Algebraic tools for the overlapping tile product

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Abstract. Overlapping tile automata and the associated notion of recognizability by means of (adequate) premorphisms in finite ordered monoids have recently been defined for coping with the collapse of classical recognizability in inverse monoids. In this paper, we investigate more in depth the associated algebraic tools that allows for a better understanding of the underlying mathematical theory. In particular, addressing the surprisingly difficult problem of language product and star, we eventually found some deep links with classical notions of inverse semigroup theory such as the notion of restricted product.

Introduction

Overlapping structures, be them linear shaped as in McAlister monoids [17], tree shaped as in free inverse monoids [22, 19] or more generally higher-dimensional (overlapping) strings as in Kellendonk's tiling monoids [13, 14], are promising high level models of system behaviors as already illustrated in musical application modeling [12] and the associated programming language proposal [11], or in distributed behavior modeling [2]. Be them for modeling/typing purposes or system analysis, there is incentive to develop the language theory of overlapping tiles.

Since Kellendonk's tiling monoids are inverse semigroups, such a language theory lies at the intersection between inverse semigroup theory [20, 16] and formal language theory. A number of studies, such as [18, 23] to mention but a few, already show deep connections between these fields. However, with Monadic Second Order logic (MSO) in the background as yardstick of expressive power (see e.g. [24]), classical language theoretic concepts and tools fail to be expressive enough [23, 9]. Adaptation of the classical theory have thus recently been proposed in order to cope with such a collapse in expressive power. The resulting concepts: tile automata and quasi-recognizability have been proved to *essentially* capture MSO both for linear tiles [10] or tree-shaped tiles [8].

Although, the resulting theory is somewhat robust – word or tree shaped tile automata are essentially non deterministic word or tree automata with adapted semantics [10, 8] – the resulting language theory remains mysteriously tricky. For instance, the product of two quasi-recognizable languages is not necessarily quasi-recognizable.

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In this paper, as an echoe of [21] in classical algebraic language theory, continuing the newly developed theory, we study the case of positive word tiles, that is birooted words where the input root never occurs after the output root – positive tiles are the word counterpart of the positive birooted trees that form the elements of free ample monoids [4]. Our interest in studying the associated algebraic language theory is that, restricted to positive birooted words or trees, the class of quasi-recognizable languages is now closed under product and star.

Indeed, this can be proved as a corollary of the obtained logical characterization of these languages. Yet, direct automata theoretic or algebraic constructions were still missing. This is the purpose of the present paper.

Interestingly, these construction remains quite difficult unless, as we propose here, we first consider the restricted product: a fundamental notion in inverse semigroup theory [16] that was so far unused in our language theoretic investigation.

As a result, the proposed study not only sheds a new light on the adequate ordered monoids that are used as recognizers in quasi-recognizability, but also strengthens quite in depth the underlying theoretical framework. The fact is that our proposed recognizer definition can be seen as a follow-up to the research track initiated by Fountain et al. [3, 15, 5, 1] on certain semigroups with local units.

Worth being mentioned, though we restrict our study to positive birooted words as studied in [9, 7, 10], it is quite clear that our constructions can be extended to the case of positive birooted trees as studied in [8]. It follows that the algebraic tools proposed here are particularly well-suited for a language theory of the free ample monoid [4] whose elements are, precisely, positive birooted trees.

Organization of the paper

In the first part, we give a formal definition of positive tiles as triplets of words, and of the set of all tiles, including or not the 0 tile.

The following part presents our automatic tools, then the algebraic ones, mainly Ehresmann monoids and premorphisms. The definition of adequate premorphisms will bring that of restricted product, i.e. the product on the condition of shared projections, which amounts for tiles to have the same domain.

The third part presents our algebraic construction : we explore all the decompositions in a restricted product of each element. For tiles, this is equivalent to going through all the possible cutting points between input and output, which simulates the existential quantifier. Since we only consider positive tiles, this amounts to only going forward, "between input and output" therefore means "on the root", while we would examine (i.e. existentially quantify on) the entirety of the domain if we were considering positive and negative tiles, therefore gaining the possibility to move forward and backward.

In the last part, we use this construction to restricted product of languages, using the fact that if a tile belongs to the restricted product of L_1 and L_2, any tile with the same set of decomposition will also have that same decomposition itself.

1 Overlapping tiles

Let A be an alphabet and let A^* be the free monoid generated by A, that is, the set of finite words equipped with the concatenation operation. The empty word is denoted by 1 and, for every two words u and $v \in A^*$, we write $u \cdot v$ or simply uv for the concatenation of the word u and v.

The set A^* is ordered by the prefix order $\leq_p$ (resp. the suffix order $\leq_s$) defined, for every word u and $v \in A^*$, by $u \leq_p v$ (resp. $u \leq_s v$) when there exists $w \in A^*$ such that $uw = v$ (resp. $wu = v$).

A defined positive overlapping unidimensional tile (or just tile in the sequel) on the alphabet A is any triple u of the form $u = (u_1, u_2, u_3) \in A^* \times A^* \times A^*$. Such a tile is depicted in Figure 2. The set of defined tiles on the alphabet A

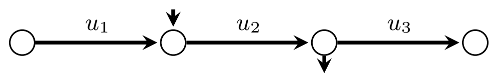

Fig. 1. A graphical representation of tile (u_1, u_2, u_3)

is denoted by $T^+(A)$. It is ordered by the *natural order* relation $\leq$ defined, for every tile $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, by

$$(u_1, u_2, u_3) \leq (v_1, v_2, v_3) \text{ when } u_1 \geq_s v_1, u_2 = v_2, u_3 \geq_p v_3$$

The (partial) product $u \cdot v$ of two such tiles u and v is defined, if it exists, as the greatest tile $w = (w_1, w_2, w_3)$ in the natural order such that $w_1 \geq_s u_1$, $w_1 u_2 \geq_s v_1$, $w_2 = u_2 v_2$, $v_2 w_3 \geq_p u_3$ and $w_3 \geq_p v_3$. Such a definition is depicted in Figure 2. Completing the set $T^+(A)$ of positive tiles by an undefined tile 0,

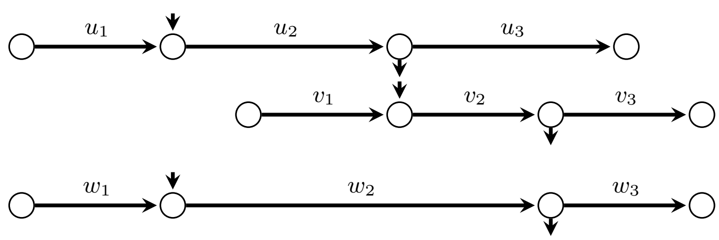

Fig. 2. A graphical representation of the product $(u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = (w_1, w_2, w_3)$

the partial product is made complete by letting $u \cdot v = 0$ when there exists no such a defined product tile w and we put $u \cdot 0 = 0 = 0 \cdot u$ for every defined or undefined tile u.

It has already been shown in [9] that the set $T_0^+(A)$ of positive tiles equipped with such a product is actually a submonoid of the (inverse) monoid of McAlister [17] with unit $1 = (1, 1, 1)$. Moreover, extending the natural order to the undefined tile by letting $0 \leq u$ for every $u \in T_0^+(A)$, the natural order is indeed a partial order relation over $T_0^+(A)$ that is also stable under product, i.e. for every tiles u, v and $w \in T_0^+(A)$, if $u \leq v$ then $w \cdot u \leq w \cot v$ and $u \cdot w \leq v \cdot w$.

For every non zero tile $u = (u_1, u_2, u_3) \in T^+(A)$, we define the *left projection* $u^L = (u_1u_2, 1, u_3)$ and the *right projection* $u^R = (u_1, 1, u_2u_3)$ of the tile u. These projections are extended to zero by taking $0^L = 0 = 0^R$. Then it can be shown [9] that for every u and $v \in T_0^+(A)$ we have $u \leq v$ if and only $u = u^R \cdot v$ if and only if $u = v \cdot v^L$. Moreover, a tile u is idempotent, that is $u \cdot u = u$ if and only if $u^L = u$ if and only if $u^R = u$ if and only if $u \leq 1$.

Let $U(T_0^+(A)) = \{u \in T_0^+(A) : u \leq 1\}$ be the set of *subunits* of the monoid of positive tiles. We have just seen that, in the monoid $T_0^+(A)$ idempotents and subunits coincide. It can also be shown that $U(T_0^+(A))$ ordered by the natural is a complete lattice with product has meet.

2 Automata and algebra for overlapping tiles

Definition 1 (Tile automata [10]). A (finite) tile automaton on the alphabet A is a triple $\mathcal{A} = \langle Q, \delta, K \rangle$ such that Q is a (finite) set of states, $\delta : A \to \mathcal{P}(Q \times Q)$ is the transition function, and $K \subseteq Q \times Q$ is the set of accepting pairs of states.

Given $\delta^* : A^* \to \mathcal{P}(Q \times Q)$ the closure of the transition function inductively defined by $\delta(1) = \{(q, q) \in Q \times Q : q \in Q\}$ and $\delta(wa) = \{(p, q) \in Q \times Q : \exists r \in Q, (p, r) \in \delta^*(w), (r, q) \in \delta(a)\}$ for every $a \in A$ and $w \in A^*$, a *run* of the tile automaton $\mathcal{A}$ on a defined tile $u = (u_1, u_2, u_3)$ is defined as a pair of states $(p, q) \in Q \times Q$, such that there is a start state $s \in Q$ and an end state $e \in Q$ such that $(s, p) \in \delta^*(u_1)$, $(p, q) \in \delta^*(u_2)$ and $(q, e) \in \delta^*(u_3)$. By convention, there exists no run of the automaton $\mathcal{A}$ on the undefined tile 0.

For every tile $u \in T_0^+(A)$, we write $\varphi_{\mathcal{A}}(u) \subseteq Q \times Q$ for the set of runs of the automaton $\mathcal{A}$ on the tile u. The language $L(\mathcal{A})$ of tiles recognized by the automaton $\mathcal{A}$ is defined by $L(\mathcal{A}) = \{u \in T_0^+(A) : \varphi_{\mathcal{A}}(u) \cap K \neq \emptyset\}$.

By definition, every such recognized language is upward closed in the natural order. We have already shown that:

Theorem 2 (Logical characterization [10]). *A language of tiles $L \subseteq T_0^+(A)$ is recognizable by a finite state tile automaton if and only if it does not contain zero, it is upward closed and definable in Monadic Second Order (MSO) logic.*

And, as a corollary:

Corollary 3 (Closure property [10]). *The union, intersection, product and star of languages recognizable by finite state tile automata are recognizable by finite state tile automata.*

Remark. For every language $L \subseteq T_0^+(A)$ recognizable by a (finite) tile automaton $\mathcal{A}$, we have $L = \varphi_{\mathcal{A}}^{-1}(\varphi_{\mathcal{A}}(L))$, i.e. the language L is recognized by the mapping $\varphi_{\mathcal{A}}$. However, although the set $\mathcal{P}(Q \times Q)$ can seen as a monoid with product $X \cdot Y = \{(p, q) \in Q \times Q : \exists r \in Q, (p, r) \in X, (r, q) \in Y\}$ for every X and

$Y \subseteq Q \times Q$, the mapping $\varphi_{\mathcal{A}} : T_0^+(A) \to \mathcal{P}(Q \times Q)$ is not a monoid morphism. Indeed, we only have $\varphi_{\mathcal{A}}(u \cdot v) \subseteq \varphi_{\mathcal{A}}(u) \cdot \varphi_{\mathcal{A}}$, that is, the mapping $\varphi_{\mathcal{A}}$ is a $\vee$-premorphism (see [6]). This observation leads us in [7] and [10] to make the properties of both the monoid $\mathcal{P}(Q \times Q)$ and the premorphism $\varphi_{\mathcal{A}}$ explicit in order to define an effective notion of algebraic recognizability called here quasi-recognizability.

The notion of quasi-recognizability itself is the main object of study in the present paper and is a refined version of the one proposed in [10] and [8].

The recognizers we use are called E-preordered monoid in reference to Ehresmann monoids defined in [15].

Definition 4 (Ehresmann preordered monoid). A *preordered monoid* is a monoid S equipped with a preorder relation $\preccurlyeq$, i.e. a reflexive and transitive relation, that is stable under product, i.e. for every $x, y, z \in S$, if $x \preccurlyeq y$ then $zx \preccurlyeq zy$ and $xz \preccurlyeq xy$.

Such a preordered monoid is said to be an *Ehresmann preordered monoid*, or just *E-monoid*, when it satisfies the following properties:

(A0) S possesses a minimum 0, i.e. for any $x \in S$, $0 \preccurlyeq x$ and if $x \preccurlyeq 0$ then $x = 0$,

(A1) relation $\preccurlyeq$ restricted to the set $U(S)$ is an order and the set $U(S) = \{x \in S \mid x \preccurlyeq 1\}$ of subunits of S ordered by $\preccurlyeq$ is a $\wedge$-semilattice with $\wedge$ as product,

(A2) the left projection $x^L = min\{y \in U(S) \mid xy = x\}$ and the right projection $x^R = min\{y \in U(S) \mid yx = x\}$ are defined for every $x \in S$,

(A3) left and right projections are monotonic, i.e. if $x \preccurlyeq y$ then $x^L \preccurlyeq y^L$ and $x^R \preccurlyeq y^R$ for every x and $y \in S$,

(A4) left and right projections induce right and left semi-congruence, i.e. we have $(xy)^L = (x^L y)^L$ and $(xy)^R = (xy^R)^R$ for every x and $y \in S$.

Remark. One easily checks that Property (A1) implies that all subunits are idempotents and commute since the product is a meet for the order induced on the subunits by the preorder on S.

One can also check that in the case where S is finite then Property (A1) implies Property (A2) since we have $x^R = \prod\{z \leq 1 : zx = x\}$ and $x^L = \prod\{z \leq 1 : xz = x\}$. In all cases, whenever $x \leq 1$ then we have $x^L = x = x^R$ hence the mappings $x \mapsto x^L$ and $x \mapsto x^R$ are indeed projections.

Surprisingly, the link between the monotonicity hypothesis (Property (A3)) and Properties (A0) to (A2) is far from being clear. Intuition says that, in the finite case at least, the Property (A3) may well be implied by the previous ones.

Property (A4) indeed equivalently says that the equivalence induced by the left (resp. right) projection is a right (resp. left) congruence. Indeed, assume that $x^L = y^L$ then, for every $z \in S$, we have $x^L z = y^L z$, and, by applying Property (A4), we have $(xz)^L = (x^L z)^L$ and $(yz)^L = (x^L z)^L$ and thus $(xz)^L = (yz)^L$. A symmetrical argument proves the right case.

Last, left and right projections are related with Green left and right preorders as follows. Recall that for a semigroup S, the left and right Green's preorders are defined, for every $x, y \in S$ by $x \leq_{\mathcal{R}} y$ when $x = yz$ for some $z \in S$ and $x \leq_{\mathcal{L}} y$ when $x = zy$ for some $z \in S$. If we assume that S is an E-monoid then one can easily check that for every x and $y \in S$ we have that if $x \leq_{\mathcal{R}} y$ then $x^R \leq y^R$ and if $x \leq_{\mathcal{L}} y$ then $x^L \leq y^L$, i.e. left and right projections are refinements of the left and right Green's classes.

Examples. Examples of E-monoids are numerous. First, every semigroup extended with a zero and trivially ordered by the relation $x \leq y$ when $x = 0$ or $x = y$ is an E-monoid. Every inverse semigroup, possibly extended with a zero, and ordered by the natural order (see [16]) is also an E-monoid with projection $x^L = x^{-1}x$ and $x^R = xx^{-1}$ for every element x. Every submonoid with a zero of an inverse monoid naturally ordered and closed under left and right projection as above is also an E-monoid. In particular, the monoid $\mathcal{T}^+(A)$ of positive tiles and naturally ordered is an E-monoid.

Less obvious, one can check that the monoid $\mathcal{P}(Q \times Q)$ of relations over the set Q, ordered by inclusion, is also an E-monoid with projection $X^L = \{(q, q) \in Q \times Q : \exists p \in Q, (p, q) \in X\}$ and $X^R = \{(p, p) \in Q \times Q : \exists q \in Q, (p, q) \in X\}$ for every $X \subseteq Q \times Q$. Every submonoid of $\mathcal{P}(Q \times Q)$ that contains $\emptyset$ (the zero for the relation product) and that is closed under the above left and right projections is an E-monoid.

Examples of E-monoids with preorders that are not partial order relations arise in the remainder of the text when defining the E-monoid of decompositions.

The following definition is an extension of the well-known notion of restricted product in inverse semigroup theory.

Definition 5 (Restricted product). Let S be an E-monoid. For every $x, y \in S$, the restricted product $x \bullet y$ of x and y is defined when $x^L = y^R$ and, in that case, it equals xy. In the sequel, we shall write $\exists x \bullet y$ to denote both the fact that the restricted product $x \bullet y$ is defined and, if needed, its value.

The restricted product is extended to subsets of S by taking $X \bullet Y = \{xy \in S : x \in X, y \in Y, \exists x \bullet y\}$.

Remark. The restricted product is associative in the sense that, for every $x, y, z \in S$, we have $\exists x \bullet (\exists y \bullet z) = \exists(\exists x \bullet y) \bullet z$. Indeed, this is a direct consequence of Property (A4) hence the fact that for every $x, y, z \in S$, we have $(x \bullet y)^R = (xy^R)^R = (xx^L)^R = x^R$ and $(y \bullet z)^L = (y^L \bullet z)^L = (z^R \bullet z)^L = z^L$. In that case, we simply write $\exists x \bullet y \bullet z$.

We can now define adequate premorphisms and quasi-recognizability .

Definition 6. Let S be an E-monoid. A premorphism $\varphi : \mathcal{T}^+(A) \to S$ is a monotonic mapping such that $\varphi(uv) \preceq \varphi(u)f(v)$. It is called adequate when, moreover, it satisfies the following properties :

- ▷ it preserves left and right projections, i.e. for every $u \in \mathcal{T}^+(A)$, we have $\varphi(u^R) = \varphi(u)^R$ and $\varphi(u^L) = \varphi(u)^L$,

- ▷ it preserves disjoint product, i.e. for every tile $u = (u_1, u_2, u_2)$ and $v = (v_1, v_2, v_3)$, if $u_3 = 1 = v_1$ then $\varphi(uv) = \varphi(u)\varphi(v)$,
- ▷ it also preserves restricted product, i.e. for every $u, v \in \mathcal{T}^+(A)$ such that $\exists u \bullet v$, we have $\varphi(u \bullet v) = \varphi(u) \bullet \varphi(v)$.

Remark. We have already seen that for very set Q, the monoid $\mathcal{P}(Q \times Q)$ ordered by inclusion is an E-monoid. One can check that the mapping $\varphi_{\mathcal{A}} : \mathcal{T}^+(A) \to \mathcal{P}(Q \times Q)$ defined out a tile automaton $\mathcal{A}$ as above in an adequate premorphism.

It must be mentioned that in [10], the adequate premorphism are not required to preserve restricted product. The fact is that, when arbitrary tiles are involved as in [10], it may be the case that the premorphism $\varphi_{\mathcal{A}}$ *does not* preserve the restricted product. So the definition of adequacy given here is really suited to the case of positive tiles.

Definition 7. A language $L \subseteq T(A)$ of non zero positive tiles is quasi-recognizable (QR) when there exists a finite E-monoid S and an adequate premorphism $\varphi : \mathcal{T}^+(A) \to S$ such that $L = \varphi^{-1}(\varphi(L))$.

One can easily drive from the analogous statement proved in [8] for birooted trees and with an automata theoretic proof:

Theorem 8 (Logical characterization [8]). *A language $L \subseteq T(A)$ of non zero positive tiles is quasi-recognizable (QR) is quasi-recognizable if and only if it is a finite boolean combination of upward closed MSO definable languages of non zero positive tiles.*

3 Restricted decompositions monoid

We aim at providing algebraic tools for the product of two languages of positive tiles. Beware that by the product of two languages X and $Y \subseteq T_0^+(A)$ we mean the product $X \cdot Y = \{xy \in T_0^+(A) : x \in X, y \in Y, xy \neq 0\}$, i.e. the undefined tile is systematically omitted from the resulting point-wise product.

Though our logical characterization of languages of positive tiles by finite state automata or finite E-monoids (and adequate premorphism) guarantees that these classes are closed under product, the need to account for all the different configurations that may arise makes such a construction lengthy and tedious.

The algebraic tools developed here are thus defined for the restricted product and, fortunately, the arbitrary product can still be expressed quite simply in terms of the restricted one.

We define in this section out of any E-monoid, the monoid of its decompositions and show how this construction preserves in some sense quasi-recognizability on positive tiles.

Then, in the next section, such a restricted decomposition monoid can used for achieving an algebraic proof that the restricted product (and henceforth the product) of two quasi-recognizable languages is quasi-recognizable.

From now on, let S be an E-monoid preordered by the relation $\preccurlyeq$. The relation $\preccurlyeq$ is extended to pairs over $S \times S$ by taking the product preorder defined by $(x,x') \preccurlyeq (y,y')$ when $x \preccurlyeq x'$ and $y \preccurlyeq y'$ for every $(x,x'),(y,y') \in S \times S$. It is then extended to $\mathcal{P}(S \times S)$ by taking $X \preccurlyeq Y$ when for every $x \in X$, there exists $y \in Y$ so that $x \preccurlyeq y$, for every $X, Y \in \mathcal{P}(S \times S)$. Similar constructions can be found in the context of ordered semigroup in [21].

Definition 9 (Restricted decompositions monoid). We define the set $\mathcal{D}^r(S) \subseteq \mathcal{P}(S \times S)$, preordered by $\preccurlyeq$, by

$$\mathcal{D}^r(S) = \{X \in \mathcal{P}(S \times S) \mid \exists c \in S,\ (c, c^L) \in X,\\ (c^R, c) \in X, \forall (x,y) \in X,\ x \bullet y = c\}.$$

The product $*$ is defined for every $(x,x'),(y,y') \in S \times S$ by:

$$(x,x') * (y,y') = \{(x(x'yy')^R, x^L x'yy'), (xx'yy'^R, (xx'y)^L y')\}.$$

and extended to $\mathcal{D}^r(S)$ in a point-wise manner, that is, for every $X, Y \in \mathcal{D}^r(S)$, by

$$X * Y = \bigcup_{\substack{(x,x') \in X \\ (y,y') \in Y}} (x,x') * (y,y').$$

Lemma 10. *The set $\mathcal{D}^r(S)$ equipped with the product $*$ and ordered by the relation $\preceq$ is an Ehresmann preordered monoid.*

Proof. The detailed proof is a little long (it is detailed in the appendix) but presents no real difficulties as soon as the appropriate definition has been found. □

Let then $L \subseteq \mathcal{T}^+(A)$ be a language recognized by adequate premorphism $\varphi : \mathcal{T}^+(A) \longrightarrow S$ into the E-monoid S. We build out of φ an adequate premorphism from $\mathcal{T}^+(A)$ to the decomposition $\mathcal{D}^r(S)$ that still recognizes L.

For that purpose, let $\psi : \mathcal{T}^+(A) \to \mathcal{D}^r(S)$ be defined, for every $u \in \mathcal{T}^+(A)$ by $\psi(u) = \{(\varphi(u_1), \varphi(u_2)) \mid u = \exists u_1 \bullet u_2\}$.

Lemma 11. *The mapping $\psi : \mathcal{T}^+(A) \to DrS$ is an adequate premorphism that recognizes L.*

Proof. The detailed proof is a little long (it is detailed in the appendix) but, again, presents no real difficulties as soon as the appropriate definitions have been found. □

4 Application to the restricted and unrestricted products of languages

In the previous section, given any adequate premorphism $\varphi : \mathcal{T}^+(A) \to S$ we have defined $\psi : \mathcal{T}^+(A) \to \mathcal{D}^r(S)$ that allows for computing φ on the two components of any restricted decomposition of a positive tile. In some sense, for every positive tile u, when u is seen as a FO-structure with edges labeled over the alphabet A, this construction allows for simulating any existential first order quantification over the vertices between and including the input root and the output root.

This intuition is used here to prove our main theorem:

Theorem 12. *Let $L_1, L_2 \subseteq \mathcal{T}^+(A)$ quasi-recognizable languages, the language $L_1 \bullet L_2$ is quasi-recognizable.*

Proof. Let S_1, S_2 E-monoids and $L_1, L_2 \subseteq \mathcal{T}^+(A)$ respectively recognized by adequate premorphisms $\varphi_1 : \mathcal{T}^+(A) \longrightarrow S_1$ and $\varphi_2 : \mathcal{T}^+(A) \longrightarrow S_2$. First, we define

$$\begin{aligned} \varphi : \mathcal{T}^+(A) &\longrightarrow S_1 \times S_2 \\ u &\longrightarrow (\varphi_1(u), \varphi_2(u)) \end{aligned}$$

Remark that $S_1 \times S_2$ is an E-monoid and φ is an adequate premorphism recognizing both L_1 and L_2.

We now consider premorphism $\psi : \mathcal{T}^+(A) \to \mathcal{D}^r(S_1 \times S_2)$ as defined in the previous section from the adequate premorphism φ. By Lemma 10, the monoid $\mathcal{D}^r(S_1 \times S_2)$ is an E-monoid and, by Lemma 11, the mapping ψ is an adequate premorphism. We will now prove that ψ recognizes $L_1 \bullet L_2$.

Let $u_1 \in L_1$ and $u_2 \in L_2$ so that $\exists u_1 \bullet u_2$, and let $v \in \mathcal{T}^+(A)$ so that $\psi(v) = p(u_1 \bullet u_2)$. So we have $(\varphi(u_1), \varphi(u_2)) \in \psi(v)$, therefore there exists $v_1, v_2 \in \mathcal{T}^+(A)$ so that $v_1 \bullet v_2 = v$ and

$$(\varphi(u_1), \varphi(u_2)) = (\varphi(v_1), \varphi(v_2)).$$

Since φ recognizes L_1 and L_2, we have $v_1 \in L_1$ and $v_2 \in L_2$. Consequently, $v = v_1 \bullet v_2 \in L_1 \bullet L_2$. ☐

We aim now at applying the restricted product case to the general product case.

Lemma 13. *Let $L_1, L_2 \subseteq \mathcal{T}^+(A)$ quasi-recognizable languages. We have*

$$\begin{aligned} L_1 L_2 = \Big((A^*)^L L_1 (A^*)^R \bullet L_2\Big) &\cup \Big(L_1 \bullet (A^*)^L L_2 (A^*)^R\Big) \\ &\cup \Big((A^*)^L L_1 \bullet L_2 (A^*)^R\Big) \cup \Big(L_1 (A^*)^R \bullet (A^*)^L L_2\Big) \end{aligned}$$

Proof. We first show that

$$\begin{aligned} L_1 L_2 \subseteq \Big((A^*)^L L_1 (A^*)^R \bullet L_2\Big) &\cup \Big(L_1 \bullet (A^*)^L L_2 (A^*)^R\Big) \\ &\cup \Big((A^*)^L L_1 \bullet L_2 (A^*)^R\Big) \cup \Big(L_1 (A^*)^R \bullet (A^*)^L L_2\Big) \end{aligned}$$

Let $u = (u_1, u_2, u_3) \in L_1$, $v = (v_1, v_2, v_3) \in L_2$, so that $uv \neq 0$. By definition of the product, we have four possibilities :

- ▷ if v_1 is a suffix of u_1u_2 and u_3 is a prefix of v_2v_3, then $wv_1 = u_1u_2$ for a $w \in A^*$ and $u_3w' = v_2v_3$ for a $w' \in A^*$, so

$$uv = (u_1, u_2, u_3w') \bullet (wv_1, v_2, v_3) \in L_1(A^*)^R \bullet (A^*)^L L_2,$$

- ▷ if u_1u_2 is a suffix of v_1 and u_3 is a prefix of v_2v_3, then $wu_1u_2 = v_1$ for a $w \in A^*$ and $u_3w' = v_2v_3$ for a $w' \in A^*$, so

$$uv = (wu_1, u_2, u_3w') \bullet (v_1, v_2, v_3) \in (A^*)^L L_1(A^*)^R \bullet L_2,$$

- ▷ if v_1 is a suffix of u_1u_2 and v_2v_3 is a prefix of u_3, then $wv_1 = u_1u_2$ for a $w \in A^*$ and $v_2v_3w' = u_3$ for a $w' \in A^*$, so

$$uv = (u_1, u_2, u_3) \bullet (wv_1, v_2, v_3w') \in L_1 \bullet (A^*)^L L_2(A^*)^R,$$

- ▷ if u_1u_2 is a suffix of v_1 and v_2v_3 is a prefix of u_3, then $wu_1u_2 = v_1$ for a $w \in A^*$ and $v_2v_3w' = u_3$ for a $w' \in A^*$, so

$$uv = (wu_1, u_2, u_3) \bullet (v_1, v_2, v_3w') \in L_1 \bullet (A^*)^L L_2(A^*)^R.$$

Conversely, let $u = (u_1, u_2, u_3) \in L_1$, $v = (v_1, v_2, v_3) \in L_2$, and $w, w' \in A^*$. If $\exists (wu_1, u_2, u_3w') \bullet (v_1, v_2, v_3) = t$, or if $\exists (u_1, u_2, u_3w) \bullet (w'v_1, v_2, v_3) = t$, or if $\exists (wu_1, u_2, u_3) \bullet (v_1, v_2, v_3w') = t$, or if $\exists (u_1, u_2, u_3) \bullet (wv_1, v_2, v_3w') = t$, then $t = uv$.

□

We then have to show that these "completions" on the right or left (the product with $(A^*)^L$ or $(A^*)^R$) preserves quasi-recognizability. First, we prove that it preserves the recognizability by automaton with simple constructions.

Lemma 14. *Let $L \subseteq \mathcal{T}^+(A)$ be a language recognized by an automaton $\mathcal{A}$, therefore there exists automatons $\mathcal{A}_r$ and $\mathcal{A}_l$ that recognize respectively $L(A^*)^R$ and $(A^*)^L L$.*

Proof. Let $\mathcal{A} = \langle Q, \delta, K \rangle$ be an automaton recognizing a language $L \subseteq \mathcal{T}^+(A)$, we define $\mathcal{A}_l = \langle Q \cup *, \delta_l, K \rangle$ and $\mathcal{A}_r = \langle Q \cup *, \delta_r, K \rangle$, with for any a, $\delta_l(a) = \delta(a) \cup \{(*, *), (*, q) \mid q \in Q\}$ and $\delta_r(a) = \delta(a) \cup \{(*, *), (q, *) \mid q \in Q\}$.
We see that any tile of the type $(a_1a_2 \dots a_ku, v, w)$, with $(u, v, w) \in L$, is recognized by $\mathcal{A}_l$, by a run of the form $*a_1 * a_2 * \dots * a_kR$, R being a run of $\mathcal{A}$ on (u, v, w).
Reciprocally, any run we have tile (u, v, w) by $\mathcal{A}_l$ is of the form $*a_1*a_2*\dots*a_kR$, where $a_1a_2 \dots a_k$ is a prefix of u, and R a $*$-less run on (u', v, w) where $u = a_1a_2 \dots a_ku'$, i.e. a run of $\mathcal{A}$ on (u', v, w). Therefore, if (u, v, w) is recognized by $\mathcal{A}_l$, then (u', v, w) is recognized by $\mathcal{A}$, so $(u, v, w) \in (A^*)^L L$.
We demonstrate symmetrically that $\mathcal{A}_r$ recognizes $L(A^*)^R$. □

We can now show that these "completions" on the right or left preserves quasi-recognizability. This is accomplished by noting that quasi-recognizable languages are combinations of upward-closed languages, that can be recognized by automatons.

Lemma 15. *Let $L \subseteq \mathcal{T}^+(A)$ be a quasi-recognizable language, therefore $L(A^*)^R$ and $(A^*)^L L$ are quasi-recognizable.*

Proof. Let $L \subseteq \mathcal{T}^+(A)$ be a quasi-recognizable language, then L is a linear combination of languages recognized by automatons. As a consequence,

$$L = \bigcup_{1 \preccurlyeq i \preccurlyeq n} (D_i \cap U_i)$$

where $n \in \mathbb{N}$ and for any i, U_1 is a quasi-recognizable upward-closed language (i.e. recognized by an automaton) and U_i is a quasi-recognizable downward-closed language (i.e. the complement of one recognized by an automaton).Therefore,

$$\begin{aligned} L(A^*)^R &= \Big(\bigcup_{1 \preccurlyeq i \preccurlyeq n} (D_i \cap U_i) \Big) (A^*)^R \\ &= \bigcup_{1 \preccurlyeq i \preccurlyeq n} \Big(D_i (A^*)^R \cap U_i (A^*)^R \Big) \end{aligned}$$

For any i, since D_i is downward closed, $D_i(A^*)^R = D_i$, and lemme 14 shows that $U_i(A^*)^R$ is quasi-recognizable. Therefore, since $\cup$ and $\cap$ preserve quasi-recognizability, $L(A^*)^R$ is quasi-recognizable. ☐

Corollary 16. *Let $L \subseteq \mathcal{T}^+(A)$ be a quasi-recognizable language, therefore $(A^*)^L L(A^*)^R$ is quasi-recognizable.*

Theorem 17. *Let $L_1, L_2 \subseteq \mathcal{T}^+(A)$ quasi-recognizable languages, therefore $L_1 L_2$ is quasi-recognizable.*

Proof. This follows directly from lemma 13, lemma 15 and corollary 16, and theorem 12. ☐

5 Conclusion

We have thus shown, by means of algebraic tools, that both the restricted product (Theorem 12) and the arbitrary product (Theorem 17) of two quasi-recognizable languages of positive tiles are quasi-recognizable. By using the notion of restricted decomposition monoid (Definition 9) we have eventually extended to positive tiles classical algebraic techniques that, over words, are used to simulate existential FO-quantification when letters are modeled by labeled graph edges. We do believe that a similar technique can be used to prove the closure under iterated product (Kleene star).

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