cross them, since \dot{A} is generally different from zero there; hence, the equations break

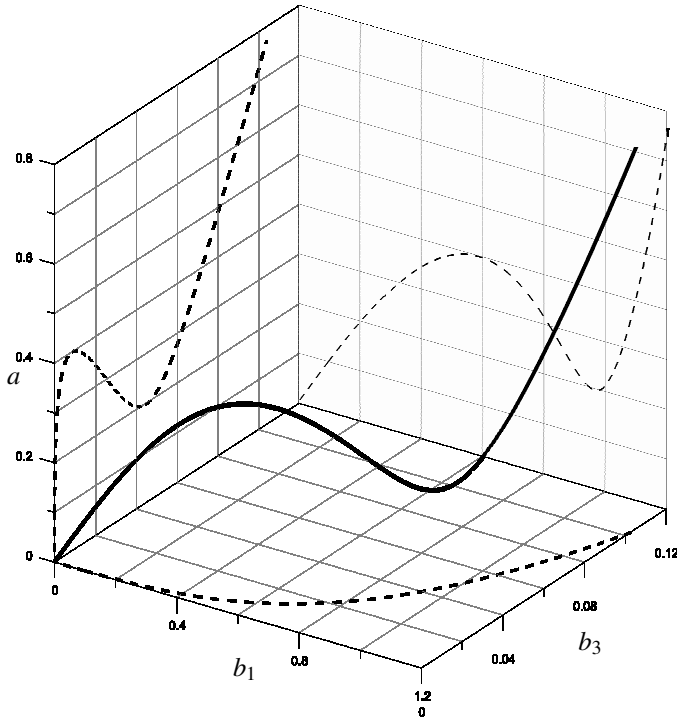


Figure 7. Typical nonlinear manifold for a NES-controlled system.

down at these singular points. To overcome this limitation, a further step must be accomplished in the perturbation analysis.

After the equation for \mathbf{x}_1 is solved and substituted in the higher-order equation for the NES (not shown for brevity), a solution of the form:

$$z_1 = \sum_{k=1}^{\infty} B_{1k}(t_1, t_2, \dots) e^{ik\omega t_0} + c.c. \quad (36)$$

is sought for via the HBM, which leads to *differential* equations of the type:

$$O(m, \xi) d_1 \mathbf{B}_1 = \mathbf{g}_1(A, \mathbf{B}_1; \mu) \quad (37)$$

where $\mathbf{B}_1 := (B_{11}, B_{12}, \dots)$ and $O(m, \xi)$ denotes coefficients of the order of the NES mass and damping coefficients, all close to zero. Here, therefore, the singular nature of the system clearly reveals itself.

By coming back to the original time, and by letting $\mathbf{B} := \mathbf{B}_0 + \mathbf{B}_1$, we finally have:

$$\begin{aligned} \dot{A} &= f(A, \mathbf{B}; \mu) \\ \varepsilon \dot{\mathbf{B}} &= \mathbf{g}(A, \mathbf{B}; \mu) \end{aligned} \quad (38)$$

If just a harmonic were used in the HBM, then the system would be four-dimensional, according to the CM theory; however, due to the singularity of the system, which call for introducing several amplitudes, the dimension is higher.

It should be remarked that, differently from the Complexification procedure by Manevitch [32], usually adopted in literature to tackle this problem, the HBM has been used here just to

solve one equation, namely the NES-equation, and not for the whole system.

The singular bifurcation equations (38) govern the well-known *relaxation oscillations*, or ‘Strongly Modulated Responses (SMR)’ [33]. They consist in alternate *slow* and *fast* regimes. Indeed, when the system moves close to the (parameter dependent) manifold of equation $\mathbf{g}(A, \mathbf{B}; \mu) = \mathbf{0}$, i.e. when $\mathbf{g}(A, \mathbf{B}; \mu) = O(\varepsilon)$, then $\dot{\mathbf{B}} = O(1)$, i.e. the motion is slow; when, in contrast, the system is far from the manifold, i.e. $\mathbf{g}(A, \mathbf{B}; \mu) = O(1)$, then $\dot{\mathbf{B}} = O(\varepsilon^{-1})$, i.e. the motion is fast.

The equations (38) were used in [11] to investigate the effectiveness of the NES in controlling the oscillations consequent to a Hopf bifurcation of a two d.o.f.’s airfoil, capable to vertically translate and to rotate. A single d.o.f. NES was attached to it, either windward or leeward. The analysis of the bifurcation equations led to the results displayed in Fig 8, where continuous lines denotes periodic solution for the original system (with and without NES) and dots numerical solutions. A small region of quasi-period solutions, consequent to secondary Hopf bifurcations, was also found (the curves denote the min/max amplitudes). It is seen that the NES has always a beneficial linear effect, since it shifts forward the critical wind velocity μ . However, while the first arrangement produces subcritical attractors, the second one entails a supercritical behavior, always reducing the amplitude of the limit-cycle.

Two phase-portraits in the amplitude-plane are reported in Fig 9, corresponding to the vertical lines in Fig 8. They show that (Fig 9-a) the stable periodic solution on the manifold is attractive of the system dynamics; in contrast (Fig 9-b), the unstable periodic motion existing between the two Hopf secondary bifurcations produces a limit cycle on the (a, b) -plane, which occurs around the manifold.

5.2 Externally excited systems

Nonlinear Energy Sinks are also employed to passively control systems under external excitation [33]; the MSM-HB also works for them [12]. As an example, let us consider a Hamiltonian system, slightly damped, subject to a harmonic excitation of frequency Ω and amplitude δ , having equation:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{J}(\mu) \mathbf{x} + \mathbf{n}_3(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \dots - (\xi \dot{z} + \kappa z^3) \hat{\mathbf{r}} + \delta \mathbf{f} \cos \Omega t \\ m(\ddot{z} - \mathbf{r}^T \ddot{\mathbf{x}}) + \xi \dot{z} + \kappa z^3 &= 0 \end{aligned} \quad (39)$$

Here $\mathbf{J}(\mu) = \mathbf{J}_0 + \mu \mathbf{J}_1$, where \mathbf{J}_0 is the Jacobian matrix of an Hamiltonian system, admitting all non-zero eigenvalues $\pm \omega_j$, and $\mu := (\sigma, \xi)$ is a vector of small parameters, including stiffness and damping coefficients. Moreover, the external frequency Ω is assumed to be resonant with a natural frequency ω_1 , and other forms of resonance are excluded.

By rescaling the external force as $\delta \rightarrow \varepsilon \delta$ and using the same arguments for the autonomous system, we obtain perturbation equations like (30) and (31), this latter accounting for the new term:

$$\begin{aligned} (d_0 - \mathbf{J}_0) \mathbf{x}_1 &= -d_1 \mathbf{x}_0 + \mu \mathbf{J}_1 \mathbf{x}_0 + \mathbf{n}_3(\mathbf{x}_0, \mathbf{x}_0, \mathbf{x}_0) \\ &\quad - (\xi d_0 z_0 + \kappa z_0^3) \hat{\mathbf{r}} + \frac{1}{2} \delta \mathbf{f} \left(e^{i\Omega t_0} + e^{-i\Omega t_0} \right) \\ m d_0^2 z_0 + \xi d_0 z_0 + \kappa z_0^3 &= m \mathbf{r}^T d_0^2 \mathbf{x}_0 \end{aligned} \quad (40)$$

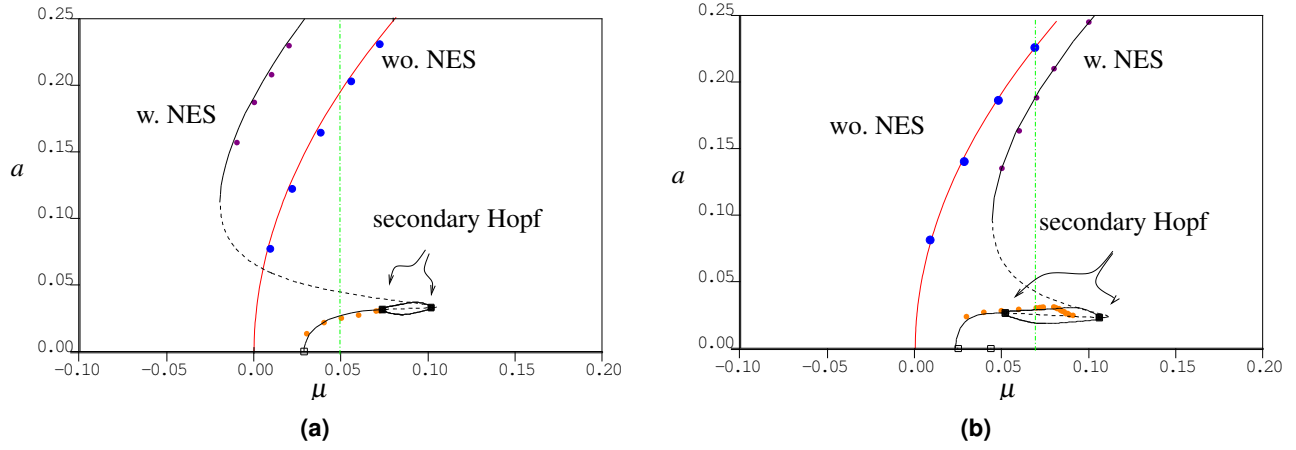


Figure 8. Periodic and quasi-periodic motions for an airfoil equipped with a NES [11], put (a) windward, and (b) leeward.

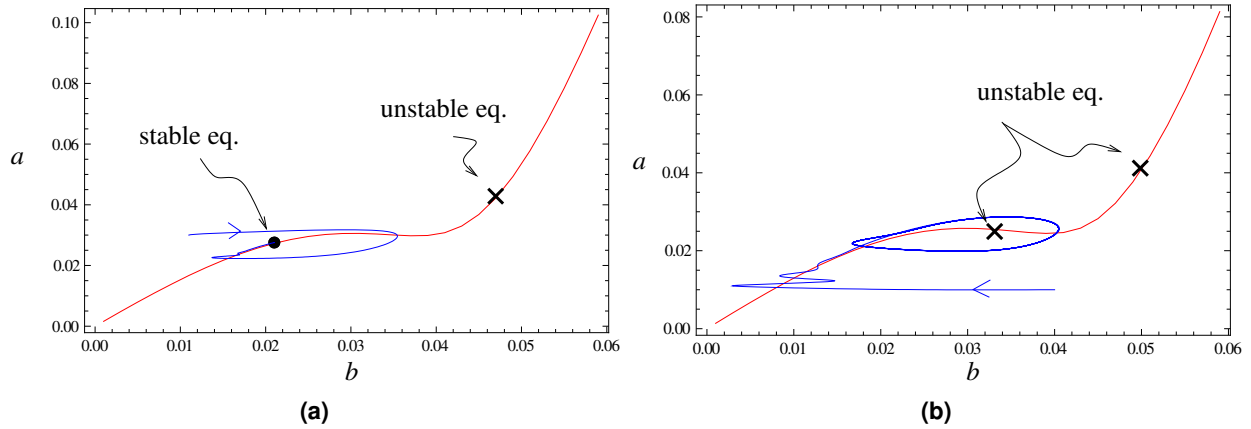


Figure 9. Phase-portrait for the airfoil [11], at the cross-sections marked in Fig 8: (a) $\mu = 0.05$, (b) $\mu = 0.07$.

Therefore, by using the generating solution (32), which takes into account only the natural mode involved in the resonance, and repeating the same steps as before, the following bifurcation equations are obtained, identical to Eqs (38), but including δ among the parameters:

$$\begin{aligned} \dot{A} &= f(A, \mathbf{B}; \mu, \delta) \\ \varepsilon \dot{\mathbf{B}} &= \mathbf{g}(A, \mathbf{B}; \mu, \delta) \end{aligned} \quad (41)$$

These equations have been used in [12] to analyze the response of a single d.o.f. system with a NES attached; some results are reported in Fig. 10. Fig. 10-a shows the amplitude-frequency curve; an unstable periodic motion (yellow circle) exists between secondary Hopf points (black squares). When the motion is plotted on the amplitude plane (Fig. 10-b) SMR are observed, occurring around the unstable periodic motion. The time-histories of the amplitude (Fig. 10-c) and that of the displacement x (Fig. 10-d) show the relaxation oscillations.

6. Conclusions

We discussed how to adapt the Multiple Scale Method in order to solve 'difficult' bifurcation problems. We obtained the following main results.

1. When the system is defective at the bifurcation, i.e. the center eigenvalue does not possess a complete set of eigenvectors, fractional power expansions and fractional times of order $\varepsilon^{1/m}$ must be used, with m the algebraic multiplicity of the eigenvalue. By systematically enforcing solvability conditions at each order, and then recombining them on the true time-scale, an m -order differential equation for the amplitude is drawn.
2. When the system is partially defective at the bifurcation, i.e. defective and non-defective eigenvalues interact, then fractional powers must be used, although some of the derivatives of the amplitudes can be zero, this denoting that the dynamics is of slow-fast type. Consequently, the bifurcation equations are differential equations of different order.
3. When the system is, in addition, singular, as it happens for NES-equipped systems, integer powers must be used, in conjunction with the Harmonic Balance Method to solve singular equations. The bifurcation equations are found to be in the form of singular equations, exhibiting relaxation oscillations.

Several applications have been shown, to display mechanical phenomenon and to test the effectiveness of the proposed meth-

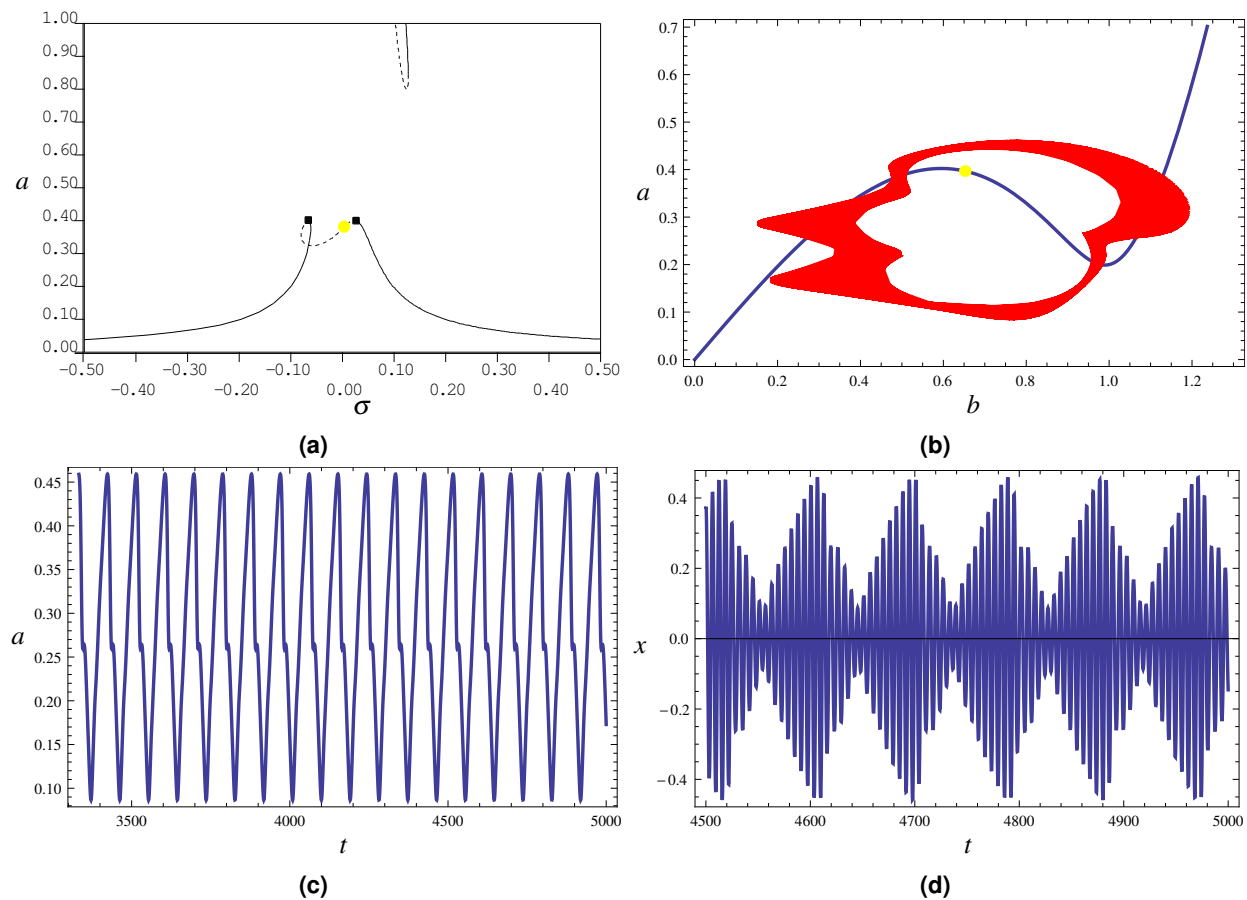


Figure 10. Response of a single-d.o.f. system equipped with NES to external excitation [12]: (a) amplitude-frequency response; (b) phase-portrait; (c), (d) time-histories.

ods. They include discrete and continuous systems, autonomous, parametrically excited and externally excited systems. Moreover, the algorithm was able to explain the transition between two bifurcations of different codimensions, when a parameter is varied.

Acknowledgments

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