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On the Use of the Multiple Scale Method in Solving 'Difficult' Bifurcation Problems

Abstract
Several algorithms consisting in 'non-standard' versions of the Multiple Scale Method are illustrated for 'difficult' bifurcation problems. Preliminary, the 'easy' case of bifurcation from a cluster of distinct eigenvalues is addressed, which requires using integer power expansions, and it leads to bifurcation equations all of the same order. Then, more complex problems are studied. The first class concerns bifurcation from a defective eigenvalue, which calls for using fractional power expansions and fractional time-scales, as well Jordan or Keldysh chains. The second class regards the interaction between defective and non-defective eigenvalues. This problem also requires fractional powers, but it leads to differential equations which are of different order for the involved amplitudes. Both autonomous and parametrically excited non-autonomous systems are studied. Moreover, the transition from a codimension-3 to a codimension-2 bifurcation is explained. As a third class of problems, singular systems possessing an evanescent mass, as Nonlinear Energy Sinks, are considered, and both autonomous systems undergoing Hopf bifurcation and non-autonomous systems under external resonant excitation, are studied. The algorithm calls for a suitable combination of the Multiple Scale Method and the Harmonic Balance Method, this latter to be applied exclusively to the singular equations. Several applications are shown, to test the effectiveness of the proposed methods. They include discrete and continuous systems, autonomous, parametrically and externally excited systems.

Keywords

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1. Introduction
Dynamical systems depend on parameters. When these are varied quasi-statically, the dynamics can, either, remain unchanged, or strongly modify. A bifurcation is said to occur at a point of the parameter space at which the dynamics abruptly changes. Here, new equilibria or cycles arise or disappear, and/or their stability changes. The bifurcation points occur on manifolds which separate the parameter space in regions in which the system possesses equivalent dynamics. Therefore bifurcations organize the scenario. The codimension of the manifold (equal to the number of independent algebraic conditions satisfied by parameters at bifurcation) is said the codimension of the bifurcation. In this paper, only bifurcations from equilibria are considered, and, although not strictly necessary, mechanical systems are addressed.

Bifurcation problems are almost universally tackled via the Center Manifold Method (CMM) and the Normal Form Theory (NFT) [1]. The CMM reduces the dimension of the dynamical system to that of the center subspace of the Jacobian (i.e. to the number of its eigenvalues having zero real part); the NFT reduces the complexity of the equations, via a quasi-identical change of coordinates, aimed to eliminate the possible largest number of nonlinear terms. The CMM is not only an algorithm, but reveals the true essence of the dynamics of a system close to bifurcation. From a geometrical point of view, it states that an attractive surface (Center Manifold, CM) exists in the (finite- or infinite-dimensional) state-space. The CM is found to be tangent at the equilibrium point to the center subspace, so that it can be tough as originated by a deformation of this subspace, caused by nonlinearities. From a mechanical point of view, the CM establishes a constraint among the state variables. Few of them, describing the motion in the tangent plane, participate with their own natural frequency, and therefore are said active coordinates; the remaining ones, describing motions orthogonal to the plane, do not contribute with their own frequencies, but are driven from the active coordinates, and therefore are said passive coordinates. The active coordinates are first-order quantities, while the passive coordinates are second- or higher-order quantities. However, they cannot be neglected, since they contribute to determine stability and to organize the dynamics.

A completely different approach to derive the normal form of the bifurcation equations, is offered by the Multiple Scale Method (MSM). Although this algorithm is very popular in nonlinear dynamics [2], it is less known, or even looked at with suspicion, in the bifurcation community. This standpoint is probably due to the fact the, while the CMM is based on a theorem, the MSM is a heuristic approach, based on formal series expansions. On the other hand, it has been shown in liter-
We introduce the MSM referring to the simplest case in which distinct eigenvalues: the 'easy' case

\[ x < 2. Bifurcation analysis from a cluster of dis-\]

However, there exist 'difficult bifurcations' for which it has been erroneously believed in literature, that the MSM (or, more in general, any perturbation method) fails in determining the bifurcation equations. In spite of this opinion, the author and his co-workers showed that, to be successful, the MSM must only be adapted to the problem; once again the method turned out to be computationally competitive with other methods.

Here, we will discuss three classes of 'difficult' problems:

1. **Defective autonomous systems**, in which the Jacobian operator at bifurcation does not posses a complete system of eigenvectors [7, 8];

2. *Partially defective autonomous systems*, in which only some of the center eigenvalues are defective [9]; or, Hamiltonian systems close to divergence, stabilized by resonant parametric excitation [10];

3. **Singular, partially defective systems**, in which an eveness mass coefficient exists, either autonomous [11] or externally excited [12].

All these problems require special treatments that will be sketched ahead; additional details can be found in the cited papers.

### 2. Bifurcation analysis from a cluster of distinct eigenvalues: the 'easy' case

We introduce the MSM referring to the simplest case in which the center eigenvalues are distinct, so that their associated eigenvectors form a complete basis for the center subspace (non-defective cases, see also [13, 14, 15]).

#### 2.1 Finite-dimensional systems

A finite-dimensional, autonomous, dynamical system is considered, whose nondimensional equations of motion, in state-form and expanded around the trivial equilibrium point \( x = \mathbf{0} \), read:

\[
\dot{x} = \mathbf{J}(\mu) x + \mathbf{n}_2(x,x) + \mathbf{n}_3(x,x,x) + \ldots
\]

Here \( x \) is the \( n \)-dimensional state-vector, \( \mathbf{J}(\mu) := \mathbf{J}_0 + \mu \mathbf{J}_1 \) is the Jacobian matrix at the origin, depending on a vector of bifurcation parameters \( \mu \), and \( \mathbf{n}_i, (\cdot) \) are vector-valued homogeneous polynomials of degree \( i = 2, 3, \ldots \); moreover, the dot denotes differentiation with respect the time \( t \). It is assumed that \( \mu = 0 \) is a bifurcation point, at which the Jacobian \( \mathbf{J}(0) = \mathbf{J}_0 \) admits one or more **distinct** center eigenvalues \( \lambda_c := (\lambda_{cj}) \). To exemplify, in a codimension-3 zero-Hopf-Hopf bifurcation (ZHH, see [6]) there are five eigenvalues in the center spectrum, namely \( \lambda_c = (0, \pm i\omega_1, \pm i\omega_2) \), while the remaining stable eigenvalues have negative real part. We want to analyze the dynamics of the system around the equilibrium point \( x = \mathbf{0} \), when \( \mu \) is varied in a ball of radius 0 < \( \varepsilon \ll 1 \), centered at \( \mu = 0 \).

First, in order a perturbation parameter appear in the equations, a rescaling is performed, namely \( x \rightarrow \varepsilon x, \mu \rightarrow \varepsilon \mu \). Then, according to the MSM, \( \dot{x}(t) = x(t_0(t), t_1(t), t_2(t), \ldots) \) is taken, where \( t_k := \varepsilon^k t \) are independent time-scales. Therefore, by the chain rule, \( \dot{x} = \sum d_k \dot{d}_k, \) where \( d_k = \partial / \partial t_k \). Finally, the state variables are expanded as \( x = \sum \varepsilon^k x_k, \) with \( x_k \) unknowns. When all these rescaling/expansions are substituted into the Eq (1) and terms with the same power of \( \varepsilon \) separately equated to zero, a set of **linear** perturbation equations is drawn, namely:

\[
(d_0 - \mathbf{J}_0) x_0 = 0 \\
(d_0 - \mathbf{J}_0) x_1 = - d_1 x_0 + \mu \mathbf{J}_1 x_0 + \mathbf{n}_2(x_0, x_0) \\
(d_0 - \mathbf{J}_0) x_2 = -(d_2 x_0 + d_1 x_1) + \mu \mathbf{J}_1 x_1 + \mathbf{n}_3(x_0, x_0, x_0) + 2 \mathbf{n}_2(x_0, x_1)
\]

The solution to Eq (2-a), when \( t \rightarrow \infty \), reads (generating solution):

\[
x_0 = A_0(t_1, t_2, \ldots) x_0 + A_1(t_1, t_2, \ldots) x_1 e^{i\omega_1 t_0} + A_2(t_1, t_2, \ldots) x_2 e^{i\omega_2 t_0} + c.c.
\]

where \( x_1, x_2 \) are the (right) eigenvectors of \( \mathbf{J}_0 \) which span its center subspace, i.e. \( (\mathbf{J}_0 - \lambda_{cj}) x_j = 0 \); the real \( A_0 \) and the complex \( A_1, A_2 \) are amplitudes depending on the slower time-scales; finally c.c. means complex conjugates. Note that Eq (3) states that the leading part of the motion belongs to the center subspace, according to the CM theory.

When the generating solution is substituted in Eq (2-b), several combinations among the eigenvalues appear in the 'forcing' terms, namely \( (0, \omega_1, \omega_2; 2\omega_1, 2\omega_2, \omega_1 \pm \omega_2) \). To avoid that \( x_1 \) diverges on the \( t_0 \)-scale, thus making the series not uniformly valid, resonant terms must be removed on the right hand member, by **solvability conditions** requiring the known term is orthogonal to the left eigenvectors \( \mathbf{v}_j \) of \( \mathbf{J}_0 \), having equal eigenvalue. In selecting the resonant terms, possible internal resonance must be accounted for; for example the 'forcing' frequency \( \omega_1 - \omega_2 \) is resonant with the natural frequency \( \omega_0 \) if \( \omega_1 = 2\omega_0 \). The solvability conditions, since they always involve \( d_1 x_0 \), are of the type:

\[
d_1 A_j = f_j(A_0, A_1, A_2; \mu), \quad j = 0, 1, 2
\]

i.e., they are first-order differential equations on the \( t_1 \)-scale. By solving Eq (2-b) \( x_1 \) is determined, to within some arbitrary quantity, that can be chosen by requiring \( x_1 \) is orthogonal to the center subspace. Thus, \( x_1 \) describes the main contribution of the passive coordinates, in the CM view.

The procedure can be carried out to higher-orders, to evaluate \( d_2 A_k, d_3 A_k, \ldots \). At the end, all these partial differential equations can be recombined in first-order ordinary differential equations, according to \( \dot{A}_j = \sum d_k A_j, \) i.e.:

\[
\dot{A}_j = f_j(A_0, A_1, A_2; \mu), \quad j = 0, 1, 2
\]
subspace. They are found to coincide with those furnished by the CMM, and already appear in their normal form.

The lower-codimension cases of zero-Hopf (ZH) and Hopf-Hopf (HH) bifurcations can easily be obtained from the previous analysis by zeroing the amplitude of the missing mode, i.e.:

$$
\dot{A}_j = \begin{cases} f_j(A_0, A_1; \mu) & j = 0, 1 \text{ for ZH} \\ f_j(A_1, A_2; \mu) & j = 1, 2 \text{ for HH} \end{cases}
$$

(6)

2.2 Infinite-dimensional systems

The method illustrated above can be straightforwardly extended to infinite-dimensional systems [16, 17, 18]. The relevant equations of motion read:

$$
\dot{w} = L(\mu)w + n_2(w, w) + n_3(w, w, w) + \ldots
$$

(7)

where $w = w(x,t)$ are state variables defined in a domain $\Omega$, $L := L_0 + \mu J_0$ is a linear differential operator, and $n_i(\cdot)$ are vector-valued homogeneous polynomials of degree $i = 2, 3, \ldots$. All the operators include field equations and boundary conditions.

By proceeding as for discrete systems, the following perturbation equations are drawn:

$$
\begin{align*}
(d_0 - L_0) w_0 &= 0 \\
(d_0 - L_0) w_1 &= -d_1 w_0 + \mu w_0 + n_2(w_0, w_0) \\
(d_0 - L_0) w_2 &= -d_2 w_1 + \mu w_1 + n_3(w_0, w_0, w_0) + 2n_2(w_0, w_1)
\end{align*}
$$

(8)

The steady generating solution, in the Hopf-Hopf-divergence case, is:

$$
\begin{align*}
x_0 = & A_0(t_1, t_2, \ldots) \phi_0(x) + A_1(t_1, t_2, \ldots) \phi_1(x) e^{i\omega_0 t_0} \\
& + A_2(t_1, t_2, \ldots) \phi_2(x) e^{i\omega_2 t_0} + \text{c.c.}
\end{align*}
$$

(9)

where $\phi_j(x)$ are the (right) eigenvectors of $L_0$ which span its center subspace, i.e. $(L_0 - \lambda_0 \mathbf{J}) \phi_j = 0$.

When this solution is substituted into the next perturbation equation, resonant terms appear. To avoid that $w_1$ diverges on the $t_0$-scale, each resonant term must be made orthogonal to the left eigenvector $\psi_j(x)$ of equal eigenvalue. These latter are solutions of the adjoint homogeneous problem $(L_0^* - \lambda_0 \mathbf{J}) \psi_j = 0$, where $L_0^*$ is the adjoint operator (including adjoint boundary conditions), supplied by the Green (or bilinear) Identity:

$$
\int_{\Omega} \overline{\psi}^T L_0^* \phi d\Omega = \int_{\Omega} \overline{\psi}^T \phi_0 d\Omega
$$

(10)

Once the solvability condition is systematically enforced at any steps, partial time-differential equations for the amplitudes, $d_0 A_j = f_j(A_0, A_1, A_2; \mu), \ j = 1, 2, \ldots$, are drawn. These equations, when recombined to come back to the true time, turn out to be formally equal to Eqs (5). Thus, an infinite-dimensional system is reduced to a 5-dimensional system.

We conclude, by the MSM, that the dimensions of the bifurcation equations do not depend on the dimensions of the original system, but rather of those of its center sub-space, as stated by the CM theory.

As an example, an elastic cantilever beam, grounded with a visco-elastic device applied at the free end, and there loaded by a follower force, was studied in [18]. The scenarios relevant to HH and ZH bifurcations are shown in Fig 1, in the plane of the control parameters $\mu = (\beta, \gamma)$, where $\beta$ is a stiffness coefficient and $\gamma$ a load parameter. The bifurcation charts display the qualitative dynamics occurring in the two-dimensional phase-space $\{a_j\} := \{ |A_j| \}$, in each region of the parameter space bounded by bifurcation loci.

3. Defective bifurcations

3.1 Finite-dimensional systems

Let us assume that the Jacobian matrix is defective at the bifurcation, i.e. the eigenvectors associated with the center eigenvalues are less than their algebraic multiplicity, so that the basis is incomplete. For example, let $\lambda_0 = (0, 0)$ be a double-zero eigenvalue, with just a (proper) eigenvector $u_0$ associated. Then a Takens-Bogdanov (or double-zero, DZ) bifurcation, of codimension-2, is said to occur.

To complete the basis, a (index-2) generalized eigenvector must be associated, so that a Jordan chain $(u_{01}, u_{02})$, with $u_{01} := u_0$ must be built-up by solving the sequence of algebraic problems:

$$
\begin{align*}
J_0 u_{01} &= 0 \\
J_0 u_{02} &= u_{01}
\end{align*}
$$

(11)

under arbitrarily chosen normalization conditions. Now, it is known from algebra, that, said $\nu_0$ the unique proper left eigenvector (satisfying $J_0^T \nu_0 = 0$), it results that $\nu_0^T u_{01} = 0$. From a geometrical point of view, since $\nu_0$ spans the kernel of the adjoint operator $J_0^T$, $u_{01}$ is in the range of the operator. Only the highest-index of the chain (in our case $u_{02}$) has a component out the the range, namely $\nu_0^T u_{02} \neq 0$.

Due to these properties, the integer power expansions used in the standard approach fail, since, in removing resonant terms, $J_0^T d_1 x_0 = 0$ is found, so that the left hand member of Eq (4) disappears! To overcome the drawback, fractional series expansions must be used, in order to create intermediate perturbation equations able to generate the highest vector of the chain, when the resonant terms appear first [7]. Thus, if the eigenvalue has multiplicity $m$, fractional powers $e^{i\omega t}/m$ must be used; here we discuss the simplest case $m = 2$.

With the same rescaling as before, we introduce fractional time scales $t_0, t_{1/2}, t_1, \ldots$, so that $d/dt = \sum_{k=0}^{\infty} e^{k/2} d_{k/2}$; moreover, we expand the state variables as $x = \sum_{k=0}^{\infty} e^{k/2} x_k$. Thus we obtain the following perturbation equations:

$$
\begin{align*}
(d_0 - J_0) x_0 &= 0 \\
(d_0 - J_0) x_{1/2} &= -d_1 x_0 \\
(d_0 - J_0) x_1 &= -(d_1 x_0 + d_{1/2} x_{1/2}) + \mu J_1 x_0 + n_2(x_0, x_0)
\end{align*}
$$

(12)

and so on.
When further steps are performed, and the solvability conditions are satisfied (as, instead, this is the case in the CMM); we will show soon, however, that this is not the case.

When the generating solution is substituted in Eq (12-b), 'resonant' 0-frequency terms appear, generated by the last two terms in the equation. However, now the operator, no solvability is required and the solution reads $x_{1/2} = - (d_{1/2}A_0) u_02$. Note that the highest-index eigenvector of the chain appeared, and that, due to the lack of solvability conditions, $(d_{1/2}A_0)$ is still unknown.

With the previous results used in Eq (12-b), 'resonant' 0-frequency terms appear, generated by the last two terms in the equation. However, now $d_{1/2}x_{1/2} = - (d_{1/2}A_0) u_02$ is available to remove them, since it is out of the range of the operator. This solvability condition is of the type:

$$d_{1/2}^2A_0 = f_{1/2}(A_0; \mu)$$ (14)

When further steps are performed, and the solvability conditions reconstituted in a whole, the following bifurcation equation is recovered:

$$\hat{A}_0 = f (A_0, \hat{A}_0; \mu)$$ (15)

Note that this is a second-order ordinary differential equation in the amplitude $A_0$; therefore, the leading motion occurs on a two-dimensional manifold, as predicted by the CMM.

### 3.2 Infinite-dimensional systems

When the system is infinite-dimensional, the perturbation analysis closely follows that for finite-dimensional systems [19, 20, 21, 22]. However, due to the occurrence of defective eigenvalues, we need to extend to continuous systems the concept of Jordan chain. This task has been accomplished by Keldysh [23, 24], who defined Keldysh chains for one-dimensional systems, similarly to the Jordan chains. Thus, if $\lambda_c = (0,0)$ is a double-zero eigenvalue, a Keldysh chain $(\phi_{01}, \phi_{02})$ is defined, satisfying the equations:

$$L_0\phi_{01} = 0$$
$$L_0\phi_{02} = \phi_{01}$$ (16)

where $\phi_{01}$ is a proper eigenvector, and $\phi_{02}$ an index-2 generalized eigenvector. The same orthogonality conditions observed for the algebraic system holds, i.e. $\int_\Omega \psi^T \phi_{01} d\Omega = 0$, $\int_\Omega \psi^T \phi_{02} d\Omega \neq 0$, where $\psi$ is the proper eigenvector which spans the kernel of $L_0^*$.

By repeating the former analysis, we find $w = w_0 + \varepsilon^{1/2}w_1 + \ldots$, i.e., with $\varepsilon$ reabsorbed:

$$w = A_0 (t_{1/2}, t_1, \ldots) \phi_{01} - \hat{A}_0 \phi_{02} + \ldots$$ (17)

where the evolution of the real amplitude is governed by the bifurcation equation (15).

As an example, when a visco-elastic externally damped cantilevered beam is loaded at the tip by a follower $\mu$ and a gravitational $v$ force, it undergoes a double-zero bifurcation for suitable combinations of the parameters [21, 22]. The relevant bifurcation chart is shown in Fig 2 for two different damping combinations, which display a non-catastrophic or catastrophic behavior (i.e. with or without an attractor in each region).

### 4. Interaction between defective and non-defective eigenvalues

#### 4.1 Autonomous systems

We saw that distinct eigenvalues call for integer power expansions, while defective eigenvalues require fractional expan-
To remove secular terms, the 0-frequency 'known term' must be avoided; consequently, all of them must be taken into account.

The generating solution, in this case, reads:

\[ x_0 = A_0 \left( t_1/2, t_1, \ldots \right) u_0 + A_1 \left( t_1/2, t_1, \ldots \right) u_1 e^{i \omega_0 t_0} + \text{c.c.} \quad (18) \]

where \( A_0 \) is real and \( A_1 \) is complex. When this is substituted in the Eq (12-b), this latter becomes:

\[ (d_0 - J_0) x_{1/2} = - (d_{1/2} A_0) u_0 - (d_{1/2} A_1) u_1 e^{i \omega_0 t_0} + \text{c.c.} \quad (19) \]

To remove secular terms, the 0-frequency 'known term' must be rendered orthonormal to the left proper eigenvector \( v_0 \), and the \( \omega_0 \)-frequency term to the left proper eigenvector \( v_1 \). However, due to the properties of the Jordan chains, while \( v_0^T u_1 \neq 0 \), it is \( v_0^T u_0 = 0 \); consequently, \( d_{1/2} A_1 = 0 \) but \( d_{1/2} A_0 \) is undetermined at this order. Therefore, as expected, \( A_1 \) follows the rule for non-defective amplitudes, while \( A_0 \) behaves as defective amplitudes. However, this is not a general rule, since, due to the interaction, this simple scheme does not hold at higher-orders. Moreover, it has been explained in [9] that, differently from the previous cases, in which the complementary solution of each perturbation equation can be ignored, here, in order to avoid inconsistencies, some of them must be taken into account. By skipping this discussion for the sake of brevity, we find \( x_{1/2} = - (d_{1/2} A_0) u_0 \). By going one step further, we obtain solvability conditions of the type:

\[ d^2_{1/2} A_0 = f_{1/2,0} (A_0, A_1; \mu) \]
\[ d_2 A_1 = f_{1/2,1} (A_0, A_1; \mu) \quad (20) \]

When the analysis is carried out to the higher-orders and the solvability conditions reconstituted, the following bifurcation equations are finally obtained:

\[ \dot{A}_0 = f_0 (A_0, \dot{A}_0, A_1; \mu) \]
\[ \dot{A}_1 = f_1 (A_0, \dot{A}_0, A_1; \mu) \quad (21) \]

They govern the dynamics of a 4-dimensional system, consistently with the dimension of the CM.

4.2 Transition from a higher- to a lower-codimension bifurcation

Let us consider a damped Hamiltonian systems at the divergence, under self-excitation causing Hopf bifurcation. There exist three non-hyperbolic eigenvalues \( \lambda_c = (0, \pm i \omega_0) \) plus a fourth hyperbolic (stable) eigenvalue \( \lambda_v = - \xi \) where \( \xi > 0 \) is a damping coefficient. If the damping is large, i.e. \( \xi = O(1) \), we can analyze this bifurcation as a (easy) codimension-2 ZH bifurcation (see Fig 3-a); however, if the damping is small, i.e. \( \xi = O(\varepsilon) \), the system is close to a (difficult) codimension-3 DZH bifurcation (see Fig 3-b). The question is: how large must be the damping in order for the fourth eigenvalue can be considered stable and not a perturbation of a zero eigenvalue? To answer the question, it needs to analyze the transition from the higher-codimension (partially-defective) to the lower-codimension (non-defective) bifurcation, by increasing \( \xi \) from zero.

A paradigmatic system introduced in [9] to analyze the problem, consists of two Duffing-Van der Pol coupled oscillators, of nondimensional equations:

\[ \ddot{x} + x - \mu \dot{x} + g(x, y, \dot{x}, \dot{y}) = 0 \]
\[ \ddot{y} - \nu y - \xi \dot{y} + h(x, y, \dot{x}, \dot{y}) = 0 \quad (22) \]

where \( \mu := (\mu, v, \xi) \rightarrow 0 \). Here, the \( x \)-oscillator, of finite stiffness, undergoes a Hopf bifurcation at \( \mu = 0 \); the \( y \)-oscillator, of evanescent stiffness \( v \) and evanescent damping \( \xi \), undergoes a DZ bifurcation at \( (v, \xi) = (0, 0) \). The nonlinear terms \( g(\cdot), h(\cdot) \) couple the two oscillators. Therefore, a DZH occurs at \( \mu = 0 \); however, if \( \xi \) is allowed to become large of order 1, a ZH bifurcation occurs.
The following bifurcation equations, of the type (21), were found by the MSM [9]:

\[
\begin{align*}
\tilde{A}_0 &= \mathcal{L}_0 \left( \frac{v A_0, A_0^2, A_1; \xi \hat{A}_0; \ldots}{\varepsilon^{3/2}} \right) \\
\tilde{A}_1 &= \mathcal{L}_1 \left( \frac{\mu A_1, A_1^2, A_1, A_0; \hat{A}_0; \ldots}{\varepsilon^{3/2}} \right)
\end{align*}
\]

where \( \mathcal{L}_i \) are linear operators. Here the order of the different terms is determined by the perturbation analysis, which states that \( A_0 = O(\varepsilon^{1/2}) \) and \( A_1 = O(\varepsilon^{1/2}) \); however, due to the fact that \( d_{1/2}A_0 \neq 0, d_{1/2}A_1 = 0 \), it happens that \( (d/dt)A_0 = O(\varepsilon^{1/2} \times \varepsilon^{1/2}) \), while \( (d/dt)A_1 = O(\varepsilon \times \varepsilon^{1/2}) \). We can conclude that the dynamic of \( A_0 \) is faster than the dynamics of \( A_1 \) (and therefore \( \hat{A}_0 \) and \( \hat{A}_1 \) are of the same order).

When, however, \( \xi \) is increased to the order-1, since a simple zero-eigenvalue is included in the center spectrum, the bifurcation becomes non-defective, so that no fractional powers can be involved in the perturbation analysis; therefore \( (d/dt)\hat{A}_j = O(\varepsilon \times \varepsilon^{1/2}) \) for both amplitudes. As a consequence, \( \hat{A}_0 \) becomes of an higher-order with respect to \( A_1 \), i.e. the dynamics of \( A_0 \) becomes slower and comparable with those of \( A_1 \). Thus, while \( \hat{A}_0 \) jumps to the higher-order terms of Eqs (23-a), \( \hat{A}_1 \) enter among the leading terms, and Eqs (23) become:

\[
\begin{align*}
\hat{\xi} \hat{A}_0 &= -\mathcal{L}_0 \left( \frac{v A_0, A_0^2, A_1; A_1 \hat{A}_0; \ldots}{\varepsilon^{3/2}} \right) \\
\hat{A}_1 &= \mathcal{L}_1 \left( \frac{\mu A_1, A_1^2, A_1, A_0; \hat{A}_0; \ldots}{\varepsilon^{3/2}} \right)
\end{align*}
\]

Note that the order of the differential equations is lowered, so that the dynamics occurs on a three-dimensional manifold. These are just the bifurcation equations for the zero-Hopf bifurcation.

Numerical solutions of Eqs (23) highlighted the existence of quasi-periodic solutions which strongly affect the dynamics close to the double-zero/Hopf bifurcation. However, when the damping is increased, they move away, thus explaining the transition phenomenon. This is illustrated in Fig 4, where two bifurcation charts, for small and large damping are represented. Lines I to III denotes bifurcations from which steady solution arise (I periodic, II buckled, III periodic around a buckled equilibrium), while QP denotes quasi-periodic solutions (in which \( A_0, A_1 \) are periodic).

4.3 Statically unstable systems under stabilizing parametric excitation

Let us consider a class of linear Hamiltonian systems, close to divergence, at which a double-zero eigenvalue occurs. It is known that, if a linear parametric excitation of frequency \( \Omega \) is applied, rendering the stiffness (or the mass) time-dependent, the statically unstable system can be re-stabilized [25, 26]. The phenomenon can clearly be explained by a perturbation analysis [10], which shows how the parametric frequency combines itself with the natural frequencies of the Hamiltonian system, to produce restabilizing time-independent terms (i.e. apparent stiffnesses), which are able (if of suitable sign) to compensate the lack of stiffness of the unexcited system.

To explain the algorithm, let us consider a Hamiltonian finite-dimensional linear system, parametrically excited by a harmonic law of small amplitude \( \delta \):

\[
\dot{x} = (J(\mu) + \delta B \cos \Omega t) x
\]

and assume that \( J_0 \) admits a double-zero eigenvalue, at which the unique eigenvector \( u_0 \equiv u_{01} \) is associated, and complex conjugate purely imaginary eigenvalues \( \pm i \omega_j \). An exhaustive analysis of the system would require considering all the resonance conditions possibly occurring between the excitation frequency \( \Omega \) and the natural frequencies \( \omega_j \) (see [10], where a specific system was analyzed); here, we limit ourselves to the 1:1 resonant case \( \Omega \approx \omega_1 \). By excluding any other form of resonance. In this case, the algorithm described above for DZH bifurcations of nonlinear autonomous systems also applies to this class of linear non-autonomous systems. Indeed, from a computational point of view, the harmonically-dependent coefficients have the same role played by nonlinearities in autonomous systems.

By performing the rescaling \( \mu \rightarrow \varepsilon^2 \mu, \delta \rightarrow \varepsilon \delta, \) and using fractional series expansions and time-scales, we obtain the following perturbation equations:

\[
\begin{align*}
(d_0 - J_0) x_0 &= 0 \\
(d_0 - J_0) x_{1/2} &= -d_{1/2} x_0 \\
(d_0 - J_0) x_1 &= - (d_1 x_0 + d_{1/2} x_{1/2}) + \frac{1}{2} \delta B x_0 \left( e^{i \omega_0} + e^{-i \omega_0} \right)
\end{align*}
\]

We take a generating solution made of two components: the static mode \( u_{01} \) and the dynamic mode \( u_1 \) in resonance with the excitation (indeed, a small damping would produce the decay of all the other components). Hence, the solution to Eq (26-a) reads as Eq (18). At the next-order, Eq (26-b), the same results are found, namely \( d_{1/2}A_2 = 0 \) and \( d_{1/2}A_2 \) undetermined; moreover \( x_{1/2} = - (d_{1/2} A_0) u_{02} \). When all these results are substituted in Eq (26-c), the parametric excitation combines with the generating solution, giving rise the forcing frequencies (\( \Omega, \Omega \pm \omega_j \)). Since \( \Omega \approx \omega_1 \), these frequencies are, in the order, resonant with
When the procedure is carried out up to the \( \varepsilon^2 \)-order, where the parameters \( \mu \) have been ordered, and the solvability conditions reconstituted, the following bifurcation equations are obtained:

\[
\begin{align*}
A_1 &= L_1 (A_0, A_1, \delta, \mu) \\
A_0 &= L_0 (A_0, A_1, \delta, \mu)
\end{align*}
\]  

which are a particular (linear) form of the more general Eqs (21).

These equations have been derived in [10] for a upright double pendulum, under gravitational forces triggering static instability, and harmonic vertical motion of amplitude \( \delta \), prescribed at the support and providing the parametric excitation. The analysis led to the results displayed in Fig 5-a, where the stability region of the equilibrium position \( x = 0 \) is shown in the frequency-amplitude plane. It is seen that, when the system is unexcited (\( \delta = 0 \)), the equilibrium is unstable; however, a frequency-dependent amplitude \( \delta \) exists, which stabilizes the equilibrium. The phenomenon, however, only takes place on the right of the resonance value, while no stable equilibrium exist on the left. Results have also been compared with numerical solutions based on the Floquet theory (Fig 5-b). It is seen that, while the accordance is excellent for small \( \delta \), it is not good for large \( \delta \). In spite of this, the MSM captures the qualitative behavior, including the existence of a narrow stability zone at higher-amplitudes.

5. Bifurcations of singular systems: the Nonlinear Energy Sinks

5.1 Autonomous systems undergoing Hopf bifurcation

The occurrence, at the bifurcation, of the center eigenvalues \( \lambda_c = (0, 0, \pm i\omega) \) is not only an accidental (and detrimental) event, but even a desired circumstance adopted in passive control. As a matter of fact, when a mechanical system undergoes a Hopf bifurcation, e.g. triggered by wind flow forces, a control strategy recently studied in literature consists in attaching to it one (or more) essentially nonlinear oscillators (i.e. single d.o.f. devices with zero linear stiffness) having evanescent mass, said Nonlinear Energy Sinks (NES, [27, 28]). Thus, the augmented system also posseses a double zero eigenvalue. The NES works as a (well-known) Tuned Mass Damper (TMD), i.e. it is able to capture the input energy by preserving the main structure of the MSM, combined with the Harmonic Balance Method (HBM), and therefore denoted by MS-HBM, was indeed shown to work in [11, 12]. This is, indeed, true, but a 'non-standard' version of the MSM, with a NES attached, whose displacement is \( y \) (Fig 6); the relevant equations of motion, in nondimensional form, read:

\[
\begin{align*}
\dot{x} &= J(\mu)x + n(x, x, x) + \ldots -(\bar{\xi}z + \kappa z^3)\hat{r} \\
(m\bar{z} - r^T\bar{x}) + \xi z^2 + \kappa z^3 &= 0
\end{align*}
\]  

where \( z := r^T x - y \) is the elongation of the NES, and \( r, \hat{r} := (0, r)^T \) are influence coefficient vectors. Moreover, \( m \ll 1 \) is the NES mass, \( \xi \ll 1 \) the damping and \( \kappa = O(1) \) the nonlinear stiffness. Since \( m \) is small, but different from zero, the system posses a

![Figure 4](image-url)
pair of (nearly) zero eigenvalues; however, since the mass is evanescent, the equations are of singular type.

\[
\begin{align*}
\sum_{k=0}^{\infty} B_{0k} \left(t_1, t_2, \ldots\right) e^{ik\omega t_0} + \text{c.c.} \\
\end{align*}
\]

where \( B_{0k} \) are complex amplitudes. The HBM furnishes a set of cubic algebraic equations of the type:

\[
g_0 \left(A, B_0\right) = 0
\]

where \( B_0 := \left(B_{01}, B_{02}, \ldots\right) \). These provide an algebraic constraint between the amplitude of oscillation \( A \) of the main system and the harmonic components of the NES elongation. They describe a (parameter independent) manifold in the amplitude space on which the motion occurs. The Figure 7 shows a typical manifold in the space of the moduli \((a, b_{01}, b_{03})\) of the amplitudes \((A, B_{01}, B_{03})\).

By substituting the \( z_0 \) solution in Eq (31-a) and removing secular terms, a differential equation governing the evolution of \( A \) on the \( t_1 \)-scale is found, having the form:

\[
d_1 A = f_1 \left(A, B_0; \mu\right)
\]

If the analysis is truncated at this order, the dynamics is ruled by this last equation, with \( B_0 \) acting as a passive variable via the constraint (34), so that the motion is restrained to the manifold. However, if this latter exhibits turning point with respect to \( A \) (as it occurs in Fig. 7), the motion cannot cross them, since \( \dot{A} \) is generally different from zero there; hence, the equations break
Nonlinear Energy Sinks are also employed to passively control systems under external excitation [33]; the MSM-HB also works for them [12]. As an example, let us consider a Hamiltonian system, slightly damped, subject to a harmonic excitation of frequency $\Omega$ and amplitude $\delta$, having equation:

$$\dot{\mathbf{x}} = \mathbf{J}(\mu) \mathbf{x} + \mathbf{n}_3(\mathbf{x}, \mathbf{x}, \mathbf{x}) + \ldots - (\xi \dot{z} + \kappa z^3) \mathbf{r} + \delta \mathbf{f} \cos \Omega t$$

(39)

Here $\mathbf{J}(\mu) = \mathbf{J}_0 + \mu \mathbf{J}_1$, where $\mathbf{J}_0$ is the Jacobian matrix of an Hamiltonian system, admitting all non-zero eigenvalues $\pm \omega_j$, and $\mu := (\sigma, \xi)$ is a vector of small parameters, including stiffness and damping coefficients. Moreover, the external frequency $\Omega$ is assumed to be resonant with a natural frequency $\omega_1$, and other forms of resonance are excluded.

By rescaling the external force as $\delta \to e^2 \delta$ and using the same arguments for the autonomous system, we obtain perturbation equations like (30) and (31), this latter accounting for the new term:

$$(d_0 - \mathbf{J}_0) \mathbf{x}_1 = -d_1 \mathbf{x}_0 + \mu \mathbf{J}_1 \mathbf{x}_0 + \mathbf{n}_3(\mathbf{x}_0, \mathbf{x}_0, \mathbf{x}_0)$$

$$- \left( \xi \dot{z}_0 + \kappa z_0^3 \right) \mathbf{r} + \frac{1}{2} \delta \mathbf{f} \left( e^{i\Omega t} + e^{-i\Omega t} \right)$$

(40)

$$m \ddot{z}_0 + \frac{\xi}{\kappa} \dot{z}_0 + \kappa z_0^3 = m r^2 \ddot{\theta} x_0$$

5.2 Externally excited systems

Nonlinear Energy Sinks are also employed to passively control systems under external excitation [33]; the MSM-HB also works for them [12]. As an example, let us consider a Hamiltonian system, slightly damped, subject to a harmonic excitation of frequency $\Omega$ and amplitude $\delta$, having equation:

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$$- \xi \dot{z}_0 + \kappa z_0^3 \frac{1}{2} \delta \mathbf{f} \left( e^{i\Omega t} + e^{-i\Omega t} \right)$$

(40)

$$m \ddot{z}_0 + \frac{\xi}{\kappa} \dot{z}_0 + \kappa z_0^3 = m r^2 \ddot{\theta} x_0$$
Therefore, by using the generating solution (32), which takes into account only the natural mode involved in the resonance, and repeating the same steps as before, the following bifurcation equations are obtained, identical to Eqs (38), but including $\delta$ among the parameters:

$$\dot{A} = f(A, B; \mu, \delta) \tag{41}$$
$$\varepsilon \dot{B} = g(A, B; \mu, \delta)$$

These equations have been used in [12] to analyze the response of a single d.o.f. system with a NES attached; some results are reported in Fig. 10. Fig. 10-a shows the amplitude-frequency curve; an unstable periodic motion (yellow circle) exists between secondary Hopf points (black squares). When the motion is plotted on the amplitude plane (Fig. 10-b) SMR are observed, occurring around the unstable periodic motion. The time-histories of the amplitude (Fig. 10-c) and that of the displacement $x$ (Fig. 10-d) show the relaxation oscillations.

6. Conclusions

We discussed how to adapt the Multiple Scale Method in order to solve ‘difficult’ bifurcation problems. We obtained the following main results.

1. When the system is defective at the bifurcation, i.e. the center eigenvalue does not possess a complete set of eigenvectors, fractional power expansions and fractional times of order $\varepsilon^{1/m}$ must be used, with $m$ the algebraic multiplicity of the eigenvalue. By systematically enforcing solvability conditions at each order, and then recombining them on the true time-scale, an $m$-order differential equation for the amplitude is drawn.

2. When the system is partially defective at the bifurcation, i.e. defective and non-defective eigenvalues interact, then fractional powers must be used, although some of the derivatives of the amplitudes can be zero, this denoting that the dynamics is of slow-fast type. Consequently, the bifurcation equations are differential equations of different order.

3. When the system is, in addition, singular, as it happens for NES-equipped systems, integer powers must be used, in conjunction with the Harmonic Balance Method to solve singular equations. The bifurcation equations are found to be in the form of singular equations, exhibiting relaxation oscillations.

Several applications have been shown, to display mechanical phenomenon and to test the effectiveness of the proposed meth-
Figure 10. Response of a single-d.o.f. system equipped with NES to external excitation [12]: (a) amplitude-frequency response; (b) phase-portrait; (c), (d) time-histories.

ods. They include discrete and continuous systems, autonomous, parametrically excited and externally excited systems. Moreover, the algorithm was able to explain the transition between two bifurcations of different codimensions, when a parameter is varied.

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References


