Speed of coming down from infinity for birth and death processes

Vincent Bansaye, Sylvie Méléard, Mathieu Richard

Speed of coming down from infinity for birth and death processes

Vincent Bansaye[*], Sylvie Méléard[†], Mathieu Richard[‡]

April 30, 2015

Abstract

We finely describe the speed of "coming down from infinity" for birth and death processes which eventually become extinct. Under general assumptions on the birth and death rates, we firstly determine the behavior of the successive hitting times of large integers. We put in light two different regimes depending on whether the mean time for the process to go from $n+1$ to n is negligible or not compared to the mean time to reach n from infinity. In the first regime, the coming down from infinity is very fast and the convergence is weak. In the second regime, the coming down from infinity is gradual and a law of large numbers and a central limit theorem for the hitting times sequence hold. By an inversion procedure, we deduce that the process is a.s. equivalent to a non-increasing function when the time goes to zero. Our results are illustrated by several examples including applications to population dynamics and population genetics. The particular case where the death rate varies regularly is studied in details.

Key words: Birth and death processes, Coming down from infinity, Hitting times, Central limit theorem.

MSC 2010: 60J27, 60J75, 60F15, 60F05, 60F10, 92D25.

1 Introduction and main results

Our goal in this paper is to finely describe the "coming down from infinity" for a birth and death process. We are motivated by the study of population dynamics and population genetics models with initially large populations. For this purpose, we first decompose the trajectory of the process with respect to the hitting times of large integers. We then study the small time behavior of the continuous time process when it comes down from infinity.

The population size is modeled by a birth and death process $(X(t), t \geq 0)$ whose birth rate (resp. death rate) at state $n \in \mathbb{N}$ is λ_n (resp. μ_n). In the whole paper, the rates λ_n are nonnegative and the rates μ_n are positive for $n \geq 1$. Moreover, we assume that $\mu_0 = \lambda_0 = 0$ for practical purpose. The latter implies that 0 is an absorbing state. Such processes have

---

[*]CMAP, Ecole Polytechnique, CNRS, route de Saclay, 91128 Palaiseau Cedex-France; E-mail: vincent.bansaye@polytechnique.edu

[†]CMAP, Ecole Polytechnique, CNRS, route de Saclay, 91128 Palaiseau Cedex-France; E-mail: sylvie.meleard@polytechnique.edu

[‡]CMAP, Ecole Polytechnique, CNRS, route de Saclay, 91128 Palaiseau Cedex-France; E-mail: mathieu.richard@cmap.polytechnique.fr

been extensively studied from the pioneering works on extinction [9] and quasi-stationary distribution [19].

It is well known [9, 10] that

$$\sum_{i\geq 1}\frac{1}{\lambda_i\pi_i}=\infty \tag{1}$$

is a necessary and sufficient condition for almost sure absorption of the process at 0, where

$$\pi_1 = \frac{1}{\mu_1} \quad \text{and for} \quad n \geq 2, \ \pi_n = \frac{\lambda_1\cdots\lambda_{n-1}}{\mu_1\cdots\mu_n}.$$

Under Condition (1), we first define the law $\mathbb{P}_\infty$ of the process starting from infinity with values in $\mathbb{N}\cup\{\infty\}$ (see Lemma 2.1) as the limit of the laws $\mathbb{P}_n$ of the process issued from n.

When the limiting process is non-degenerate, it hits finite values in finite time with positive probability. This behavior is captured by the notion of "coming down from infinity". A key role is played by the decreasing sequence $(T_n)_{n\geq 0}$ of hitting times defined as

$$T_n := \inf\{t \geq 0, X(t) = n\}.$$

As proved in [17, p.384] and in [2, Chap.3],

$$\mathbb{E}_1(T_0)=\sum_{i\geq 1}\pi_i \quad \text{and} \quad \mathbb{E}_{n+1}(T_n)=\frac{1}{\lambda_n\pi_n}\sum_{i\geq n+1}\pi_i=\sum_{i\geq n+1}\frac{\lambda_{n+1}\cdots\lambda_{i-1}}{\mu_{n+1}\cdots\mu_i}, \quad \text{for } n \geq 1. \tag{2}$$

Remark that in case of pure-death process, the law of T_n under $\mathbb{P}_{n+1}$ is exponential with parameter μ_{n+1} and for $n \in \mathbb{N}$, $\mathbb{E}_{n+1}(T_n) = \frac{1}{\mu_{n+1}}$.

Characterizations of the coming down from infinity have been given in [3, 7]. They rely on the convergence of the mean time of absorption when the initial condition goes to infinity or equivalently to the convergence of the series

$$S = \lim_{n\to\infty}\mathbb{E}_n(T_0)=\sum_{i\geq 1}\pi_i+\sum_{n\geq 1}\frac{1}{\lambda_n\pi_n}\sum_{i\geq n+1}\pi_i=\sum_{n\geq 0}\left(\sum_{i\geq n+1}\frac{\lambda_{n+1}\cdots\lambda_{i-1}}{\mu_{n+1}\cdots\mu_i}\right)<+\infty. \tag{3}$$

This is equivalent to the existence and uniqueness of the quasi-stationary distribution at 0 (see [19], [7]) and to the finiteness of some exponential moments of T_0. Furthermore, monotonicity properties allow us to show that this is also equivalent to instantaneous almost-sure coming down from infinity (Proposition 2.5).

In the whole paper, we suppose that Assumption (1) holds and from Section 3 onward, we assume that (3) is satisfied, that is, the process instantaneously comes down from infinity. It guarantees the finiteness of all moments of T_n under $\mathbb{P}_{n+1}$ and under $\mathbb{P}_\infty$, for which we have an explicit expression (Proposition 2.2). In Section 3, we put in light two different regimes for the asymptotic behavior of $T_n/\mathbb{E}_\infty(T_n)$, depending on whether the mean time to go from $n+1$ to n is negligible or not compared to the mean time to reach n from ∞. In the first regime, the coming down from infinity is very fast and the limit is random. In the second one, the coming down is gradual and due to the accumulation of small independent contributions, which leads to a law of large numbers. More precisely, we assume that

$$\frac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)}=\frac{\frac{1}{\lambda_n\pi_n}\sum_{i\geq n+1}\pi_i}{\sum_{j\geq n}\frac{1}{\lambda_j\pi_j}\sum_{i\geq j+1}\pi_i}\xrightarrow[n\to+\infty]{}\alpha.$$

In the first (fast) regime ($\alpha > 0$) and under the additional assumption, which is stronger than (1):

$$l := \lim_{n \to +\infty} \frac{\lambda_n}{\mu_n} < 1, \tag{4}$$

we prove that $T_n/\mathbb{E}_\infty(T_n)$ converges in law to a non-degenerate random variable whose distribution is characterized by l and α.

In the second (gradual) regime ($\alpha = 0$), we prove a weak law of large numbers under the following second moment assumption

$$\sup_{n \geq 0} \frac{\mathbb{E}_{n+1}(T_n^2)}{(\mathbb{E}_{n+1}(T_n))^2} < +\infty. \tag{5}$$

More precisely, we prove that the sequence $(T_n/\mathbb{E}_\infty(T_n))$ converges in probability to 1. Under some additional variance assumptions, we also obtain a central limit theorem.

Thanks to (2) and to forthcoming (8), we note that both expectations in (5) can be written in terms of the birth and death rates. Condition (5) is fulfilled in many cases we have in mind. For instance, it holds for pure death processes.

We will see in the next section some more tractable conditions ensuring (1), (3), (4) and (5).

In the second regime, under (5) and the following additional condition

$$\sum_{n \geq 0} \left(\frac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)} \right)^2 < +\infty, \tag{6}$$

which means that the convergence of $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n)$ to 0 is fast enough, one also get a strong law of large numbers for $T_n/\mathbb{E}_\infty(T_n)$.

We then derive in Section 4 the small time behavior of the process X. We prove that

$$\lim_{t \to 0} \frac{X(t)}{v(t)} = 1,$$

where v is the generalized inverse function of $n \mapsto \mathbb{E}_\infty(T_n) = \sum_{j \geq n} \frac{1}{\lambda_j \pi_j} \sum_{i \geq j+1} \pi_i$:

$$v(t) = \inf\{n \geq 0; \ \mathbb{E}_\infty(T_n) \leq t\}.$$

The limit holds in probability in the first regime. Remark that in this fast case, $T_n/\mathbb{E}_\infty(T_n)$ converges in law (an not in probability) to a random variable but nevertheless, $X(t)$ behaves as $v(t)$ for t small. That is due to the fact that $\mathbb{E}_\infty(T_{[nx]})$ is negligible with respect to $\mathbb{E}_\infty(T_n)$ for any $x > 1$ and for large n. In the second regime, one needs some additional assumptions and almost sure convergence can be obtained. The proof relies on two ingredients: the short time behavior of the non-increasing process equal to n on $[T_n, T_{n-1}[$ and the control of the height of the excursion of the process X during the time interval $[T_n, T_{n-1})$. Technical assumptions are required in the second regime to estimate the variations of $\mathbb{E}_\infty(T_n)$ and to deduce the behavior of X from that of $(T_n)_n$, by a non trivial inversion procedure. Our motivations and applications from population dynamics and population genetics meet these assumptions. Thus, our results cover general birth and death models including many different ecological scenarios,

as competition models with polynomial death rates [16] or Allee effect [12]. Lambert [13] characterizes the distribution of the absorption time for the logistic branching process starting from infinity. Our work extends in different way the case of Kingman coalescent, for which speed of coming down from infinity has already been obtained by Aldous [1]. More generally, in the gradual regime, the behavior of the process coming down from in finity is similar to that of Λ coalescent obtained in [4, 14]. We also note that our approach relies on the decomposition of the trajectory of X with respect to the reaching times of the successive integers and our results could be extended to processes with several births. Another motivation for the results below is the study of birth and death process in random environment and in particular the study of the regulation of the population during unfavorable periods. This latter is a work in progress.

The paper is organized as follows. In the next section, we work under the absorption assumption (1) and prove the existence of the law of the process starting from infinity. Thus we gather general characterizations of the coming down from infinity and we show the equivalence to the a.s. instantaneous coming down. Focusing in Section 3 on birth and death processes satisfying (4) or (5), we describe the hitting times of large integers. In Section 4, we obtain a law of large numbers describing the small time behavior of the process X. Examples and applications are provided all along the paper and illustrate the different regimes. The last Section 5 focuses on regularly varing death rate and provides our main application to small time behavior for population dynamics and population genetics one-dimensional processes coming down from infinity.

2 Preliminaries and coming down from infinity

2.1 Preliminaries

The first lemma allows us to define the law of the process starting from infinity. It is based on monotonicity arguments following Donnelly [8]. We set $\overline{\mathbb{N}} := \{0, 1, \ldots\} \cup \{\infty\}$ and for any $T > 0$, we denote by $\mathbb{D}_{\overline{\mathbb{N}}}([0, T])$ the Skorohod space of càdlàg functions on $[0, T]$ with values in $\overline{\mathbb{N}}$.

Lemma 2.1. *Under* (1), *the sequence* $(\mathbb{P}_n)_n$ *converges weakly in the space of probability measures on* $\mathbb{D}_{\overline{\mathbb{N}}}([0, T])$ *to a probability measure* $\mathbb{P}_\infty$.

At this point, the limiting process is not assumed to be finite for positive times.

Proof. We follow the tightness argument given in the first part of the proof of Theorem 1 by Donnelly in [8]. Indeed, no integer is an instantaneous state for the process ($\lambda_n, \mu_n < \infty$ for each $n \geq 0$) and the process is stochastically monotone with respect to the initial condition. It ensures that Assumption (A1) of [8] holds. In addition, Assumption (1) ensures that the process almost surely does not explode and (A2) of [8, Thm. 1] is also satisfied by denoting B_n^N the birth and death process X issued from n and stopped in N.

Then the tightness holds and we identify the finite marginal distributions by noticing that for $k \geq 1$, for $t_1, \ldots, t_k \geq 0$ and for $a_1, \ldots, a_k \in \mathbb{N}$, the quantities $\mathbb{P}_n(X(t_1) \leq a_1, \cdots, X(t_k) \leq a_k)$ are non-increasing with respect to $n \in \mathbb{N}$ (and thus converge). $\qquad\square$

4

We now focus on the time spent by the process $(X(t), t \geq 0)$ to go from level $n+1$ to level n. For $n \geq 0$, we introduce the function

$$G_n(a) := \mathbb{E}_{n+1}(\exp(-aT_n)), \ a > 0.$$

Proposition 2.2. *Suppose that* (1) *holds. For any $a > 0$ and $n \geq 1$, we have*

$$G_n(a) = 1 + \frac{\mu_n + a}{\lambda_n} - \frac{\mu_n}{\lambda_n}\frac{1}{G_{n-1}(a)}. \tag{7}$$

Moreover, for every $n \geq 0$,

$$\mathbb{E}_{n+1}(T_n^2) = \frac{2}{\lambda_n \pi_n} \sum_{i \geq n} \lambda_i \pi_i \mathbb{E}_{i+1}(T_i)^2, \quad \mathbb{E}_{n+1}(T_n^3) = \frac{6}{\lambda_n \pi_n} \sum_{i \geq n} \lambda_i \pi_i \mathbb{E}_{i+1}(T_i)\text{Var}_{i+1}(T_i). \tag{8}$$

Proof. We denote by τ_n a random variable distributed as T_n under $\mathbb{P}_{n+1}$ and consider the Laplace transform of τ_n. Following [3, p. 264] and by the Markov property, we have

$$\tau_{n-1} \overset{(d)}{=} \mathbf{1}_{\{Y_n=-1\}}E_n + \mathbf{1}_{\{Y_n=1\}}\left(E_n + \tau_n + \tau'_{n-1}\right)$$

where Y_n, E_n, τ'_{n-1} and τ_n are independent random variables, E_n is an exponential random variable with parameter $\lambda_n + \mu_n$ and τ'_{n-1} is distributed as τ_{n-1} and $\mathbb{P}(Y_n = 1) = 1 - \mathbb{P}(Y_n = -1) = \lambda_n/(\lambda_n + \mu_n)$. Hence, we get

$$G_{n-1}(a) = \frac{\lambda_n + \mu_n}{a + \lambda_n + \mu_n}\left(G_n(a)G_{n-1}(a)\frac{\lambda_n}{\lambda_n + \mu_n} + \frac{\mu_n}{\lambda_n + \mu_n}\right)$$

and (7) follows.

Differentiating (7) twice at $a = 0$, we get

$$\mathbb{E}_n(T_{n-1}^2) = \frac{\lambda_n}{\mu_n}\mathbb{E}_{n+1}(T_n^2) + 2\mathbb{E}_n(T_{n-1})^2, \quad n \geq 1.$$

In the particular case when $\lambda_N = 0$ for some $N > n$, a simple induction gives

$$\mathbb{E}_{n+1}(T_n^2) = \frac{2}{\lambda_n \pi_n} \sum_{n \leq i \leq N-1} \lambda_i \pi_i \mathbb{E}_{i+1}(T_i)^2 \tag{9}$$

and (8) is proved. In the general case, let $N > n$. Thanks to Assumption (1), T_0 is finite and the process a.s. does not explode in finite time for any initial condition. Then T_n is finite and $T_N \to +\infty$ $\mathbb{P}_{n+1}$-a.s., where we use the convention $T_N = +\infty$ on the event $\{\forall t \geq 0 : X(t) \neq N\}$. The monotone convergence theorem yields

$$\mathbb{E}_{n+1}(T_n^2; T_n \leq T_N) \underset{N\to+\infty}{\longrightarrow} \mathbb{E}_{n+1}(T_n^2).$$

Let us consider a birth and death process X^N with birth and death rates $(\lambda_k^N, \mu_k^N : k \geq 0)$ such that $(\lambda_k^N, \mu_k^N) = (\lambda_k, \mu_k)$ for $k \neq N$ and $\lambda_N^N = 0, \mu_N^N = \mu_N$.

Since $(X_t : t \leq T_N)$ and $(X_t^N : t \leq T_N^N)$ have the same distribution under $\mathbb{P}_{n+1}$, we get

$$\mathbb{E}_{n+1}\left(T_n^2; T_n \leq T_N\right) = \mathbb{E}_{n+1}\left((T_n^N)^2; T_n^N \leq T_N^N\right),$$

which yields

$$\mathbb{E}_{n+1}(T_n^2) = \lim_{N \to \infty} \mathbb{E}_{n+1}\left((T_n^N)^2; T_n^N \leq T_N^N\right) \leq \lim_{N \to \infty} \mathbb{E}_{n+1}\left((T_n^N)^2\right),$$

where the convergence of the last term is due to the stochastic monotonicity of T_n^N with respect to N under $\mathbb{P}_{n+1}$. Since T_n^N is stochastically smaller than T_n under $\mathbb{P}_{n+1}$, we have also

$$\mathbb{E}_{n+1}((T_n)^2) \geq \mathbb{E}_{n+1}((T_n^N)^2).$$

We deduce that

$$\mathbb{E}_{n+1}((T_n)^2) = \lim_{N \to \infty} \mathbb{E}_{n+1}((T_n^N)^2) = \lim_{N \to \infty} \frac{2}{\lambda_n \pi_n} \sum_{n \leq i \leq N-1} \lambda_i \pi_i \mathbb{E}_{i+1}(T_i^N)^2,$$

where the last identity comes from (9). Adding that $\mathbb{E}_{i+1}(T_i^N)$ is non-decreasing with respect to N yields the expected expression for $\mathbb{E}_{n+1}((T_n)^2)$ by monotone convergence.

The third moment is obtained similarly by differentiating (7) three times, which gives the recurrence equation

$$\mathbb{E}_n(T_{n-1}^3) = \frac{\lambda_n}{\mu_n} \mathbb{E}_{n+1}(T_n^3) + 6\mathbb{E}_n(T_{n-1})\mathrm{Var}_n(T_{n-1}), \quad n \geq 1.$$

The coupling argument we have used above allows us to conclude. $\qquad\square$

Remark 2.3. *Using Proposition 2.2, (5) writes*

$$\sup_{n \geq 0} \sum_{i \geq n} \frac{\lambda_i \pi_i}{\lambda_n \pi_n} \left(\frac{\mathbb{E}_{i+1}(T_i)}{\mathbb{E}_{n+1}(T_n)}\right)^2 < +\infty. \tag{10}$$

2.2 Instantaneous coming down from infinity

We now define a strong notion of coming down from infinity corresponding to the behavior of birth and death processes under (1) and (3): the process comes down instantaneously almost surely.

Definition 2.4. *The process $(X(t), t \geq 0)$ instantaneously comes down from infinity if for any $t > 0$,*

$$\lim_{m \to \infty} \lim_{k \to +\infty} \mathbb{P}_k(T_m < t) = 1. \tag{11}$$

Note that (11) is equivalent to

$$\mathbb{P}_\infty(\forall t > 0, X(t) < +\infty) = 1.$$

Let us now show that (11) is satisfied under (1) and (3). In fact we give several necessary and sufficient conditions for $(X(t), t \geq 0)$ to come down from infinity. The first two ones are directly taken from [7]. We add here an exponential moment criterion. We also mention that it is equivalent to the existence (cf. [19]) and uniqueness (cf. [7]) of a quasi-stationary distribution for the process X.

Proposition 2.5. *Under condition* (1)*, the following assertions are equivalent:*

  (i) *The process* $(X(t), t \geq 0)$ *instantaneously comes down from infinity.*

  (ii) *Assumption* (3) *is satisfied:* $S < +\infty$.

  (iii) $\sup_{k \geq 0} \mathbb{E}_k[T_0] < +\infty$.

  (iv) *For all* $a > 0$*, there exists* $k_a \in \mathbb{N}$ *such that* $\sup_{k \geq k_a} \mathbb{E}_k \left(\exp(aT_{k_a}) \right) < +\infty$.

Proposition 2.5 implies in particular that under (1) and (3), the moments of T_n under $\mathbb{P}_n$ and $\mathbb{P}_\infty$ are finite. Moreover their explicit expression can be derived from Proposition 2.2 and will be useful in the rest of the paper.

Remark 2.6. *It is proved in the next Lemma 2.7 (i), that the equivalence between Assertion (ii), and then (i), (iii), (iv), is equivalent to the convergence of the series* $\sum_{i \geq 1} 1/\mu_i$*, as soon as forthcoming Assumption* (12) *is satisfied.*

Proof of Proposition 2.5. Assertion (i) implies (ii), and (ii) and (iii) are equivalent according to [7, Prop 7.10]. We now prove that (iv) is equivalent to (ii) and that (ii) implies (i).

First, we check that (iv) implies that X comes down from infinity, which means that $+\infty$ is an entrance boundary. Then it well known that (ii) holds (see Section 8.1 in [3] or Proposition 7.10 in [7]). Indeed, taking $a = 1$ in (iv), we have $M := \sup_{k \geq k_1} \mathbb{E}_k \left(\exp(T_{k_1}) \right) < +\infty$. Then, Markov inequality ensures that for all $k \geq k_1$ and $t \geq 0$, $\mathbb{P}_k(T_{k_1} < t) \geq 1 - \exp(-t)M$. Choosing t large enough ensures that the process comes down from infinity.

We then prove that (ii) implies (iv) by adapting the proof of [7, Prop 7.6] to the discrete setting. We fix $a > 0$ and using $S < +\infty$, there exists $k_a > 1$ such that

$$\sum_{n \geq k_a - 1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i \leq \frac{1}{a}.$$

We now define the Lyapounov function J_a as

$$J_a(m) := \begin{cases} \displaystyle\sum_{n=k_a-1}^{m-1} \frac{1}{\lambda_n \pi_n} \sum_{i \geq n+1} \pi_i & \text{if } m \geq k_a\,, \\ 0 & \text{if } m < k_a\,. \end{cases}$$

We notice that J_a is non-decreasing and bounded and we introduce the infinitesimal generator L of X, defined by

$$L(f)(n) = (f(n+1) - f(n))\, \lambda_n + (f(n-1) - f(n))\, \mu_n,$$

for any bounded function f and any $n \geq 1$. Then, the process

$$M_t := e^{at} J_a(X(t)) - \int_0^t e^{au} \left(aJ_a(X(u)) + LJ_a(X(u)) \right) \mathrm{d}u, \qquad (t \geq 0)$$

7

is a martingale with respect to the natural filtration of X. Adding that $LJ_a(m) = -1$ for any $m \geq k_a$ and that $J_a(X(u)) \leq J_a(\infty) \leq 1/a$, we get for all $k \geq k_a$ and $t \geq 0$,

$$
\begin{aligned}
\mathbb{E}_k \left(e^{at \wedge T_{k_a}} J_a(X(t \wedge T_{k_a})) \right) &= \mathbb{E}_k \left(\int_0^{t \wedge T_{k_a}} e^{au} \left(a J_a(X(u)) + L J_a(X(u)) \right) \mathrm{d}u \right) + J_a(k) \\
&= \mathbb{E}_k \left(\int_0^{t \wedge T_{k_a}} e^{au} \left(a J_a(X(u)) - 1 \right) \mathrm{d}u \right) + J_a(k) \\
&\leq J_a(k).
\end{aligned}
$$

Adding that for any $k \geq k_a$, $\mathbb{P}_k$-a.s. $J_a(X(t \wedge T_{k_a})) \geq J_a(k_a)$, we get $\mathbb{E}_k \left(e^{at \wedge T_{k_a}} \right) \leq \frac{J_a(k)}{J_a(k_a)}$. Then (iv) follows from the monotone convergence theorem and Assumption (ii).

It remains to show that (ii) implies (i). On the one hand, according to (2), $\mathbb{E}_\infty(T_n) = \sum_{i \geq n} \mathbb{E}_{i+1}(T_i)$ and Assumption (ii) entails that $\mathbb{E}_\infty(T_n)$ vanishes as $n \to \infty$ as the rest of the finite series S. On the other hand, under $\mathbb{P}_\infty$, the sequence $(T_n)_{n \geq 0}$ decreases to some random variable $T_{[0,\infty)}$. Then, from the monotone convergence theorem, $\mathbb{E}_\infty(T_n)$ decreases to $\mathbb{E}_\infty(T_{[0,\infty)})$ and $\mathbb{E}_\infty(T_{[0,\infty)}) = 0$. It ensures that $T_{[0,\infty)} = 0$ $\mathbb{P}_\infty$ a.s. and X instantaneously comes down from infinity. The proof is then complete. $\qquad\square$

2.3 More tractable conditions

Let us give some tractable conditions ensuring (1), (3), (4) or (5), which will be useful for examples and applications.

Lemma 2.7. (i) *Under Assumption*

$$
\sup_{n,i \geq 1} \frac{\mu_n}{\mu_{n+i}} < +\infty, \qquad \limsup_{n \to \infty} \frac{\lambda_n}{\mu_n} < 1, \tag{12}
$$

Condition (3) *holds if and only if*

$$
\sum_{n \geq 1} \frac{1}{\mu_n} < +\infty. \tag{13}
$$

(ii) *Assuming that*

$$
\sup_{n,i \geq 1} \frac{\mu_n}{\mu_{n+i}} < +\infty, \qquad \frac{\lambda_n}{\mu_n} \xrightarrow[n \to \infty]{} 0, \qquad \sum_{n \geq 1} \frac{1}{\mu_n} < +\infty, \tag{14}
$$

then (1), (3), (4) *and* (5) *are satisfied and*

$$
\mathbb{E}_{n+1}(T_n^k) \underset{n \to +\infty}{\sim} \frac{k!}{\mu_{n+1}^k}, \quad \text{for} \ \ k = 1, 2, 3.
$$

Criterion (13) can be seen as the discrete counterpart of the criterion in [7, p.1953] stating that the Feller diffusion process Z defined by $\mathrm{d}Z_t = \sqrt{\gamma Z_t} \mathrm{d}B_t + Z_t(r - f(Z_t))\mathrm{d}t$ (for a suitable function f and $r > 0$), comes down from infinity if and only if $\int_1^\infty \frac{\mathrm{d}x}{x f(x)} < +\infty$.

Proof. We begin with the proof of point (i). Coming back to (2), the first term of the series giving $\mathbb{E}_{n+1}(T_n)$ is $1/\mu_{n+1}$, hence $\mathbb{E}_{n+1}(T_n) \geq 1/\mu_{n+1}$. Moreover, using the second part of Assumption (12), there is $l' < 1$ such that for n large enough, $\lambda_n/\mu_n \in [0, l')$ and then for n large enough

$$\frac{1}{\mu_{n+1}} \leq \mathbb{E}_{n+1}(T_n) \leq \sum_{j \geq 1} l'^{j-1} \frac{1}{\mu_{n+j}} \leq \frac{1}{1-l'} \frac{1}{\mu_{n+1}} \sup_{n \geq 1, k \geq 0} \frac{\mu_n}{\mu_{n+k}}.$$

Then the first part of Assumption (12) allows to get (i).

Under the assumptions (14), the properties (1), (4) and (5) are obvious, whereas (3) is a consequence of point (i) of the lemma.

To get the asymptotic behavior of the moments of T_n, we use the expression of $\mathbb{E}_{n+1}(T_n^k)$ provided in Proposition 2.2 and the fact that λ_n/μ_n goes to 0. Then, for $k \in \{1, 2, 3\}$, by induction we can write

$$\mathbb{E}_{n+1}(T_n^k) = k! b_{n,k}(1 + A_{n,k}),$$

where $b_{n,1} = 1/\mu_{n+1}$, $b_{n,2} = (\mathbb{E}_{n+1}(T_n))^2$ and $b_{n,3} = \mathbb{E}_{n+1}(T_n)\mathrm{Var}_{n+1}(T_n)$ and $A_{n,k} \to 0$ as $n \to \infty$, which will complete the proof. Indeed, for $k = 1$, we know from (2) that $\mathbb{E}_{n+1}(T_n) = \frac{1}{\mu_{n+1}}\left[1 + \sum_{i \geq n+2} \frac{\lambda_{n+1} \cdots \lambda_{i-1}}{\mu_{n+2} \cdots \mu_i}\right]$. Moreover for every $l' \in (0, 1)$ and n large enough, we have $\lambda_n/\mu_n < l'$ and

$$\frac{\lambda_{n+1} \cdots \lambda_{i-1}}{\mu_{n+2} \cdots \mu_i} \leq l'^{i-(n+1)} \cdot \sup_{n \geq 1, k \geq 0} \frac{\mu_n}{\mu_{n+k}},$$

which ensures that $\mathbb{E}_{n+1}(T_n) \sim 1/\mu_{n+1}$. Combining this equivalence and the expression of $\mathbb{E}_{n+1}(T_n^2)$ provided in Proposition 2.2 yields similarly the asymptotic behavior of the second moment ($k = 2$) and then the third moment ($k = 3$). $\square$

Remark 2.8. *Our original motivations for considering the coming down from infinity of birth and death processes are the regulation of large populations due to competition and the short time behavior of branching coalescing models (see e.g. [15] for some motivations for ancestral graphs). In this context, the birth rate is usually linear, which corresponds to independent reproduction events, or even zero for pure coalescing models. The death rate is often quadratic such as for Kingman coalescent and logistic competition, but polynomial death rate may be relevant, see in particular [16] for a statistical study of the death rate due to competition. Thus, we are interested in the particular case $\lambda_n \leq Cn$ for some $C > 0$ and $\mu_n = n^\rho \log^\gamma n$ with $\rho > 1$. In this case, Assumption (14) is obviously satisfied and then (1), (3), (4) and (5) hold. Proposition 2.5 ensures that the process comes down a.s. instantaneously from infinity and has bounded exponential moments. We refer to Section 5 for the fine description of this coming down from infinity.*

3 Asymptotic behavior of T_n under $\mathbb{P}_\infty$

From now on, we consider sequences $(\lambda_n)_{n \geq 0}$ and $(\mu_n)_{n \geq 0}$ satisfying the hypotheses (1) and (3). Thus, according to Lemma 2.1 and Proposition 2.5, $\mathbb{P}_\infty$ is well-defined and X strongly comes down from infinity. Moreover $T_n < +\infty$ $\mathbb{P}_\infty$ a.s. for any $n \geq 0$. In this section, we study the

asymptotic behavior of T_n as $n \to +\infty$ under $\mathbb{P}_\infty$. Let us recall that $\mathbb{E}_\infty(T_n) = \sum_{i \geq n} \mathbb{E}_{i+1}(T_i)$, so that (2) yields

$$\mathbb{E}_\infty(T_n) = \sum_{i \geq n} \frac{1}{\lambda_i \pi_i} \sum_{j \geq i+1} \pi_j.$$

Then, $S < +\infty$ ensures that $\mathbb{E}_\infty(T_n)$ decreases to 0 as $n \to +\infty$.

In the following two subsections, we compare T_n to its mean $\mathbb{E}_\infty(T_n)$ as $n \to +\infty$. Two regimes appear depending on whether the ratio of mean times $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n)$ converges to a non-degenerate value or vanishes. In the first case (fast regime - Theorem 3.1), the process comes down very quickly from infinity, T_n is then essentially the time spent close to n and renormalizing T_n by its mean yields a random limit. In the second case (gradual regime - Theorem 3.3), T_n can be seen as the contribution of a large number of independent random variables and the limit equals 1.

In both cases, the proofs rely on the fact that $T_n = \sum_{i \geq n} \tau_i$ $\mathbb{P}_\infty$-a.s., where for $n \geq 0$, the random variable τ_n is the time spent between T_{n+1} and T_n :

$$\tau_n := \inf\{t > T_{n+1}; X(t) = n\} - T_{n+1}.$$

By the strong Markov property, the random variables $(\tau_i)_{i \geq 0}$ are independent (under $\mathbb{P}_\infty$) and τ_i is distributed as T_i under $\mathbb{P}_{i+1}$. In the sequel of the section, we use for $n \geq 0$ the notation

$$m_n := \mathbb{E}(\tau_n) = \mathbb{E}_{n+1}(T_n), \quad r_n := \frac{m_n}{\mathbb{E}_\infty(T_n)} = \frac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)}.$$

Examples which illustrate the two regimes and the two convergences are provided in forthcoming Section 3.3, while an application to the regularly varying case is developed in Section 5.

3.1 The fast regime

Theorem 3.1. *We assume that* (1), (3) *and* (4) *hold and*

$$\frac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)} \xrightarrow[n \to +\infty]{} \alpha,$$

with $\alpha \in (0, 1]$. Then,

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \xrightarrow[n \to +\infty]{(d)} Z := \sum_{k \geq 0} \alpha (1 - \alpha)^k Z_k,$$

where $(Z_k)_k$ is a sequence of i.i.d. random variables whose common Laplace transform $G(a) := \mathbb{E}_\infty(\exp(-aZ_0))$ is the unique function $[0, +\infty) \to [0, 1]$ that satisfies

$$\forall a > 0, \quad G(a) \left[l\big(1 - G(a(1 - \alpha))\big) + 1 + a(1 - l(1 - \alpha)) \right] = 1. \tag{15}$$

We note that when $\alpha = 1$, $Z = Z_0$ is an exponential random variable with parameter 1.

Example: If $\mu_n = (n!)^\gamma$ with $\gamma > 0$, $\mathbb{E}_\infty(T_n) \sim ((n+1)!)^{-\gamma}$. Hence, $\lim_{n \to +\infty} \mathbb{E}(\tau_n)/\mathbb{E}_\infty(T_n) = 1$ and Theorem 3.1 (i) yields

$$((n+1)!)^\gamma T_n \xrightarrow[n \to +\infty]{(d)} E,$$

where E is an exponential r.v. with parameter 1. Another example is studied in forthcoming Section 4.2.

Before proving Theorem 3.1, let us show the following key lemma, which focuses on the asymptotic behavior of the distribution of $(\tau_n)_n$.

Lemma 3.2. *If* $\lim_{n \to +\infty} r_n = \alpha \in (0, 1]$, *we have*

$$\frac{\tau_n}{m_n} \xrightarrow[n \to +\infty]{(d)} \zeta,$$

where the Laplace transform of ζ *is the unique solution of* (15).

Proof. Recalling $\lambda_n/\mu_n \to l$ as $n \to \infty$, let us first check that

$$\lim_{n \to +\infty} \frac{\mathbb{E}_\infty(T_{n+1})}{\mathbb{E}_\infty(T_n)} = \lim_{n \to +\infty} \frac{m_{n+1}}{m_n} = 1 - \alpha, \qquad \lim_{n \to +\infty} \mu_n m_{n-1} = \frac{1}{1 - l(1 - \alpha)}. \tag{16}$$

The first part of (16) comes from $\mathbb{E}_\infty(T_{n+1})/\mathbb{E}_\infty(T_n) = 1 - r_n$ and

$$\frac{m_{n+1}}{m_n} = \frac{r_{n+1}}{r_n} \frac{\mathbb{E}_\infty(T_{n+1})}{\mathbb{E}_\infty(T_n)}.$$

Moreover, differentiating (7) at $a = 0$ yields

$$1 = \frac{\lambda_n}{\mu_n} \frac{m_n}{m_{n-1}} + \frac{1}{\mu_n m_{n-1}}$$

and using (4) gives the second part of (16).

Let us prove the uniqueness of the function satisfying (15). For any bounded function $g : [0, +\infty) \to [0, 1]$, we define the function $H(g) : [0, +\infty) \to [0, 1]$ as

$$H(g) : a \longmapsto [1 + a(1 - l(1 - \alpha)) + l(1 - g(a(1 - \alpha)))]^{-1}.$$

For two functions g_1 and g_2 and any $a > 0$, we have

$$|H(g_1)(a) - H(g_2)(a)| = H(g_1)(a)H(g_2)(a)l\,|g_1(a(1 - \alpha)) - g_2(a(1 - \alpha))|$$

and using that for any $a > 0$, $H(g_1)(a) \leq 1$,

$$\|H(g_1) - H(g_2)\|_\infty \leq l\|g_1 - g_2\|_\infty, \tag{17}$$

which ensures the expected uniqueness since $l < 1$.

11

We now prove the convergence in distribution of τ_n/m_n as $n \to +\infty$. For $n \geq 0$, let $F_n : [0, +\infty) \longrightarrow [0, 1]$ be defined as

$$F_n(a) := \mathbb{E}(\exp(-a\tau_n/m_n)) = \mathbb{E}_{n+1}(\exp(-aT_n/m_n)) = G_n\left(\frac{a}{m_n}\right), \quad (a > 0).$$

By (7), for all $a > 0$ and $n \geq 1$, we have

$$G_{n-1}\left(\frac{a}{m_{n-1}}\right) = \left[1 + \frac{a}{\mu_n m_{n-1}} + \frac{\lambda_n}{\mu_n}\left(1 - G_n\left(\frac{a}{m_{n-1}}\right)\right)\right]^{-1},$$

which we rewrite as

$$F_{n-1} = H_n(F_n), \tag{18}$$

where for every function $f : [0, \infty) \to [0, 1]$, $n \geq 1$ and $a \geq 0$,

$$H_n(f)(a) = \left[1 + \frac{a}{\mu_n m_{n-1}} + \frac{\lambda_n}{\mu_n}\left(1 - f\left(a\frac{m_n}{m_{n-1}}\right)\right)\right]^{-1}.$$

Using (16), we have for every $a \geq 0$,

$$\sup_{f \in \mathcal{C}_1^1} |H_n(f)(a) - H(f)(a)| \xrightarrow[n \to +\infty]{} 0,$$

with $\mathcal{C}_1^1 := \{f \in \mathcal{C}^1([0, \infty), [0, 1]) : \|f'\|_\infty \leq 1\}$. Moreover, $F_n = H_{n+1} \circ \ldots \circ H_{n+k}(F_{n+k})$ and by triangle inequality

$$\left|F_n(a) - H^{\circ k}(F_{n+k})(a)\right| \leq \left|H_{n+1}(F_{n+1})(a) - H(F_{n+1})(a)\right|$$
$$+ \left|H(F_{n+1})(a) - H(H^{\circ k-1}(F_{n+k}))(a)\right|.$$

Adding that for every n, $F_n \in \mathcal{C}_1^1$ and recalling (17), we get by induction over $k \geq 0$ that

$$F_n(a) - H^{\circ k}(F_{n+k})(a) \xrightarrow[n \to +\infty]{} 0$$

for all $a \geq 0$ and $k \geq 0$. We use again (17) to obtain that

$$\|H^{\circ k}(F_{n+k}) - H^{\circ k}(\mathbf{1})\|_\infty \leq l^k \|F_{n+k} - 1\|_\infty \leq l^k.$$

Recalling that $l < 1$, we can combine the two last displays and for each $\epsilon > 0$, we can find k such that for n large enough

$$|F_n(a) - H^{\circ k}(\mathbf{1})(a)| \leq 2\epsilon.$$

Thus, $(F_n(a) : n \geq 0)$ is a Cauchy sequence and $F_n(a)$ converges to $F(a)$ on $[0, \infty)$. The fact that $F_n \in \mathcal{C}_1^1$ ensures that this convergence is uniform in each compact set. Letting $n \to \infty$ in (18) then yields $F = H(F)$, which means that F satisfies (15).

Finally, we check that F is the Laplace transform of some random variable by proving that $F(0^+) = \lim_{a \to 0} F(a) = 1$. From (15), $F(0^+)$ is a solution of $l\, F(0^+)^2 - (1+l)F(0^+) + 1 = 0$. If $l = 0$, this equation has the unique root 1. If $l > 0$, the two roots are 1 and $1/l$. But $1/l > 1$ and obviously $F(0^+) \leq 1$, so that $F(0^+) = 1$. That ends the proof of the weak convergence of τ_n/m_n. $\qquad\square$

We can now proceed with the following proof.

Proof of Theorem 3.1. Let $Z = \sum_{k \geq 0} \alpha(1-\alpha)^k Z_k$ be defined as in the statement of the theorem. We need the following elementary result which can be proved thanks to a simple induction: for every $a > 0$ and for all complex numbers $z_1, z_2, \ldots, z_n, u_1, \ldots u_n$ with modulus less than 1

$$\left| \prod_{i=1}^{n} z_i - \prod_{i=1}^{n} u_i \right| \leq \sum_{i=1}^{n} |z_i - u_i| . \tag{19}$$

Then, recalling that $T_n = \sum_{k \geq n} \tau_k$ where the τ_k's are independent,

$$\left| \mathbb{E}_\infty \left(\exp\left(-a \frac{T_n}{\mathbb{E}_\infty(T_n)} \right) \right) - \mathbb{E}\left(\exp\left(-aZ \right) \right) \right|$$

$$= \left| \prod_{k \geq n} \mathbb{E}\left(\exp\left(-a \frac{\tau_k}{\mathbb{E}_\infty(T_n)} \right) \right) - \prod_{k \geq 0} \mathbb{E}\left(\exp\left(-a\alpha(1-\alpha)^k Z_k \right) \right) \right|$$

$$\leq \sum_{k \geq 0} \left| \mathbb{E}\left(\exp\left(-a \frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} \right) \right) - \mathbb{E}\left(\exp\left(-a\alpha(1-\alpha)^k Z_k \right) \right) \right| . \tag{20}$$

From Lemma 3.2, we know that in $\mathbb{P}_\infty$-distribution, τ_n/m_n converges to ζ. Then, thanks to (16) and the fact that $r_n \to \alpha$, we have for $k \geq 0$

$$\frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} = \frac{m_{n+k}}{\mathbb{E}_\infty(T_{n+k})} \prod_{i=1}^{k} \frac{\mathbb{E}_\infty[T_{n+i}]}{\mathbb{E}_\infty[T_{n+i-1}]} \cdot \frac{\tau_{k+n}}{m_{n+k}} \xrightarrow[n \to +\infty]{(d)} \alpha(1-\alpha)^k \zeta.$$

The uniqueness in (15) ensures that the variables $(Z_k)_k$ are distributed as ζ. Then, with the last display, we get that all the terms of the sum in (20) vanish as $n \to +\infty$. We proceed by bounded convergence. Using that $1 - \exp(-x) \leq x$ for any $x \geq 0$, we get for $k, n \geq 0$

$$\left| \mathbb{E}\left(\exp\left(-a \frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} \right) \right) - \mathbb{E}\left(\exp\left(-a\alpha(1-\alpha)^k Z_k \right) \right) \right|$$

$$\leq \left| 1 - \mathbb{E}\left(\exp\left(-a \frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} \right) \right) \right| + \left| 1 - \mathbb{E}\left(\exp\left(-a\alpha(1-\alpha)^k Z_k \right) \right) \right|$$

$$\leq a \frac{m_{k+n}}{\mathbb{E}_\infty(T_n)} + a\alpha(1-\alpha)^k \mathbb{E}(Z_0). \tag{21}$$

By differentiating (15) at 0, one finds $\mathbb{E}_\infty[Z_0] = 1$. Moreover,

$$\frac{m_{n+k}}{\mathbb{E}_\infty(T_n)} = \frac{\mathbb{E}_\infty(T_{n+1})}{\mathbb{E}_\infty(T_n)} \frac{\mathbb{E}_\infty(T_{n+2})}{\mathbb{E}_\infty(T_{n+1})} \cdots \frac{\mathbb{E}_\infty(T_{n+k})}{\mathbb{E}_\infty(T_{n+k-1})} \frac{m_{n+k}}{\mathbb{E}_\infty(T_{n+k})}.$$

Since $m_{k+n}/\mathbb{E}_\infty[T_{k+n}] \leq 1$ and $\mathbb{E}_\infty(T_{n+1})/\mathbb{E}_\infty(T_n) \to 1 - \alpha < 1$ as $n \to +\infty$, there exist $n_0 \in \mathbb{N}$, $\beta < 1$ and $C > 0$ such that $m_{k+n}/\mathbb{E}_\infty(T_n) \leq C\beta^k$ for all $k \geq 0, n \geq n_0$. Thus, coming back to (21), for $n \geq n_0$, we have

$$\left| \mathbb{E}\left(\exp\left(-a \frac{\tau_{k+n}}{\mathbb{E}_\infty(T_n)} \right) \right) - \mathbb{E}\left(\exp\left(-a\alpha(1-\alpha)^k Z_k \right) \right) \right| \leq C\beta^k + a\alpha(1-\alpha)^k.$$

Since the r.h.s. in the last display is summable, the proof is complete. $\square$

3.2 The gradual regime

We now focus on the second regime and specify the fluctuations of T_n. In this case, we will obtain a weak and a strong law of large numbers.

Theorem 3.3. *We assume that* (1), (3) *and* (5) *hold.*

(i) *If* $\dfrac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)} \underset{n\to+\infty}{\longrightarrow} 0$, *then*

$$\frac{T_n}{\mathbb{E}_\infty(T_n)} \underset{n\to+\infty}{\longrightarrow} 1 \qquad in\ \mathbb{P}_\infty - probability.$$

(ii) *If*

$$\lim_{n\to+\infty} \frac{\mathrm{Var}_{n+1}(T_n)}{\mathrm{Var}_\infty(T_n)} = 0 \tag{22}$$

and if

$$\lim_{n\to+\infty} \mathrm{Var}_\infty(T_n)^{-3/2} \sum_{k\geq n} \mathbb{E}_{k+1}(|T_k - \mathbb{E}_{k+1}(T_k)|^3) = 0, \tag{23}$$

we have

$$\frac{T_n - \mathbb{E}_\infty(T_n)}{\mathrm{Var}_\infty(T_n)^{1/2}} \underset{n\to+\infty}{\overset{(d)}{\longrightarrow}} \mathcal{N},$$

where $\mathcal{N}$ follows a standard normal distribution.

Using Cauchy-Schwarz inequality, we note that under (5), Assumption (22) implies that $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \to 0$, as n tends to infinity.

Proof of Theorem 3.3. (i) - We suppose here that $r_n \to 0$. Let $\varepsilon > 0$. Using Bienaymé-Tchebychev inequality and the independence of the random variables $(\tau_n)_n$, we have

$$\mathbb{P}_\infty\left(\left|\frac{T_n}{\mathbb{E}_\infty(T_n)} - 1\right| > \varepsilon\right) \leq \frac{\mathrm{Var}_\infty(T_n)}{\varepsilon^2 \mathbb{E}_\infty(T_n)^2} = \frac{\sum_{k\geq n} \mathrm{Var}(\tau_k)}{\varepsilon^2 \mathbb{E}_\infty(T_n)^2}. \tag{24}$$

As $\mathbb{E}_\infty(T_{n+1})/m_n = 1/r_n - 1 \to +\infty$ as $n \to +\infty$, for all $A > 0$, there exists an integer n_0 such that, for $n \geq n_0$, $\mathbb{E}_\infty(T_{n+1}) \geq A\,m_n$ and

$$\mathbb{E}_\infty(T_n)^2 = \left(\sum_{k\geq n} m_k\right)^2 \geq 2\sum_{k\geq n} m_k \sum_{l>k} m_l \geq 2A\sum_{k\geq n} m_k^2,$$

since $\sum_{l>k} m_l = \mathbb{E}_\infty(T_{k+1}) \geq Am_k$. Coming back to (24), for $n \geq n_0$, we have

$$\mathbb{P}_\infty\left(\left|\frac{T_n}{\mathbb{E}_\infty(T_n)} - 1\right| > \varepsilon\right) \leq \frac{1}{2A\varepsilon^2} \frac{\displaystyle\sum_{k\geq n} \mathrm{Var}(\tau_k)}{\displaystyle\sum_{k\geq n} m_k^2}. \tag{25}$$

By assumption (5), there exists $C > 0$ such that $\mathrm{Var}(\tau_n) \leq Cm_n^2$. Hence, the r.h.s. of (25) goes to 0 as $A \to +\infty$ and the proof of the convergence in probability is complete.

(ii) - We follow classical ideas for the proof of central limit theorem for partial sums of independent random variables (see Theorem 27.2 in Billingsley [5]).
We first note that $\mathrm{Var}_\infty(T_n) \geq \mathrm{Var}_\infty(T_{n+k})$ for any $n, k \geq 0$ since $T_n = \sum_{i \geq n} \tau_i$ and the r.v. $(\tau_i : i \geq 0)$ are independent. Using (22), it ensures that

$$\sup_{k \geq 0} \frac{\mathrm{Var}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)} \leq \sup_{k \geq 0} \frac{\mathrm{Var}(\tau_{k+n})}{\mathrm{Var}_\infty(T_{n+k})} \xrightarrow[n \to +\infty]{} 0. \tag{26}$$

This convergence being uniform with respect to $k \geq 0$, we have

$$\sum_{k \geq 0} \log \left(1 - \frac{t^2}{2} \frac{\mathrm{Var}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)} \right) \underset{n \to +\infty}{\sim} -\frac{t^2}{2} \sum_{k \geq 0} \frac{\mathrm{Var}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)} = -\frac{t^2}{2}.$$

Therefore,

$$\exp(-t^2/2) = \lim_{n \to \infty} \prod_{k \geq 0} \left(1 - \frac{t^2}{2} \frac{\mathrm{Var}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)} \right). \tag{27}$$

Let us now prove that $\frac{T_n - \mathbb{E}_\infty(T_n)}{\mathrm{Var}_\infty(T_n)^{1/2}}$ converges in distribution as $n \to +\infty$ toward a standard normal random variable. By Lévy's theorem and (27), it suffices to prove that for any fixed t,

$$U_n = \mathbb{E}_\infty \left(\exp \left(it \frac{T_n - \mathbb{E}_\infty(T_n)}{\mathrm{Var}_\infty(T_n)^{1/2}} \right) \right) - \prod_{k \geq 0} \left(1 - \frac{t^2}{2} \frac{\mathrm{Var}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)} \right)$$

vanishes as $n \to +\infty$. First, since the τ_n's are independent, for all $t \in \mathbb{R}, n \geq 0$

$$|U_n| = \left| \prod_{k \geq 0} \mathbb{E} \left(\exp \left(it \frac{\tau_{k+n} - \mathbb{E}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)^{1/2}} \right) \right) - \prod_{k \geq 0} \left(1 - \frac{t^2}{2} \frac{\mathrm{Var}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)} \right) \right|. \tag{28}$$

According to (26), for n large enough and for any k, all the factors of the second product of (28) are less than 1. Hence, thanks to (19), we have the inequality

$$|U_n| \leq \sum_{k \geq 0} \left| \mathbb{E} \left(\exp \left(it \frac{\tau_{k+n} - \mathbb{E}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)^{1/2}} \right) \right) - 1 + \frac{t^2}{2} \frac{\mathrm{Var}(\tau_{k+n})}{\mathrm{Var}_\infty(T_n)} \right|. \tag{29}$$

According to equation (27.11) in [5, p.369], for any centered random variable ξ with a finite second moment, we have $\left| \mathbb{E}(\exp(it\xi)) - 1 + \mathrm{Var}(\xi)t^2/2 \right| \leq \mathbb{E}\left(\min(|t\xi|^2, |t\xi|^3) \right)$ for $t \geq 0$. Using this inequality with the random variables $[\tau_{n+k} - \mathbb{E}(\tau_{n+k})]/\mathrm{Var}_\infty(T_n)^{3/2}$, we obtain from (29) that

$$|U_n| \leq |t|^3 \sum_{k \geq 0} \frac{\mathbb{E}\left(|\tau_{k+n} - \mathbb{E}(\tau_{k+n})|^3 \right)}{\mathrm{Var}_\infty(T_n)^{3/2}}.$$

and using (23), U_n goes to 0 as $n \to +\infty$. This completes the proof. $\qquad \square$

Let us now state a strong law of large numbers.

Theorem 3.4. *We assume that* (1), (3), (5) *and* (6) *hold. Then the sequence* $(\frac{T_n}{\mathbb{E}_\infty(T_n)})_n$ *converges to 1,* $\mathbb{P}_\infty$*-almost surely.*

Proof of Theorem 3.4. We prove the a.s. convergence when (6) holds. According to the law of large numbers of Proposition 1 in [11], we just need to check that

$$\sum_{n \geq 0} \frac{\mathrm{Var}(\tau_n)}{\mathbb{E}_\infty(T_n)^2} < +\infty. \tag{30}$$

Using $\mathrm{Var}(\tau_n) \leq \widehat{C}(\mathbb{E}_{n+1}(T_n))^2$ thanks to (5) and Assumption (6) ensure (30) and the proof is complete. $\qquad\square$

Let us illustrate this result with the example $\lambda_n = 0$ and $\mu_n = n \log^\gamma n$ $(\gamma > 1)$. One can check that

$$\mathbb{E}_{n+1}(T_n) = \frac{1}{(n+1)\log^\gamma(n+1)}$$

and

$$\mathbb{E}_\infty(T_n) = \sum_{k \geq n+1} \frac{1}{k \log^\gamma k} \underset{n \to +\infty}{\sim} \frac{1}{(\gamma - 1)\log^{\gamma-1} n}.$$

Using Lemma 2.7 (ii), we know that (1), (3), (5) hold. It's also easy to check that (6) is true. Then we can apply Theorem 3.4 to get that $T_n/\mathbb{E}_\infty(T_n)$ converges a.s. to 1 as $n \to \infty$.

Other examples will be developed in Section 5.

3.3 Comments and examples

In the statement of the three previous theorems appear different assumptions. Let us show here that the choice of these assumptions is very subtile and illustrate our results.

1- One can exhibit a situation of gradual regime where the assumptions of Theorem 3.3 are satisfied, while (4) fails.

We assume that for each $n \geq 0$, $\mu_n = n^2$ and

$$\lambda_n = \frac{n^2}{2} \quad \text{if } n \in \mathbb{N} - 4\mathbb{N}; \qquad \lambda_n = 2n^2 \quad \text{if } n \in 4\mathbb{N}.$$

Then

$$\limsup_{n \to \infty} \lambda_n/\mu_n > 1.$$

For each $n \geq 1$,

$$\frac{1}{2^{1+n/2}} \leq \lambda_n \pi_n \leq \frac{1}{2^{n/2}}, \quad \frac{1}{n^2 2^{2+n/2}} \leq \pi_n \leq \frac{2}{n^2 2^{n/2}},$$

so that (1) is satisfied and (2) gives

$$\mathbb{E}_{n+1}(T_n) = \mathcal{O}\left(\frac{1}{n^2}\right).$$

16

Then (3) also holds, $\mathbb{E}_{i+1}(T_i)/\mathbb{E}_{n+1}(T_n)$ is bounded for $i \geq n$, and (10) can be easily checked since $\frac{\lambda_i \pi_i}{\lambda_n \pi_n} \leq \frac{2}{2^{(i-n)/2}}$. Then (5) is also fulfilled. Thus the assumptions of Theorem 3.3 are satisfied.

2 - The assumptions for the weak law of large numbers in Theorem 3.3 are not sufficient to obtain the strong law of large numbers (Theorem 3.4). Let us consider a pure death process with $\mu_n = \exp(n/\log n) \log n$ and prove that the convergence holds in probability but not almost surely.

Here $l = 0$ and
$$\mathbb{E}(\tau_n) \underset{n \to \infty}{=} \frac{1}{\mu_{n+1}}, \qquad \mathbb{E}_\infty(T_n) = s_{n+1} = \sum_{k \geq n+1} \frac{1}{\mu_k}.$$

Moreover, as μ_n is non-decreasing,
$$\int_n^\infty \frac{e^{-x/\log(x)}}{\log x} \mathrm{d}x \leq s_n \leq \int_n^\infty \frac{e^{-x/\log(x)}}{\log x} \mathrm{d}x + \frac{e^{-n/\log(n)}}{\log n}$$

and
$$\int_n^\infty \frac{e^{-x/\log(x)}}{\log x} \mathrm{d}x \underset{n \to +\infty}{\sim} \int_n^\infty \left(\frac{1}{\log x} + \frac{1}{(\log x)^2} \right) e^{-x/\log(x)} \mathrm{d}x = e^{-n/\log(n)}.$$

Combining the two last displays and recalling $r_n = \mathbb{E}(\tau_n)/\mathbb{E}_\infty(T_n)$, we have
$$s_n \sim \exp(-n/\log n), \qquad r_n \sim 1/\log n, \qquad r_n \to 0,$$

so that T_n/s_{n+1} goes to 1 in probability.

We prove now that the almost sure convergence does not hold and proceed by contradiction. Thus, we assume now that $V_n := T_n/s_{n+1}$ does converge a.s. toward 1. We have
$$V_{n+1} - V_n = V_{n+1} \left(1 - \frac{s_{n+2}}{s_{n+1}} \right) - \frac{\tau_n}{s_{n+1}}.$$

By hypothesis, the left hand side of the latter a.s. vanishes as $n \to +\infty$. Moreover, simple computations lead to $s_{n+1}/s_n \to 1$ and the first term in the r.h.s. of the last display a.s. goes to 0 since our assumption implies that a.s. $(V_n)_n$ is bounded. Hence, putting all pieces together, the term τ_n/s_{n+1} has to go to 0 a.s.

To get a contradiction thanks to Borel-Cantelli's lemma, it suffices to prove that for ε small enough,
$$\sum_{n \geq 0} \mathbb{P}(\tau_n/s_{n+1} > \varepsilon) = \infty,$$

recalling that the random variables τ_n are independent. The law of τ_n is exponential with parameter μ_{n+1}. Then, $\mathbb{P}(\tau_n/s_{n+1} > \varepsilon) = \exp(-\varepsilon \mu_{n+1} s_{n+1})$. Since $\mu_n s_n \sim \log n$ as $n \to +\infty$, there exists $C > 0$ such that
$$\mathbb{P}(\tau_n/s_{n+1} > \varepsilon) \geq e^{-\varepsilon C \log n} = \frac{1}{n^{C\varepsilon}},$$

which completes the proof since $\sum_{n \geq 0} \mathbb{P}(\tau_{n+1}/s_n > \varepsilon)$ is infinite as soon as ε is small enough.

3 - In Theorems 3.1 and 3.3, we did not consider the case where $r_n = \mathbb{E}(\tau_n)/\mathbb{E}_\infty(T_n)$ does not converge. In such a case, one can only state analogous results along the convergent subsequences. For instance, if $\mu_{2n} = \mu_{2n+1} = 3^{2n}$, we have

$$r_{2n} \underset{n\to+\infty}{\longrightarrow} \frac{4}{9} \quad \text{and} \quad r_{2n+1} \underset{n\to+\infty}{\longrightarrow} \frac{4}{5}.$$

Theorem 3.1 then still holds but the subsequences $(T_{2n}/\mathbb{E}_\infty(T_{2n}))_n$ and $(T_{2n+1}/\mathbb{E}_\infty(T_{2n+1}))_n$ converge in distribution to different limits.

One can also find examples where $0 = \liminf_n r_n < \limsup_n r_n$. Then, $(T_n)_n$ has two subsequences satisfying the two regimes of Theorems 3.1 and 3.3.

4 Speed of coming down from infinity

In this section, we use the asymptotic behavior of $T_n/\mathbb{E}_\infty(T_n)$ obtained in the previous Section to derive the short time behavior of $X(t)$. We prove that X behaves as the following non-increasing function that tends to infinity as $t \to 0$

$$v(t) := \inf\{n \geq 0;\ \mathbb{E}_\infty(T_n) \leq t\}.$$

The function v is a càd-làg step function defined on $\mathbb{R}_+$, decreasing from $+\infty$ to 0, it equals n between $\mathbb{E}_\infty(T_n)$ and $\mathbb{E}_\infty(T_{n-1})$, and 0 after $\mathbb{E}_\infty(T_0)$.

The short time behavior of X relies on the inversion of the asymptotic behavior of T_n (see forthcoming Lemma 4.2) and the control of the excursion of X between two successive stopping times $[T_{n+1}, T_n]$ (see forthcoming Lemma 4.1). The latter is true under the assumption

$$\limsup_{n\to\infty} \frac{\lambda_n}{\mu_n} < 1. \tag{31}$$

This assumption is already necessary for Theorem 3.1 but not for Theorem 3.3, as developed in Subsection 3.3 Example 1.

The proof is organized as follows. We introduce the a.s. non-increasing process Y defined by

$$Y(t) = n \quad \text{if} \quad t \in [T_n, T_{n-1})$$

In the next Section, we prove that this (more regular) process comes down from infinity at speed $v(t)$ and we compare the processes $X(t)$ and $Y(t)$ as $t \to 0$ by the study of the height of the excursions of the process X.

4.1 Height of the excursions and non-increasing process.

We first compare the processes $X(t)$ and $Y(t)$ by estimating the number of birth events between the times T_n and T_{n-1}:

$$H_n = \#\{s \in [T_n, T_{n-1}) : X(s) - X(s-) > 0\}, \quad n \geq 1.$$

Lemma 4.1. *(i) We have*

$$0 \leq \frac{X(t)}{v(t)} - \frac{Y(t)}{v(t)} \leq \frac{H_{Y(t)}}{Y(t)} \frac{Y(t)}{v(t)} \tag{32}$$

(ii) Under Assumption (31), $\dfrac{H_n}{n} \to 0$ $\mathbb{P}_\infty$ *a.s.*

Proof. (i) For any $t \in [T_n, T_{n-1})$, $Y(t) = n$ and $0 \leq X(t) - Y(t) \leq H_n$, so the first part is obvious.

(ii) Let us first notice that H_n equals the number of positive jumps between time T_n and T_{n-1} of a random walk whose transition probabilities are given by $p_{i,i+1} = \lambda_i/(\lambda_i + \mu_i)$, $p_{i,i-1} = \mu_i/(\lambda_i + \mu_i)$ for $i \geq 1$. Using (31), we can choose n_0 large enough so that $p = \sup_{n \geq n_0} \lambda_n/(\lambda_n + \mu_n) < 1/2$. Then, for $n \geq n_0$, H_n is stochastically dominated by T, the hitting time of $n - 1$ by a simple random walk starting at n, with probability transitions $(1 - p, p)$. Since $p < 1/2$, $\mathbb{E}(T^2) < +\infty$. Hence $\sup_{n \geq n_0} \mathbb{E}_\infty \left(H_n^2 \right) < +\infty$ and the sequences $(\mathbb{E}_\infty (H_n))_n$ and $(\mathbb{E}_\infty (H_n^2))_n$ are bounded.

Let us now consider the Laplace transform of H_n given by $\widehat{G}_n(a) = \mathbb{E}_\infty (\exp(-aH_n))$. In the same vein as we have obtained (7) and by applying the strong Markov property at the first time when X jumps after T_n, we get the recursion formula

$$\widehat{G}_n(a) = \frac{\mu_n}{\lambda_n + \mu_n} + \frac{\lambda_n}{\lambda_n + \mu_n} e^{-a} \widehat{G}_n(a) \widehat{G}_{n+1}(a), \quad a \geq 0, \ n \geq 1. \tag{33}$$

Differentiating (33) twice at $a = 0$, the second moment of H_n satisfies the following recursion formula

$$\frac{\mu_n}{\lambda_n} \mathbb{E}_\infty \left(H_n^2 \right) = \mathbb{E}_\infty \left(H_{n+1}^2 \right) + 1 + 2 \left(\mathbb{E}_\infty (H_n) + \mathbb{E}_\infty (H_{n+1}) + \mathbb{E}_\infty (H_n) \mathbb{E}_\infty (H_{n+1}) \right).$$

We have seen that the right hand side of the latter is uniformly bounded in $n \geq 0$. It entails that there is $C > 0$ such that

$$\mathbb{E}_\infty \left(H_n^2 \right) \leq C \frac{\lambda_n}{\mu_n}, \quad n \geq 1. \tag{34}$$

Finally, $\mathbb{E}_\infty \left(\sum_{n \geq 1} \left(\frac{H_n}{n} \right)^2 \right) \leq C \sum_{n \geq 1} \frac{1}{n^2} \frac{\lambda_n}{\mu_n} < +\infty$ using again (31). In particular, it turns out that the sequence $(\frac{H_n}{n})_n$ almost surely goes to 0 as $n \to +\infty$. $\square$

Let us now introduce the quantity

$$R(x, y) := \mathbb{E}_\infty(T_{[x]})/\mathbb{E}_\infty(T_{[y]})$$

and study the behavior of $Y(t)/v(t)$ as t tends to 0.

Proposition 4.2. *(i) If $T_n/\mathbb{E}_\infty(T_n)$ is tight on $(0, \infty)$ and for every $x > 1$, $\lim_{n \to \infty} R(nx, n) = 0$ then $Y(t)/v(t) \to 1$ in $\mathbb{P}_\infty -$ probability.*

(ii) If $T_n/\mathbb{E}_\infty(T_n) \to 1$ in $\mathbb{P}_\infty -$ probability and if for every $x > 1$, $\limsup_{n \to \infty} R(nx, n) < 1$, then $Y(t)/v(t) \to 1$ in $\mathbb{P}_\infty -$ probability.

19

(iii) *If* $\lim_{n \to \infty} R(n+1, n) = 1$ *and* $\limsup_{n \to \infty} R(nx, n) < 1$ *for every* $x > 1$, *then*

$$\lim_{a \to 1} \limsup_{t \to 0} \left| \frac{v(at)}{v(t)} - 1 \right| = 0. \tag{35}$$

If additionally $T_n / \mathbb{E}_\infty(T_n) \to 1$ *a.s., then* $Y(t)/v(t) \to 1$ $\mathbb{P}_\infty$ *a.s.*

Proof. (i) Under the tightness assumption, for any $\epsilon > 0$, there exist $0 < A \le B$ such that for every $n \ge 0$,

$$\mathbb{P}_\infty(A \le T_n / \mathbb{E}_\infty(T_n) \le B) \ge 1 - \epsilon. \tag{36}$$

Moreover, for every $x > 1$ and n large enough,

$$R(nx, n) \le \min(1/(2B), A/2). \tag{37}$$

By the definition of the function v, we have

$$\mathbb{E}_\infty \left(T_{v(t)} \right) \le t < \mathbb{E}_\infty \left(T_{v(t)-1} \right).$$

It implies that for any $t > 0$,

$$\mathbb{P}_\infty \left(T_{[xv(t)]} \le \frac{t}{2} \right) \ge \mathbb{P}_\infty \left(T_{[xv(t)]} \le \frac{\mathbb{E}_\infty \left(T_{v(t)} \right)}{2} \right) = \mathbb{P}_\infty \left(\frac{T_{[xv(t)]}}{\mathbb{E}_\infty \left(T_{[xv(t)]} \right)} \le \frac{R(v(t), xv(t))}{2} \right).$$

Hence, using (36) and (37) and for t small enough,

$$\mathbb{P}_\infty(T_{[xv(t)]} \le t/2) \ge \mathbb{P}_\infty \left(\frac{T_{[xv(t)]}}{\mathbb{E}_\infty \left(T_{[xv(t)]} \right)} \le B \right) \ge 1 - \epsilon.$$

We similarly get that for t small enough

$$\mathbb{P}_\infty(T_{[v(t)/x]} \ge 2t) \ge \mathbb{P}_\infty \left(\frac{T_{[v(t)/x]}}{\mathbb{E}_\infty \left(T_{[v(t)/x]} \right)} \ge A \right) \ge 1 - \epsilon.$$

Then, we have for t small enough

$$\mathbb{P}_\infty(T_{[xv(t)]} \le t/2, T_{[v(t)/x]} \ge 2t) \ge 1 - 2\epsilon.$$

Since Y is non-increasing, that implies $\mathbb{P}_\infty(Y(t) \in [v(t)/x, v(t)x]) \ge 1 - 2\epsilon$ and ensures that $Y(t)/v(t)$ tends to 1 in probability as $t \to 0$.

The proof of (ii) follows the same steps as the one of (i). Since $T_n / \mathbb{E}_\infty(T_n) \to 1$ in $\mathbb{P}_\infty -$ probability as n tends to infinity, we can choose for any $x > 1$, A and B close enough to 1 and $a < 1$ such that

$$\mathbb{P}_\infty \left(A \le \frac{T_{[v(t)/x]}}{\mathbb{E}_\infty \left(T_{[v(t)/x]} \right)} \le B \right) \ge 1 - \epsilon, \quad R(nx, n) \le a \min(1/B, A)$$

for t small enough and n large enough. We conclude as previously using now $\mathbb{P}_\infty(T_{[xv(t)]} \le at, T_{[v(t)/x]} \ge t/a) \ge 1 - 2\epsilon$.

Let us now prove (iii). We first note that v is non-increasing. Let us first prove that

$$\lim_{a\downarrow 1}\liminf_{t\to 0}\frac{v(at)}{v(t)}=1.$$

Then for all $t>0$ and $a>1$,

$$\frac{v(at)}{v(t)}\leq 1. \tag{38}$$

Moreover,

$$\mathbb{E}_\infty(T_{v(at)})\leq ta,\qquad \mathbb{E}_\infty(T_{v(t)})\,R(v(t)-1,v(t))=\mathbb{E}_\infty(T_{v(t)-1})\geq t.$$

For all $\eta>1$, the first assumption of (iii) yields $R(v(t)-1,v(t))\leq\eta$ for t small enough and we get

$$\mathbb{E}_\infty(T_{v(at)})\leq a\,\eta\mathbb{E}_\infty(T_{v(t)}),$$

which implies that

$$\lim_{a\downarrow 1}\limsup_{t\to 0}R(v(at),v(t))\leq 1.$$

Moreover the second assumption of (iii) ensures that $\liminf_{n\to\infty}R(nx,n)>1$ for every $x<1$. We add that for each t such that $v(at)\leq xv(t)$, we have

$$R(v(at),v(t))\geq R(xv(t),v(t))$$

Combining the three last displays ensures that for any $x<1$ ensures that

$$\lim_{a\downarrow 1}\liminf_{t\to 0}\frac{v(at)}{v(t)}\geq x.$$

Letting $x\to 1$ and recalling (38) yields

$$\lim_{a\downarrow 1}\limsup_{t\to 0}\Big|\frac{v(at)}{v(t)}-1\Big|=0.$$

To conclude to the first part of (iii), it remains to consider $a<1$, which is simply derived from the previous limit by changing t into t/a.

We assume now that $T_n/\mathbb{E}_\infty(T_n)\to 1$ a.s. and (35) enables us to compose the equivalence by v:

$$\lim_{n\to\infty}\frac{v(T_n)}{v(\mathbb{E}_\infty(T_n))}=1=\lim_{n\to\infty}\frac{v(T_n)}{n}\quad a.s.$$

since $v(\mathbb{E}_\infty(T_n))=n$ by definition of v and $n\to\mathbb{E}_\infty(T_n)$ is decreasing. Noting that T_n goes to 0 a.s. and

$$\frac{n}{v(T_n)}\leq\frac{Y(t)}{v(t)}\leq\frac{n}{v(T_{n-1})}$$

a.s. on the event $t\in[T_n,T_{n-1})$ ends up the proof of (iii). $\qquad\square$

21

4.2 Fast coming down from infinity

We state the convergence in probability inherited from Theorem 3.1 ($\alpha > 0$).

Theorem 4.3. *We assume that* (1), (3) *and* (4) *hold and*

$$\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \underset{n\to+\infty}{\longrightarrow} \alpha \in (0,1].$$

Then

$$\lim_{t\to 0} \frac{X(t)}{v(t)} = 1 \ in \ \mathbb{P}_\infty - probability.$$

Proof. Under the assumptions of Theorem 3.1, the sequence $T_n/\mathbb{E}_\infty(T_n)$ converges in law to a random variable whose law is supported on $(0,\infty)$, since $G(a) \to 0$ as $a \to \infty$. Then this sequence is tight on $(0,\infty)$. Let us fix $x > 1$ and show that

$$\lim_{n\to\infty} R(nx,n) = \lim_{n\to\infty} \frac{\mathbb{E}_\infty(T_{[nx]})}{\mathbb{E}_\infty(T_n)} = 0.$$

Indeed, as seen in (16), $\lim_{n\to+\infty} \frac{\mathbb{E}_\infty(T_{n+1})}{\mathbb{E}_\infty(T_n)} = 1 - \alpha$. Then for n large enough,

$$\lim_{n\to\infty} \frac{\mathbb{E}_\infty(T_{[nx]})}{\mathbb{E}_\infty(T_n)} = \lim_{n\to\infty} \prod_{j=n}^{[nx]-1} \frac{\mathbb{E}_\infty(T_{j+1})}{\mathbb{E}_\infty(T_j)} = \lim_{n\to\infty} (1-\alpha)^{[nx]-n} = 0.$$

Then by Proposition 4.2-(i), $Y(t)/v(t) \to 1$ in $\mathbb{P}_\infty -$ probability. Lemma 4.1 will then allow to conclude. $\square$

Example. Let us consider a pure death process with $\mu_n = e^{\beta n}$ with $\beta > 0$ and thus $\mathbb{E}_\infty(T_n) \sim e^{-\beta(n+1)}/(1 - e^{-\beta})$. Hence, the conditions of Theorem 3.1 hold true with $\alpha = 1 - e^{-\beta}$ and

$$e^{\beta(n+1)} T_n \xrightarrow[n\to+\infty]{(d)} \sum_{k\geq 0} e^{-\beta k} E_k$$

where the E_k's are independant exponential random variables with parameter 1. In that case, we can explicitly determine the speed v of Theorem 4.3 and we get $X(t) \sim -(\log t)/\beta$ as $t \to 0$ in probability.

4.3 Gradual coming down from infinity

We give now the speed of convergence and describe the fluctuations of X in the case $\alpha = 0$.

Theorem 4.4. *We assume that* (1), (3), (5) *hold and* $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \underset{n\to+\infty}{\longrightarrow} 0$.
We assume also that (31) *holds and that for every* $x > 1$,

$$\limsup_{n\to\infty} \frac{\mathbb{E}_\infty(T_{[nx]})}{\mathbb{E}_\infty(T_n)} < 1. \tag{39}$$

Then,

$$\lim_{t\to 0} \frac{X(t)}{v(t)} = 1 \qquad in \ \ \mathbb{P}_\infty - probability.$$

Assuming further that $\sum_n \left(\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \right)^2 < \infty$, *this convergence holds a.s.*

The additional assumption (39) is required for the inversion (Proposition 4.2 (ii)). Indeed the quantity $\mathbb{E}_\infty(T_n)$ should not tend too slowly to 0. The example $\mu_n = n\log^\gamma n$ ($\gamma > 1$) and $\lambda_n = 0$ shows that (39) may fail while the other assumptions hold (see Section 3.2 for details).

Proof. Let us deal with the convergence in probability and work under $\mathbb{P}_\infty$. The first four assumptions allow us to apply Theorem 3.3, so that $T_n/\mathbb{E}_\infty(T_n) \to 1$ in probability as $n \to \infty$. Using (39), we can apply Proposition 4.2 (ii) to get that $Y(t)/v(t) \to 1$ in probability. Moreover Assumption (31) enables us to use Lemma 4.1. It ensures that $H_{Y(t)}/Y(t) \to 0$ a.s. since $Y(t) \to +\infty$ a.s as $t \to 0$ and $X(t)/v(t) - Y(t)/v(t) \to 0$ in probability and then the convergence in probability of $X(t)/v(t)$ to 1.

We note now that

$$R(n+1, n) = 1 - \frac{\mathbb{E}_{n+1}(T_n)}{\mathbb{E}_\infty(T_n)}.$$

Thus, $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \to 0$ yields $R(n+1, n) \to 1$ as $n \to \infty$. Then the a.s. convergence is obtained similarly combining Theorem 3.4, Lemma 4.1 and Proposition 4.2 (iii). $\qquad\square$

Remark 4.5. *We remark that if $\lambda_n = 0$ and $\mu_n = n(n-1)/2$, $X(t)$ is the number of blocks of the Kingman coalescent at time t. In this case,*

$$\mathbb{E}_\infty(T_n) = \sum_{i \geq n} \frac{2}{i(i+1)} = \frac{2}{n}.$$

Then, $v(t) = \frac{2}{t}$ and we recover from Theorem 4.4 the speed of coming down from infinity for this process, obtained by Aldous in [1]-paragraph 4.2.: $tX(t) \underset{t\to 0}{\longrightarrow} 2$ a.s..

We refer to the next section for more general examples, where we also provide the fluctuations of X under $\mathbb{P}_\infty$ for t close to 0 using the following result.

Proposition 4.6. *We assume that (1), (3) and (5) hold and that (22) and (23) hold. We also assume that $\sum_{n\in\mathbb{N}} \frac{1}{n}\frac{\lambda_n}{\mu_n} < \infty$ and (39) and that for every $x \in \mathbb{R}$,*

$$\frac{t - \mathbb{E}_\infty\left(T_{s(x,t)}\right)}{\sqrt{\mathrm{Var}_\infty\left(T_{s(x,t)}\right)}} \xrightarrow[t\to 0]{} x, \tag{40}$$

where $s(x,t) = [v(t) + x\sqrt{v(t)}]$. Then,

$$\sqrt{v(t)}\left(\frac{X(t)}{v(t)} - 1\right) \xrightarrow[t\to 0]{(d)} \mathcal{N}, \tag{41}$$

where $\mathcal{N}$ follows a standard normal distribution.

Proof of Proposition 4.6. The proof follows the same steps than the previous theorem. We use the C.L.T theorem for T_n to firstly establish a central limit theorem for the a.s. non-increasing process Y.

As Y is non-increasing, we can follow the proof of the central limit theorem for renewal processes (as suggested by Aldous for Kingman's coalescent, cf. [1]). More precisely, for any $t > 0, x \in \mathbb{R}$, we use $\mathbb{P}_\infty(Y(t) > s(x,t)) = \mathbb{P}_\infty\big(T_{s(x,t)} > t\big)$ to get

$$\mathbb{P}_\infty\left(\sqrt{v(t)}\left(\frac{Y(t)}{v(t)} - 1\right) > x\right) = \mathbb{P}_\infty\left(\widetilde{Z}_{s(x,t)} > \frac{t - \mathbb{E}_\infty\big(T_{s(x,t)}\big)}{\sqrt{\mathrm{Var}_\infty\big(T_{s(x,t)}\big)}}\right),$$

where

$$\widetilde{Z}_n = \frac{T_n - \mathbb{E}_\infty(T_n)}{\sqrt{\mathrm{Var}_\infty(T_n)}}$$

All the assumptions of Theorem 3.3 (ii) are met, so $\widetilde{Z}_n$ converges weakly to a standard normal variable. Using (40), we obtain the C.L.T. (41) for Y.

We end the proof by deducing the C.L.T for X thanks to the decomposition

$$\sqrt{v(t)}\left(\frac{X(t)}{v(t)} - 1\right) = \sqrt{v(t)}\left(\frac{Y(t)}{v(t)} - 1\right) + \frac{X(t) - Y(t)}{\sqrt{v(t)}}. \tag{42}$$

From (32), we almost surely have

$$0 \leq \frac{X(t) - Y(t)}{\sqrt{v(t)}} \leq \frac{H_{Y(t)}}{\sqrt{Y(t)}} \frac{\sqrt{Y(t)}}{\sqrt{v(t)}}. \tag{43}$$

Using (34), there exists C such that

$$\mathbb{E}_\infty\left(\sum_{n\geq 1}\left(\frac{H_n}{\sqrt{n}}\right)^2\right) \leq C \sum_{n\geq 1} \frac{1}{n}\frac{\lambda_n}{\mu_n}.$$

Since this series converges by hypothesis, $H_n/\sqrt{n}$ a.s. goes to 0 as $n \to +\infty$. Recalling that (5) and (22) ensure that $\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n) \to 0$, all the assumptions of Theorem 4.4 are fulfilled, so $Y(t) \sim X(t) \sim v(t)$ as $t \to 0$ in probability. Then the right hand side of (43) vanishes as $t \to 0$ in probability and (42) allows us to derive (41) from the C.L.T for Y established above. $\qquad\square$

5 Application for regularly varying death rates

As mentioned before (see in particular the end of Section 2.3), the following class of birth and death processes is particulary relevant for population dynamics and population genetics models:

$$\lambda_n \leq Cn, \qquad \mu_n = n^\rho \log^\gamma n, \text{ where } C > 0, \rho > 1, \gamma \in \mathbb{R}. \tag{44}$$

This is a particular case of a main class that we can attain with our results. In what follows, we will suppose that the birth rate is sub-linear as assumed in (44). The main assumption is that the death rate varies regularly. Our previous theorems apply in this general context.

Recall that a sequence of real non-zero numbers $(u_n)_{n\geq 0}$ varies regularly with index $\rho \neq 0$ if for all $a > 0$,

$$\lim_{n\to+\infty} \frac{u_{[an]}}{u_n} = a^\rho.$$

A function $g : [0, +\infty) \to (0, +\infty)$ *varies regularly* at 0 with index $\rho \neq 0$ if for all $a > 0$,

$$\lim_{x\to 0} \frac{g(ax)}{g(x)} = a^\rho.$$

Theorem 5.1. *Suppose that $\lim_{n\to+\infty} \lambda_n/\mu_n = 0$ and that $(\mu_n)_n$ varies regularly with index $\rho > 1$. Then,*

$$\lim_{n\to\infty} \frac{\mu_{n+1}(\rho - 1)}{n} T_n = \lim_{t\to 0} \frac{X(t)}{v(t)} = 1 \qquad \mathbb{P}_\infty - a.s. \tag{45}$$

where v is regularly varying at 0 with index $1/(1 - \rho)$. Further,

$$\frac{\sqrt{2\rho - 1}}{\sqrt{n}} \left(T_n - \frac{n}{\mu_{n+1}(\rho - 1)} \right) \xrightarrow[n\to+\infty]{(d)} \mathcal{N},$$

Assuming further that $\sum_{n\geq 1} \frac{1}{n}\frac{\lambda_n}{\mu_n} < +\infty$, we also get that

$$\sqrt{(2\rho - 1)v(t)} \left(\frac{X(t)}{v(t)} - 1 \right)$$

converges in law, as t tends to 0, to a standard normal distribution.

We recover the central limit theorem for the Kingman coalescent. We also mention that [14] provides Gaussian limits for more general Λ coalescent processes whose "Kingman part" is non trivial.

Corollary 5.2. *Assume* (44). *Then,*

$$\lim_{n\to\infty} T_n (\rho - 1) n^{\rho - 1} \log^\gamma n = 1 \qquad \mathbb{P}_\infty - a.s.$$

and

$$\frac{\sqrt{2\rho - 1}}{\sqrt{n}} \left(T_n - \frac{1}{(\rho - 1)n^{\rho - 1} \log^\gamma n} \right) \xrightarrow[n\to+\infty]{(d)} \mathcal{N}$$

where $\mathcal{N}$ follows a standard normal distribution.

Proof of Theorem 5.1. Let us first remark that since $(\mu_n)_n$ varies regularly with index $\rho > 1$, then $(1/\mu_n)_n$ (resp. $(1/\mu_n^2)_n$) varies regularly with index $-\rho < -1$ (resp. $-2\rho < -2$). Under the assumptions of Theorem 5.1, we note that conditions (14) are satisfied. Indeed, Theorem 1.5.3 in [6] shows that the sequence (μ_n) is equivalent to a non-decreasing sequence and Lemma

A.1 applies directly to $\sum 1/\mu_n$. Lemma 2.7 (ii) can be applied, so that (1), (3), (4) and (5) are satisfied and

$$\mathbb{E}_{n+1}(T_n^k) \underset{n \to +\infty}{\sim} \frac{k!}{\mu_{n+1}^k}, \quad \text{for} \quad k = 1, 2, 3. \tag{46}$$

Then by forthcoming Lemma A.1,

$$\mathbb{E}_\infty(T_n) \underset{n \to +\infty}{\sim} \frac{n}{\mu_{n+1}(\rho - 1)}.$$

Thus for $x > 1$,

$$\limsup_{n \to \infty} \frac{\mathbb{E}_\infty(T_{[nx]})}{\mathbb{E}_\infty(T_n)} = x^{1-\rho}.$$

Since $1 - \rho < 0$, then $x^{1-\rho} < 1$. Moreover, $\sum_n (\mathbb{E}_{n+1}(T_n)/\mathbb{E}_\infty(T_n))^2$ converges and the assumptions for Theorem 3.4 and Theorem 4.4 are satisfied, implying (45).

To prove the C.L.T. for (T_n), we use again (46) for $k = 1, 2$ to get $\operatorname{Var}(\tau_n) \sim 1/\mu_{n+1}^2$ as $n \to +\infty$, which implies that $(\operatorname{Var}(\tau_n))_n$ varies regularly with index -2ρ. Then $\operatorname{Var}_\infty(T_n) = \sum_{i=n+1}^\infty \operatorname{Var}(\tau_n)$ and forthcoming Lemma A.1 ensures that $\operatorname{Var}_\infty(T_n)$ varies regularly with index $1 - 2\rho$ and

$$\operatorname{Var}_\infty(T_n) \sim \frac{n}{(2\rho - 1)\mu_{n+1}^2}. \tag{47}$$

Therefore we have $\frac{\operatorname{Var}(\tau_n)}{\operatorname{Var}_\infty(T_n)} \underset{n \to +\infty}{\sim} \frac{2\rho-1}{n}$, which entails (22).

Moreover, by the triangle inequality and the binomial theorem, we have

$$\mathbb{E}_{k+1}(|T_k - \mathbb{E}_{k+1}(T_k)|^3) \leq \mathbb{E}_{k+1}(T_k^3) + 3\mathbb{E}_{k+1}(T_k)\mathbb{E}_{k+1}(T_k^2) + 4\mathbb{E}_{k+1}(T_k)^3.$$

Thanks to (46), all the terms of the r.h.s. are of order of magnitude $1/\mu_{n+1}^3$ as $n \to +\infty$. Thus, using again Lemma A.1 and (47), there is a positive constant C' such that

$$\frac{\sum_{k \geq n} \mathbb{E}_{k+1}(|T_k - \mathbb{E}_{k+1}(T_k)|^3)}{\operatorname{Var}_\infty(T_n)^{3/2}} \leq \frac{C'}{\sqrt{n}}.$$

This latter vanishes as $n \to +\infty$ and (23) is satisfied. Hence we apply Theorem 3.3 and

$$\frac{\sqrt{2\rho - 1}}{\sqrt{n}} \left(T_n - \frac{n}{\mu_{n+1}(\rho - 1)} \right) \underset{n \to +\infty}{\overset{(d)}{\longrightarrow}} \mathcal{N},$$

where $\mathcal{N}$ follows a standard normal distribution.

Let us now prove the last assertion of Theorem 5.1. To apply Proposition 4.6, we need to prove that

$$\frac{t - \mathbb{E}_\infty(T_{s(x,t)})}{\sqrt{\operatorname{Var}_\infty(T_{s(x,t)})}} \underset{t \to 0}{\longrightarrow} x\sqrt{2\rho - 1} \tag{48}$$

and we first consider the case $x > 0$. We note that

$$\mathbb{E}_\infty(T_{v(t)}) - \mathbb{E}_\infty(T_{s(x,t)}) \leq t - \mathbb{E}_\infty(T_{s(x,t)}) \leq \mathbb{E}_\infty(T_{v(t)-1}) - \mathbb{E}_\infty(T_{s(x,t)}), \tag{49}$$

and we handle the two sides similarly. For the left hand side, we have

$$\mathbb{E}_\infty\left(T_n\right) - \mathbb{E}_\infty\left(T_{[n+x\sqrt{n}]}\right) = \sum_{k=n}^{[n+x\sqrt{n}]-1} \mathbb{E}_{k+1}(T_k) \underset{n\to+\infty}{\sim} \frac{x\sqrt{n}}{\mu_{n+1}}, \qquad (50)$$

using forthcoming Lemma A.3 with $u_n = [n + x\sqrt{n}]$. Moreover, applying forthcoming Lemma A.2 with $f(y) = y$, $g(y) = [y + u\sqrt{y}]$ and $h(n) = \mathrm{Var}_\infty\left(T_n\right)$ and using (47), we get

$$\mathrm{Var}_\infty\left(T_{[n+x\sqrt{n}]}\right) \underset{n\to+\infty}{\sim} \mathrm{Var}_\infty\left(T_n\right) \underset{n\to+\infty}{\sim} \frac{n}{(2\rho - 1)\mu_{n+1}^2}.$$

Combining this equivalence with (50) and (49) yields (48) for $x > 0$, while the case $x < 0$ can be handled similarly. It ends up the proof. $\qquad\square$

Proof of Corollary 5.2. We easily remark that

$$\mathbb{E}_{n+1}(T_n) \underset{n\to+\infty}{\sim} \frac{1}{n^\rho \log^\gamma n} \quad ; \quad \mathbb{E}_\infty\left(T_n\right) \underset{n\to+\infty}{\sim} \sum_{k\geq n+1} \frac{1}{k^\rho \log^\gamma k} \underset{n\to+\infty}{\sim} \frac{1}{(\rho - 1)n^{\rho-1}\log^\gamma n}$$

and

$$\mathrm{Var}_\infty\left(T_n\right) \underset{n\to+\infty}{\sim} \frac{1}{\left((2\rho - 1)n^{2\rho-1}\log^{2\gamma} n\right)}.$$

The assumptions of Theorem 5.1 can then be easily checked to get the result. $\qquad\square$

A   Appendix : regularly varying functions

The proofs of the previous section rely on the following technical results on regularly varying functions.

Lemma A.1. *Let g be a function that varies regularly at $+\infty$ with index $\rho' < -1$. Then the series $\sum_{k\geq 0} g(k)$ converges and $R(n) = \sum_{k\geq n} g(k)$ varies regularly with index $\rho' + 1$ and*

$$\sum_{k\geq n} g(k) \underset{n\to+\infty}{\sim} -\frac{ng(n)}{\rho' + 1}.$$

Proof. First, since $\rho' < -1$, $\sum_{k\geq 0} g(k)$ and $\int_0^{+\infty} g(x)\mathrm{d}x$ are both convergent. Moreover, thanks to [6, Thm 1.5.3], a regularly varying function with negative index is equivalent to a non-increasing function. Then, without loss of generality, one can suppose that g is non-increasing. Then, if $I_n := \int_n^{+\infty} g(x)\mathrm{d}x$, a classical comparison between series and integrals entails that $1 \leq \frac{R_n}{I_n} \leq 1 + \frac{g(n)}{I_n}$. Using that g varies regularly and according to [6, Thm 1.5.11],

$$\lim_{n\to+\infty} \frac{ng(n)}{I_n} = -(\rho' + 1). \qquad (51)$$

Hence, $I_n \sim R_n$ as $n \to +\infty$. We also see from (51) that I varies regularly at $+\infty$ with index $\rho' + 1$. Since I and R are equivalent, R also varies regularly with the same index. $\qquad\square$

Lemma A.2. *Let $x_0 \in [0, +\infty]$ and let f and g be two positive functions such that*

$$f(x) \underset{x \to x_0}{\longrightarrow} L \in \{0, +\infty\}, \qquad \frac{f(x)}{g(x)} \underset{x \to x_0}{\longrightarrow} 1.$$

If h varies regularly at L, then

$$\frac{h(f(x))}{h(g(x))} \underset{x \to x_0}{\longrightarrow} 1.$$

Moreover, if $f(x) = f(x, t) = g(x)(1 + t\varepsilon(x))$ with $\lim_{x \to x_0} \varepsilon(x) = 0$, the previous convergence holds uniformly in t in any compact subset of $\mathbb{R}$.

Proof. We only prove the case $L = 0$ and fix $\varepsilon > 0$. Thanks to Theorem 1.5.1 p.22 in [6], the convergence given above in the definition of regularly varying function can be taken uniform with respect to a in some compact set. Then, there exist $\eta, \eta' > 0$ such that for every $a \in [1 - \eta, 1 + \eta]$ and $y \in (0, \eta')$,

$$1 - \varepsilon \leq \frac{h(ay)}{h(y)} \leq 1 + \varepsilon.$$

Furthermore, for x close enough to x_0, we have $g(x) \leq \eta'$ and $(1 - \eta) \leq f(x)/g(x) \leq (1 + \eta)$, so that

$$\left| \frac{h\left(g(x) \cdot \frac{f(x)}{g(x)}\right)}{h(g(x))} - 1 \right| \leq \varepsilon,$$

which ends up the first part of the proof. The second part follows in the same way since $1 + t\varepsilon(x)$ goes to 1 uniformly in t in any compact set. $\qquad\square$

Lemma A.3. *Let $(m_n)_n$ be a regularly varying sequence and $(u_n)_n$ a sequence of integers such that $u_n \to \infty$ and $u_n/n \to 0$ as $n \to \infty$. Then*

$$\sum_{k=n}^{n+u_n-1} m_k \underset{n \to +\infty}{\sim} u_n m_n.$$

Proof. We write

$$\left| \sum_{k=n}^{n+u_n-1} m_k - u_n m_n \right| \leq m_n \sum_{k=n}^{n+u_n-1} \left| \frac{m_k}{m_n} - 1 \right| \leq u_n m_n \sup_{t \in [0,1]} \left| \frac{m_{\lfloor n(1+tu_n/n) \rfloor}}{m_n} - 1 \right|.$$

Using the second part of Lemma A.2 with the regularly varying sequence $(m_n)_n$, the last term vanishes, which ends up the proof. $\qquad\square$

References

[1] D. J. Aldous. Deterministic and stochastic models for coalescence (aggregation and co-agulation): a review of the mean-field theory for probabilists. *Bernoulli*, 5(1):3–48, 1999.

[2] L. J. S. Allen. *An introduction to stochastic processes with applications to biology.* CRC Press, Boca Raton, FL, second edition, 2011.

[3] W. J. Anderson. *Continuous-time Markov chains.* Springer Series in Statistics: Probability and its Applications. Springer-Verlag, New York, 1991. An applications-oriented approach.

[4] J. Berestycki, N. Berestycki, and V. Limic. The Lambda-coalescent speed of coming down from infinity. *Ann. Probab.*, 38(1):207–233, 2010.

[5] P. Billingsley. *Probability and measure.* Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, second edition, 1986.

[6] N. H. Bingham, C. M. Goldie, and J. L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1989.

[7] P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, 37(5):1926–1969, 2009.

[8] P. Donnelly. Weak convergence to a Markov chain with an entrance boundary: ancestral processes in population genetics. *Ann. Probab.*, 19(3):1102–1117, 1991.

[9] S. Karlin and J. L. McGregor. The differential equations of birth-and-death processes, and the Stieltjes moment problem. *Trans. Amer. Math. Soc.*, 85:489–546, 1957.

[10] S. Karlin and H. M. Taylor. *A first course in stochastic processes.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, second edition, 1975.

[11] O. I. Klesov. The rate of convergence of series of random variables. *Ukrain. Mat. Zh.*, 35(3):309–314, 1983.

[12] M. Kot. *Elements of mathematical ecology.* Cambridge University Press, Cambridge, 2001.

[13] A. Lambert. The branching process with logistic growth. *Ann. Appl. Probab.*, 15(2):1506–1535, 2005.

[14] V. Limic, A. Talarczyk (2014). Diffusion limits for mixed with Kingman coalescents at small times. Avialable via http://arxiv.org/abs/1409.6200.

[15] P. F. Slade, J. Wakeley. The Structured Ancestral Selection Graph and the Many-Demes Limit. *Genetics.* 2005 169(2): 1117-1131.

[16] R. M. Sibly, D. Barker, M. C. Denham, J. Hone, and M. Pagel. On the regulation of populations of mammals, birds, fish, and insects. *Science*, 309(5734):607–610, 2005.

[17] H. M. Taylor and S. Karlin. *An introduction to stochastic modeling.* Academic Press, Inc., San Diego, CA, third edition, 1998.

[18] J. van den Broek and H. Heesterbeek. Nonhomogeneous birth and death models for epidemic outbreak data. *Biostatistics*, 8(2):453–467, 2007.

[19] E. A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, 23(4):683–700, 1991.