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WEAK DIMENSION OF FP-INJECTIVE MODULES OVER
CHAIN RINGS

FRANÇOIS COUCHOT

Abstract. It is proven that the weak dimension of each FP-injective module
over a chain ring which is either Archimedean or not semicoherent is less
or equal to 2. This implies that the projective dimension of any countably
generated FP-injective module over an Archimedean chain ring is less or equal
to 3.

By [7, Theorem 1], for any module \(G\) over a commutative arithmetical ring \(R\) the
weak dimension of \(G\) is 0, 1, 2 or \(\infty\). In this paper we consider the weak dimension
of almost FP-injective modules over a chain ring. This class of modules contains
the one of FP-injective modules and these two classes coincide if and only if the
ring is coherent. If \(G\) is an almost FP-injective module over a chain ring \(R\) then its
weak dimension is possibly infinite only if \(R\) is semicoherent and not coherent. In
the other cases the weak dimension of \(G\) is at most 2. Moreover this dimension is
not equal to 1 if \(R\) is not an integral domain. Theorem 15 summarizes main results
of this paper.

We complete this short paper by considering almost FP-injective modules over
local \(f\)-qp-rings. This class of rings was introduced in [1] by Abuhlail, Jarrar and
Kabbaj. It contains the one of arithmetical rings. It is shown the weak dimension
of \(G\) is infinite if \(G\) is an almost FP-injective module over a local \(f\)-qp-ring which is
not a chain ring (see Theorem 23).

All rings in this paper are unitary and commutative. A ring \(R\) is said to be a
chain ring\(^1\) if its lattice of ideals is totally ordered by inclusion. Chain rings
are also called valuation rings (see [9]). If \(M\) is an \(R\)-module, we denote by
\(\text{w.d.}(M)\) its weak dimension and \(\text{p.d.}(M)\) its projective dimension. Recall
that \(\text{w.d.}(M) \leq n\) if \(\text{Tor}_n^{R}(M, N) = 0\) for each \(R\)-module \(N\). For any ring \(R\), its
global weak dimension \(\text{w.gl.d}(R)\) is the supremum of \(\text{w.d.}(M)\) where \(M\) ranges
over all (finitely presented cyclic) \(R\)-modules. Its finitistic weak dimension
\(\text{f.w.d.}(R)\) is the supremum of \(\text{w.d.}(M)\) where \(M\) ranges over all \(R\)-modules of finite
weak dimension.

A ring is called coherent if all its finitely generated ideals are finitely presented.
As in [14], a ring \(R\) is said to be semicoherent if \(\text{Hom}_R(E, F)\) is a submodule of a
flat \(R\)-module for any pair of injective \(R\)-modules \(E, F\). A ring \(R\) is said to be \(\text{IF}\)
(semi-regular in [14]) if each injective \(R\)-module is flat. If \(R\) is a chain ring, we
denote by \(P\) its maximal ideal, \(Z\) its subset of zerodivisors which is a prime ideal
and \(Q(= R_Z)\) its fraction ring. If \(x\) is an element of a module \(M\) over a ring \(R\), we
denote by \((0 : x)\) the annihilator ideal of \(x\) and by \(E(M)\) the injective hull of \(M\).

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\(^1\)we prefer “chain ring” to “valuation ring” to avoid confusion with “Manis valuation ring”.

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Some preliminary results are needed to prove Theorem 15 which is the main result of this paper.

**Proposition 1.** [7, Proposition 4]. Let \( R \) be a chain ring. The following conditions are equivalent:

1. \( R \) is semicoherent;
2. \( Q \) is an IF-ring;
3. \( Q \) is coherent;
4. either \( Z = 0 \) or \( Z \) is not flat;
5. \( E(Q/Qa) \) is flat for each nonzero element \( a \) of \( Z \);
6. there exists \( 0 \neq a \in Z \) such that \( (0 : a) \) is a finitely generated over \( Q \).

An exact sequence of \( R \)-modules \( 0 \to F \to E \to G \to 0 \) is **pure** if it remains exact when tensoring it with any \( R \)-module. Then, we say that \( F \) is a **pure submodule** of \( E \). When \( F \) is flat, it is well known that \( G \) is flat if and only if \( F \) is a pure submodule of \( E \). An \( R \)-module \( E \) is **FP-injective** if \( \operatorname{Ext}^1_R(F,E) = 0 \) for any finitely presented \( R \)-module \( F \). We recall that a module \( E \) is FP-injective if and only if it is a pure submodule of every overmodule. We define a module \( G \) to be **almost FP-injective** if there exist a FP-injective module \( E \) and a pure submodule \( D \) such that \( G \cong E/D \). By [15, 35.9] the following theorem holds:

**Theorem 2.** A ring \( R \) is coherent if and only if each almost FP-injective \( R \)-module is FP-injective.

**Proposition 3.** For any ring \( R \) the following conditions are equivalent:

1. \( R \) is self FP-injective;
2. each flat \( R \)-module is almost FP-injective.

**Proof.** (1) \( \Rightarrow \) (2). Each flat module \( G \) is of the form \( F/K \) where \( F \) is free and \( K \) is a pure submodule of \( F \). Since \( F \) is FP-injective then \( G \) is almost FP-injective.

(2) \( \Rightarrow \) (1). We use the fact that \( R \) is projective to show that \( R \) is a direct summand of a FP-injective module. \( \square \)

**Proposition 4.** Let \( R \) be a chain ring and \( 0 \to L \xrightarrow{u} M \to N \to 0 \) a exact sequence of \( R \)-modules. Then \( \operatorname{w.d.}(M) = \operatorname{max}(\operatorname{w.d.}(L), \operatorname{w.d.}(N)) \).

**Proof.** Let \( m = \operatorname{w.d.}(M) \) and let \( G \) be an \( R \)-module. We consider the following flat resolution of \( G \):

\[
F_p \xrightarrow{u_p} F_{p-1} \to \cdots F_2 \xrightarrow{u_2} F_1 \xrightarrow{u_1} F_0 \to G \to 0.
\]

For each positive integer \( p \) let \( G_p \) be the image of \( u_p \). We have the following exact sequence:

\[
\operatorname{Tor}_1^R(G_m, M) \to \operatorname{Tor}_1^R(G_m, N) \to G_m \otimes_R L \xrightarrow{id_G \otimes v} G_m \otimes R M.
\]

Since \( \operatorname{Tor}_1^R(G_m, M) \cong \operatorname{Tor}_m^R(G, M) \cong 0 \) and \( id_G \otimes v \) is a monomorphism we deduce that \( \operatorname{Tor}_{m+1}^R(G, N) \cong 0 \). So, \( \operatorname{w.d.}(N) \leq m \). Now it is easy to show first, that \( \operatorname{w.d.}(L) \leq m \) too, and then, if \( q = \operatorname{max}(\operatorname{w.d.}(L), \operatorname{w.d.}(N)) \), that \( \operatorname{Tor}_{q+1}^R(G, M) = 0 \) for each \( R \)-module \( G \). Hence \( m = q \). \( \square \)

**Proposition 5.** Let \( R \) be a chain ring. Then:

1. \( \operatorname{w.d.}(R/Z) = \infty \) if \( R \) is semicoherent and \( Z \neq 0 \);
2. \( \operatorname{w.d.}(R/Z) = 1 \) if \( R \) is not semicoherent.
Proof. We consider the following exact sequence: \( 0 \rightarrow R/Z \rightarrow Q/Z \rightarrow Q/R \rightarrow 0 \). If \( R \neq Q \) then \( \text{w.d.}(Q/R) = 1 \).

(1). By [4, Propositions 8 and 14] applied to \( R \) and Proposition 1 \( \text{w.d.}(Q/Z) = \infty \). By using the previous exact sequence when \( R \neq Q \) we get \( \text{w.d.}(R/Z) = \infty \).

(2). By Proposition 1 \( \text{w.d.}(Q/Z) = 1 \). When \( R \neq Q \) we conclude by using the previous exact sequence. \( \square \)

**Proposition 6.** Let \( R \) be a chain ring. If \( R \) is IF, let \( H = E(R/rR) \) where \( 0 \neq r \in P \). If \( R \) is not IF, let \( x \in E(R/Z) \) such that \( Z = (0 : x) \) and \( H = E(R/Z)/Rx \). Then:

- (1) \( H \) is FP-injective;
- (2) for each \( 0 \neq r \in R \) there exists \( h \in H \) such that \( (0 : h) = Rr \);
- (3) \( \text{w.d.}(H) = \infty \) if \( R \) is semi-coherent and \( R \neq Q \);
- (4) \( \text{w.d.}(H) = 2 \) if \( R \) is not semi-coherent.

**Proof.** (1). When \( R \neq Q \), \( H \) is FP-injective by [4, Proposition 6]. If \( R = Q \) is not IF then \( H \cong E(R/rR) \) for each \( 0 \neq r \in P \) by [4, Proposition 14].

(2). Let \( 0 \neq r \in R \). Then \( (0 : r) \subseteq Z = (0 : x) \). From the injectivity of \( E(R/Z) \) we deduce that there exists \( y \in E(R/Z) \) such that \( x = ry \). Now, if we put \( h = y + Rr \) it is easy to check that \( (0 : h) = (Rx : y) = Rr \).

(3) and (4). By [4, Proposition 8] \( E(R/Z) \) is flat. We use the exact sequence \( 0 \rightarrow R/Z \rightarrow E(R/Z) \rightarrow H \rightarrow 0 \) and Proposition 5 to conclude. \( \square \)

The following proposition is a slight modification of [4, Proposition 9].

**Proposition 7.** Let \( R \) be a chain ring and \( G \) an injective module. Then there exists a pure exact sequence: \( 0 \rightarrow K \rightarrow I \rightarrow G \rightarrow 0 \), such that \( I \) is a direct sum of submodules isomorphic to \( R \) or \( H \).

**Proof.** There exist a set \( \Lambda \) and an epimorphism \( \varphi : L = R^\Lambda \rightarrow G \). We put \( \Delta = \text{Hom}_R(H,G) \) and \( \rho : H^{(\Delta)} \rightarrow G \) the morphism defined by the elements of \( \Delta \). Thus \( \psi \) and \( \rho \) induce an epimorphism \( \phi : I = R^{(\Lambda)} \oplus H^{(\Delta)} \rightarrow G \). Since, for every \( r \in P, r \neq 0 \), each morphism \( g : R/Re \rightarrow G \) can be extended to \( H \rightarrow G \), we deduce that \( K = \ker \phi \) is a pure submodule of \( I \).

**Lemma 8.** Let \( G \) be an almost FP-injective module over a chain ring \( R \). Then for any \( x \in G \) and \( a \in R \) such that \( (0 : a) \subseteq (0 : x) \) there exists \( y \in G \) such that \( x = ay \).

**Proof.** We have \( G = \text{E.D} \) where \( E \) is a FP-injective module and \( D \) a pure submodule. Let \( e \in E \) such that \( x = e + D \). Let \( b \in (0 : x) \setminus (0 : a) \). Then \( be \in D \). So, we have \( b(e - d) = 0 \) for some \( d \in D \). Whence \( (0 : a) \subseteq Rb \subseteq (0 : e - d) \). This is a contradiction as \( F \)-injectivity of \( E \) we deduce that \( e - d = az \) for some \( z \in E \). Hence \( x = ay \) where \( y = z + D \).

Let \( M \) be a non-zero module over a ring \( R \). We set:

\[ M_1 = \{ s \in R \mid \exists \neq x \in M \text{ such that } sx = 0 \} \quad \text{and} \quad M^5 = \{ s \in R \mid sM \subseteq M \}. \]

Then \( R \setminus M_1 \) and \( R \setminus M^5 \) are multiplicative subsets of \( R \). If \( M \) is a module over a chain ring \( R \) then \( M_1 \) and \( M^5 \) are prime ideals called respectively the **bottom prime ideal** and the **top prime ideal** associated with \( M \).

**Proposition 9.** Let \( G \) be a module over a chain ring \( R \). Then:
(1) $G_1 \subseteq Z$ if $G$ is flat;
(2) $G^2 \subseteq Z$ if $G$ is almost FP-injective;
(3) $G^3 \subseteq Z \cap G_2$ if $G$ is FP-injective. So, $G$ is a module over $R_{G_1}$;
(4) $G$ is flat and FP-injective if $G$ is almost FP-injective and $G_1 \cup G^2 \subset Z$. In this case $G^3 \subseteq G_1$.

Proof. (1). Let $a \in G_1$. There exists $0 \neq x \in G$ such that $ax = 0$. The flatness of $G$ implies that $x \in (0 : a)G$. So, $(0 : a) \neq 0$ and $a \in Z$.

(2). Let $s \in R \setminus Z$. Then for each $x \in G$, $0 = (0 : s) \subseteq (0 : x)$. If $G$ is FP-injective then $x = sy$ for some $y \in G$. If $G$ is almost FP-injective then it is factor of a FP-injective module, so, the multiplication by $s$ in $G$ is surjective.

(3). Let $a \in R \setminus G_1$ and $x \in G$. Let $b \in (0 : a)$. Then $abx = 0$, whence $bx = 0$. So, $(0 : a) \subseteq (0 : x)$. It follows that $x = ay$ for some $y \in G$ since $G$ is FP-injective. Hence $a \notin G^2$.

(4). Let $0 \to X \to E \to G \to 0$ be a pure exact sequence where $E$ is FP-injective, and $L = G^2 \cup G_2$. Then $0 \to X_L \to E_L \to G \to 0$ is a pure exact sequence and by [5, Theorem 3] $E_L$ is FP-injective. By [4, Theorem 11] $R_L$ is an IF-ring. Hence $G$ is FP-injective and flat.

Example 10. Let $D$ be a valuation domain. Assume that $D$ contains a non-zero prime ideal $L \neq P$ and let $0 \neq a \in L$. By [7, Example 11 and Corollary 9] $R = D/\alpha aL$ is semicoherent and not coherent. Since $R$ is not self FP-injective then $E_R(R/P)$ is not flat by [3, Proposition 2.4].

Example 11. Let $R$ be a semicoherent chain ring which is not coherent and $G$ an FP-injective module which is not flat. Then $G_Z$ is almost FP-injective over $R$ but not over $Q$, and $(G_Z)_1 \cup (G_Z)^2 = Z$.

Proof. Let $G'$ be the kernel of the canonical homomorphism $G \to G_Z$. By Proposition 9 the multiplication in $G$ and $G/G'$ by any $s \in R \setminus Z$ is surjective. So, $G_Z \cong G/G'$. For each $s \in P \setminus Z$ we put $G_{(s)} = G/(0 :_G s)$. Then $G_{(s)}$ is FP-injective because it is isomorphic to $G$. Since $G' = \cup_{s \in P \setminus Z} (0 :_G s)$ then $G_Z \cong \lim_{\leftarrow \substack{s \in P \setminus Z}} G_{(s)}$. By [15, 33.9(2)] $G_Z$ is factor of the FP-injective module $\oplus_{s \in P \setminus Z} G_{(s)}$ modulo a pure submodule. Hence $G_Z$ is almost FP-injective over $R$. Since $Q$ is IF and $G_Z$ is not FP-injective by [5, Theorem 3], from Theorem 2 we deduce that $G_Z$ is not almost FP-injective over $Q$. We complete the proof by using Proposition 9(4).

The following lemma is a consequence of [10, Lemma 3] and [13, Proposition 1.3].

Lemma 12. Let $R$ be a chain ring for which $Z = P$, and $A$ an ideal of $R$. Then
$A \subseteq (0 : (0 : A))$ if and only if $P$ is faithful and there exists $t \in R$ such that $A = tP$ and $(0 : (0 : A)) = tR$.

Proposition 13. Let $R$ be a chain ring for which $Z \neq 0$. Then w.d.(G) $\neq 1$ for any almost FP-injective $R$-module $G$.

Proof. By way of contradiction assume there exists an almost FP-injective $R$-module $G$ with w.d.(G) = 1. There exists an exact sequence $0 \to K \to F \to G \to 0$ which is not pure, where $F$ is free and $K$ is flat.

First we assume that $R = Q$, whence $P = Z$. So there exist $b \in R$ and $x \in F$ such that $bx \in K \setminus bK$. We put $B = (K : x)$. From $b \in B$ we deduce that $(0 : B) \subseteq (0 : b)$. We investigate the following cases:
1. \((0 : B) \subset (0 : b)\).

Let \(a \in (0 : b) \setminus (0 : B)\). Then \((0 : a) \subset (0 : (0 : B))\). If \(P\) is not faithful then \((0 : a) \subset B\) by Lemma 12. If \(P\) is faithful then \(B \neq (0 : (0 : B))\) if \(B = Pt\) for some \(t \in R\). But, in this case \((0 : a)\) is not of the form \(Ps\) for some \(s \in R\). So, \((0 : a) \subset B\). By Lemma 8 there exist \(y \in F\) and \(z \in K\) such that \(x = ay + z\). It follows that \(bx = bz \in bK\), whence a contradiction.

2.1. \((0 : B) = (0 : b) \subset P\).

Let \(r \in P \setminus (0 : b)\). Then \((0 : r) \subset Rb\). Let \(0 \neq c \in (0 : r)\). There exists \(t \in P\) such that \(c = tb\). So, \((0 : b) \subset (0 : c)\). If \(cx \in cK\), then \(txc = tby\) for some \(y \in K\). Since \(K\) is flat we get that \((bx - by) \in (0 : t)K \subset bK\) \((c = bt \neq 0\) implies that \((0 : t) \subset bR\), whence \(bx \in bK\), a contradiction. Hence \(cx \notin cK\). So, if we replace \(b\) with \(c\) we get the case 1.

2.2.1. \((0 : B) = (0 : b) = P = bR\).

Since \(b^2 = 0\) we have \(b(bx) = 0\). The flatness of \(K\) implies that \(bx \in bK\), a contradiction.

2.2.2. \((0 : B) = (0 : b) = P \neq bR\).

Let \(c \in P \setminus bR\). Then \(cx \notin K\) and there exists \(s \in P\) such that \(b = sc\). We have \((K : cx) = Rs \neq Rb\). So, if \(sex \notin sK\) we get the case 2.1 by replacing \(b\) with \(s\) and \(x\) with \(cx\). Suppose that \(sex \in sK\). We get \(s(cx - y) = 0\) for some \(y \in K\). The flatness of \(F\) implies that \((cx - y) \in (0 : s)F \subset cF\). Whence \(y = c(x + z)\) for some \(z \in F\). If \(y = cx\) for some \(v \in K\) then \(bx = sy = scv = bv \in bK\). This is false. Hence \(c(x + z) \in K \setminus cK\) and \((0 : (K : x + z)) \subset (0 : c) \subset Rs \subset P\). So, we get either the case 1 or the case 2.1 by replacing \(x\) with \((x + z)\) and \(b\) with \(c\).

Now, we assume that \(R \neq Q\). First, we show that \(G_Z\) is flat. Since \(K\) and \(F\) are flat then so are \(K_Z\) and \(F_Z\). If \(Q\) is coherent, then \(K_Z\) is \(FP\)-injective. So, it is a pure submodule of \(F_Z\) and consequently \(G_Z\) is flat. If \(Q\) is not coherent, then \(Z\) is flat, and by using [5, Theorem 3] it is easy to show that \(G_Z\) is almost \(FP\)-injective. Since \(w.d.(G_Z) \leq 1\), from above we deduce that \(G_Z\) is flat. Let \(G'\) be the kernel of the canonical homomorphism \(G \to G_Z\). As in the proof of Proposition 9 we have \(G_Z \cong G/G'\). So, \(G'\) is a pure submodule of \(G\). For each \(x \in G\) there exists \(s \in P \setminus Z\) such that \(sx = 0\). Hence \(G'\) is a module over \(R/Z\). But if \(0 \neq a \in Z\), \((0 : a) \subset Z \subset (0 : x)\) for any \(x \in G'\). By Lemma 8 \(x = ay\) for some \(y \in G\), and since \(G'\) is a pure submodule, we may assume that \(y \in G\). Hence \(G' = 0, G \cong G_Z\) and \(G\) is flat.

Example 14. Let \(D\) be a valuation domain. Let \(0 \neq a \in P\). By [4, Theorem 11] \(R = D/aD\) is an \(IF\)-ring.

Now it is possible to state and to prove our main result.

Theorem 15. For any almost \(FP\)-injective module \(G\) over a chain ring \(R\):

1. \(w.d.(G) = 0\) if \(R\) is an \(IF\)-ring;
2. if \(R\) is a valuation domain which is not a field then:
   a. \(w.d.(G) = 0\) if \(G^2 = 0\) (\(G\) is torsionfree);
   b. \(w.d.(G) = 1\) if \(G^2 \neq 0\);
3. if \(R\) is semicoherent but not coherent then:
   a. \(w.d.(G) = \infty\) if \(Z \subset G_1\);
   b. \(w.d.(G) = 0\) if \(G_2 \cup G^2 \subset Z\);
   c. \(G_2 \cup G^2 = Z\) either \(w.d.(G) = 0\) if \(G\) is \(FP\)-injective or \(w.d.(G) = \infty\) if \(G\) is not \(FP\)-injective;
Corollary 16. For any almost FP-injective module $G$ over an Archimedean chain ring $R$: 

(1) $\text{w.d.}(G) = 0$ if $R$ is an IF-ring;
(2) $\text{w.d.}(G) \leq 1$ if $R$ is a valuation domain;
(3) either $\text{w.d.}(G) = 2$ or $\text{w.d.}(G) = 0$ if $R$ is not coherent. More precisely $G$ is flat if and only if, for any $0 \neq x \in G$, $(0 : x)$ is not of the form $Ra$ for some $0 \neq a \in P$. Moreover $P \otimes_R G$ is flat and almost FP-injective.

Let us observe that $\text{w.d.}(G) \leq 2$ for any almost FP-injective module $G$ over an Archimedean chain ring.

Example 17. Let $D$ be an Archimedean valuation domain. Let $0 \neq a \in P$. Then $D/aD$ is Archimedean and it is IF by Example 14.

Corollary 18. Let $R$ be an Archimedean chain ring. For any almost FP-injective $R$-module $G$ which is either countably generated or uniserial.
(1) p.d.(G) ≤ 1 if R is an IF-ring;
(2) p.d.(G) ≤ 2 if R is a valuation domain;
(3) p.d.(G) ≤ 3 and p.d.(P ⊗_R G) ≤ 1 if R is not coherent.

Proof. By [6, Proposition 16] each uniserial module (ideal) is countably generated. So, this corollary is a consequence of Corollary 16 and [12, Lemmas 1 and 2]. □

Remark 19. If R is an Archimedean valuation domain then w.d.(G) ≤ 1 and p.d.(G) ≤ 2 for any R-module G.

Let R be a ring, M an R-module. A R-module V is M-projective if the natural homomorphism Hom_R(V, M) → Hom_R(V, M/X) is surjective for every submodule X of M. We say that V is quasi-projective if V is V-projective. A ring R is said to be an fqp-ring if every finitely generated ideal of R is quasi-projective.

The following theorem can be proven by using [1, Lemmas 3.8, 3.12 and 4.5].

Theorem 20. [8, Theorem 4.1]. Let R a local ring and N its nilradical. Then R is a fqp-ring if and only if either R is a chain ring or R/N is a valuation domain and N is a divisible torsionfree R/N-module.

Lemma 21. Let R be a local ring and P its maximal ideal. Assume that (0 : P) ≠ 0. If R is a FP-injective module then (0 : P) is a simple module.

Proof. Let 0 ≠ a ∈ (0 : P). By [11, Corollary 2.5] Ra = (0 : (0 : a)). But (0 : a) = P. So, Ra = (0 : P). □

Example 22. Let D be a valuation domain which is not a field and E a non-zero divisible torsionfree D-module which is not uniserial. Then R = D × E the trivial extension of D by E is a local fqp-ring which is not a chain ring by [8, Corollary 4.3(2)].

Theorem 23. For any non-zero almost FP-injective module G over a local fqp-ring R which is not a chain ring, w.d.(G) = ∞.

Proof. By [8, Proposition 5.2] f.w.d.(R) ≤ 1. If N is the nilradical of R then its quotient ring Q is R_N and this ring is primary.

First assume that R = Q. Then each flat module is free and f.w.d.(R) = 0 by [2, Theorems P and 6.3]. If G is free, it follows that R is FP-injective. By Lemma 21 this is possible only if N is simple, whence R is a chain ring. Hence w.d.(G) = ∞.

Now assume that R ≠ Q and w.d.(G) ≤ 1. Then w.d.(G_N) ≤ 1. Since f.w.d.(Q) = 0 it follows that G_N is flat. As in the proof of Proposition 13, with the same notations we show that G_N ≅ G/G' and G' is a module over R/N. But if 0 ≠ a ∈ N, (0 : a) = N ⊂ (0 : x) for any x ∈ G'. As in the proof of Lemma 8 we show that x = ay for some y ∈ G. Since G' is a pure submodule, we may assume that y ∈ G'. Hence G' = 0, G ≅ G_N and G is flat. We use the first part of the proof to conclude. □

References


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