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Problems in number theory from busy beaver competition

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Abstract

By introducing the busy beaver competition of Turing machines, in 1962, Rado defined noncomputable functions on positive integers. The study of these functions and variants leads to many mathematical challenges. This article takes up the following one: How can a small Turing machine manage to produce very big numbers? It provides the following answer: mostly by simulating Collatz-like functions, that are generalizations of the famous 3x+1 function. These functions, like the 3x+1 function, lead to new unsolved problems in number theory.

Keywords: Turing machine, busy beaver, Collatz-like functions.

Mathematics Subject Classification (2000, 2010): Primary 03D10, Secondary 68Q05, 11B83.

1 Introduction

1.1 A well defined noncomputable function

It is easy to define a noncomputable function on nonnegative integers. Indeed, given a programming language, you produce a systematic list of the programs for functions, and, by diagonalization, you define a function whose output, on input n, is different from the output of the nth program. This simple definition raises many problems: Which programming language? How to list the programs? How to choose the output?

In 1962, Rado [38] gave a practical solution by defining the busy beaver game, also called now the busy beaver competition. Consider all Turing machines on one infinite tape, with n
states (plus a special halting state), and two symbols (1, and the blank symbol 0), and launch all of them on a blank tape. Define $S(n)$ as the maximum number of computation steps made by such a machine before it stops, and define $\Sigma(n)$ as the maximum number of symbols 1 left on the tape by a machine when it stops. Then functions $S$ and $\Sigma$ are noncomputable, and, moreover, grow faster than any computable function, that is, for any computable function $f$, there exists an integer $N$ such that, for any $n \geq N$, $S(n) > \Sigma(n) > f(n)$.

More than fifty years later, no better choice has been found for a practical noncomputable function. Only variants of Rado’s definition have been proposed. So, in 1988, Brady [5] defined similar functions $S(n, m)$ and $\Sigma(n, m)$ for $n \times m$ Turing machines, that is Turing machines with $n$ states and $m$ symbols. He also introduced analogous functions for two-dimensional Turing machines and “turingN machines”, later resumed and expanded by Tim Hutton [13]. Bátfai [1, 2] relaxed the rule about head moving, by allowing the head to stand still.

In this article, we will consider functions $S(n, m)$ and $\Sigma(n, m)$. Recall that $S(n) = S(n, 2)$ and $\Sigma(n) = \Sigma(n, 2)$.

1.2 Computing the values of noncomputable functions

The busy beaver functions $S$ and $\Sigma$ are explicitly defined, and it is possible to compute $S(n, m)$ and $\Sigma(n, m)$ for small $n$ and $m$. In the first article on busy beavers, Rado [38] gave $\Sigma(2) = 2$ and $\Sigma(3) \geq 6$. These results show that two problems are at stake:

- **Problem 1**: To give lower bounds on $S(n, m)$ and $\Sigma(n, m)$ by finding Turing machines with high scores.
- **Problem 2**: To compute $S(n, m)$ and $\Sigma(n, m)$ by proving that no Turing machines do better than the best known ones.

Problem 1 can be tackled either by hand search, as did, for example, Green [11] and Lynn [26], or by computer search, using acceleration techniques of computation and, for example, simulated annealing, as did T. and S. Ligocki [21].

Solving Problem 2 requires more work to be done: clever enumeration of $n \times m$ Turing machines, simulation of computation with acceleration techniques, proofs of non-halting for the machines that do not halt.

Currently, the following results are known (see Michel [32, 33] for a historical survey):

- $S(2) = 6$ and $\Sigma(2) = 4$ (Rado [38]),
- $S(3) = 21$ and $\Sigma(3) = 6$ (Lin and Rado [25]),
- $S(4) = 107$ and $\Sigma(4) = 13$ (Brady [3, 4], Machlin and Stout [27]),
- $S(5) \geq 47,176,870$ and $\Sigma(5) \geq 4098$ (Marxen and Buntrock [29]),
- $S(6) > 7.4 \times 10^{36534}$ and $\Sigma(6) > 3.5 \times 10^{18267}$ (P. Kropitz in 2010),
- $S(2, 3) = 38$ and $\Sigma(2, 3) = 9$ (Lafitte and Papazian [16]),
- $S(3, 3) > 1.1 \times 10^{17}$ and $\Sigma(3, 3) \geq 347,676,383$ (T. and S. Ligocki in 2007),
• $S(4,3) > 1.0 \times 10^{14072}$ and $\Sigma(4,3) > 1.3 \times 10^{7036}$ (T. and S. Ligocki in 2008),
• $S(2,4) \geq 3,932,964$ and $\Sigma(2,4) \geq 2050$ (T. and S. Ligocki in 2005),
• $S(3,4) > 5.2 \times 10^{13036}$ and $\Sigma(3,4) > 3.7 \times 10^{9518}$ (T. and S. Ligocki in 2007),
• $S(2,5) > 1.9 \times 10^{704}$ and $\Sigma(2,5) > 1.7 \times 10^{1352}$ (T. and S. Ligocki in 2007),
• $S(2,6) > 2.4 \times 10^{9866}$ and $\Sigma(2,6) > 1.9 \times 10^{4933}$ (T. and S. Ligocki in 2008).

In order to achieve these results, many computational and mathematical challenges had to be taken up.

A. Computational challenges.

A1. To generate all $n \times m$ Turing machines, or rather, to treat all cases without having to generate all $n \times m$ Turing machines.

A2. To simulate the computation of a machine by using acceleration techniques (see Marxen and Buntrock [29, Marxen [28]]).

A3. To gave automatic proofs that non-halting machines do not halt (see Brady [4], Marxen and Buntrock [29, Machlin and Stout [27, Hertel [12], Lafitte and Pappazian [16]]).

B. Mathematical challenges.

B1. To prove by hand that a non-halting machine that resists the computational proof does not halt.

B2. To understand how the Turing machines that reach high scores manage to do it.

1.3 Facing open problems in number theory

Let us come back to mathematical challenge B1. For example, the computational study of $5 \times 2$ Turing machines by Marxen and Buntrock [29], Skelet [10] and Hertel [12] left holdouts that needed to be analyzed by hand. Marxen and Buntrock [29] gave an unsettled $5 \times 2$ Turing machine, named #4, that turned out to never halt, by an intricate analysis.

Actually, the halting problem for Turing machines launched on a blank tape is m-complete, and this implies that this problem is as hard as the problem of the provability of a mathematical statement in a logical theory such as ZFC (Zermelo Fraenkel set theory with axiom of choice). So, when Turing machines with more and more states and symbols are studied, potentially all theorems of mathematics will be met. When more and more non-halting Turing machines are studied to be proved non-halting, one has to expect to face hard open problems in mathematics, that is problems that current mathematical knowledge can’t settle.

Consider now mathematical challenge B2, which is the very subject of this article.

From 1983 to 1989, several $5 \times 2$ Turing machines with high scores were discovered by Uwe Schult, by George Uhing, and by Heiner Marxen and Jürgen Buntrock. Michel [30, 31] analyzed some of these machines and found that their behavior is Collatz-like, which implies that the halting problems on general inputs for these machines are open problems in number theory (see Table 1).
From 2005 and 2007, many $3 \times 3$, $2 \times 4$ and $2 \times 5$ Turing machines with high scores were discovered, mainly by two teams: the French one of Grégory Lafitte and Christophe Papazian, and the father-and-son collaboration of Terry and Shawn Ligocki. Collatz-like behavior of these champions seems to be the rule (see Tables 3, 4 and 5).

However, the behaviors of $6 \times 2$ Turing machines display some variety. Many machines were discovered, from 1990 to 2010, by Heiner Marxen and Jürgen Buntrock, by Terry and Shawn Ligocki and by Pavel Kropitz. The analyses of some of these machines, by Robert Munafo, Clive Tooth, Shawn Ligocki and the author, show that the behaviors can be Collatz-like, exponential Collatz-like, loosely Collatz-like, or definitely not Collatz-like. Almost all of them raise open problems (see Table 2).

Note: The Turing machines listed in Tables 1–5 are those for which an analysis is known by the author. The machines without references for the study of behavior were analyzed by the author [32, 34]. Many other machines are waiting for their analyses.

1.4 Collatz functions, Collatz-like functions and other functions

The $3x+1$ function, or Collatz function, is the function $T$ on positive integers defined by

$$T(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{(3n + 1)}{2} & \text{if } n \text{ is odd} \end{cases}$$

This function is famous because, when it is iterated on a positive integer, it seems to lead to the loop $2, 1, 2, 1, \ldots$. Is it always true? This is an open problem. See Lagarias [17, 18, 19, 20] for more information.

It is natural to generalize the definition of the $3x+1$ function by replacing $n \equiv 0, \ldots, d-1 \pmod d$, and by replacing $n/2, (3n + 1)/2$ by $an + b$ for rational numbers $a, b$. Unfortunately, no name for such functions is currently taken for granted. Formal definitions were given by Rawsthorne [39], who proposed Collatz-type iteration functions, by Buttsworth and Matthews [6], who proposed generalized Collatz mappings, by Kaščák [14], who proposed one-state linear operator algorithms (OLOA), and by Kohl [15], who proposed residue-class-wise affine functions (RCWA). Without giving a formal definition, Lagarias [17] proposed periodically linear functions, and Wagon [41] proposed Collatz-like functions.

We will choose the following definitions.

**Definition 1.1** A mapping $f : \mathbb{Z} \to \mathbb{Z}$ is a generalized Collatz mapping if there exists an integer $d \geq 2$ such that the following three equivalent conditions are satisfied:

(i) (see [42, p.14]) There exist rational numbers $q_0, \ldots, q_{d-1}, r_0, \ldots, r_{d-1}$, such that, for all $i, 0 \leq i \leq d-1$, we have $q_i \in \mathbb{Z}, q_i + r_i \in \mathbb{Z}$, and, for all $n \in \mathbb{Z}$, $f(n) = q_in + r_i$ if $n \equiv i \pmod d$.

(ii) (see [6]) There exist integers $m_0, \ldots, m_{d-1}, p_0, \ldots, p_{d-1}$, such that, for all $i, 0 \leq i \leq d-1$, we have $p_i \equiv im_i \pmod d$ and, for all $n \in \mathbb{Z}$, $f(n) = (m_in - p_i)/d$ if $n \equiv i \pmod d$.

(iii) There exist integers $a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1}$, such that we have, for all $i, 0 \leq i \leq d-1$, for all $n \in \mathbb{Z}$, $f(dn + i) = a_in + b_i$. 

4
These definitions are easily seen to be equivalent: we have \( a_i = m_i = q_id \) and \( b_i = (im_i - p_i)/d = q_i + r_i \).

The definitions above concern total functions, but, in this article, we always deal with partial functions and functions with parameters, so we introduce the following definitions.

**Definition 1.2** A partial function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) is a generalized Collatz function, or a pure Collatz-like function (without parameter) if, in the previous definition, \( f(dn + i) \) can be undefined for one or many \( i, 0 \leq i \leq d - 1 \).

**Definition 1.3** A partial function \( f : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} \) is a pure Collatz-like function with parameter if there exist an integer \( d \geq 2 \), integers \( a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1} \), a set \( S \) of integers and a function \( p : \{0, \ldots, d - 1\} \times S \rightarrow S \) such that, for all \( i, 0 \leq i \leq d - 1 \), for all \( n \in \mathbb{Z} \), for all \( s \in S \), \( f dn + i, s) = (a_i n + b_i, p(i, s)) \) or is undefined.

**Definition 1.4** If, in the definitions above, \( a_i = a \) for all \( i, 0 \leq i \leq d - 1 \), we say that \( f \) is pure Collatz-like of type \( d \rightarrow a \).

We also need to define a new type of function, as follows.

**Definition 1.5** A partial function \( f : \mathbb{Z} \rightarrow \mathbb{Z} \) is an exponential Collatz-like function if there exist integers \( d, p \geq 2 \), integers \( a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1}, c_0, \ldots, c_{d-1} \), such that, for all \( i, 0 \leq i \leq d - 1 \), all \( n \in \mathbb{Z} \), \( f d n + i = (a_i n^p + b_i)/c_i \) or is undefined. In this definition, integers \( p, a_i, b_i, c_i \) are chosen such that \((a_i n^p + b_i)/c_i \) is an integer for all \( n \in \mathbb{Z} \).

Currently, no study of this type of function is known. Note that iterates \( f(n), f^2(n), \ldots \) grow much faster than for pure Collatz-like functions.

### 1.5 From Collatz-like functions to high scores

The Turing machines studied in this article have behaviors modeled on iterations of functions, where halting configurations correspond to undefined values of functions.

In Section 3, we present a \( 3 \times 3 \) Turing machine \( M_1 \) whose behavior is pure Collatz-like, of type \( 8 \rightarrow 14 \). In Section 4, we present a \( 2 \times 4 \) Turing machine \( M_2 \) whose behavior is pure Collatz-like with parameter, of type \( 3 \rightarrow 5 \). In Section 5, we present a \( 2 \times 5 \) Turing machine \( M_3 \) whose behavior is pure Collatz-like with parameter, of type \( 2 \rightarrow 3 \). Thus, the halting problem for machines \( M_1, M_2 \) and \( M_3 \) depends on an open problem about iterating Collatz-like functions.

In Section 6, we present a \( 6 \times 2 \) Turing machine \( M_4 \) whose behavior is exponential Collatz-like.

In Section 7, we present a \( 6 \times 2 \) Turing machine \( M_5 \) whose behavior depends on iterating a partial function \( g_5(n, p) \). Without being Collatz-like, this function seems to share some properties with Collatz-like functions.

In Section 8, we present a \( 6 \times 2 \) Turing machine \( M_6 \) whose behavior looks like a loosely Collatz-like behavior with parameter, of type \( 2 \rightarrow 5 \). The novelty is that a potentially infinite set of rules seems to be necessary to describe the behavior of the machine on inputs \( 00x, x \in \{0, 1\}^* \). A string \( x \in \{0, 1\}^* \) ending with symbol 1 can be taken as the binary writing of a number \( p \), read in the opposite direction, so \( x = R(\text{bin}(p)) \), where \( \text{bin}(p) \) is the usual binary
writing of $p$, and $R(w)$ is the reverse of string $w$, that is $R(w_1 \ldots w_n) = w_n \ldots w_1$. In Table 2 we write "R(bin$(p)$)" to indicate the machines with a behavior involving an infinite set of rules. Of course, only a finite subset of these rules are used when the machine is launched on a blank tape.

In Section 9, we present a $6 \times 2$ Turing machine $M_2$ whose behavior on the blank tape depends on configurations $C(n)$ all of them provably leading to a halting configuration. We present such a machine to show how a Turing machine can take a long time to stop without calling for Collatz-like functions.

## 2 Preliminaries

A Turing machine involved in the busy beaver competition is defined as follows. It has a tape made of cells, infinite in both directions. Each cell contains a symbol, and a head can read and write a symbol on a cell. The Turing machine can be in a finite number of states. A computation of the Turing machine is a sequence of steps. In a step of computation, according to the current state and the symbol read by the head on the current cell, the head writes a symbol on the cell, moves to the next cell on the right side or on the left side, and the machine enters a new state.

Formally, a Turing machine $M = (Q, \Gamma, \delta)$ has a finite set of states $Q = \{A, B, \ldots\}$, a finite set of symbols $\Gamma = \{0, 1, \ldots\}$, and a transition function (or next move function) $\delta$, which is a mapping

$$\delta : Q \times \Gamma \to (\Gamma \times \{L, R\} \times Q) \cup \{(1, R, H)\}.$$  

If $\delta(q,a) = (b, D, q')$, then the Turing machine, when it is in state $q$ reading symbol $a$ on the current cell, writes symbol $b$ instead of $a$ on this cell, moves one cell left if $D = L$, one cell right if $D = R$, and comes in state $q'$. The transition function is usually given by a transition table.

There a special state $A$, called the initial state, and a special symbol 0, called the blank symbol. In the busy beaver competition, at the beginning of a computation, the Turing machine is in state $A$, and the tape is blank, that is all the cells of the tape contain the blank symbol. There is another state $H$, the halting state, not in the set $Q$ of states. When a Turing machine comes in this state, the computation stops. We impose that, at the last step, the machine writes 1, moves right, and comes in state $H$.

A word is a finite string of symbols. The set of words with symbols in the set $\Gamma$ is denoted by $\Gamma^*$. The number of symbols in a word $x \in \Gamma^*$ is called the length of $x$ and is denoted by $|x|$. The empty word is the word of length zero, denoted by $\lambda$. If $x \in \Gamma^*$, and $n \geq 0$, $x^n$ is the word $xx \ldots x$, where $x$ is repeated $n$ times, that is, formally: $x^0 = \lambda$, $x^1 = x$ and $x^{n+1} = x^n x$. An infinite to the left string of 0 is denoted by $\omega 0$, and an infinite to the right string of 0 is denoted by $0^\omega$.

A configuration is a way to encode the symbols on the tape, the state, and the cell currently read by the head. The Turing machine is in configuration $\omega 0x(Sa)y0^\omega$, with $S \in Q \cup \{H\}$, $a \in \Gamma$, $x, y \in \Gamma^*$, if the word $xay$ is written on the tape, the state is $S$, and the head is reading symbol $a$. Since, at the beginning of the computation, the state is $A$ and the tape is blank, the initial configuration is $\omega 0(A0)0^\omega$. If the state is $H$, the configuration is halting. We also consider configurations $x(Sa)y$ with finite length. If the computation from
<table>
<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 1983 Uwe Schult</td>
<td>Pure Collatz-like (4 → 9)</td>
<td>Robinson, cited in [9].</td>
</tr>
<tr>
<td></td>
<td>without parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7 transitions</td>
<td></td>
</tr>
<tr>
<td>December 1984 George Uhing</td>
<td>Pure Collatz-like (3 → 8)</td>
<td>Michel [30]</td>
</tr>
<tr>
<td></td>
<td>with parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>9 transitions</td>
<td></td>
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<tr>
<td>February 1986 George Uhing</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>without parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>5 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>11 transitions</td>
<td></td>
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<tr>
<td>August 1989 Marxen, Buntrock</td>
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<td>Michel [30]</td>
</tr>
<tr>
<td></td>
<td>without parameter</td>
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</tr>
<tr>
<td></td>
<td>3 rules</td>
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<tr>
<td></td>
<td>15 transitions</td>
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<td>3 rules</td>
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<td>15 transitions</td>
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<tr>
<td>September 1989 Marxen, Buntrock</td>
<td>Pure Collatz-like (3 → 5)</td>
<td>Michel [31]</td>
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<tr>
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<tr>
<td></td>
<td>3 rules</td>
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</tr>
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<td></td>
<td>15 transitions</td>
<td></td>
</tr>
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</table>

Table 1: Study of behavior of $5 \times 2$ machines
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<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>September 1997 Marxen and Buntrock $s &gt; 8.6 \times 10^{15}$</td>
<td>Pure Collatz-like (4 → 10) without parameter 5 rules 21 transitions</td>
<td>Munafo [36]</td>
</tr>
<tr>
<td>October 2000 Marxen and Buntrock (machine o) $s &gt; 6.1 \times 10^{119}$</td>
<td>R(bin(p)) (2 → 3) 9 rules 337 transitions</td>
<td></td>
</tr>
<tr>
<td>October 2000 Marxen and Buntrock (machine q) $s &gt; 6.1 \times 10^{925}$</td>
<td>All C(n) stop 4 rules 5 transitions</td>
<td>Munafo [37] Section 9</td>
</tr>
<tr>
<td>March 2001 Marxen and Buntrock $s &gt; 3.0 \times 10^{1730}$</td>
<td>R(bin(p)) (2 → 3) 20 rules 4911 transitions</td>
<td>Tooth [40]</td>
</tr>
<tr>
<td>November 2007 T. and S. Ligocki $s &gt; 8.9 \times 10^{1762}$</td>
<td>R(bin(p)) (2 → 5) 12 rules 3346 transitions</td>
<td>Section 8</td>
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<td>December 2007 T. and S. Ligocki $s &gt; 2.5 \times 10^{2879}$</td>
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<tr>
<td>May 2010 Pavel Kropitz $s &gt; 3.8 \times 10^{21132}$</td>
<td>Unclassifiable 6 rules 22158 transitions</td>
<td>S. Ligocki [24] Section 7</td>
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<tr>
<td>June 2010 Pavel Kropitz $s &gt; 7.4 \times 10^{36534}$</td>
<td>Exponential Collatz-like without parameter 4 rules 5 transitions</td>
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Table 2: Study of behavior of $6 \times 2$ machines
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<tr>
<td>December 2004 Brady</td>
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<td>$s = 92,649,163$</td>
<td>with parameter 5 rules</td>
<td>11 transitions</td>
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<tr>
<td>July 2005 Souris</td>
<td>Pure Collatz-like (2 → 5)</td>
<td></td>
</tr>
<tr>
<td>$\sigma = 36089$</td>
<td>with parameter 5 rules</td>
<td>12 transitions</td>
</tr>
<tr>
<td>$s = 310,341,163$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>July 2005 Souris</td>
<td>Pure Collatz-like (3 → 7)</td>
<td></td>
</tr>
<tr>
<td>$s = 544,884,219$</td>
<td>with parameter 7 rules</td>
<td>12 transitions</td>
</tr>
<tr>
<td>August 2005 Lafitte and Papazian</td>
<td>Pure Collatz-like (4 → 7)</td>
<td></td>
</tr>
<tr>
<td>$s &gt; 4.9 \times 10^9$</td>
<td>with parameter 8 rules</td>
<td>21 transitions</td>
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<td>$s &gt; 9.8 \times 10^{11}$</td>
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<td>24 transitions</td>
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<td>April 2006 Lafitte and Papazian</td>
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<tr>
<td>$s &gt; 4.1 \times 10^{12}$</td>
<td>with parameter 5 rules</td>
<td>16 transitions</td>
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<td>August 2006 T. and S. Ligocki</td>
<td>Pure Collatz-like (2 → 5)</td>
<td>S. Ligocki [23]</td>
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<tr>
<td>$s &gt; 4.3 \times 10^{15}$</td>
<td>with parameter 4 rules</td>
<td>20 transitions</td>
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<td>Section 3</td>
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<td>$s &gt; 1.1 \times 10^{17}$</td>
<td>without parameter 9 rules</td>
<td>34 transitions</td>
</tr>
</tbody>
</table>

Table 3: Study of behavior of $3 \times 3$ machines
<table>
<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>1988 Brady</td>
<td>Pure Collatz-like (3 → 5) with parameter</td>
<td></td>
</tr>
<tr>
<td></td>
<td>6 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>7 transitions</td>
<td></td>
</tr>
<tr>
<td>February 2005</td>
<td>Pure Collatz-like (3 → 5) with parameter</td>
<td>Section 4</td>
</tr>
<tr>
<td>T. and S. Ligocki</td>
<td>7 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>14 transitions</td>
<td></td>
</tr>
</tbody>
</table>

Table 4: Study of behavior of $2 \times 4$ machines

<table>
<thead>
<tr>
<th>Machine</th>
<th>Behavior</th>
<th>Study of behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>October 2005</td>
<td>Pure Collatz-like (2 → 5) with parameter</td>
<td></td>
</tr>
<tr>
<td>Lafitte, Papazian</td>
<td>7 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>15 transitions</td>
<td></td>
</tr>
<tr>
<td>December 2005</td>
<td>Pure Collatz-like (2 → 5) with parameter</td>
<td></td>
</tr>
<tr>
<td>Lafitte, Papazian</td>
<td>5 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>14 transitions</td>
<td></td>
</tr>
<tr>
<td>May 2006</td>
<td>Pure Collatz-like (3 → 4) with parameter</td>
<td></td>
</tr>
<tr>
<td>Lafitte, Papazian</td>
<td>7 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>45 transitions</td>
<td></td>
</tr>
<tr>
<td>June 2006</td>
<td>Pure Collatz-like (2 → 3) with parameter</td>
<td></td>
</tr>
<tr>
<td>Lafitte, Papazian</td>
<td>9 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>36 transitions</td>
<td></td>
</tr>
<tr>
<td>August 2006</td>
<td>Pure Collatz-like (2 → 5) with parameter</td>
<td>S. Ligocki [22]</td>
</tr>
<tr>
<td>T. and S. Ligocki</td>
<td>9 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>30 transitions</td>
<td></td>
</tr>
<tr>
<td>November 2007</td>
<td>Pure Collatz-like (2 → 3) with parameter</td>
<td>Section 5</td>
</tr>
<tr>
<td>T. and S. Ligocki</td>
<td>17 rules</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2002 transitions</td>
<td></td>
</tr>
</tbody>
</table>

Table 5: Study of behavior of $2 \times 5$ machines
configuration $C_1$ to configuration $C_2$ takes $t$ steps, we write $C_1 \vdash (t) C_2$, and $t$ is said to be the time taken by the machine to go from $C_1$ to $C_2$. If $C_2$ is a halting configuration, we also write $C_1 \vdash ( )$ END. We write $C_1 \vdash ( ) C_2$ if the time is not specified. If $C_1$ and $C_2$ are configurations with finite length, then they refer to the same part of the tape. For example, $(A 0) 0 \vdash (1) 1 (B 0)$ if $\delta(A, 0) = (1, R, B)$.

A Turing machine $M$ computes a partial function $f_M : \Gamma^* \rightarrow \Gamma^*$ as follows. Let $x = x_1 \ldots x_n \in \Gamma^*$. Then $x$ becomes an input for $M$ by considering the computation of $M$ on initial configuration $^0 (A x_1) x_2 \ldots x_n 0^*$. If $M$ never stops on this configuration, then $f_M(x)$ is undefined. If $M$ stops, in configuration $^0 y (H a) z 0^*$, with $a \in \Gamma, y, z \in \Gamma^*$, then the output $f_M(x)$ is defined from this configuration by a suitable convention. The halting set is $\{ x \in \Gamma^* : f_M(x) \text{ is defined} \}$. The halting problem for machine $M$ is the problem consisting in determining the halting set. Note that the Turing machines with two symbols 0 and 1 are powerful enough to compute any computable function, and their halting sets can be any computably enumerable (also called recursively enumerable) set.

A Turing machine with $n$ states and $m$ symbols is called an $n \times m$ machine. The set of $n \times m$ machines is denoted by $\text{TM}(n, m)$. With our definition of the transition function, there are $(2nm + 1)^{nm}$ machines in the set $\text{TM}(n, m)$. In the busy beaver competition, for fixed numbers of states $n$ and symbols $m$, all the $(2nm + 1)^{nm}$ Turing machines in $\text{TM}(n, m)$ are launched on the blank tape. Some of them never stop. Those which stop are called busy beavers. Each busy beaver takes some time to stop, and leaves some non-blank symbols on the tape, so busy beavers are involved in two competitions: to take the longest time before stopping, and to leave the greatest number of non-blank symbols on the tape when stopping. The time taken by Turing machine $M$ to stop is denoted by $s(M)$, and the number of non-blank symbols left by $M$ when it stops is denoted by $\sigma(M)$. The busy beaver functions are defined by

$$S(n, m) = \max \{ s(M) : M \text{ is a busy beaver with } n \text{ states and } m \text{ symbols} \}$$

$$\Sigma(n, m) = \max \{ \sigma(M) : M \text{ is a busy beaver with } n \text{ states and } m \text{ symbols} \}$$

Rado [38] initially defined functions $S(n) = S(n, 2)$ and $\Sigma(n) = \Sigma(n, 2)$ for Turing machines with $n$ states and two symbols.

## 3 Pure Collatz-like behavior

Let $M_1$ be the $3 \times 3$ Turing machine defined by Table 6.

We have $s(M_1) = 119,112,334,170,342,540$ and $\sigma(M_1) = 374,676,383$.

This machine is the current champion for the busy beaver competition for $3 \times 3$ machines. It was discovered in November 2007 by Terry and Shawn Ligocki, who wrote (email on

<table>
<thead>
<tr>
<th>$M_1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1RB</td>
<td>2LA</td>
<td>1LC</td>
</tr>
<tr>
<td>B</td>
<td>0LA</td>
<td>2RB</td>
<td>1LB</td>
</tr>
<tr>
<td>C</td>
<td>1RH</td>
<td>1RA</td>
<td>1RC</td>
</tr>
</tbody>
</table>

Table 6: Machine $M_1$ discovered in November 2007 by T. and S. Ligocki
November, 9th) that they enumerated all the $3 \times 3$ machines and applied the techniques of acceleration and proof systems originally developed by Marxen and Buntrock.

The following theorem gives the rules that enable Turing machine $M_1$ to reach a halting configuration from a blank tape.

**Theorem 3.1** Let $C(n) = \omega^0(A0)2^n0^\omega$. Then

(a) $\omega^0(A0)0^\omega \vdash (3 \ C(1))$,

and, for all $k \geq 0$,

(b) $C(8k + 1) \vdash (112k^2 + 116k + 13) \ C(14k + 3)$,

(c) $C(8k + 2) \vdash (112k^2 + 144k + 38) \ C(14k + 7)$,

(d) $C(8k + 3) \vdash (112k^2 + 172k + 54) \ C(14k + 8)$,

(e) $C(8k + 4) \vdash (112k^2 + 200k + 74) \ C(14k + 9)$,

(f) $C(8k + 5) \vdash (112k^2 + 228k + 97) \omega^0(1H1)2^{14k+9}0^\omega$,

(g) $C(8k + 6) \vdash (112k^2 + 256k + 139) \ C(14k + 14)$,

(h) $C(8k + 7) \vdash (112k^2 + 284k + 169) \ C(14k + 15)$,

(i) $C(8k + 8) \vdash (112k^2 + 312k + 203) \ C(14k + 16)$.

**Proof.** A direct inspection of the transition table gives

(1) $0(A0)0 \vdash (3 \ (A0)20)$,

(2) $0^3(A0)2^5 \vdash (53 \ (B1)1^8)$,

(3) $0(A1) \vdash (1) \ (A0)2$,

(4) $1(A1) \vdash (1) \ (A1)2$,

(5) $02(A1) \vdash (3) \ 1(H1)2$,

(6) $12(A1) \vdash (4) \ (A1)22$,

(7) $22(A1) \vdash (8) \ (A1)22$,

(8) $2(B1)0^2 \vdash (7) \ 11(A1)0$,

(9) $(B1)1 \vdash (1) \ 2(B1)$,

(10) $(B1)2 \vdash (1) \ 2(B2)$,

(11) $0^3(B2) \vdash (14) \ 1^3(B1)$,

(12) $1(B2) \vdash (1) \ (B1)1$,

(13) $2(B2) \vdash (1) \ (B2)1$.

From this point, $k$ will be an integer, $k \geq 0$.

Iterating, respectively, (4), (7), (9) and (13) gives

(14) $1^k(A1) \vdash (k) \ (A1)2^k$,

(15) $2^k(A1) \vdash (8k) \ (A1)2^{2k}$,

(16) $(B1)1^k \vdash (k) \ 2^k(B1)$,

(17) $2^k(B2) \vdash (k) \ (B2)1^k$.

Using consecutively (16), (10), (17) and (12), we get

(18) $1(B1)1^k2 \vdash (2k + 3) \ (B1)1^{k+2}$.

Using (16), (10), (17) and (11), we get

(19) $0^3(B1)1^k2 \vdash (2k + 16) \ 1^3(B1)1^{k+1}$.

Using (19) and three times (18), we get

(20) $0^3(B1)1^k2^4 \vdash (8k + 43) \ (B1)1^{k+7}$.

For any $n \geq 0$, by induction on $k$, using (20), we get

(21) $0^{3k}(B1)1^n2^{4k} \vdash (28k^2 + (8n + 15)k) \ (B1)1^{7k+n}$.
By taking \( n = 8 \) in (21), we get
\[(22) \quad 0 \vdash (B1)N^2 {\uparrow} (28k^2 + 79k) \quad (B1)17k+8.\]

Using (8), (14) and (15), we get
\[(23) \quad 2^2k+1(B1)0^2 \vdash (8k + 9) \quad (A1)2^2k+2.0.\]

We are now ready to prove the results of the theorem.

Using (2), (22) and (16), we get
\[(24) \quad 0 \vdash (A0)2^2k+5 \vdash (28k^2 + 86k + 61) \quad 2^7k+8(B1).\]

Using (24), (23) and (5) we get
\[(25) \quad 0^3k^2+4(A0)2^3k+50^2 \vdash (112k^2 + 228k + 97) \quad 1(H1)2^4k+9.0.\]

Using (24), (23) and (3) we get
\[0^6k+7(A0)2^6k+90^2 \vdash (112k^2 + 340k + 241) \quad (A0)2^{4k+17}0,\]
and this result is still true for \( k = -1 \), so we have
\[(26) \quad 0^6k+1(A0)2^6k+10^2 \vdash (112k^2 + 116k + 13) \quad (A0)2^{4k+3}0.\]

Using (2), (22), (19) and (16) we get
\[(27) \quad 0^3k^2+6(A0)2^3k+6 \vdash (28k^2 + 100k + 94) \quad 1^22^7k+9(B1).\]

Using (27), (23), (14) and (3) we get
\[(28) \quad 0^6k+7(A0)2^6k+60^2 \vdash (112k^2 + 256k + 139) \quad (A0)2^{4k+14}.0.\]

Using (27), (23), (6), (14) and (3) we get
\[0^6k+10(A0)2^6k+100^2 \vdash (112k^2 + 368k + 294) \quad (A0)2^{4k+21}0,\]
and this result is still true for \( k = -1 \), so we have
\[(29) \quad 0^6k+4(A0)2^6k+20^2 \vdash (112k^2 + 144k + 38) \quad (A0)2^{4k+7}0.\]

Using (2), (22), (19), (18), (16), (8) and (14) we get
\[(30) \quad 0^3k^2+6(A0)2^3k+60^2 \vdash (28k^2 + 114k + 126) \quad 1^22^7k+10(A1)220.\]

Using (30), (15), (14) and (3) we get
\[(31) \quad 0^6k+7(A0)2^6k+70^2 \vdash (112k^2 + 284k + 169) \quad (A0)2^{4k+15}0.\]

Using (30), (15), (6), (4) and (3) we get
\[0^6k+10(A0)2^6k+110^2 \vdash (112k^2 + 396k + 338) \quad (A0)2^{4k+22}0,\]
and this result is still true for \( k = -1 \), so we have
\[(32) \quad 0^6k+4(A0)2^6k+30^2 \vdash (112k^2 + 172k + 54) \quad (A0)2^{4k+8}0.\]

Using (2), (22), (19), (18), (16), (8) and (14) we get
\[(33) \quad 0^3k^2+6(A0)2^3k+50^2 \vdash (28k^2 + 128k + 153) \quad 12^7k+12(A1)220.\]

Using (33), (15), (4) and (3) we get
\[(34) \quad 0^6k+7(A0)2^6k+90^2 \vdash (112k^2 + 312k + 203) \quad (A0)2^{4k+16}0.\]

Using (33), (15), (6) and (3) we get
\[0^6k+10(A0)2^6k+120^2 \vdash (112k^2 + 424k + 386) \quad (A0)2^{4k+23}0,\]
and this result is still true for \( k = -1 \), so we have
\[(35) \quad 0^6k+4(A0)2^6k+40^2 \vdash (112k^2 + 200k + 74) \quad (A0)2^{4k+9}0.\]

The results (1), (26), (29), (32), (35), (25), (28), (31) and (34) give, respectively, the results (a)–(i) of the theorem. \( \square \)

Using the rules of this theorem, we have, in 34 transitions,
\[\omega^0(A0)0^\omega \vdash (3) \quad C(1) \vdash (13) \quad C(3) \vdash ( ) \ast \vdash ( ) \quad \omega^0(1(H1)2^{374.676.381}0^\omega.\]
Let \( g_1 \) be the pure Collatz-like function defined by: for \( k \geq 0 \),
\[
\begin{align*}
g_1(8k + 1) &= 14k + 3, \\
g_1(8k + 2) &= 14k + 7, \\
g_1(8k + 3) &= 14k + 8, \\
g_1(8k + 4) &= 14k + 9, \\
g_1(8k + 5) &= \text{undefined,} \\
g_1(8k + 6) &= 14k + 14, \\
g_1(8k + 7) &= 14k + 15, \\
g_1(8k + 8) &= 14k + 16.
\end{align*}
\]

Then \( g_1^{33}(1) \) is undefined.

The theorem gives immediately the following proposition.

**Proposition 3.2** The behavior of Turing machine \( M_1 \), on inputs \( 02^n, n \geq 1 \), depends on the behavior of iterated \( g_1^k(n) \), \( k \geq 1 \).

Since the behavior of iterated \( g_1^k(n) \) is an open problem in mathematics, the halting problem for Turing machine \( M_1 \) is so.

Let \( h_1(n) = \min\{k : g_1^k(n) \text{ is undefined}\} \). We have seen that \( h_1(1) = 33 \). We also have \( h_1(144) = 41 \), \( h_1(270) = 51 \).

## 4 Collatz-like with parameter: first example

Let \( M_2 \) be the 2 \( \times \) 4 Turing machine defined by Table 7.

We have \( s(M_2) = 3,932,964 \) and \( \sigma(M_2) = 2050 \).

This machine is the current champion for the busy beaver competition for 2 \( \times \) 4 machines. It was discovered in February 2005 by Terry and Shawn Ligocki, who wrote (email on February, 13th) that they found this machine using simulated annealing.

The following theorem gives the rules that enable Turing machine \( M_2 \) to reach a halting configuration from a blank tape.

**Theorem 4.1** Let
\[
\begin{align*}
C(n, 1) &= \omega^{0(A0)2^n10^n}, \\
C(n, 2) &= \omega^{0(A0)2^n110^n}.
\end{align*}
\]

Then
\[
(a) \quad \omega^{0(A0)0^n} \vdash (6) \quad C(1, 2),
\]
and, for all \( k \geq 0 \),
\[(b)\] \[C(3k, 1) \vdash (15k^2 + 9k + 3) \quad C(5k + 1, 1),\]
\[(c)\] \[C(3k + 1, 1) \vdash (15k^2 + 24k + 13) \quad \vdash 013k^{k+1}(H1)0^\omega,\]
\[(d)\] \[C(3k + 2, 1) \vdash (15k^2 + 29k + 17) \quad C(5k + 4, 2),\]
\[(e)\] \[C(3k, 2) \vdash (15k^2 + 11k + 3) \quad C(5k + 1, 2),\]
\[(f)\] \[C(3k + 1, 2) \vdash (15k^2 + 21k + 7) \quad C(5k + 3, 1),\]
\[(g)\] \[C(3k + 2, 2) \vdash (15k^2 + 36k + 23) \quad \vdash 013k^{k+1}(H1)0^\omega.\]

**Proof.** A direct inspection of the transition table gives
\[(1)\] \[0^2(A0)0 \vdash (6) \quad (A0)211,\]
\[(2)\] \[1(A0)0 \vdash (3) \quad (A1)11,\]
\[(3)\] \[(A0)2 \vdash (1) \quad (1B2),\]
\[(4)\] \[0(A1) \vdash (1) \quad (A0)2,\]
\[(5)\] \[1(A1) \vdash (1) \quad (A1)2,\]
\[(6)\] \[3(A1) \vdash (2) \quad 1(A2),\]
\[(7)\] \[(A2)0 \vdash (1) \quad (A0),\]
\[(8)\] \[(A2)1 \vdash (1) \quad (A1),\]
\[(9)\] \[(A2)2 \vdash (1) \quad (A2),\]
\[(10)\] \[(B2)0 \vdash (3) \quad (H1),\]
\[(11)\] \[(B2)1 \vdash (4) \quad (A1),\]
\[(12)\] \[(B2)2 \vdash (1) \quad (3(B2)).\]

Iterating, respectively, (5), (9) and (12) gives
\[(13)\] \[1^k(A1) \vdash (k) \quad (A1)2^k,\]
\[(14)\] \[(A2)2^k \vdash (k) \quad 1^k(A2),\]
\[(15)\] \[(B2)2^k \vdash (k) \quad 3^k(B2).\]

Using (2), (13) and (4) we get
\[01^{k+1}(A0)0 \vdash (k + 4) \quad (A0)2^{k+1}11,\]
and this result is still true for \(k = -1\), so we have
\[(16)\] \[01^k(A0)0 \vdash (k + 3) \quad (A0)2^k11.\]

Using (2), (13), (6), (14), (8), (13) and (4) we get
\[(17)\] \[013^{k+1}(A0)0 \vdash (3k + 10) \quad (A0)2^{k+1}1.\]

Using (2), (13), (6), (14), (8), (13), (6), (14), (8), (13) and (4) we get
\[(18)\] \[0133^{k+1}(A0)0 \vdash (5k + 19) \quad (A0)2^{k+6}.\]

Using (3), (15) and (10) we get
\[(19)\] \[(A0)2^{k+1}0 \vdash (k + 4) \quad 13k1(H1).\]

Using (3), (15), (11), (6), (9) and (7) we get
\[(20)\] \[(A0)2^{k+2}10 \vdash (k + 10) \quad 13^k13(A0).\]

Using (3), (15), (11), (6), (9), (8), (13), (6), (14) and (7) we get
\[(21)\] \[(A0)2^{k+3}110 \vdash (k + 20) \quad 13^{k+1}(A0).\]

Using (2), (13), (6), (14), (8), (13), (6), (14), (8), (13), (6), (14) and (7) we get
\[(22)\] \[3^11^{k+1}(A0)0^2 \vdash (6k + 24) \quad 1^{k+6}(A0).\]

By induction on \(k\), from (22), we get
\[(23)\] \[3^{1n+1}(A0)0^2 \vdash (15k^2 + 6nk + 9k) \quad 1^{5k+n+1}(A0),\]
so we have, for \(n = 2\) and \(n = 4\)
We are now ready to prove the theorem.

Using (20), (24) and (17) we get
\[ 0(A0)^{2k+3}10^{2k+2} \vdash (15k^2 + 39k + 27) \] (A0)^{2k+6}1, 
and the result is still true for \( k = -1 \), so we have
\[ 0(A0)^{2k+3}10^{2k} \vdash (15k^2 + 9k + 3) \] (A0)^{2k+1}. 

Using (20), (24), (18) and (19) we get
\[ 0(A0)^{2k+4}10^{2k+3} \vdash (15k^2 + 54k + 52) \] 13^{5k+7}1(H1), 
and the result is still true for \( k = -1 \), so we have
\[ 0(A0)^{2k+1}10^{2k+1} \vdash (15k^2 + 24k + 13) \] 13^{5k+21}(H1). 

Using (20), (24) and (16) we get
\[ 0(A0)^{2k+2}10^{2k+2} \vdash (15k^2 + 29k + 17) \] (A0)^{2k+4}11. 

Using (21), (25) and (16) we get
\[ 0(A0)^{2k+3}110^{2k+2} \vdash (15k^2 + 41k + 29) \] (A0)^{2k+6}11, 
and the result is still true for \( k = -1 \), so we have
\[ 0(A0)^{2k+1}110^{2k} \vdash (15k^2 + 11k + 3) \] (A0)^{2k+1}11. 

Using (21), (25) and (17) we get
\[ 0(A0)^{2k+4}110^{2k+2} \vdash (15k^2 + 51k + 43) \] (A0)^{2k+8}1, 
and the result is still true for \( k = -1 \), so we have
\[ 0(A0)^{2k+1}110^{2k} \vdash (15k^2 + 21k + 7) \] (A0)^{2k+3}1. 

Using (21), (25), (18) and (19) we get
\[ 0(A0)^{2k+5}110^{2k+3} \vdash (15k^2 + 66k + 74) \] 13^{5k+9}1(H1), 
and the result is still true for \( k = -1 \), so we have
\[ 0(A0)^{2k+2}110^{2k+1} \vdash (15k^2 + 36k + 23) \] 13^{5k+4}1(H1). 

The theorem comes from results (1) and (26)–(31). \( \square \)

Using the rules of this theorem, we have, in 14 transitions,
\[ \omega 0(A0)^{2} \vdash (6) \ \ C(1, 2) \vdash (7) \ \ C(3, 1) \vdash (\cdot \cdot \cdot \vdash (\omega 13^{2047}1(H1)0^\omega). \]

Let \( g_2 \) be the pure Collatz-like function with parameter defined by: for \( k \geq 0 \),
\[
\begin{align*}
g_2(3k, 1) &= (5k + 1, 1), \\
g_2(3k + 1, 1) &= \text{undefined}, \\
g_2(3k + 2, 1) &= (5k + 4, 2), \\
g_2(3k, 2) &= (5k + 1, 2), \\
g_2(3k + 1, 2) &= (5k + 3, 1), \\
g_2(3k + 2, 2) &= \text{undefined}.
\end{align*}
\]

Then \( g_1^{13}(1, 2) \) is undefined.
The theorem gives immediately the following proposition.

**Proposition 4.2** The behavior of Turing machine \( M_2 \), on inputs \( 02^n1^i, \ n \geq 1, \ i \in \{1, 2\} \) depends on the behavior of iterated \( g_2^k(n, i), \ k \geq 1 \).
Table 8: Machine $M_3$ discovered in November 2007 by T. and S. Ligocki

Since the behavior of iterated $g^k_2(n, i)$ is an open problem in mathematics, the halting problem for Turing machine $M_2$ is so.

Let $h_2(n, i) = \min\{k : g^k_2(n, i) \text{ is undefined}\}$. We have seen that $h_2(1, 2) = 13$. We also have $h_2(137, 1) = 16$, $h_2(210, 2) = 20$.

5 Collatz-like with parameter: second example

Let $M_3$ be the $2 \times 5$ Turing machine defined by Table 8.

We have $s(M_3) > 1.9 \times 10^{704}$ and $\sigma(M_3) > 1.7 \times 10^{352}$.

This machine is the current champion for the busy beaver competition for $2 \times 5$ machines. It was discovered in November 2007 by Terry and Shawn Ligocki, who wrote (email on November, 9th) that, as they did for $3 \times 3$ machine $M_1$, they enumerated all the $2 \times 5$ machines and applied the techniques of acceleration and proof systems originally developed by Marxen and Buntrock.

The following theorem gives the rules that enable Turing machine $M_3$ to reach a halting configuration from a blank tape.

Theorem 5.1 Let

$C(n, 1) = \omega 013^n (B0)0\omega$, 
$C(n, 2) = \omega 023^n (B0)0\omega$, 
$C(n, 3) = \omega 03^n (B0)0\omega$, 
$C(n, 4) = \omega 04113^n (B0)0\omega$, 
$C(n, 5) = \omega 04123^n (B0)0\omega$, 
$C(n, 6) = \omega 0413^n (B0)0\omega$, 
$C(n, 7) = \omega 0423^n (B0)0\omega$, 
$C(n, 8) = \omega 043^n (B0)0\omega$.

Then

(a) $\omega 0(A0)0\omega \vdash (1) C(0, 1)$, 
and, for all $k \geq 0$, 

and the result is still true for 

Using (9), (22), (6), (19), (23) and (7) we get 

and the result is still true for }k = -1\text{, so we have}
Using (28), (21) and (29) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (22) and (6) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (30), (5), (6), (13) and (29) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (14), (5), (6), (13) and (29) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (13) and (29) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (24) and (11) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (3), (24) and (11) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (14), (4) and (29) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (14), (4) and (29) we get
\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (14), (5), (6), (13) and (29) we get
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\[ k \quad \text{and the result is still true for} \quad k \]
Using (28), (14), (5), (6), (13) and (29) we get
\[ k \quad \text{and the result is still true for} \quad k \]
We are now ready to prove the theorem.
Results (1) and (31)–(46) give results (a)–(p) of the theorem. □

Using (28), (17), (24) and (11) we get
\[13^{3k+4}(B0)0^{k+1} \vdash (3k^2 + 14k + 15) \quad 13^{3k+1}(B0),\]
and the result is still true for \(k = -1\), so we have
\[(38) \quad 043^{2k}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 13^{3k+1}(B0).
\]

Using (28), (16), (2) and (29) we get
\[0^{3,2k+3}(B0)0^{k+1} \vdash (3k^2 + 14k + 33) \quad 41^23^{3k+4}(B0),\]
and the result is still true for \(k = -1\), so we have
\[(41) \quad 04113^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 22) \quad 41^23^{3k+4}(B0).
\]

Using (39) we get
\[(42) \quad 04113^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 04123^{3k+1}(B0).
\]

Using (40) we get
\[(43) \quad 04123^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 0413^{3k+2}(B0).
\]

Using (39) we get
\[(44) \quad 0413^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 0423^{3k+1}(B0).
\]

Using (40) we get
\[(45) \quad 0423^{2k+1}(B0)0^k \vdash (3k^2 + 8k + 4) \quad 043^{3k+2}(B0).
\]

Using (28) and (20) we get
\[43^{3k+3}(B0)0^{k+1} \vdash (3k^2 + 11k + 11) \quad 1(H2)^23^{3k+30},\]
and the result is still true for \(k = -1\), so we have
\[(46) \quad 43^{2k+1}(B0)0^k \vdash (3k^2 + 5k + 3) \quad 1(H2)^23^{3k}0.
\]

Results (1) and (31)–(46) give results (a)–(p) of the theorem. □

Using the rules of this theorem, we have, in 2002 transitions,
\[\omega(0, A0)0^\omega \vdash (1) \quad C(0, 1) \vdash (4) \quad C(1, 1) \vdash (\ldots \vdash (\ldots END.
\]

Let \(g_3\) be the pure Collatz-like function with parameter defined by: for \(k \geq 0,\)
\[
g_3(2k, 1) = (3k + 1, 1) \quad \quad g_3(2k, 1) = (3k + 1, 2)
g_3(2k, 2) = (3k + 2, 1) \quad \quad g_3(2k, 2) = (3k + 2, 2)
g_3(2k, 3) = (3k, 1) \quad \quad g_3(2k, 3) = (3k + 1, 4)
g_3(2k, 4) = (3k + 3, 1) \quad \quad g_3(2k, 4) = (3k + 1, 5)
g_3(2k, 5) = (3k + 4, 1) \quad \quad g_3(2k, 5) = (3k + 2, 6)
g_3(2k, 6) = (3k + 2, 1) \quad \quad g_3(2k, 6) = (3k + 1, 7)
g_3(2k, 7) = (3k + 3, 1) \quad \quad g_3(2k, 7) = (3k + 2, 8)
g_3(2k, 8) = (3k + 1, 1) \quad \quad g_3(2k + 1, 8) \quad \text{undefined}
\]

Then \(g_3^{2001}(0, 1)\) is undefined.
Table 9: Machine $M_4$ discovered in June 2010 by P. Kropitz

<p>| | | |</p>
<table>
<thead>
<tr>
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<tr>
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<td>E</td>
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<td>0RD</td>
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<tr>
<td>F</td>
<td>1RH</td>
<td>1RC</td>
</tr>
</tbody>
</table>

Proposition 5.2 The behavior of Turing machine $M_3$, on inputs $02^n$, $n \geq 1$, depends on the behavior of iterated $g_3^k(n, 1)$, $k \geq 1$.

Proof. We have $\omega_0(0(0A)2^n0^n) \vdash (n + 1)$ $\omega_013^n(B0)0^n = C(n, 1)$. □

Since the behavior of iterated $g_3^k(n, 1)$ is an open problem in mathematics, the halting problem for Turing machine $M_3$ is so.

Note that the way by which a high score is obtained is particularly clear for machine $M_3$. The parameter $p$, $1 \leq p \leq 8$, can be seen as a state. If $n$ is odd, $g_3(n, p) = (n', p + 1)$, the state goes from $p$ to $p + 1$, and the computation stops when state 8 is reached. If $n$ is even, $g_3(n, p) = (n', 1)$, and the state goes back to 1.

6 Exponential Collatz-like

Let $M_4$ be the $6 \times 2$ Turing machine defined by Table 9.

We have $s(M_4) > 7.4 \times 10^{36534}$ and $\sigma(M_4) > 3.5 \times 10^{18267}$.

This machine is the current champion for the busy beaver competition for $6 \times 2$ machines.

It was discovered in June 2010 by Pavel Kropitz.

The following theorem gives the rules observed by Turing machine $M_4$.

Theorem 6.1 Let $C(n) = \omega_0(0(0A)1^n0^n)$. Then

(a) $C(0) \vdash (29)$ $C(9)$,
(b) $C(2) \vdash (36)$ $C(11)$,
(c) $C(3) \vdash (48)$ $C(13)$,
(d) $C(3k + 1) \vdash (3k + 3)$
(e) $C(9k + 5) \vdash ((4802 \times 16^k + 6370 \times 4^k + 2280k - 25362)/270)$ $C((98 \times 4^k - 11)/3)$,
(f) $C(9k + 6) \vdash ((125 \times 16^k + 575 \times 4^k + 228k - 2226)/27)$ $C((50 \times 4^k + 1 - 59)/3)$,
(g) $C(9k + 8) \vdash ((4802 \times 16^k + 6370 \times 4^k + 2280k - 11592)/270)$ $C((98 \times 4^k + 1)/3)$,
(h) $C(9k + 9) \vdash ((125 \times 16^k + 325 \times 4^k + 228k - 2289)/27)$ $C((50 \times 4^k + 1 - 11)/3)$,
(i) $C(9k + 11) \vdash ((4802 \times 16^k + 1270 \times 4^k + 2280k - 22452)/270)$ $C((98 \times 4^k + 1 - 59)/3)$,
(j) $C(9k + 12) \vdash ((125 \times 16^k + 325 \times 4^k + 912)/27)$ $C((50 \times 4^k + 1)/3)$.

Note that the behavior of this Turing machine on the blank tape involves only items (a), (d), (h) and (j).
Proof. A direct inspection of the transition table gives

(1) \(0^3(A0)0^6 \vdash (29) (A0)1^9\),
(2) \(0^3(A0)1^20^5 \vdash (36) (A0)1^{11}\),
(3) \(0^3(A0)1^30^6 \vdash (48) (A0)1^{13}\),
(4) \((A0)1 \vdash (1) 1(B1)\),
(5) \(01(E0) \vdash (2) (E0)1,\)
(6) \(0(E0) \vdash (1) (A0)1,\)
(7) \(11(E0) \vdash (2) (E1)1,\)
(8) \(01(C0) \vdash (2) (C0)01,\)
(9) \(11(C0) \vdash (2) (C1)01,\)
(10) \(0(C0) \vdash (2) 1(E1),\)
(11) \((E1)01 \vdash (2) 01(E1),\)
(12) \((E1)00 \vdash (2) 01(E0),\)
(13) \((E1)1 \vdash (2) (C0)0,\)
(14) \((B1)^1^3 \vdash (3) 110(B1),\)
(15) \((B1)00 \vdash (2) 11(H0),\)
(16) \((B1)10 \vdash (6) 01(C1),\)
(17) \((B1)1100 \vdash (12) (01)^2(C1),\)
(18) \((C1)01 \vdash (2) 01(C1),\)
(19) \((C1)00 \vdash (2) 01(C0),\)
(20) \((C1)^{1^6}0^6 \vdash (44) 01(E1)^{1^10},\)
(21) \((C1)^{1^8}0^{1^1} \vdash (113) 1(01)^{5}(E1)^{1^8}.

Iterating, respectively, (5), (8), (11), (14) and (18) gives

(22) \((01)^k(E0) \vdash (2k) (E0)^{1^{2k}},\)
(23) \((01)^k(C0) \vdash (2k) (C0)(01)^k,\)
(24) \((E1)(01)^k \vdash (2k) (01)^k(E1),\)
(25) \((B1)^{1^3}k \vdash (3k) (110)^k(B1),\)
(26) \((C1)(01)^k \vdash (2k) (01)^k(C1).\)

Using (19), (23), (10) and (24), we get

(27) \(0(01)^k(C1)00 \vdash (4k + 8) 1(01)^{k+1}(E1).\)

Using (12), (22) and (6), we get

(28) \(0(01)^k(E1)00 \vdash (2k + 5) (A0)^{1^{2k+3}}.\)

Using (12), (22) and (7), we get

(29) \(11(01)^k(E1)0 \vdash (2k + 6) (E1)^{1^{2k+4}}.\)

Using (13), (23), (10) and (24), we get

(30) \(0(01)^k(E1)1 \vdash (4k + 4) 1(01)^k(E1)0.\)

Using (13), (23), (9) and (26), we get

(31) \(11(01)^k(E1)1 \vdash (4k + 6) (01)^{k+1}(C1)0.\)

Using (20), (30), (11), (31) and (18), we get

(32) \(10(01)^k(C1)1^60^6 \vdash (8k + 70) (01)^{k+4}(C1)^1.\)

By induction on \(n\), using (32), we get

(33) \((10)^n(01)^k(C1)^{1^6}0^{6n} \vdash (16n^2 + 8kn + 54n) (01)^{4n+k}(C1)^1.\)

Using (30) and (11), we get

(34) \(00(01)^k(E1)11 \vdash (4k + 6) (01)^{k+2}(E1).\)
By induction on $n$, using (34), we get

\[ 0^{2n}(01)^k(E1)1^{2n} \vdash (4n^2 + 4kn + 2n) \ (01)^{2n+k}(E1). \]

Using (20), (31), (18), (21), (31), (18) and (33), we get

\[ 11(01)^k(110)_{0+2}^{3(1)} (110)^{0+2} + (16k^2 + 178k + 481) \ (01)^{4k+15}(C1)1^6. \]

Using (20), (35) and (28), we get

\[ 0^{11}(01)^k(C1)1^{0+8} \vdash (22k + 201) \ (A0)^{12k+25}. \]

Using (20), (35), (31), (18), (32) and (36), we get

\[ (110)^3(01)^k(110)_{0+101}^{3} (110)^{0+101} \vdash (16k^2 + 514k + 4045) \ (01)^{4k+55}(C1)1^6. \]

By induction on $k$, using (38) we get

\[ (110)^{3k}(01)^k(110)_{1+0}^{k} (110)^{1+0} \vdash (T) \ (01)^{k}(C1)1^6, \]

with $a = (2(3n + 55)4^k - 6n - 27k - 110)/3$, $b = ((3n + 55)4^k - 55)/3$, and $T = \frac{16(3n + 55)^2}{180} 4^k - \frac{3}{2}k - \frac{18(3n + 55)^2}{180} + 218(3n + 55)/27$.

Using (4), (25), (17), (27), (29), (31), (18), (21), (31), (18) and (33), we get

\[ (40) \ (A0)^{12k+190} \vdash (3k + 484) \ 1(110)^6(01)^k(C1)1^6. \]

Using (40), (39), (32) and (37), we get

\[ (39) \ (110)^{3k}(01)^k(110)_{1+3}^{k} (110)^{1+3} \vdash (T) \ (A0)^{1b}, \]

with $a = (50 \times 4^k + 11)/3 - 9k - 20$, $b = (50 \times 4^k + 11)/3$, and $T = (125 \times 16k^2 + 325 \times 4^k^2 + 228k - 2289)/27$.

Using (40), (39), (20), (35), (31), (18) and (37), we get

\[ (42) \ 0^k(1)(A0)^{14k+120} \vdash (T) \ (A0)^{1b}, \]

with $a = (50 \times 4^k + 1)/3 - 9k - 24$, $b = (50 \times 4^k + 1)/3$, and $T = (125 \times 16k^2 + 325 \times 4^k^2 + 228k - 912)/27$.

Using (4), (25) and (15), we get

\[ (43) \ (A0)^{13k+100} \vdash (3k + 3) \ 111(01)^k(H0). \]

Results (1), (43), (41) and (42) are results (a), (d), (h) and (j) of the theorem. They are sufficient to analyze the behavior of the Turing machine on a blank tape. The following gives its behavior from configurations $0(01)^k0^n$, where $n = 9k + m$, $m \in \{5, 6, 8, 11\}$.

Using (4), (25), (16), (27), (29), (31), (18), (32) and (36), we get

\[ (44) \ (A0)^{13k+140} \vdash (3k + 3076) \ 1(110)^6(01)^k(C1)1^6. \]

Using (44), (39), (32) and (37), we get

\[ 0^1(1)(A0)^{13k+140} \vdash (T) \ (A0)^{1b}, \]

with $b = (98 \times 4^k + 11)/3$ and $T = (4802 \times 16k^2 + 6370 \times 4^k + 22800k + 1) - 25362)/270$, and this result is still true for $k = -1$, since

\[ 0^1(1)(A0)^{13k+10} \vdash (285) \ (A0)^{1b}, \]

so we get

\[ (45) \ 0^1(1)(A0)^{13k+5} \vdash (T) \ (A0)^{1b}, \]

with $b = (98 \times 4^k - 11)/3$ and $T = (4802 \times 16k^2 + 6370 \times 4^k + 2280k + 25362)/270$.

Using (20), (35), (31), (18), (32), (36) and (37), we get

\[ (46) \ 0^1(1)(A0)^{13k+140} \vdash (T) \ (A0)^{1b}, \]

with $b = (98 \times 4^k + 11)/3$ and $T = (4802 \times 16k^2 + 6370 \times 4^k + 22800k + 1) - 25362)/270$.

Using (40), (39) and (46), we get

\[ 0^10(A0)^{13k+150} \vdash (T) \ (A0)^{1b}, \]

23
with $b = (50 \times 4^{k+2} - 59)/3$ and $T = (125 \times 16^{k+3} - 575 \times 4^{k+3} + 228(k+1) - 2226)/27$, and this result is still true for $k = -1$, since

$$0^{10}(A0)16031 \vdash (762) \quad (A0)1^{47},$$

so we get

$$(47) \quad 0^{10}(A0)1^{9k+6}a \vdash (T) \quad (A0)1^b,$$

with $b = (50 \times 4^{k+1} - 59)/3$ and $T = (125 \times 16^{k+2} - 575 \times 4^{k+2} + 228k - 2226)/27$.

Using (20), (30), (11), (31), (18) and (37), we get

$$0^{12}11100(01)k(C1)^{16}0^{14} \vdash (30k + 407) \quad (A0)1^{2k+37}.$$  

Using (44), (39) and (48), we get

$$0^{12}(A0)1^{9k+17}0^a \vdash (T) \quad (A0)1^b,$$

with $b = (98 \times 4^{k+1} + 1)/3$ and $T = (4802 \times 16^{k+2} + 6370 \times 4^{k+2} + 2280(k+1) - 11592)/270$, and this result is still true for $k = -1$, since

$$0^{12}(A0)1^{8}0^{13} \vdash (336) \quad (A0)1^{43},$$

so we get

$$(49) \quad 0^{12}(A0)1^{9k+8}0^a \vdash (T) \quad (A0)1^b,$$

with $b = (98 \times 4^{k+1} + 1)/3$ and $T = (4802 \times 16^{k+1} + 6370 \times 4^{k+1} + 2280k - 11592)/270$.

Using (44), (39) and (46), we get

$$0^{10}(A0)1^{9k+20}0^a \vdash (T) \quad (A0)1^b,$$

with $b = (98 \times 4^{k+2} - 59)/3$ and $T = (4802 \times 16^{k+3} - 11270 \times 4^{k+3} + 2280(k+1) - 22452)/270$, and this result is still true for $k = -1$, since

$$0^{10}(A0)1^{11}0^90 \vdash (3802) \quad (A0)1^{111},$$

so we get

$$(50) \quad 0^{10}(A0)1^{9k+11}0^a \vdash (T) \quad (A0)1^b,$$

with $b = (98 \times 4^{k+1} - 59)/3$ and $T = (4802 \times 16^{k+2} - 11270 \times 4^{k+2} + 2280k - 22452)/270$.

Results (45), (47), (49) and (50) are results (e), (f), (g) and (i) of the theorem.

Results (1), (2) and (3) give special cases (a), (b) and (c) of the theorem.

Using the rules of this theorem, we have,

$$\omega(0)(A0)0^a \vdash (29) \quad C(9) \vdash (1293) \quad C(63) \vdash (19, 884, 896, 677)$$

$$C(273063) \vdash (T_1) \quad C((50 \times 4^{3034} + 1)/3) \vdash (T_2) \quad \omega(111)(011)K(H10)0^a,$$

with $T_1 = (125 \times 16^{3034} + 325 \times 4^{3034} + 6916380)/27$, $T_2 = (50 \times 4^{3034} + 7)/3$, $K = (50 \times 4^{3034} - 2)/9$.

The total time is $s(M_1) = (125 \times 16^{3034} + 1750 \times 4^{3034} + 15)/27 + 19, 885, 154, 163$, and the final number of symbols 1 is $\sigma(M_4) = (25 \times 4^{3034} + 23)/9$.

Let $g_4$ be the exponential Collatz-like function defined by: for $k \geq 0,$
Table 10: Machine $M_5$ discovered in May 2010 by P. Kropitz

<table>
<thead>
<tr>
<th>$M_5$</th>
<th>0</th>
<th>1</th>
</tr>
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<tbody>
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<td>A</td>
<td>1RB</td>
<td>0LD</td>
</tr>
<tr>
<td>B</td>
<td>1RC</td>
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</tr>
<tr>
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<tr>
<td>E</td>
<td>1LA</td>
<td>0RB</td>
</tr>
<tr>
<td>F</td>
<td>0RC</td>
<td>0RE</td>
</tr>
</tbody>
</table>

Then $g_5^1(0)$ is undefined.

Theorem 7.1 Let $C(k, n) = ^{"010^n1(C1)1^k0"}$. Then

(a) $^{"0(A0)0"} \vdash (47) \ C(2, 5),$ and, for all $k \geq 0$,
Using (12), (14), (16), (8), (7) and (17), we get

\[ k \]

By induction on \( k \), using (16), (17) and (10), we get

\[ 01(B1)1^{3k+1}0 \]

Using (12), (10), (16), (6), (19), (16) and (17), we get

\[ 01(C1)1^{3k}00 \]

By induction on \( k \), using (18), we get

\[ 01(B1)1^{3k+1}0 \]

Using (12), (10), (16), (6), (19), (16) and (17), we get

\[ 01(C1)1^{3k}00 \]

Using (14), (16), (8), (7) and (17), we get

\[ 101(C1)1^{3k+1}0 \]

Using (12), (10), (16), (6), (19), (16) and (17), we get

\[ 01(C1)1^{3k}00 \]

Using the rules of this theorem, we have, in 22158 transitions,

\[ \omega 0(A0)0^\omega \vdash (47) \quad C(2, 5) \vdash (430) \quad C(11, 1) \vdash ( ) \cdots \vdash ( ) \quad \text{END}. \]

Let \( g_5 \) be the partial function defined by: for \( k, n \geq 0 \),
Then $g_5^{2157}(2, 5)$ is undefined.

**Proposition 7.2** The behavior of Turing machine $M_5$, on inputs $01^{3n+3}$, $n \geq 0$, depends on the behavior of iterated $g_5^k(n, p)$, $k \geq 1$, $n, p \geq 0$.

**Proof.** We have $\omega_0(0A1)13n+3p0 \vdash (3n + 30) - 1 = 01^{3n+21}(C1)160^0 = C(2, 3n + 2)$. \(\square\)

Since the behavior of iterated $g_5^k(n, p)$ is an open problem in mathematics, the halting problem for Turing machine $M_5$ is so.

The following proposition shows that some configurations take a long time to halt.

**Proposition 7.3** $C(9, 1) \vdash (T) \text{ END with } T > 10^{10^{10^{10^{520}}}}$.

**Proof.** By induction on $n$, using Theorem 7.1 (f), it is easy to prove that, if $n \geq 0$, $0 \leq r \leq 3$, we have

$$C(2, 4n + r) \vdash (u_n) \quad C(u_n, r),$$

with $u_n = (3^n + 5)/2$ and $t_n = (3 \times 9^{n+3} - 80 \times 3^n + 584n - 27)/32$.

By induction on $k$, it is easy to prove that, if $k \geq 2$, we have

$$3^{2^{k-1}} \equiv 2^{k+1} + 1 \pmod{2^{k+2}}$$

so the multiplicative order of 3 modulo $2^{k+2}$ is $2^k$ for $k \geq 1$. Thus we can prove that, for $k \geq 1$, $n, m \geq 0$, we have

$$n \equiv m \pmod{2^k} \iff u_n \equiv u_m \pmod{2^{k+1}}.$$  

Now, suppose that, for $a \in \{1, 3\}$, $n, n' \geq 1$, $q, q' \geq 1$, $0 \leq r, r' \leq 3$, we have

$$C(n, a) \vdash (3n + 27 + 10a) \quad C(2, 3a + 3a - 1) = C(2, 4q + r) \vdash (t_q) \quad C(u_q, r),$$

and

$$C(n', a) \vdash (3n' + 27 + 10a) \quad C(2, 3n' + 3a - 1) = C(2, 4q' + r') \vdash (t_{q'}) \quad C(u_{q'}, r'),$$

and let $k \geq 2$ such that $n \equiv n' \pmod{2^{k+1}}$. Then it is easy to prove that $r = r'$ and $u_q \equiv u_{q'} \pmod{2^k}$. So the behavior of configurations $C(n, a)$ is mirrored by the behavior of configurations $C(n', a)$ with $n' \leq 2^k$ for suitable $k$.

In the following computation on $C(9, 1)$:

| $C(9, 1)$ | $\vdash$ | $C(2, 4 \times 7 + 1)$ | $\vdash$ | $(t_2)$ |
| $C(9839, 1)$ | $\vdash$ | $C(2, 4 \times 7379 + 3)$ | $\vdash$ | $(t_{7379})$ |
| $C(u_{7379}, 3)$ | $\vdash$ | $C(2, 4 \times q_3 + r_3)$ | $\vdash$ | $(t_{q_3})$ |
| $C(u_{q_3}, r_3)$ | $\vdash$ | $C(2, 4 \times q_4 + r_4)$ | $\vdash$ | $(t_{q_4})$ |
| $C(u_{q_4}, r_4)$ | $\vdash$ | $C(2, 4 \times q_5 + r_5)$ | $\vdash$ | $(t_{q_5})$ |
| $C(u_{q_5}, r_5)$ | $\vdash$ | $C(2, 4 \times q_6 + r_6)$ | $\vdash$ | $(t_{q_6})$ |
| $C(u_{q_6}, r_6)$ | $\vdash$ | $(3)$ | $\text{ END}$ |
Table 11: Machine $M_6$ discovered in November 2007 by T. and S. Ligocki

we know that $r_6 = 0$ because we have

- $u_{q_1} = u_7 \equiv 47 \pmod{64}$, $3 \times 47 + 2 = 4 \times 35 + 3$, $q_2 = 35$, $r_2 = 3$,
- $u_{q_2} \equiv u_{q_1} \equiv 23 \pmod{32}$, $3 \times 23 + 8 = 4 \times 19 + 1$, $q_3 = 19$, $r_3 = 1$,
- $u_{q_3} \equiv u_{q_2} \equiv 7 \pmod{16}$, $(3 \times 7) + 2 = 4 \times 5 + 3$, $q_4 = 5$, $r_4 = 3$,
- $u_{q_4} \equiv u_{q_3} \equiv 3 \pmod{8}$, $(3 \times 3) + 8 = 4 \times 4 + 1$, $q_5 = 4$, $r_5 = 1$,
- $u_{q_5} \equiv u_{q_4} \equiv 2 \pmod{4}$, $(3 \times 2) + 2 = 4 \times 2 + 0$, $q_6 = 2$, $r_6 = 0$.

It is easy to see that, if $a \in \{1, 3\}$, $n \geq 0$, if

$$C(n, a) \vdash (3n + 27 + 10a) \quad C(2, 3n + 3a - 1) = C(2, 4q + r) \vdash (t_q) \quad C(u_q, r),$$

then $q \geq (3n - 1)/4$ and $u_q > (3^{3/4})^n > 2^n$.

And we also have $n \geq 5 \Rightarrow t_n > 68 \times 9^n$, so, if $C(9, 1) \vdash (T) \ END$, we have

$$T > t_{q_6} > 9^{q_6} > 9^{3u_{q_6}/4} > 5^{u_{q_6}}.$$}

and $u_{q_6} > 2^{u_{q_5}}$, $u_{q_5} > 2^{u_{q_4}}$, $u_{q_4} > 2^{u_{q_3}} = 2^{u_{7379}}$, so $T > 5^{2^{2^{37379}}}$. Using $u_{7379} > 10^{10^{521}}$, and, for $x \geq 1$, $210^x > 10^{10^{x-53}}$, $2^{10^{10^x}} > 10^{10^{10^x-3}}$, $2^{10^{10^{10^x}}} > 10^{10^{10^{10^x-33}}}$ and $5^{10^{10^{10^x}}} > 10^{10^{10^{10^x-33}}}$, we are done. □

8 An infinite set of rules

Let $M_6$ be the $6 \times 2$ Turing machine defined by Table 11.

We have $s(M_6) > 8.9 \times 10^{1762}$ and $\sigma(M_6) > 2.5 \times 10^{881}$.

This machine was discovered in November 2007 by Terry and Shawn Ligocki. It was the champion for the busy beaver competition for $6 \times 2$ machines from November to December 2007.

The following theorem gives the rules that enable Turing machine $M_6$ to reach a halting configuration from a blank tape.

Recall that bin($p$) is the usual binary writing of number $p$, and R($w_1 \ldots w_n$) = $w_n \ldots w_1$.

**Theorem 8.1** Let $C(n, p) = \omega(0(10)^n R(bin(p))0^x)$, so that $C(k, 4m + 1) = C(k + 1, m)$. Then

(a) $\omega(0A0)0^x \vdash (6) \ C(0, 15)$,
and, for all \( k, m \geq 0 \),
\[
\begin{align*}
(b) & \quad C(k, 4m + 3) \vdash (4k + 6) \quad C(k + 2, m), \\
(c) & \quad C(2k, 4m) \vdash (30k^2 + 20k + 15) \quad C(5k + 2, 2m + 1), \\
(d) & \quad C(2k + 1, 4m) \vdash (30k^2 + 40k + 25) \quad C(5k + 2, 32m + 20), \\
(e) & \quad C(k, 8m + 2) \vdash (8k + 20) \quad C(k + 3, 2m + 1), \\
(f) & \quad C(2k, 16m + 6) \vdash (30k^2 + 40k + 23) \quad C(5k + 2, 32m + 20), \\
(g) & \quad C(2k + 1, 16m + 6) \vdash (30k^2 + 80k + 63) \quad C(5k + 7, 2m + 1), \\
(h) & \quad C(k, 32m + 14) \vdash (4k + 18) \quad C(k + 3, 2m + 1), \\
(i) & \quad C(2k, 128m + 94) \vdash (30k^2 + 40k + 39) \quad C(5k + 2, 256m + 84), \\
(j) & \quad C(2k + 1, 128m + 94) \vdash (30k^2 + 80k + 79) \quad C(5k + 9, m), \\
(k) & \quad C(k, 256m + 190) \vdash (4k + 34) \quad C(k + 5, m), \\
(l) & \quad C(k, 512m + 30) \vdash (2k + 43) \quad \omega_0(10^k 1(H0)(10)^2(01)^2 R(bin(m))0 \omega).
\end{align*}
\]

**Proof.** A direct inspection of the transition table gives
\[
\begin{align*}
(1) & \quad 0^3(A0)0 \vdash (6) (F0)1^40, \\
(2) & \quad 0(F0)00 \vdash (9) (F0)1^3, \\
(3) & \quad (F0)10 \vdash (2) 10(F0), \\
(4) & \quad (F0)11 \vdash (4) (F1)10, \\
(5) & \quad (F0)01 \vdash (4) 10(C1), \\
(6) & \quad 10(E1) \vdash (2) (E1)10, \\
(7) & \quad 1(E1) \vdash (2) 1(H0), \\
(8) & \quad 0^4(E1) \vdash (3) (F0)110, \\
(9) & \quad 00(F1) \vdash (2) (F0)10, \\
(10) & \quad 10(F1) \vdash (2) (F1)10, \\
(11) & \quad 1(F1) \vdash (1) (E1)0, \\
(12) & \quad (C1)1 \vdash (1) 0(C1), \\
(13) & \quad (C1)0 \vdash (1) 0(C0), \\
(14) & \quad 0^3(C0) \vdash (3) (F0)1^3, \\
(15) & \quad 100(C0) \vdash (3) (F1)1^3.
\end{align*}
\]

Iterating, respectively, (3), (10), (6) and (12) gives
\[
\begin{align*}
(16) & \quad (F0)(10)^k \vdash (2k) (10)^k(F0), \\
(17) & \quad (10)^k(F1) \vdash (2k) (F1)(10)^k, \\
(18) & \quad (10)^k(E1) \vdash (2k) (E1)(10)^k, \\
(19) & \quad (C1)^k \vdash (k) 0^k(C1).
\end{align*}
\]

Using (5), (19) and (13), we get
\[
(20) \quad (F0)01^{k+2}0 \vdash (k + 5) 10^{k+2}(C0).
\]

Using (16), (4), (17) and (9), we get
\[
(21) \quad 0^2(F0)(10)^k11 \vdash (4k + 6) (F0)(10)^{k+2}.
\]

Using (16), (20), (15), (17) and (21), we get
\[
(22) \quad 0^4(F0)(10)^k010 \vdash (8k + 20) (F0)(10)^{k+3}.
\]

Using (22) and (21), we get
\[
(23) \quad 0^6(F0)(10)^k(01)^2 \vdash (12k + 38) (F0)(10)^{k+5}.
\]

By induction on \( n \), using (23), we get
\[
(24) \quad 0^{3n}(F0)(10)^k(01)^{2n} \vdash (30n^2 + 12kn + 8n) (F0)(10)^{5n+k}.
\]
Using (24), with \( k = 2 \), we get
\[
(25) \quad 0^{6k}(F0)(10)^2(01)^2 \vdash (30k^2 + 32k) \quad (F0)(10)^{5k+2}.\]

Using (16), (2), (4), (11), (18), (8) and (21), we get
\[
(26) \quad 0^4(F0)(10)^{k+1}0^2 \vdash (4k + 25) \quad (F0)(10)^2(01)^k00101.\]

Using (26) and (25), we get
\[
(27) \quad 0^{6k+5}(F0)(10)^{2k+1}0^2 \vdash (30k^2 + 40k + 25) \quad (F0)(10)^{5k+2}00101.\]

Using (25) and (22), we get
\[
(28) \quad 0^{6k+4}(F0)(10)^2(01)^{2k+1}0 \vdash (30k^2 + 72k + 36) \quad (F0)(10)^{5k+5}.\]

Using (26) and (28), we get
\[
0^{6k+9}(F0)(10)^{2k+2}0^2 \vdash (30k^2 + 80k + 65) \quad (F0)(10)^{5k+7},\]
and the result is still true for \( k = -1 \), so we have
\[
(29) \quad 0^{6k+3}(F0)(10)^{2k+2}0^2 \vdash (30k^2 + 20k + 15) \quad (F0)(10)^{5k+2}.\]

Using (16), (20), (14), (4), (11), (18), (8) and (21), we get
\[
(30) \quad 0^2(F0)(10)^601^20 \vdash (4k + 23) \quad (F0)(10)^2(01)^800101.\]

Using (30) and (25), we get
\[
(31) \quad 0^{6k+5}(F0)(10)^{2k+2}0 \vdash (30k^2 + 40k + 23) \quad (F0)(10)^{5k+2}00101.\]

Using (30) and (28), we get
\[
(32) \quad 0^{6k+9}(F0)(10)^{2k+1}0^2 \vdash (30k^2 + 80k + 63) \quad (F0)(10)^{5k+7}.\]

Using (16), (20), (14), (4), (17) and (9), we get
\[
(33) \quad 0^2(F0)(10)^601^20 \vdash (4k + 18) \quad (F0)(10)^{5k+3}.\]

Using (16), (20), (14), (4), (9) and (16), we get
\[
(34) \quad (F0)(10)^601^20 \vdash (2k + 21) \quad (10)^k1(10)^k(F0)01.\]

Using (34), (4), (17), (11), (18), (8) and (21), we get
\[
(35) \quad 0^2(F0)(10)^601^20 \vdash (4k + 39) \quad (F0)(10)^2(01)^400(10)^3.\]

Using (35) and (25), we get
\[
(36) \quad 0^{6k+5}(F0)(10)^{2k}0^14^01 \vdash (30k^2 + 40k + 39) \quad (F0)(10)^{5k+2}0^2(10)^3.\]

Using (35), (25) and (22), we get
\[
(37) \quad 0^{6k+9}(F0)(10)^{2k+1}0^14^01 \vdash (30k^2 + 80k + 79) \quad (F0)(10)^{5k+9}.\]

Using (16), (20), (14), (4), (9) and (16), we get
\[
(38) \quad (F0)(01)^k0^14^01 \vdash (2k + 22) \quad (10)^{k+3}(F0)01.\]

Using (38), (4), (17) and (9), we get
\[
(39) \quad 0^2(F0)(10)^601^20 \vdash (4k + 34) \quad (F0)(10)^{k+5}.\]

Using (34), (3), (2), (4), (11), (18) and (7), we get
\[
(40) \quad (F0)(10)^601^40^3 \vdash (2k + 43) \quad (10)^2(1)(H0)(10)^2(01)^2.\]

Results (1), (21), (29), (27), (22), (31), (32), (33), (36), (37), (39) and (40) give results (a)–(l) of the theorem. \(\square\)

Note that the rules (a)–(l) are written in their order of occurrence in the computation of Turing machine \(M_6\) on the blank tape.

Using the rules of this theorem, we have, in 3346 transitions,
\[
\omega 0(40)0^\omega \vdash (6) \quad C(0, 15) \vdash (6) \quad C(2, 3) \vdash (\ldots \vdash (\quad \text{END}).
\]

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<tr>
<td>F</td>
<td>1RH</td>
<td>1LE</td>
</tr>
</tbody>
</table>

Table 12: Machine $M_7$ discovered in October 2000 by Marxen and Buntrock

We have
\[ \omega^0(A0)0R(bin(p))0^\omega \vdash (6) \]
\[ \omega^0(F0)1^40R(bin(p))0^\omega \]
\[ = C(0, 32p + 15) \vdash (6) \]
\[ C(2, 8p + 3) \vdash (14) \]
\[ C(4, 2p), \]
so the behavior of Turing machine $M_6$ on inputs $00x$, $x \in \{0, 1\}^*$, depends on the behavior of configurations $C(n, p)$, and the halting problem for Turing machine $M_6$ depends on this behavior.

9 Configurations provably stopping

Let $M_7$ be the $6 \times 2$ Turing machine defined by Table 12.

We have $s(M_7) > 6.1 \times 10^{225}$ and $\sigma(M_7) > 6.4 \times 10^{462}$.

This machine was discovered in October 2000 by Heiner Marxen and Jürgen Buntrock. It was the champion for the busy beaver competition for $6 \times 2$ machines from October 2000 to March 2001.

The following theorem was initially obtained by Munaño [37]. It gives the rules observed by Turing machine $M_7$.

**Theorem 9.1** Let $C(n) = \omega^01^n(B0)0^\omega$. Then

(a) $\omega^0(A0)0^\omega \vdash (1) \ C(1)$,
(b) $C(3k) \vdash (54 \times 4^{k+1} - 27 \times 2^{k+3} + 26k + 86)$ $\ C(9 \times 2^{k+1} - 8)$,
(c) $C(3k + 1) \vdash (2048 \times (4^k - 1)/3 - 3 \times 2^{k+7} + 26k + 792)$ $\ C(2^{k+5} - 8)$,
(d) $C(3k + 2) \vdash (3k + 8)$ $\ C(\omega^01(H1)(011)^k010^\omega)$.

**Proof.** A direct inspection of the transition table gives

(1) $11(B0)00 \vdash (6) \ (D1)0101$,
(2) $00(B0)01 \vdash (8) \ (B0)1^4$,
(3) $11(B0)01 \vdash (6) \ (B1)01^3$,
(4) $\ 0(B0)1 \vdash (3) \ 1(B0)$,
(5) $\ 0(B1) \vdash (1) \ (B0)1$,
(6) $\ 1(B1) \vdash (1) \ (B1)1$,
(7) $\ 0(D1) \vdash (2) \ 1(H1)$,
(8) $\ 0^31(D1) \vdash (6) \ (B0)1^4$,
(9) $\ 0^31^2(D1) \vdash (8) \ (B0)1^201^2$,
(10) $\ 1^3(D1) \vdash (3) \ (D1)011$. 

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Iterating, respectively, (4), (6) and (10) gives

\[(11) \quad B_0 1^k, \quad k \in \mathbb{N}. \]
\[(12) \quad 1^k B_1, \quad k \in \mathbb{N}. \]
\[(13) \quad 1^{3k} D_1, \quad k \in \mathbb{N}. \]

Using (3), (12), (5) and (11), we get

\[01^{k+2} k \in \mathbb{N}, \quad 1^{k+1} k \in \mathbb{N}, \]
and the result is still true for \(k = 1\), so we have

\[(14) \quad 01^{k+1} k \in \mathbb{N}, \quad 1^{k+1} k \in \mathbb{N}. \]

By induction on \(k\), using (14), we get

\[(15) \quad 0^k 1^k B_0 k \in \mathbb{N}, \quad 1^k B_0 k \in \mathbb{N}. \]

Using (1), (13) and (7), we get

\[(16) \quad 01^{k+2}, 0^2 \in \mathbb{N}, \quad 1^k k \in \mathbb{N}. \]

Using (1), (13) and (9), we get

\[0^4 1^{3k+4} k \in \mathbb{N}, \quad (B_0) 1^k k \in \mathbb{N}, \]
and the result is still true for \(k = 1\), so we have

\[(17) \quad 0^4 1^{3k+1} k \in \mathbb{N}, \quad (B_0) 1^k k \in \mathbb{N}. \]

Using (1), (13) and (8), we get

\[(18) \quad 0^4 1^{3k+3} k \in \mathbb{N}, \quad (B_0) 1^4 k \in \mathbb{N}. \]

Using (11), (15) and (2), we get

\[(19) \quad 0^k 2^2, 1^k k \in \mathbb{N}, \quad 1^k B_0 k \in \mathbb{N}. \]

By induction on \(k\), using (19), we get

\[(20) \quad 0^k 2^{(n+5)/5}, 0^k 1^k k \in \mathbb{N}, \quad 1^k k \in \mathbb{N}, \]
with \(T = 2(n+5) (4^k - 1)/(n+5) (2^k - 1) + 23k.\)

Using (20), for \(n = 3\) and \(n = 4\), we get respectively

\[(21) \quad 0^4 1^{3k-8} k \in \mathbb{N}, \quad (B_0) 1^k k \in \mathbb{N}, \quad 1^k k \in \mathbb{N}, \]
\[(22) \quad 0^4 1^{3k-9} k \in \mathbb{N}, \quad (B_0) 1^k k \in \mathbb{N}, \quad 1^k k \in \mathbb{N}. \]

Using (11), (15), (2), (11), (15), (2) and (11), we get

\[(23) \quad 0^k 8, 0^k 1^k k \in \mathbb{N}, \quad 1^k B_0 k \in \mathbb{N}. \]

Using (17), (21) and (23), we get

\[(24) \quad 0^k 8, 0^k 1^k k \in \mathbb{N}, \quad 1^k B_0 k \in \mathbb{N}. \]

Using (18), (22) and (23), we get

\[0^9 2^{2k+3} k \in \mathbb{N}, \quad (B_0) 2^{2k+4} k \in \mathbb{N}, \quad (B_0) 2^{2k+4} k \in \mathbb{N}, \]
and the result is still true for \(k = 1\), so we have

\[(25) \quad 0^9 2^{2k+3} k \in \mathbb{N}, \quad (B_0) 2^{2k+4} k \in \mathbb{N}, \quad (B_0) 2^{2k+4} k \in \mathbb{N}. \]

Results (25), (24) and (16) give results (b)-(d) of the theorem. \(\square\)

Using the rules of this theorem, we have

\[\omega 0(A0) 0^2, 0^3(C1) 0^3, C(1) 0^3, C(408) 0^3, C(24) 0^3, C(14100774) 0^3, \]
\[C(4600) 0^3, C(2^{1538} - 8) 0^3, C(2^{1538} - 2) 0^3, 01(H1)(011) 0^3, \]
with \(T = 2048 \times 4^1 1^3 - 3 - 3 \times 2^{1540} + 40650\) and \(p = (2^{1538} - 10)/3.\)
So the total time is $s(M_7) = 2048 \times (4^{1533} - 1)/3 - 11 \times 2^{1538} + 14141831$, and the final number of symbols 1 is $\sigma(M_7) = 2 \times (2^{1538} - 10)/3 + 4$.

Note that

$$C(6k+1) \vdash ( ) \quad C(3m) \vdash ( ) \quad C(6p+4) \vdash ( ) \quad C(3q+2) \vdash ( ) \quad \text{END},$$

with $m = (2^{2k+5} - 8)/3$, $p = 3 \times 2^n - 2$, $q = (2^{2p+6} - 10)/3$. So all configurations $C(n)$ lead to a halting configuration. Those taking the most time are $C(6k+1)$. For example, $C(7) \vdash (t)$ END with $t > 10^{1.9 \times 10^{12}}$. More generally, $C(6k+1) \vdash (t(k)) \text{END}$ with $t(k) > 10^{10^{4^{3k+2}+2}}$.

10 Conclusion

We discuss two questions as a conclusion to this article.

A. How simulating Collatz-like functions allows Turing machines to achieve high scores?

Lagarias [17] noted that the successive iterates of the $3x+1$ function $T$ have an irregular behavior. For example, 7 iterations of function $T$ on $n = 26$ lead to the value 1, but 70 iterations are necessary on $n = 27$. It seems that many Collatz-like functions have the same irregular behavior. Iterating them on small numbers may produce very long runs before stopping.

Adding parameters may increase the number of iterations by allowing the iterated values to range the set of parameters before stopping. The pure Collatz-like function with parameter $g_3(n, p)$ presented in Section 5 is particularly illustrative.

Another way to high scores is given by exponential Collatz-like functions such as function $g_4$ in Section 6. Only five iterations are performed on a blank tape, but exponential growth ensures a high score.

Irregular behavior is a condition for a Collatz-like function to be eligible to the busy beaver competition. Another condition is, of course, being computable by a very small Turing machine.

B. Are some universal devices more natural than others?

Conway [7] proved that there is no algorithm that, given as inputs a Collatz-like function $g$ and two integers $n, p$, outputs an answer yes or no to the question: Does there exist a positive integer $k$ such that $g^k(n) = p$? Conway [7, 8] also proved that Collatz-like functions can be used to simulate all computable (also called recursive) functions. These properties can be summed up by writing that Collatz-like functions provide a universal model of computation with a m-complete decision problem.

Many universal models of computation are known: Turing machines, tag-systems, cellular automata, Diophantine equations, etc. (see [35]). Of course, any universal model can simulate and be simulated by any other universal model. But it is Collatz-like functions, and not another model, that appear naturally in this study. Their unexpectedly pervasive presence leads to wonder about the significance of their status among mathematical beings.

References


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