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Stability properties for quasilinear parabolic equations with measure data and applications

Marie-Françoise BIDAUT-VERON∗  Hung NGUYEN QUOC†

Abstract

Let Ω be a bounded domain of \(\mathbb{R}^N\), and \(Q = \Omega \times (0,T)\). We first study problems of the model type

\[
\begin{align*}
  u_t - \Delta_p u &= \mu & \text{in } Q, \\
  u &= 0 & \text{on } \partial \Omega \times (0,T), \\
  u(0) &= u_0 & \text{in } \Omega,
\end{align*}
\]

where \(p > 1\), \(\mu \in \mathcal{M}_b(Q)\) and \(u_0 \in L^1(\Omega)\). Our main result is a stability theorem extending the results of Dal Maso, Murat, Orsina, Prignet, for the elliptic case, valid for quasilinear operators \(u \mapsto -\nabla \cdot (A(x,t,\nabla u))\).

As an application, we consider perturbed problems of type

\[
\begin{align*}
  u_t - \Delta_p u + G(u) &= \mu & \text{in } Q, \\
  u &= 0 & \text{on } \partial \Omega \times (0,T), \\
  u(0) &= u_0 & \text{in } \Omega,
\end{align*}
\]

where \(G(u)\) may be an absorption or a source term. In the model case \(G(u) = \pm |u|^{q-1} u (q > p - 1)\), or \(G\) has an exponential type. We give existence results when \(q\) is subcritical, or when the measure \(\mu\) is good in time and satisfies suitable capacity conditions.

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1 Introduction

Let $\Omega$ be a bounded domain of $\mathbb{R}^N$, and $Q = \Omega \times (0, T)$, $T > 0$. We denote by $\mathcal{M}_b(\Omega)$ and $\mathcal{M}_b(Q)$ the sets of bounded Radon measures on $\Omega$ and $Q$ respectively. We are concerned with the problem

$$\begin{cases}
    u_t - \text{div}(A(x, t, \nabla u)) = \mu & \text{in } Q, \\
    u = 0 & \text{on } \partial\Omega \times (0, T), \\
    u(0) = u_0 & \text{in } \Omega,
\end{cases}$$

(1.1)

where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$ and $A$ is a Caratheodory function on $Q \times \mathbb{R}^N$, such that for a.e. $(x, t) \in Q$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x, t, \xi) \cdot \xi \geq c_1 |\xi|^p, \quad |A(x, t, \xi)| \leq a(x, t) + c_2 |\xi|^{p-1}, \quad c_1, c_2 > 0, a \in L^p(Q),$$

(1.2)

$$A(x, t, \xi) - A(x, t, \zeta) \cdot (\xi - \zeta) > 0 \quad \text{if } \xi \neq \zeta.$$

(1.3)

This includes the model problem

$$\begin{cases}
    u_t - \Delta_p u = \mu & \text{in } Q, \\
    u = 0 & \text{on } \partial\Omega \times (0, T), \\
    u(0) = u_0 & \text{in } \Omega,
\end{cases}$$

(1.4)

where $\Delta_p$ is the $p$-Laplacian defined by $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ with $p > 1$.

As an application, we consider problems with a nonlinear term of order 0:

$$\begin{cases}
    u_t - \text{div}(A(x, \nabla u)) + G(u) = \mu & \text{in } Q, \\
    u = 0 & \text{on } \partial\Omega \times (0, T), \\
    u(0) = u_0 & \text{in } \Omega,
\end{cases}$$

(1.5)

where $A$ is a Caratheodory function on $\Omega \times \mathbb{R}^N$, such that, for a.e. $x \in \Omega$, and any $\xi, \zeta \in \mathbb{R}^N$,

$$A(x, \xi) \cdot \xi \geq c_1 |\xi|^p, \quad |A(x, \xi)| \leq c_2 |\xi|^{p-1}, \quad c_3, c_4 > 0,$$

(1.6)

$$A(x, \xi) - A(x, \zeta) \cdot (\xi - \zeta) > 0 \quad \text{if } \xi \neq \zeta,$$

(1.7)

and $G(u)$ may be an absorption or a source term, and possibly depends on $(x, t) \in Q$. The model problem is the case where $G$ has a power-type $G(u) = \pm |u|^{q-1}u \quad (q > p - 1)$, or an exponential type.

First make a brief survey of the elliptic associated problem:

$$\begin{cases}
    -\text{div}(A(x, \nabla u)) = \mu & \text{in } \Omega, \\
    u = 0 & \text{on } \partial\Omega,
\end{cases}$$

with $\mu \in \mathcal{M}_b(\Omega)$ and assumptions (1.6), (1.7). When $p = 2$, $A(x, \nabla u) = \nabla u$ existence and uniqueness are proved for general elliptic operators by duality methods in [58]. For $p > 2 - 1/N$, the existence of solutions in the sense of distributions is obtained in [23] and [24]. The condition
on $p$ ensures that the gradient $\nabla u$ is well defined in $(L^1(\Omega))^N$. For general $p > 1$, new classes of solutions are introduced, first when $\mu \in L^1(\Omega)$, such as entropy solutions, and renormalized solutions, see [13], and also [57], and existence and uniqueness is obtained. For any $\mu \in \mathcal{M}_b(\Omega)$ the main work is done in [32, Theorems 3.1, 3.2], where not only existence is proved, but also a stability result, fundamental for applications. Uniqueness is still an open problem.

Next we make a brief survey about problem (1.1).

The first studies concern the case $\mu \in L^p'(Q)$ and $u_0 \in L^2(\Omega)$, where existence and uniqueness is obtained by variational methods, see [45]. In the general case $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$, the pioneer results come from [23], proving the existence of solutions in the sense of distributions for $p > p_1 = 2 - \frac{1}{N+1}$, \begin{equation} \tag{1.8} \end{equation}

see also [55], [56], and [26]. The approximated solutions of (1.1) lie in Marcinkiewicz spaces $u \in L^{p_c,\infty}(Q)$ and $|\nabla u| \in L^{m_c,\infty}(Q)$, where $p_c = p - 1 + \frac{p}{N}$, \begin{equation} \tag{1.9} \end{equation}

This condition (1.8) ensures that $u$ and $|\nabla u|$ belong to $L^1(Q)$, since $m_c > 1$ means $p > p_1$ and $p_c > 1$ means $p > 2N/(N + 1)$. Uniqueness follows in the case $p = 2$, $A(x,t,\nabla u) = \nabla u$, by duality methods, see [48].

For $\mu \in L^1(Q)$, uniqueness is obtained in new classes of solutions: entropy solutions, and renormalized solutions, see [19], [54], see also [3] for a semi-group approach.

Then a class of regular measures is studied in [33], where a notion of parabolic capacity $c^Q_p$ is introduced, defined by $c^Q_p(E) = \inf \left\{ \inf_{E \subset U \text{ open} \subset Q} \{|u|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q \} \right\}$, for any Borel set $E \subset Q$, where $X = L^p(0,T; W_{0}^{1,p}(\Omega) \cap L^2(\Omega))$, $W = \{ z : z \in X, \quad z_t \in X' \}$, embedded with the norm $||u||_W = ||u||_X + ||u_t||_{X'}$.

Let $\mathcal{M}_0(Q)$ be the set of Radon measures $\mu$ on $Q$ that do not charge the sets of zero $c^Q_p$-capacity: $\forall E$ Borel set $\subset Q$, $c^Q_p(E) = 0 \implies |\mu(E)| = 0$.

Then existence and uniqueness of renormalized solutions holds for any measure $\mu \in \mathcal{M}_b(Q) \cap \mathcal{M}_0(Q)$, called regular (or diffuse) and $u_0 \in L^1(\Omega)$, and $p > 1$. The equivalence with the notion of entropy solutions is shown in [34]; see also [20] for more general equations.

Next consider any measure $\mu \in \mathcal{M}_b(Q)$. Let $\mathcal{M}_s(Q)$ be the set of all bounded Radon measures on $Q$ with support on a set of zero $c^Q_p$-capacity, also called singular. Let $\mathcal{M}_b^+(Q), \mathcal{M}_b^+(Q), \mathcal{M}_s^+(Q)$
be the positive cones of $\mathcal{M}_b(Q), \mathcal{M}_0(Q), \mathcal{M}_s(Q)$. From [33], $\mu$ can be written (in a unique way) under the form
\[
\mu = \mu_0 + \mu_s, \quad \mu_0 \in \mathcal{M}_0(Q), \quad \mu_s = \mu_s^+ - \mu_s^-, \quad \mu_s^+, \mu_s^- \in \mathcal{M}_s^+(Q),
\]
and $\mu_0 \in \mathcal{M}_0(Q)$ admits (at least) a decomposition under the form
\[
\mu_0 = f - \text{div} \ g + h_t, \quad f \in L^1(Q), \quad g \in (L^p(Q))^N, \quad h \in X,
\]
and we write $\mu_0 = (f, g, h)$. The solutions of (1.1) are searched in a renormalized sense linked to this decomposition, introduced in [19], [49]. In the range (1.8) the existence of a renormalized solution relative to the decomposition (1.11) is proved in [49], using suitable approximations of $\mu_0$ and $\mu_s$.

Next consider the problem (1.5). First we consider the case of an absorption term: $\mathcal{G}(u)u \geq 0$. Let us recall the case $p = 2$, $A(x, \nabla u) = \nabla u$ and $\mathcal{G}(u) = |u|^{q-1}u$ ($q > 1$). The first results concern the case $\mu = 0$ and $u_0$ is a Dirac mass in $\Omega$, see [28]: existence holds if and only if $q < (N + 2)/N$. Then optimal results are given in [7], for any $\mu \in \mathcal{M}_b(Q)$ and $u_0 \in \mathcal{M}_b(\Omega)$. Here two capacities are involved: the elliptic Bessel capacity $C_{\alpha,k}$, $(\alpha, k > 1)$ defined, for any Borel set $E \subset \mathbb{R}^N$, by
\[
C_{\alpha,k}(E) = \inf \{ \| \varphi \|_{L^k(\mathbb{R}^N)} : \varphi \in L^k(\mathbb{R}^N), \ G_\alpha * \varphi \geq \chi_E \},
\]
where $G_\alpha$ is the Bessel kernel of order $\alpha$, and a capacity $c_{G,k}$ ($k > 1$) adapted to the operator of the heat equation of kernel $G(x,t) = \chi_{(0,\infty)}(4\pi t)^{-N/2}e^{-|x|^2/4t}$: for any Borel set $E \subset \mathbb{R}^{N+1}$,
\[
c_{G,k}(E) = \inf \{ \| \varphi \|_{L^k(\mathbb{R}^{N+1})} : \varphi \in L^k(\mathbb{R}^{N+1}), \ G * \varphi \geq \chi_E \}.
\]
From [7], there exists a solution if and only if $\mu$ does not charge the sets of $c_{G,k}(E)$ capacity zero and $u_0$ does not charge the sets of $C_{2/qd'}$, capacity zero. Observe that one can reduce to a zero initial data, by considering the measure $\mu + u_0 \otimes \delta_0$ in $\Omega \times (-T, T)$, where $\otimes$ is the tensorial product and $\delta_0$ is the Dirac mass in time at 0.

For $p \neq 2$ such a linear parabolic capacity cannot be used. Most of the contributions are relative to the case $\mu = 0$ with $\Omega$ bounded, or $\Omega = \mathbb{R}^N$. The case where $u_0$ is a Dirac mass in $\Omega$ is studied in [36], [40] when $p > 2$, and [29] when $p < 2$. Existence and uniqueness hold in the subcritical case $q < p_c$. If $q \geq p_c$ and $q > 1$, there is no solution with an isolated singularity at $t = 0$. For $q < p_c$, and $u_0 \in \mathcal{M}_b^+(\Omega)$, the existence is obtained in the sense of distributions in [60], and for any $u_0 \in \mathcal{M}_b(\Omega)$ in [16]. The case $\mu \in L^1(Q)$, $u_0 = 0$ is treated in [30], and $\mu \in L^1(Q)$, $u_0 = L^1(\Omega)$ in [4] where $\mathcal{G}$ can be multivalued. The case $\mu \in \mathcal{M}_0(Q)$ is studied in [50], with a new formulation of the solutions, and existence and uniqueness is obtained for any function $\mathcal{G} \in C(\mathbb{R})$ such that $\mathcal{G}(u)u \geq 0$. Up to our knowledge, up to now no existence results have been obtained for a general measure $\mu \in \mathcal{M}_b(Q)$.

The case of a source term $\mathcal{G}(u) = -u^q$ with $u \geq 0$ has been treated in [6] for $p = 2$, where optimal conditions are given for existence. As in the absorption case the arguments of proofs cannot be extended to general $p$. 

4
2 Main results

In all the sequel we suppose that \( p \) satisfies (1.8). Then

\[
X = L^p(0,T; W^{1,p}_0(\Omega)), \quad X' = L^{p'}(0,T; W^{-1,p'}(\Omega)).
\]

We first study problem (1.1). In Section 3 we give some approximations of \( \mu \in \mathcal{M}_b(Q) \), useful for the applications. In Section 4 we recall the definition of renormalized solutions, that we call R-solutions of (1.1), relative to the decomposition (1.11) of \( \mu_0 \), and study some of their properties.

Our main result is a stability theorem for problem (1.1), proved in Section 5, extending to the parabolic case the stability result of [32, Theorem 3.4], and improving the result of [49]:

**Theorem 2.1** Let \( A : Q \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) satisfying (1.2), (1.3). Let \( u_0 \in L^1(\Omega) \), and

\[
\mu = f - \text{div} \, g + h, \quad \mu^+_n - \mu^-_n \in \mathcal{M}_b(Q),
\]

with \( f \in L^1(Q), g \in (L^{p'}(Q))^N, \ h \in X \) and \( \mu^+_n, \mu^-_n \in \mathcal{M}_s^+(Q) \). Let \( u_{0,n} \in L^1(\Omega) \),

\[
\mu_n = f_n - \text{div} \, g_n + h_n + \rho_n - \eta_n \in \mathcal{M}_b(Q),
\]

with \( f_n \in L^1(Q), g_n \in (L^{p'}(Q))^N, h_n \in X, \) and \( \rho_n, \eta_n \in \mathcal{M}_b(Q) \), such that

\[
\rho_n = \rho_n^1 - \text{div} \, \rho_n^2 + \rho_{n,s}, \quad \eta_n = \eta_n^1 - \text{div} \, \eta_n^2 + \eta_{n,s},
\]

with \( \rho_n^1, \eta_n^1 \in L^1(Q), \rho_n^2, \eta_n^2 \in (L^{p'}(Q))^N \) and \( \rho_{n,s}, \eta_{n,s} \in \mathcal{M}_s^+(Q) \). Assume that

\[
\sup_n |\mu_n| (Q) < \infty,
\]

and \( \{u_{0,n}\} \) converges to \( u_0 \) strongly in \( L^1(\Omega) \), \( \{f_n\} \) converges to \( f \) weakly in \( L^1(Q) \), \( \{g_n\} \) converges to \( g \) strongly in \( (L^{p'}(Q))^N \), \( \{h_n\} \) converges to \( h \) strongly in \( X \), \( \{\rho_n\} \) converges to \( \mu^+_n \) and \( \{\eta_n\} \) converges to \( \mu^-_n \) in the narrow topology of measures; and \( \{\rho_n^1\}, \{\eta_n^1\} \) are bounded in \( L^1(Q) \), and \( \{\rho_n^2\}, \{\eta_n^2\} \) bounded in \( (L^{p'}(Q))^N \). Let \( \{u_n\} \) be a sequence of R-solutions of

\[
\begin{aligned}
\left\{ \begin{array}{l}
u_{n,t} - \text{div}(A(x,t,\nabla u_n)) = \mu_n \quad \text{in } Q, \\
u_n = 0 \quad \text{on } \partial Q \times (0,T), \\
u_n(0) = u_{0,n} \quad \text{in } \Omega.
\end{array} \right.
\end{aligned}
\tag{2.1}
\]

relative to the decomposition \( \{f_n + \rho_n^1 - \eta_n^1, g_n + \rho_n^2 - \eta_n^2, h_n\} \) of \( \mu_{n,0} \). Let \( v_n = u_n - h_n \). Then up to a subsequence, \( \{u_n\} \) converges a.e. in \( Q \) to a R-solution \( u \) of (1.1), and \( \{v_n\} \) converges a.e. in \( Q \) to \( v = u - h \). Moreover, \( \{\nabla u_n\}, \{\nabla v_n\} \) converge respectively to \( \nabla u, \nabla v \) a.e. in \( Q \), and \( \{T_k(u_n)\}, \{T_k(v_n)\} \) converge to \( T_k(u), T_k(v) \) strongly in \( X \) for any \( k > 0 \).

In Section 6 we give applications to problems of type (1.5).

We first give an existence result of subcritical type, valid for any measure \( \mu \in \mathcal{M}_b(Q) \):
Theorem 2.2 Let $A : Q \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.2), (1.3) with $a \equiv 0$. Let $(x, t, r) \mapsto G(x, t, r)$ be a Caratheodory function on $Q \times \mathbb{R}$ and $G \in C(\mathbb{R}^+)$ be a nondecreasing function with values in $\mathbb{R}^+$, such that

$$|G(x, t, r)| \leq G(|r|) \quad \text{for a.e. } (x, t) \in Q \text{ and any } r \in \mathbb{R},$$

$$\int_1^\infty G(s)s^{-1-p}ds < \infty.$$  

(i) Suppose that $G(x, t, r)r \geq 0$, for a.e. $(x, t)$ in $Q$ and any $r \in \mathbb{R}$. Then, for any $\mu \in M_b(Q)$ and $u_0 \in L^1(\Omega)$, there exists a $R$-solution $u$ of problem

$$\begin{cases}
  u_t - \text{div}(A(x, t, \nabla u)) + G(u) = \mu & \text{in } Q, \\
  u = 0 & \text{in } \partial Q \times (0, T), \\
  u(0) = u_0 & \text{in } \Omega.
\end{cases}$$

(ii) Suppose that $G(x, t, r)r \leq 0$, for a.e. $(x, t) \in Q$ and any $r \in \mathbb{R}$, and $u_0 \geq 0, \mu \geq 0$. There exists $\varepsilon > 0$ such that for any $\lambda > 0$, any $\mu \in M_b(Q)$ and $u_0 \in L^1(\Omega)$ with $\lambda + |\mu|(Q) + ||u_0||_{L^1(\Omega)} \leq \varepsilon$, problem

$$\begin{cases}
  u_t - \text{div}(A(x, t, \nabla u)) + \lambda G(u) = \mu & \text{in } Q, \\
  u = 0 & \text{in } \partial Q \times (0, T), \\
  u(0) = u_0 & \text{in } \Omega,
\end{cases}$$

admits a nonnegative $R$-solution.

In particular for any $0 < q < p_c$, if $G(u) = |u|^{q-1}u$, existence holds for any measure $\mu \in M_b(Q)$; if $G(u) = -|u|^{q-1}u$, existence holds for $\mu$ small enough. In the supercritical case $q \geq p_c$, the class of "admissible" measures, for which there exist solutions, is not known.

Next we give new results relative to measures that have a good behaviour in $t$, based on recent results of [17] relative to the elliptic case. We recall the notions of (truncated) Wölf potential for any nonnegative measure $\omega \in \mathcal{M}^+(\mathbb{R}^N)$ any $R > 0, x_0 \in \mathbb{R}^N$,

$$\mathcal{W}^R_{1,\rho}[\omega](x_0) = \int_0^R (t^{p-N}\omega(B(x_0, t)))^{\frac{1}{p-1}} dt.$$  

Any measure $\omega \in M_b(\Omega)$ is identified with its extension by 0 to $\mathbb{R}^N$. In case of absorption, we obtain the following:

Theorem 2.3 Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.6), (1.7). Let $p < N$, $q > p - 1$, $\mu \in M_b(Q)$, $f \in L^1(Q)$ and $u_0 \in L^1(\Omega)$. Assume that

$$|\mu| \leq \omega \otimes F, \quad \text{with } \omega \in \mathcal{M}^+_b(\Omega), F \in L^1((0, T)), F \geq 0,$$

and $\omega$ does not charge the sets of $C_p\frac{q}{q-1-p}\text{-capacity zero}$. Then there exists a $R$-solution $u$ of problem

$$\begin{cases}
  u_t - \text{div}(A(x, \nabla u)) + |u|^{q-1}u = f + \mu & \text{in } Q, \\
  u = 0 & \text{on } \partial \Omega \times (0, T), \\
  u(0) = u_0 & \text{in } \Omega.
\end{cases}$$
We show that some of these measures may not lie in $\mathcal{M}_0(Q)$, which improves the existence results of [50], see Proposition 3.3 and Remark 6.7. Otherwise our result can be extended to a more general function $\mathcal{G}$, see Remark 6.9. We also consider a source term:

**Theorem 2.4** Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ satisfying (1.6), (1.7). Let $p < N$, $q > p - 1$. Let $\mu \in \mathcal{M}_b^+(Q)$, and $u_0 \in L^\infty(\Omega)$, $u_0 \geq 0$. Assume that

$$\mu \leq \omega \otimes \chi_{(0,T)}, \quad \text{with } \omega \in \mathcal{M}_b^+(\Omega).$$

Then there exist $\lambda_0 = \lambda_0(N, p, q, c_3, c_4, \text{diam}\Omega)$ and $b_0 = b_0(N, p, q, c_3, c_4, \text{diam}\Omega)$ such that, if

$$\omega(E) \leq \lambda_0 c_{p,q-1}^{-1}(E), \quad \forall E \text{ compact } \subset \mathbb{R}^N, \quad \|u_0\|_{\infty, \Omega} \leq b_0,$$

there exists a nonnegative $R$-solution $u$ of problem

$$u_t - \text{div}(A(x, \nabla u)) = u^q + \mu \quad \text{in } Q,$$

$$u = 0 \quad \text{on } \partial \Omega \times (0, T),$$

$$u(0) = u_0 \quad \text{in } \Omega,$$

which satisfies, a.e. in $Q$,

$$u(x, t) \leq C \mathcal{W}^{2 \text{diam}\Omega}[\omega](x) + 2\|u_0\|_{L^\infty},$$

where $C = C(N, p, c_3, c_4)$.

Corresponding results in case where $\mathcal{G}$ has exponential type are given at Theorems 6.10 and 6.15.

### 3 Approximations of measures

For any open set $\omega$ of $\mathbb{R}^m$ and $F \in (L^k(\omega))^\nu$, $k \in [1, \infty]$, $m, \nu \in \mathbb{N}^*$, we set $\|F\|_{k, \omega} = \|F\|_{(L^k(\omega))^\nu}$.

First we give approximations of nonnegative measures in $\mathcal{M}_0(Q)$. We recall that any measure $\mu \in \mathcal{M}_0(Q) \cap \mathcal{M}_b(Q)$ admits a decomposition under the form $\mu = (f, g, h)$ given by (1.11). Conversely, any measure of this form, such that $h \in L^\infty(Q)$, lies in $\mathcal{M}_0(Q)$, see [50, Proposition 3.1].

**Lemma 3.1** Let $\mu \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$ and $\varepsilon > 0$.

(i) Then, we can find a decomposition $\mu = (f, g, h)$ with $f \in L^1(Q), g \in (L^{p'}(Q))^N, h \in X$ such that

$$\|f\|_{1,Q} + \|g\|_{p',Q} + \|h\|_{X} \leq (1 + \varepsilon)\mu(Q), \quad \|g\|_{p',Q} + \|h\|_{X} \leq \varepsilon.$$  \hspace{1cm} (3.1)

(ii) Furthermore, there exists a sequence of measures $\mu_n = (f_n, g_n, h_n)$, such that $f_n, g_n, h_n \in C^\infty_c(Q)$ and strongly converge to $f, g, h$ in $L^1(Q), (L^{p'}(Q))^N$ and $X$ respectively, and $\mu_n$ converges to $\mu$ in the narrow topology, and satisfying

$$\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_{X} \leq (1 + 2\varepsilon)\mu(Q), \quad \|g_n\|_{p',Q} + \|h_n\|_{X} \leq 2\varepsilon.$$  \hspace{1cm} (3.2)
Proof. (i) Step 1. Case where $\mu$ has a compact support in $Q$. By [33], we can find a decomposition $\mu = (f, g, h)$ with $f, g, h$ have a compact support in $Q$. Let $\{\varphi_n\}$ be sequence of mollifiers in $\mathbb{R}^{N+1}$. Then $\mu_n = \varphi_n * \mu \in C_c^\infty(Q)$ for $n$ large enough. We see that $\mu_n(Q) = \mu(Q) \mu_n$ admits the decomposition $\mu_n = (f_n, g_n, h_n) = (\varphi_n * f, \varphi_n * g, \varphi_n * h)$. Since $(f_n), \{g_n\}, \{h_n\}$ strongly converge to $f, g, h$ in $L^1(Q), (L^p(Q))^N$ and $X$ respectively, we have for $n_0$ large enough,
\[
\|f - f_{n_0}\|_{1,Q} + \|g - g_{n_0}\|_{p',Q} + \|h - h_{n_0}\|_X \leq \varepsilon \min\{\mu(Q), 1\}.
\]
Then we obtain a decomposition $\mu = (\hat{f}, \hat{g}, \hat{h}) = (\mu_{n_0} + f - f_{n_0}, g - g_{n_0}, h - h_{n_0})$, such that
\[
\|\hat{f}\|_{1,Q} + \|\hat{g}\|_{p',Q} + \|\hat{h}\|_X \leq (1 + \varepsilon) \mu(Q), \quad \|\hat{g}\|_{p',Q} + \|\hat{h}\|_X \leq \varepsilon.
\]
(3.3)

Step 2. General case. Let $\{\theta_n\}$ be a nonnegative, nondecreasing sequence in $C_c^\infty(Q)$ which converges to 1, a.e. in $Q$. Set $\tilde{\mu}_0 = \theta_0 \mu$, and $\tilde{\mu}_n = (\theta_n - \theta_{n-1}) \mu$, for any $n \geq 1$. Since $\tilde{\mu}_n \in \mathcal{M}_0(Q) \cap \mathcal{M}_b^+(Q)$ has compact support, by Step 1, we can find a decomposition $\tilde{\mu}_n = (\tilde{f}_n, \tilde{g}_n, \tilde{h}_n)$ such that
\[
\|\tilde{f}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq (1 + \varepsilon) \mu_n(Q), \quad \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq 2^{-n-1} \varepsilon.
\]
Let $\tilde{T}_n = \sum_{k=0}^n \tilde{f}_k, \tilde{g}_n = \sum_{k=0}^n \tilde{g}_k$ and $\tilde{h}_n = \sum_{k=0}^n \tilde{h}_k$. Clearly, $\theta_n \mu = (\tilde{T}_n, \tilde{g}_n, \tilde{h}_n)$, and $\{\tilde{T}_n\}, \{\tilde{g}_n\}, \{\tilde{h}_n\}$ converge strongly to some $f, g, h$, respectively, in $L^1(Q), (L^p(Q))^N, X$, with
\[
\|\tilde{T}_n\|_{1,Q} + \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq (1 + \varepsilon) \mu(Q), \quad \|\tilde{g}_n\|_{p',Q} + \|\tilde{h}_n\|_X \leq \varepsilon.
\]
Therefore, $\mu = (f, g, h)$ and (3.1) holds.

(ii) We take a sequence $\{m_n\}$ in $\mathbb{N}$ such that $f_n = \varphi_{m_n} * \tilde{T}_n, g_n = \varphi_{m_n} * \tilde{g}_n, h_n = \varphi_{m_n} * \tilde{h}_n \in C_c^\infty(Q)$ and
\[
\|f_n - \tilde{T}_n\|_{1,Q} + \|g_n - \tilde{g}_n\|_{p',Q} + \|h_n - \tilde{h}_n\|_X \leq \frac{\varepsilon}{n+1} \min\{\mu(Q), 1\}.
\]
Let $\mu_n = \varphi_{m_n} * (\theta_n \mu) = (f_n, g_n, h_n)$. Therefore, $(f_n), \{g_n\}, \{h_n\}$ strongly converge to $f, g, h$ in $L^1(Q), (L^p(Q))^N$ and $X$ respectively. And (3.2) holds. Furthermore, $\{\mu_n\}$ converges weak-* to $\mu$, and $\mu_n(Q) = \int_Q \theta_n d\mu$ converges to $\mu(Q)$, thus $\{\mu_n\}$ converges in the narrow topology. ■

As a consequence, we get an approximation property for any measure $\mu \in \mathcal{M}_b^+(Q)$:

Proposition 3.2 Let $\mu \in \mathcal{M}_b^+(Q)$ and $\varepsilon > 0$. Let $\{\mu_n\}$ be a nondecreasing sequence in $\mathcal{M}_b^+(Q)$ converging to $\mu$ in $\mathcal{M}_b(Q)$. Then, there exist $f_n, f \in L^1(Q), g_n, g \in (L^p(Q))^N$ and $h_n, h \in X, \mu_n,s, \mu_s \in \mathcal{M}_b^+(Q)$ such that
\[
\mu = f - \text{div} g + h_t + \mu_s, \quad \mu_n = f_n - \text{div} g_n + (h_n)_t + \mu_n,s,
\]
and $(f_n), \{g_n\}, \{h_n\}$ strongly converge to $f, g, h$ in $L^1(Q), (L^p(Q))^N$ and $X$ respectively, and $\{\mu_n,s\}$ converges to $\mu_s$ (strongly) in $\mathcal{M}_b(Q)$ and
\[
\|f_n\|_{1,Q} + \|g_n\|_{p',Q} + \|h_n\|_X + \mu_n,s(\Omega) \leq (1 + \varepsilon) \mu(Q), \quad \text{and} \quad \|g_n\|_{p',Q} + \|h_n\|_X \leq \varepsilon.
\]
(3.4)
Proposition 3.3

Let \( f \in L^1((0, T)) \) with \( \int_0^T F(t) dt \neq 0 \), and \( \mu \in M_b(\Omega) \),
then \( \omega \in M_{0,e}(\Omega) \) if and only if \( \omega \otimes F \in M_0(\Omega) \).

Proof. Assume that \( \omega \otimes F \in M_0(\Omega) \). Then, there exist \( f \in L^1(\Omega) \), \( g \in \left( L^p'(\Omega) \right)^N \) and \( h \in X \), such that

\[
\int_{\Omega} \varphi(x, t) F(t) d\omega(x) dt = \int_{\Omega} \varphi(x, t) f(x, t) dx dt + \int_{\Omega} g(x, t) \nabla \varphi(x, t) dx dt - \int_{\Omega} h(x, t) \varphi_t(x, t) dx dt,
\]
for all \( \varphi \in C_c^\infty(\Omega \times [0, T]) \), see [50, Lemma 2.24 and Theorem 2.27]. By choosing \( \varphi(x, t) = \varphi(x) \in C_c^\infty(\Omega) \) and using Fubini’s Theorem, (3.5) is rewritten as

\[
\int_{\Omega} \varphi(x) d\omega(x) = \int_{\Omega} \varphi(x) \tilde{f}(x) dx + \int_{\Omega} \tilde{g}(x) \nabla \varphi(x) dx,
\]
where \( \tilde{f}(x) = \left( \int_0^T F(t) dt \right)^{-1} \int_0^T f(x, t) dt \in L^1(\Omega) \) and \( \tilde{g}(x) = \left( \int_0^T F(t) dt \right)^{-1} \int_0^T g(x, t) dt \in \left( L^p'(\Omega) \right)^N \); hence \( \omega \in M_{0,e}(\Omega) \).
Conversely, assume that \( \omega = \tilde{f} - \text{div} \, \tilde{g} \in M_{0,e}(\Omega) \), with \( \tilde{f} \in L^1(\Omega) \) and \( \tilde{g} \in \left( L^p(\Omega) \right)^N \). So \( \omega \otimes T_n(F) = f_n - \text{div} \, g_n \), with \( f_n = \tilde{f} T_n(F) \in L^1(Q) \) and \( g_n = \tilde{g} T_n(F) \in \left( L^p(\Omega) \right)^N \). Then \( \omega \otimes T_n(F) \) admits the decomposition \( (f_n, g_n, h) \), with \( h = 0 \in L^\infty(Q) \), thus \( \omega \otimes T_n(F) \in M_0(Q) \). And \( \{ \omega \otimes T_n(F) \} \) converges to \( \omega \otimes F \) strongly in \( M_b(Q) \), since \( \| \omega \otimes (F - T_n(F)) \|_{M_b(Q)} \leq \| \omega \|_{M_{0b}(\Omega)} \| F - T_n(F) \|_{L^1((0,T))} \). Then \( \omega \otimes F \in M_0(Q) \), since \( M_0(Q) \cap M_b(Q) \) is strongly closed in \( M_b(Q) \).

4 Renormalized solutions of problem (1.1)

4.1 Notations and Definition

For any function \( f \in L^1(Q) \), we write \( \int_Q f \) instead of \( \int_Q f \, dxdt \), and for any measurable set \( E \subset Q \), \( \int_E f \) instead of \( \int_E f \, dxdt \).

We set \( T_k(r) = \max \{ \min \{ r, k \}, -k \} \), for any \( k > 0 \) and \( r \in \mathbb{R} \). We recall that if \( u \) is a measurable function defined and finite a.e. in \( Q \), such that \( T_k(u) \in X \) for any \( k > 0 \), there exists a measurable function \( w \) from \( Q \) into \( \mathbb{R}^N \) such that \( \nabla T_k(u) = \chi_{|u|\leq k} \, w \), a.e. in \( Q \), and for any \( k > 0 \). We define the gradient \( \nabla u \) of \( u \) by \( w = \nabla u \).

Let \( \mu = \mu_0 + \mu_s \in M_b(Q) \), and \((f, g, h)\) be a decomposition of \( \mu_0 \) given by (1.11), and \( \tilde{\mu}_0 = \mu_0 - h_t = f - \text{div} \, g \). In the general case \( \tilde{\mu}_0 \notin M(Q) \), but we write, for convenience,

\[
\int_Q \omega d\tilde{\mu}_0 := \int_Q (fw + g, \nabla w), \quad \forall w \in X \cap L^\infty(Q).
\]

Definition 4.1 Let \( u_0 \in L^1(\Omega) \), \( \mu = \mu_0 + \mu_s \in M_b(Q) \). A measurable function \( u \) is a renormalized solution, called R-solution of (1.1) if there exists a decomposition \((f, g, h)\) of \( \mu_0 \) such that

\[
v = u - h \in L^\sigma(0, T; W_{0,t}^{1,\sigma}(\Omega) \cap L^\infty(0, T; L^1(\Omega))), \quad \forall \sigma \in [1, m_c); \quad T_k(v) \in X, \quad \forall k > 0, \quad (4.1)
\]

and:

(i) for any \( S \in W^{2,\infty}(\mathbb{R}) \) such that \( S' \) has compact support on \( \mathbb{R} \), and \( S(0) = 0 \),

\[
- \int_\Omega S(u_0) \varphi(0) \, dx - \int_Q \varphi_t S(v) + \int_Q S'(v) A(x, t, \nabla u) \cdot \nabla \varphi + \int_Q S''(v) \varphi A(x, t, \nabla u) \cdot \nabla \varphi = \int_Q S'(v) \varphi \, d\tilde{\mu}_0,
\]

for any \( \varphi \in X \cap L^\infty(Q) \) such that \( \varphi_t \in X' \cap L^1(Q) \) and \( \varphi(T, \cdot) = 0 \);

(ii) for any \( \phi \in C(\overline{Q}) \),

\[
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \phi A(x, t, \nabla u) \cdot \nabla v = \int_Q \phi \, d\mu_s^+.
\]
\[
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \phi A(x, t, \nabla u). \nabla v = \int_Q \phi \mu_+^\infty.
\] (4.4)

**Remark 4.2** As a consequence, \( S(v) \in C([0, T]; L^1(\Omega)) \) and \( S(v)(0, \cdot) = S(u_0) \) in \( \Omega \); and \( u \) satisfies the equation
\[
(S(v))_t - \text{div}(S'(v)A(x, t, \nabla u)) + S''(v)A(x, t, \nabla u). \nabla v = fS'(v) - \text{div}(gS'(v)) + S''(v)g. \nabla v,
\] (4.5)
in the sense of distributions in \( Q \), see [49, Remark 3]. Moreover
\[
\|S(v)_t\|_{X' + L^1(\Omega)} \leq \left\|\text{div}(S'(v)A(x, t, \nabla u))\right\|_{X'} + \left\|S''(v)A(x, t, \nabla u). \nabla v\right\|_{1, Q} + \left\|S'(v)f\right\|_{1, Q} + \left\|gS''(v)\nabla v\right\|_{1, Q} + \left\|\text{div}(S'(v)g)\right\|_{X'}.
\]

Thus, if \([-M, M] \supset \text{supp} S'\),
\[
\left\|S''(v)A(x, t, \nabla u). \nabla v\right\|_{1, Q} \leq \left\|S\right\|_{W^{2, \infty}(\mathbb{R})} \left(\|A(x, t, \nabla u)\chi_{|v| \leq M}\|_{P, Q}^p + \|\nabla T_M(v)\|_{p, Q}^p\right)
\leq C \left\|S\right\|_{W^{2, \infty}(\mathbb{R})} \left(\|\nabla u\|_1^p \chi_{|v| \leq M}\|_{1, Q} + \|a\|_{P, Q}^p + \|\nabla T_M(v)\|_{p, Q}^p\right)
\]

thus
\[
\|S(v)_t\|_{X' + L^1(\Omega)} \leq C \left\|S\right\|_{W^{2, \infty}(\mathbb{R})} \left(\|\nabla u\|_1^p \chi_{|v| \leq M}\|_{1, Q} + \|a\|_{P, Q}^p + \|f\|_{1, Q} + \|g\|_{P, Q} \|\nabla u\|_1^{1/p} \chi_{|v| \leq M}\|_{1, Q}^{1/p} + \|g\|_{P, Q}\right)
\] (4.6)

We also deduce that, for any \( \varphi \in X \cap L^\infty(\Omega) \), such that \( \varphi \in X' + L^1(\Omega) \),
\[
\int_\Omega S(v(T)) \varphi(T) dx - \int_\Omega S(u_0) \varphi(0) dx - \int_Q \varphi \cdot S(v) + \int_Q S'(v)A(x, t, \nabla u). \nabla \varphi
+ \int_Q S''(v)A(x, t, \nabla u). \nabla \varphi = \int_Q S'(v) \varphi d\mu_0.
\] (4.7)

**Remark 4.3** Let \( u, v \) satisfying (4.1). It is easy to see that the condition (4.3) ( resp. (4.4) ) is equivalent to
\[
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \phi A(x, t, \nabla u). \nabla u = \int_Q \phi d\mu_+^\infty
\] (4.8)
resp.
\[
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \phi A(x, t, \nabla u). \nabla u = \int_Q \phi d\mu_-^\infty.
\] (4.9)

In particular, for any \( \varphi \in L^p(\Omega) \) there holds
\[
\lim_{m \to \infty} \frac{1}{m} \int_{m \leq |v| < 2m} |\nabla u| \varphi = 0, \quad \lim_{m \to \infty} \frac{1}{m} \int_{m \leq |v| < 2m} |\nabla v| \varphi = 0.
\] (4.10)
Remark 4.4 (i) Any function $U \in X$ such that $U_t \in X' + L^1(Q)$ admits a unique $c^0\nu$-quasi continuous representative, defined $c^0\nu$-quasi a.e. in $Q$, still denoted $U$. Furthermore, if $U \in L^\infty(Q)$, then for any $\mu_0 \in \mathcal{M}_0(Q)$, there holds $U \in L^\infty(Q, d\mu_0)$, see [49, Theorem 3 and Corollary 1].

(ii) Let $u$ be any $R$-solution of problem (1.1). Then, $v = u - h$ admits a $c^0\nu$-quasi continuous functions representative which is finite $c^0\nu$-quasi a.e. in $Q$, and $u$ satisfies definition 4.1 for every decomposition $(\tilde{f}, \tilde{g}, \tilde{h})$ such that $h - \tilde{h} \in L^\infty(Q)$, see [49, Proposition 3 and Theorem 4].

4.2 Steklov and Landes approximations

A main difficulty for proving Theorem 2.1 is the choice of admissible test functions $(S, \varphi)$ in (4.2), valid for any $R$-solution. Because of a lack of regularity of these solutions, we use two ways of approximation adapted to parabolic equations:

Definition 4.5 Let $\varepsilon \in (0, T)$ and $z \in L^1_{\text{loc}}(Q)$. For any $l \in (0, \varepsilon)$ we define the Steklov time-averages $[z]_l, [z]_{-l}$ of $z$ by

$$[z]_l(x, t) = \frac{1}{l} \int_{t}^{t+l} z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (0, T - \varepsilon),$$

$$[z]_{-l}(x, t) = \frac{1}{l} \int_{t-l}^{t} z(x, s) ds \quad \text{for a.e. } (x, t) \in \Omega \times (\varepsilon, T).$$

The idea to use this approximation for R-solutions can be found in [22]. Recall some properties, given in [50]. Let $\varepsilon \in (0, T)$, and $\varphi_1 \in C^\infty_c(\overline{\Omega} \times [0, T])$, $\varphi_2 \in C^\infty_c(\overline{\Omega} \times (0, T])$ with $\text{Supp} \varphi_1 \subset \overline{\Omega} \times [0, T - \varepsilon]$, $\text{Supp} \varphi_2 \subset \overline{\Omega} \times [\varepsilon, T]$. There holds

(i) If $z \in X$, then $\varphi_1[z]_l$ and $\varphi_2[z]_{-l} \in W$.

(ii) If $z \in X$ and $z_t \in X' + L^1(Q)$, then, as $l \to 0$, $(\varphi_1[z]_l)$ and $(\varphi_2[z]_{-l})$ converge respectively to $\varphi_1 z$ and $\varphi_2 z$ in $X$, and a.e. in $Q$; and $(\varphi_1[z]_l)_t, (\varphi_2[z]_{-l})_t$ converge to $(\varphi_1 z)_t, (\varphi_2 z)_t$ in $X' + L^1(Q)$.

(iii) If moreover $z \in L^\infty(Q)$, then from any sequence $\{l_n\} \to 0$, there exists a subsequence $\{l_{n'}\}$ such that $\{[z]_{l_{n'}}\}, \{[z]_{-l_{n'}}\}$ converge to $z$, $c^0\nu$-quasi everywhere in $Q$.

Next we recall the approximation introduced in [42], used in [30], [26], [21]:

Definition 4.6 Let $\mu \in \mathcal{M}_0(Q)$ and $u_0 \in L^1(\Omega)$. Let $u$ be a $R$-solution of (1.1), and $v = u - h$ given at (4.1), and $k > 0$. For any $\nu \in \mathbb{N}$, the Landes-time approximation $(T_k(v))_\nu$ of the truncate function $T_k(v)$ is defined in the following way:
Let \( \{ z_{\nu} \} \) be a sequence of functions in \( W^{1,p}_{0}(\Omega) \cap L^\infty(\Omega) \), such that \( \| z_{\nu} \|_{\infty,\Omega} \leq k \), \( \{ z_{\nu} \} \) converges to \( T_k(u_0) \) a.e. in \( \Omega \), and \( \nu^{-1}\| z_{\nu} \|_{W^{1,p}_{0}(\Omega)} \) converges to 0. Then, \( (T_k(v))_\nu \) is the unique solution of the problem

\[
((T_k(v))_\nu)_t = \nu (T_k(v) - (T_k(v))_\nu) \quad \text{in the sense of distributions,} \quad (T_k(v))_\nu(0) = z_{\nu}, \quad \text{in } \Omega.
\]

Therefore, \( (T_k(v))_\nu \in X \cap L^\infty(Q) \) and \( (T_k(v))_\nu)_t \in X \), see [42]. Furthermore, up to subsequences, \( \{ (T_k(v))_\nu \} \) converges to \( T_k(v) \) strongly in \( X \) and a.e. in \( Q \), and \( \| (T_k(v))_\nu \|_{L^\infty(Q)} \leq k \).

### 4.3 First properties

In the sequel we use the following notations: for any function \( J \in W^{1,\infty}(\mathbb{R}) \), nondecreasing with \( J(0) = 0 \), we set

\[
\overline{J}(r) = \int_0^r J(\tau)d\tau, \quad \underline{J}(r) = \int_0^r J'(\tau)\tau d\tau.
\]

(4.11)

It is easy to verify that \( \overline{J}(r) \geq 0 \),

\[
\overline{J}(r) + \underline{J}(r) = J(r)r, \quad \text{and} \quad \overline{J}(r) - \underline{J}(s) \geq s(J(r) - J(s)) \quad \forall r, s \in \mathbb{R}.
\]

(4.12)

In particular we define, for any \( k > 0 \), and any \( r \in \mathbb{R} \),

\[
\overline{T_k}(r) = \int_0^r T_k(\tau)d\tau, \quad \underline{T_k}(r) = \int_0^r T'_k(\tau)\tau d\tau.
\]

(4.13)

and we use several times a truncature used in [32]:

\[
H_m(r) = \chi_{[-m,m]}(r) + \frac{2m - |s|}{m} \chi_{m < |s| \leq 2m}(r), \quad \overline{H_m}(r) = \int_0^r H_m(\tau)d\tau.
\]

(4.14)

The next Lemma allows to extend the range of the test functions in (4.2). Its proof, given in the Appendix, is obtained by Steklov approximation of the solutions.

**Lemma 4.7** Let \( u \) be a R-solution of problem (1.1). Let \( J \in W^{1,\infty}(\mathbb{R}) \) be nondecreasing with \( J(0) = 0 \), and \( \overline{J} \) defined by (4.11). Then,

\[
\int_Q S'(v)A(x,t,\nabla u)\cdot \nabla (\xi J(S(v))) + \int_Q S''(v)A(x,t,\nabla u)\cdot \nabla v \xi J(S(v))
\]

\[
- \int_\Omega \xi(0)J(S(u_0))S(u_0) - \int_\Omega \xi_t \overline{J}(S(v))
\]

\[
\leq \int_Q S'(v)\xi J(S(v))d\mu_0,
\]

(4.15)

for any \( S \in W^{2,\infty}(\mathbb{R}) \) such that \( S' \) has compact support on \( \mathbb{R} \) and \( S(0) = 0 \), and for any \( \xi \in C^1(Q) \cap W^{1,\infty}(Q) \), \( \xi \geq 0 \).
Next we give estimates of the gradient, following the first estimates of [26], see also [33], [49, Proposition 2], [43].

**Proposition 4.8** If \( u \) is a R-solution of problem (1.1), then there exists \( c = c(p) \) such that, for any \( k \geq 1 \) and \( \ell \geq 0 \),

\[
\int_{\ell \leq |v| \leq \ell + k} |\nabla u|^p + \int_{\ell \leq |v| \leq \ell + k} |\nabla v|^p \leq ckM \tag{4.16}
\]

and

\[
\|v\|_{L^\infty((0,T);L^1(\Omega))} \leq c(M + |\Omega|), \tag{4.17}
\]

where

\[
M = \|u_0\|_{1,\Omega} + |\mu_s| (Q) + \|f\|_{1,Q} + \|g\|_{\mu',Q}^P + \|h\|_X + \|a\|_{\mu',Q}^P.
\]

As a consequence, for any \( k \geq 1 \),

\[
\text{meas } \{ |v| > k \} \leq C_1M_1k^{-p_c}, \quad \text{meas } \{ |\nabla v| > k \} \leq C_2M_2k^{-m_c}, \tag{4.18}
\]

\[
\text{meas } \{ |u| > k \} \leq C_3M_3k^{-p_c}, \quad \text{meas } \{ |\nabla u| > k \} \leq C_4M_2k^{-m_c}, \tag{4.19}
\]

where \( C_i = C_i(N,p,c_1,c_2), \) \( i = 1-4 \), and \( M_1 = (M+|\Omega|)^{\frac{p}{p-1}}M \) and \( M_2 = M_1 + M \).

**Proof.** Set for any \( r \in \mathbb{R} \), and \( m, k, \ell > 0 \),

\[
T_{k,\ell}(r) = \max\{\min\{r - \ell, k\}, 0\} + \min\{\max\{r + \ell, -k\}, 0\}.
\]

For \( m > k + \ell \), we can choose \((J, S, \xi) = (T_{k,\ell}(\overline{H}_m), \xi)\) as test functions in (4.15), where \( \overline{H}_m \) is defined at (4.14) and \( \xi \in C^1([0, T]) \) with values in \([0, 1]\), independent on \( x \). Since \( T_{k,\ell}(\overline{H}_m(r)) = T_{k,\ell}(r) \) for all \( r \in \mathbb{R} \), we obtain

\[
- \int_\Omega \xi(0) T_{k,\ell}(u_0) \overline{H}_m(u_0) - \int_Q \xi T_{k,\ell}(\overline{H}_m(v)) + \int_{\{\ell \leq |v| < \ell + k\}} \xi A(x, t, \nabla u) \cdot \nabla v - \frac{k}{m} \int_{\{\ell \leq |v| < 2m\}} \xi A(x, t, \nabla u) \cdot \nabla v \leq \int_Q \overline{H}_m(v) \xi T_{k,\ell}(v) \, d\mu_0.
\]

And

\[
\int_Q \overline{H}_m(v) \xi T_{k,\ell}(v) \, d\mu_0 = \int_Q \overline{H}_m(v) \xi T_{k,\ell}(v) f + \int_{\{\ell \leq |v| < \ell + k\}} \xi \nabla v \cdot g - \frac{k}{m} \int_{\{\ell \leq |v| < 2m\}} \xi \nabla v \cdot g.
\]

Let \( m \to \infty \); then, for any \( k \geq 1 \), since \( v \in L^1(Q) \) and from (4.3), (4.4), and (4.10), we find

\[
- \int_Q \xi T_{k,\ell}(v) + \int_{\{\ell \leq |v| < \ell + k\}} \xi A(x, t, \nabla u) \cdot \nabla v \leq \int_{\{\ell \leq |v| < \ell + k\}} \xi \nabla v \cdot g + k(\|u_0\|_{1,\Omega} + |\mu_s| (Q) + \|f\|_{1,Q}). \tag{4.20}
\]
Next, we take \( \xi \equiv 1 \). We verify that there exists \( c = c(p) \) such that

\[
A(x, t, \nabla u) \cdot \nabla v - \nabla v \cdot g \geq C_1 \left( |\nabla u|^p + |\nabla v|^p \right) - c(|g|^p + |\nabla h|^p + |a|^p)
\]

where \( c_1 \) is the constant in (1.2). Hence (4.16) follows. Thus, from (4.20) and the Hölder inequality, we get, with another constant \( c \), for any \( \xi \in C^1([0, T]) \) with values in \([0, 1]\),

\[
- \int_Q \xi t \frac{\partial v}{\partial t} \leq c k M
\]

Thus \( \int_\Omega T_k(v)(t) \leq c k M \), for a.e. \( t \in (0, T) \). We deduce (4.17) by taking \( k = 1, \ell = 0 \), since \( T_{1,0}(r) = T_1(r) \geq |r| - 1 \), for any \( r \in \mathbb{R} \).

Next, from the Gagliardo-Nirenberg embedding Theorem, we have

\[
\int_Q \left| T_k(v) \right|^\frac{p(N+1)}{N} \leq C_1 \| v \|_{L^\infty((0,T);L^1(\Omega))}^\frac{N}{p} \int_Q |\nabla T_k(v)|^p,
\]

where \( C_1 = C_1(N, p) \). Then, from (4.16) and (4.17), we get, for any \( k \geq 1 \),

\[
\text{meas} \{ |v| > k \} \leq k^{-\frac{p(N+1)}{N}} \int_Q |T_k(v)|^\frac{p(N+1)}{N} \leq C \| v \|_{L^\infty((0,T);L^1(\Omega))}^\frac{N}{p} k^{-\frac{p(N+1)}{N}} \int_Q |\nabla T_k(v)|^p \leq C_2 M_1 k^{-p c},
\]

with \( C_2 = C_2(N, p, c_1, c_2) \). We obtain

\[
\text{meas} \{ |\nabla v| > k \} \leq \frac{1}{k^p} \int_0^{k^p} \text{meas} \left( \{|\nabla v|^p > s\} \right) ds
\]

\[
\leq \text{meas} \left\{ |v| > k^\frac{N}{N+p} \right\} + \frac{1}{k^p} \int_0^{k^p} \text{meas} \left( \{|\nabla v|^p > s, |v| \leq k^\frac{N}{N+p}\} \right) ds
\]

\[
\leq C_3 M_1 k^{-m c} + \frac{1}{k^p} \int_{|v| \leq k^\frac{N}{N+p}} |\nabla v|^p \leq C_2 M_2 k^{-m c},
\]

with \( C_3 = C_3(N, p, c_1, c_2) \). Furthermore, for any \( k \geq 1 \),

\[
\text{meas} \{ |h| > k \} + \text{meas} \{ |\nabla h| > k \} \leq C_4 k^{-p} \| h \|_{L^p}^p,
\]

where \( C_4 = C_4(N, p, c_1, c_2) \). Therefore, we easily get (4.19). \( \square \)

**Remark 4.9** If \( \mu \in L^1(Q) \) and \( a = 0 \) in (1.2), then (4.16) holds for all \( k > 0 \) and the term \(|\Omega|\) in inequality (4.17) can be removed where \( M = \| u_0 \|_{L^1(\Omega)} + |\mu|(Q) \). Furthermore, (4.19) is stated as follows:

\[
\text{meas} \{ |u| > k \} \leq C_3 M^{-\frac{p+N}{N}} k^{-p c}, \quad \text{meas} \{ |\nabla u| > k \} \leq C_4 M^{-\frac{N+2}{N+4}} k^{-m c}, \forall k > 0.
\]
To see last inequality, we do in the following way:

\[
\text{meas}\{|\nabla v| > k\} \leq \frac{1}{k^p} \int_0^{k^p} \text{meas}\{|\nabla v|^p > s\}\,ds
\]

\[
\leq \text{meas}\{|v| > M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\} + \frac{1}{k^p} \int_0^{k^p} \text{meas}\{|\nabla v|^p > s, |v| \leq M^{\frac{1}{N+1}} k^{\frac{N}{N+1}}\}\,ds
\]

\[
\leq C_4 M^{\frac{N+1}{N}} k^{-m_c}.
\]

**Proposition 4.10** Let \(\{\mu_n\} \subset \mathcal{M}_b(Q)\), and \(\{u_{0,n}\} \subset L^1(\Omega)\), with

\[
\sup_n |\mu_n| (Q) < \infty, \text{ and } \sup_n \|u_{0,n}\|_{1,\Omega} < \infty.
\]

Let \(u_n\) be a \(R\)-solution of (1.1) with data \(\mu_n = \mu_{n,0} + \mu_{n,s}\) and \(u_{0,n}\), relative to a decomposition \((f_n, g_n, h_n)\) of \(\mu_{n,0}\), and \(v_n = u_n - h_n\). Assume that \(\{f_n\}\) is bounded in \(L^1(\Omega)\), \(\{g_n\}\) bounded in \((L^p(\Omega))^N\) and \(\{h_n\}\) bounded in \(X\).

Then, up to a subsequence, \(\{v_n\}\) converges a.e. to a function \(v\), such that \(T_k(v) \in X\) and \(v \in L^2((0,T);W^{1,p}(\Omega)) \cap L^\infty((0,T);L^1(\Omega))\) for any \(\sigma \in [1, m_c]\). And

(i) \(\{v_n\}\) converges to \(v\) strongly in \(L^\sigma(\Omega)\) for any \(\sigma \in [1, m_c]\), and \(\sup_n \|v_n\|_{L^\infty((0,T);L^1(\Omega))} < \infty\),

(ii) \(\sup_{k \geq 0} \sup_n \frac{1}{k+1} \int_Q |\nabla T_k(v_n)|^p < \infty\),

(iii) \(\{T_k(v_n)\}\) converges to \(T_k(v)\) weakly in \(X\), for any \(k > 0\),

(iv) \(\{A(x, t, \nabla (T_k(v_n) + h_n))\}\) converges to some \(F_k\) weakly in \((L^p(\Omega))^N\).

**Proof.** Take \(S \in W_{2,\infty}(\mathbb{R})\) such that \(S'\) has compact support on \(\mathbb{R}\) and \(S(0) = 0\). We combine (4.6) with (4.16), and deduce that \(\{S(v_n)\}_k\) is bounded in \(X' + L^1(\Omega)\) and \(\{S(v_n)\}_k\) bounded in \(X\). Hence, \(\{S(v_n)\}_k\) is relatively compact in \(L^1(\Omega)\). On the other hand, we choose \(S = S_k\) such that \(S_k(z) = z\), if \(|z| < k\) and \(S(z) = 2k - z\) sign \(z\), if \(|z| > 2k\). Thanks to (4.17), we obtain

\[
\text{meas}\{|v_n - v_m| > \sigma\} \leq \text{meas}\{|v_n| > k\} + \text{meas}\{|v_m| > k\} + \text{meas}\{|S_k(v_n) - S_k(v_m)| > \sigma\}
\]

\[
\leq \frac{1}{k} (|v_n|_{1,Q} + |v_m|_{1,Q}) + \text{meas}\{|S_k(v_n) - S_k(v_m)| > \sigma\}
\]

\[
\leq C + \text{meas}\{|S_k(v_n) - S_k(v_m)| > \sigma\}.
\]

(4.22)

Thus, up to a subsequence \(\{u_n\}\) is a Cauchy sequence in measure, and converges a.e. in \(Q\) to a function \(u\). Thus, \(\{T_k(v_n)\}\) converges to \(T_k(v)\) weakly in \(X\), since \(\sup_n \|T_k(v_n)\|_X < \infty\) for any \(k > 0\). And \(\{\nabla (T_k(v_n) + h_n)\}\) converges to some \(F_k\) weakly in \((L^p(\Omega))^N\). Furthermore, from (4.18), \(\{v_n\}\) converges to \(v\) strongly in \(L^\sigma(\Omega)\), for any \(\sigma < p_c\).
5 The convergence theorem

We first recall some properties of the measures, see [49, Lemma 5], [32].

**Proposition 5.1** Let \( \mu_s = \mu^+_s - \mu^-_s \in \mathcal{M}_b(Q) \), where \( \mu^+_s \) and \( \mu^-_s \) are concentrated, respectively, on two disjoint sets \( E^+ \) and \( E^- \) of zero \( \mathcal{C}_p^Q \)-capacity. Then, for any \( \delta > 0 \), there exist two compact sets \( K^+_\delta \subseteq E^+ \) and \( K^-\delta \subseteq E^- \) such that

\[
\mu^+_s(E^+ \setminus K^+_\delta) \leq \delta, \quad \mu^-_s(E^- \setminus K^-\delta) \leq \delta,
\]

and there exist \( \psi_\delta^+, \psi_\delta^- \in C_c^1(Q) \) with values in \([0,1]\), such that \( \psi_\delta^+, \psi_\delta^- = 1 \) respectively on \( K^+_\delta, K^-\delta \), and \( \text{supp}(\psi_\delta^+) \cap \text{supp}(\psi_\delta^-) = \emptyset \), and

\[
\|\psi_\delta^+\|_X + \|\psi_\delta^-\|_{X' + L^1(Q)} \leq \delta, \quad \|\psi_\delta^-\|_X + \|\psi_\delta^+\|_{X + L^1(Q)} \leq \delta.
\]

There exist decompositions \( (\psi_\delta^+)_t = (\psi_{\delta_1}^+)_t + (\psi_{\delta_2}^+)_t \) and \( (\psi_\delta^-)_t = (\psi_{\delta_1}^-)_t + (\psi_{\delta_2}^-)_t \) in \( X' + L^1(Q) \), such that

\[
\left\| (\psi_{\delta_1}^+)_t \right\|_{X'} \leq \frac{\delta}{3}, \quad \left\| (\psi_{\delta_2}^+)_t \right\|_{1,Q} \leq \frac{\delta}{3}, \quad \left\| (\psi_{\delta_1}^-)_t \right\|_{X'} \leq \frac{\delta}{3}, \quad \left\| (\psi_{\delta_2}^-)_t \right\|_{1,Q} \leq \frac{\delta}{3}. \tag{5.1}
\]

Both \( \{\psi_\delta^+\} \) and \( \{\psi_\delta^-\} \) converge to 0, *-weakly in \( L^\infty(Q) \), and strongly in \( L^1(Q) \) and up to subsequences, a.e. in \( Q \), as \( \delta \) tends to 0.

Moreover if \( \rho_n \) and \( \eta_n \) are as in Theorem 2.1, we have, for any \( \delta, \delta_1, \delta_2 > 0 \),

\[
\int_Q \psi^-_\delta \, d\rho_n + \int_Q \psi^+_\delta \, d\eta_n = \omega(n, \delta), \quad \int_Q \psi^-_\delta \, d\mu^+_s \leq \delta, \quad \int_Q \psi^+_\delta \, d\mu^-_s \leq \delta, \tag{5.2}
\]

\[
\int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) \, d\rho_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^+ \psi_{\delta_2}^+) \, d\mu^+_s \leq \delta_1 + \delta_2, \tag{5.3}
\]

\[
\int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) \, d\eta_n = \omega(n, \delta_1, \delta_2), \quad \int_Q (1 - \psi_{\delta_1}^- \psi_{\delta_2}^-) \, d\mu^-_s \leq \delta_1 + \delta_2. \tag{5.4}
\]

Hereafter, if \( n, \varepsilon, ..., \nu \) are real numbers, and a function \( \phi \) depends on \( n, \varepsilon, ..., \nu \) and eventual other parameters \( \alpha, \beta, ..., \gamma \), and \( n \to n_0, \varepsilon \to \varepsilon_0, ..., \nu \to \nu_0 \), we write \( \phi = \omega(n, \varepsilon, ..., \nu) \), then this means \( \lim_{n \to n_0} \lim_{\varepsilon \to \varepsilon_0} \lim_{\nu \to \nu_0} \lim_{n \to n_0} |\phi| = 0 \), when the parameters \( \alpha, \beta, ..., \gamma \) are fixed. In the same way, \( \phi \leq \omega(n, \varepsilon, \delta, ..., \nu) \) means \( \lim_{n \to n_0} \lim_{\varepsilon \to \varepsilon_0} \lim_{\delta \to \delta_0} \lim_{n \to n_0} \phi \leq 0 \), and \( \phi \geq \omega(n, \varepsilon, ..., \nu) \) means \( -\phi \leq \omega(n, \varepsilon, ..., \nu) \).

**Remark 5.2** In the sequel we use a convergence property, consequence of the Dunford-Pettis theorem, still used in [32]: If \( \{a_n\} \) is a sequence in \( L^1(Q) \) converging to a weakly in \( L^1(Q) \) and \( \{b_n\} \) a bounded sequence in \( L^\infty(Q) \) converging to \( b \), a.e. in \( Q \), then \( \lim_{n \to \infty} \int_Q a_n b_n = \int_Q ab \).
Next we prove Theorem 2.1.

**Scheme of the proof.** Let \( \{\mu_n\}, \{u_{0,n}\} \) and \( \{u_n\} \) satisfying the assumptions of Theorem 2.1. Then we can apply Proposition 4.10. Setting \( v_n = u_n - h_n \), up to subsequences, \( \{u_n\} \) converges \( a.e. \) in \( Q \) to some function \( u \), and \( \{v_n\} \) converges \( a.e. \) to \( v = u - h \), such that \( T_k(v) \in X \) and \( v \in L^\sigma((0,T); W_0^{1,\sigma}(\Omega)) \cap L^\infty((0,T); L^1(\Omega)) \) for every \( \sigma \in [1,m_c) \). And \( \{v_n\} \) satisfies the conclusions (i) to (iv) of Proposition 4.10. We have

\[
\begin{aligned}
\mu_n &= \left( f_n - \text{div} g_n + (h_n)_t \right) + \left( \rho_n^1 - \text{div} \rho_n^2 \right) - \left( \eta_n^1 - \text{div} \eta_n^2 \right) + \rho_{n,s} - \eta_{n,s} \\
&= \mu_{n,0} + (\rho_{n,s} - \eta_{n,s})^+ - (\rho_{n,s} - \eta_{n,s})^-,
\end{aligned}
\]

where

\[
\mu_{n,0} = \lambda_{n,0} + \rho_{n,0} - \eta_{n,0}, \quad \text{with} \quad \lambda_{n,0} = f_n - \text{div} g_n + (h_n)_t, \quad \rho_{n,0} = \rho_n^1 - \text{div} \rho_n^2, \quad \eta_{n,0} = \eta_n^1 - \text{div} \eta_n^2.
\]

Hence

\[
\rho_{n,0}, \eta_{n,0} \in M^+_b(Q) \cap M_0(Q), \quad \text{and} \quad \rho_n \geq \rho_{n,0}, \quad \eta_n \geq \eta_{n,0}.
\]

Let \( E^+, E^- \) be the sets where, respectively, \( \mu_+^\sigma \) and \( \mu_-^\sigma \) are concentrated. For any \( \delta_1, \delta_2 > 0 \), let \( \psi_{\delta_1}^+, \psi_{\delta_2}^+ \) and \( \psi_{\delta_1}^-, \psi_{\delta_2}^- \) as in Proposition 5.1 and set

\[
\Phi_{\delta_1, \delta_2} = \psi_{\delta_1}^+ \psi_{\delta_2}^+ + \psi_{\delta_1}^- \psi_{\delta_2}^-.
\]

Suppose that we can prove the two estimates, near \( E \)

\[
I_1 := \int_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla (v_n - (T_k(v))_\nu) \leq \omega(n, \nu, \delta_1, \delta_2),
\]

and far from \( E \),

\[
I_2 := \int_{\{|v_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla (v_n - (T_k(v))_\nu) \leq \omega(n, \nu, \delta_1, \delta_2).
\]

Then it follows that

\[
\overline{\lim}_{n, \nu} \int_{\{|v_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (v_n - (T_k(v))_\nu) \leq 0,
\]

which implies

\[
\overline{\lim}_{n \to \infty} \int_{\{|v_n| \leq k\}} A(x, t, \nabla u_n) \cdot \nabla (v_n - T_k(v)) \leq 0,
\]

since \( \{|T_k(v)_\nu\} \) converges to \( T_k(v) \) in \( X \). On the other hand, from the weak convergence of \( \{|T_k(v_n)\} \) to \( T_k(v) \) in \( X \), we verify that

\[
\int_{\{|v_n| \leq k\}} A(x, t, \nabla (T_k(v) + h_n)) \cdot \nabla (T_k(v_n) - T_k(v)) = \omega(n).
\]
Thus we get
\[
\int_{\{\lvert v_n \rvert \leq k\}} (A(x, t, \nabla u_n) - A(x, t, \nabla (T_k(v) + h_n))) \cdot \nabla (u_n - (T_k(v) + h_n)) = \omega(n).
\]

Then, it is easy to show that, up to a subsequence,
\[
\{\nabla u_n\} \text{ converges to } \nabla u, \quad \text{a.e. in } Q.
\] (5.11)

Therefore, \{\{A(x, t, \nabla u_n)\} converges to \(A(x, t, \nabla u)\) weakly in \((L^p(Q))^N\); and from (5.10) we find
\[
\lim_{n \to \infty} \int_Q A(x, t, \nabla u_n) \nabla T_k(v_n) \leq \int_Q A(x, t, \nabla u) \nabla T_k(v).
\]

Otherwise, \{\{A(x, t, \nabla (T_k(v_n) + h_n))\} converges weakly in \((L^p(Q))^N\) to some \(F_k\), from Proposition 4.10, and we obtain that \(F_k = A(x, t, \nabla (T_k(v) + h))\). Hence
\[
\lim_{n \to \infty} \int_Q A(x, t, \nabla (T_k(v_n) + h_n)) \cdot \nabla (T_k(v_n) + h_n) \leq \lim_{n \to \infty} \int_Q A(x, t, \nabla u_n) \nabla T_k(v)
\]
\[
+ \lim_{n \to \infty} \int_Q A(x, t, \nabla (T_k(v_n) + h_n)) \cdot \nabla h_n
\]
\[
\leq \int_Q A(x, t, \nabla (T_k(v) + h)) \cdot \nabla (T_k(v) + h).
\]

As a consequence
\[
\{T_k(v_n)\} \text{ converges to } T_k(v), \text{ strongly in } X, \quad \forall k > 0.
\] (5.12)

Then to finish the proof we have to check that \(u\) is a solution of (1.1). \[\blacksquare\]

In order to prove (5.7) we need a first Lemma, inspired of [32, Lemma 6.1], extending [49, Lemma 6 and Lemma 7]:

**Lemma 5.3** Let \(\psi_{1,\delta}, \psi_{2,\delta} \in C^1(Q)\) be uniformly bounded in \(W^{1,\infty}(Q)\) with values in \([0,1]\), such that \(\int_Q \psi_{1,\delta} d\mu_s^+ \leq \delta\) and \(\int_Q \psi_{2,\delta} d\mu_s^- \leq \delta\). Then, under the assumptions of Theorem 2.1,
\[
\frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta),
\]
\[
\frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(n, m, \delta),
\] (5.13)
\[
\frac{1}{m} \int_{-2m < v_n \leq -m} |\nabla u_n|^p \psi_{1,\delta} = \omega(n, m, \delta),
\]
\[
\frac{1}{m} \int_{-2m < v_n \leq -m} |\nabla v_n|^p \psi_{1,\delta} = \omega(n, m, \delta),
\] (5.14)
and for any \(k > 0\),
\[
\int_{\{m \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta),
\]
\[
\int_{\{m \leq v_n < m+k\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(n, m, \delta),
\] (5.15)
functions in (4.15) for $A$

Proof. (i) Proof of (5.13), (5.14). Set for any $r \in \mathbb{R}$ and any $m, \ell \geq 1$

$$S_{m,\ell}(r) = \int_0^r \left( -\frac{m + \tau}{m} \chi_{[m,2m]}(\tau) + \chi_{[2m,2m+\ell]}(\tau) + \frac{4m + 2h - \tau}{2m + \ell} \chi_{(2m+\ell,4m+2h]}(\tau) \right) d\tau,$$

$$S_m(r) = \int_0^r \left( -\frac{m + \tau}{m} \chi_{[m,2m]}(\tau) + \chi_{(2m,\infty)}(\tau) \right) d\tau.$$}

Note that $S_{m,\ell}' = \chi_{[m,2m]}/m - \chi_{[2m+\ell,2(2m+\ell)]}/(2m+\ell)$. We choose $(\xi, I, S) = (\psi_2, \delta, T_1, S_{m,\ell})$ as test functions in (4.15) for $u_n$, and observe that, from (5.5),

$$\mu_n = \mu_{n,0} - (h_n)_{1} = \frac{x_n}{\rho} + \rho_n - \eta_n = \psi_n + \rho_n - \eta_n$$

Thus we can write $\sum_{i=1}^6 A_i \leq \sum_{i=7}^{12} A_i$, where

$$A_1 = -\int_{\Omega} \psi_2, (0) T_1 (S_{m,\ell}(u_{0,n})) S_{m,\ell}(u_{0,n}), \quad A_2 = -\int_{Q} (\psi_2, \delta) T_1 (S_{m,\ell}(v_0)),$$

$$A_3 = \int_{Q} S_{m,\ell}'(v_0) T_1 (S_{m,\ell}(v_0)) A(x, t, \nabla u_n) \nabla \psi_2,,$$

$$A_4 = \int_{Q} S_{m,\ell}'(v_0)^2 \psi_2, T_1 (S_{m,\ell}(v_0)) A(x, t, \nabla u_n) \nabla v_n,$$

$$A_5 = \frac{1}{m} \int_{\{m \leq n \leq 2m\}} \psi_2, T_1 (S_{m,\ell}(v_0)) A(x, t, \nabla u_n) \nabla v_n,$$

$$A_6 = \frac{1}{2m + \ell} \int_{\{2m + \ell \leq n < 2(2m + \ell)\}} \psi_2, \delta A(x, t, \nabla u_n) \nabla v_n,$$

$$A_7 = \int_{Q} S_{m,\ell}'(v_0) T_1 (S_{m,\ell}(v_0)) \psi_2, f_n, \quad A_8 = \int_{Q} S_{m,\ell}'(v_0) T_1 (S_{m,\ell}(v_0)) g_n \nabla \psi_2,,$$

$$A_9 = \int_{Q} (S_{m,\ell}'(v_0))^2 T_1 (S_{m,\ell}(v_0)) \psi_2, \delta g_n \nabla v_n, \quad A_{10} = \frac{1}{m} \int_{m \leq n \leq 2m} T_1 (S_{m,\ell}(v_0)) \psi_2, \delta g_n \nabla v_n,$$

$$A_{11} = \frac{1}{2m + \ell} \int_{\{2m + \ell \leq n < 2(2m + \ell)\}} \psi_2, \delta g_n \nabla v_n, \quad A_{12} = \int_{Q} S_{m,\ell}'(v_0) T_1 (S_{m,\ell}(v_0)) \psi_2, \delta d (\rho_n - \eta_n).$$

Since $||S_{m,\ell}(u_{0,n})||_{1, \Omega} \leq \int_{\{m \leq u_{0,n}\}} u_{0,n} dx$, we find $A_1 = \omega(\ell, n, m)$. Otherwise

$$|A_2| \leq ||\psi_2, ||_{W^{1,\infty}(Q)} \int_{\{m \leq v_n\}} v_n, \quad |A_3| \leq ||\psi_2, ||_{W^{1,\infty}(Q)} \int_{\{m \leq v_n\}} (|a| + c_2 ||\nabla u_n||_{p-1}).$$

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which implies \( A_2 = \omega(\ell, n, m) \) and \( A_3 = \omega(\ell, n, m) \). Using (4.3) for \( u_n \), we have

\[
A_6 = - \int_Q \psi_{2,\delta} d(\rho_{n,s} - \eta_{n,s})^+ + \omega(\ell) = \omega(\ell, n, m, \delta).
\]

Hence \( A_6 = \omega(\ell, n, m, \delta) \), since \( (\rho_{n,s} - \eta_{n,s})^+ \) converges to \( \mu_\delta^+ \) as \( n \to \infty \) in the narrow topology, and \( \int_Q \psi_{2,\delta} d\mu_\delta^+ \leq \delta \). We also obtain \( A_{11} = \omega(\ell) \) from (4.10).

Now \( \left\{ S'_{m,\ell}(v_n)T_1(S_m(v_n)) \right\}_n \) converges to \( S'_m(v)T_1(S_m(v)) \). Using the Holder inequality we have

\[
\int_Q \int_{S(m,v_n)} \psi_{2,\delta} \leq \int_Q \psi_{2,\delta} d\mu_\delta^+ \leq \delta.
\]

Similarly we also show that \( A \) implies

\[
A_{12} \leq \int_Q \psi_{2,\delta} d\rho_n, \quad \text{and} \quad \int_Q \psi_{2,\delta} d\rho_n \text{ converges to } \int_Q \psi_{2,\delta} d\mu_\delta^+, \text{ hence } A_{12} \leq \omega(\ell, n, m, \delta).
\]

Using the Holder inequality and the condition (1.2) we have

\[
g_n \nabla v_n - A(x, t, \nabla u_n) \nabla v_n \leq C_1 \left( |g_n|^p' + |\nabla h_n|^p + |a|^p' \right)
\]

with \( C_1 = C_1(p, c_2) \), which implies

\[
A_9 - A_4 \leq C_1 \int_Q \left( S'_{m,\ell}(v_n) \right)^2 T_1(S_m(v_n)) \psi_{2,\delta} \left( |g_n|^p' + |h_n|^p + |a|^p' \right) = \omega(\ell, n, m).
\]

Similarly we also show that \( A_{10} - A_5/2 \leq \omega(\ell, n, m) \). Combining the estimates, we get \( A_{5}/2 \leq \omega(\ell, n, m, \delta) \). Using the Holder inequality we have

\[
A(x, t, \nabla u_n) \nabla v_n \geq C_2 \int_Q \left( |\nabla u_n|^p - C_2(|a|^p' + |\nabla h_n|^p) \right).
\]

with \( C_2 = C_2(p, c_1, c_2) \), which implies

\[
\frac{1}{m} \int_{\{m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_m(v_n)) = \omega(\ell, n, m, \delta).
\]

Note that for all \( m > 4 \), \( S_m(\ell) \geq 1 \) for any \( r \in [\frac{3}{2}m, 2m] \); hence \( T_1(S_m(\ell) = 1 \). So,

\[
\frac{1}{m} \int_{\{\frac{3}{2}m \leq v_n < 2m\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).
\]
Since $|\nabla v_n|^p \leq 2^{p-1}(|\nabla u_n|^p + 2^{p-1} |\nabla h_n|^p)$, there also holds
\[
\frac{1}{m} \int_{\{\frac{1}{2}m \leq v_n < 2m\}} |\nabla v_n|^p \psi_{2,\delta} = \omega(\ell, n, m, \delta).
\]

We deduce (5.13) by summing on each set $\{(\frac{1}{2})^\nu m \leq v_n \leq (\frac{1}{2})^{\nu+1}m\}$ for $\nu = 0, 1, 2$. Similarly, we can choose $(\xi, \psi, S) = (\psi_1, \delta, T_1, \tilde{S}_{m,\ell})$ as test functions in (4.15) for $u_n$, where $\tilde{S}_{m,\ell}(r) = S_{m,\ell}(-r)$, and we obtain (5.14).

(ii) Proof of (5.15), (5.16). We set, for any $k, m, \ell \geq 1$,
\[
S_{k,m,\ell}(r) = \int_0^r \left( T_k(\tau - T_m(\tau)) \chi_{[m,k+m+\ell]} + k\frac{2(k + \ell + m) - \tau}{k + m + \ell} \chi_{[k+m+\ell,2(k+m+\ell)]} \right) d\tau.
\]
\[
S_{k,m}(r) = \int_0^r T_k(\tau - T_m(\tau)) \chi_{[m,\infty)} d\tau.
\]

We choose $(\xi, \psi, S) = (\psi_2, \delta, T_1, S_{k,m,\ell})$ as test functions in (4.15) for $u_n$. In the same way we also obtain
\[
\int_{\{m \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} T_1(S_{k,m,\ell}(v_n)) = \omega(\ell, n, m, \delta).
\]

Note that $T_1(S_{k,m,\ell}(r)) = 1$ for any $r \geq m + 1$, thus
\[
\int_{\{m+1 \leq v_n < m+k\}} |\nabla u_n|^p \psi_{2,\delta} = \omega(n, m, \delta),
\]
which implies (5.15) by changing $m$ into $m - 1$. Similarly, we obtain (5.16).

Next we look at the behaviour near $E$.

**Lemma 5.4** Estimate (5.7) holds.

**Proof.** There holds
\[
I_1 = \int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(v_n) - \int_{\{\{v_n\} \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(v)_{\nu}.
\]

From Proposition 4.10, (iv), $\{A(x, t, \nabla (T_k(v_n) + h_n)) \cdot \nabla T_k(v)_{\nu}\}$ converges weakly in $L^1(Q)$ to $F_k \nabla T_k(v)_{\nu}$, and $\{\chi_{\{|v_n| \leq k\}}\}$ converges to $\chi_{|v| \leq k}$, a.e. in $Q$, and $\Phi_{\delta_1, \delta_2}$ converges to 0 a.e. in $Q$ as $\delta_1 \to 0$, and $\Phi_{\delta_1, \delta_2}$ takes its values in $[0, 1]$. Thanks to Remark 5.2, we have
\[
\int_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \cdot \nabla T_k(v)_{\nu}
\]
\[
= \int_Q \chi_{\{|v_n| \leq k\}} \Phi_{\delta_1, \delta_2} A(x, t, \nabla (T_k(v_n) + h_n)) \cdot \nabla T_k(v)_{\nu}
\]
\[
= \int_Q \chi_{|v| \leq k} \Phi_{\delta_1, \delta_2} F_k \nabla T_k(v)_{\nu} + \omega(n) = \omega(n, \nu, \delta_1).
\]

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Therefore, if we prove that
\[ \int_Q \Phi_{\delta_1, \delta_2} A(x, t, \nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2), \]  
(5.18)
then we deduce (5.7). As noticed in [32], [49], it is precisely for this estimate that we need the double cut \( \psi_{\delta_1}^+ \psi_{\delta_2}^+ \). To do this, we set, for any \( m > k > 0 \), and any \( r \in \mathbb{R} \),
\[ \hat{S}_{k,m}(r) = \int_0^r (k - T_k(\tau)) H_m(\tau) d\tau, \]
where \( H_m \) is defined at (4.14). Hence \( \text{supp} \hat{S}_{k,m} \subset [-2m, k] \); and \( \hat{S}_{k,m}' = -\chi_{[-k,k]} + \frac{2k}{m} \chi_{[-2m,-m]} \).
We choose \( (\varphi, S) = (\psi_{\delta_1}^+ \psi_{\delta_2}^+, \hat{S}_{k,m}) \) as test functions in (4.2). From (5.17), we can write
\[ A_1 + A_2 - A_3 + A_4 + A_5 + A_6 = 0, \]
where
\[ A_1 = -\int_Q (\psi_{\delta_1}^+ \psi_{\delta_2}^+)_t \hat{S}_{k,m}(v_n), \quad A_2 = \int_Q (k - T_k(v_n)) H_m(v_n) A(x, t, \nabla u_n) \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+), \]
\[ A_3 = \int_Q \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \nabla T_k(v_n), \quad A_4 = \frac{2k}{m} \int_{-2m < v_n \leq -m} \psi_{\delta_1}^+ \psi_{\delta_2}^+ A(x, t, \nabla u_n) \nabla v_n, \]
\[ A_5 = -\int_Q (k - T_k(v_n)) H_m(v_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d\lambda_{n,0}, \quad A_6 = \int_Q (k - T_k(v_n)) H_m(v_n) \psi_{\delta_1}^+ \psi_{\delta_2}^+ d(\eta_{n,0} - \rho_{n,0}); \]
and we estimate \( A_3 \). As in [49, p.585], since \( \{ \hat{S}_{k,m}(v_n) \} \) converges to \( \hat{S}_{k,m}(v) \) weakly in \( X \), and \( \hat{S}_{k,m}(v) \in L^\infty(Q) \), and from (5.1), there holds
\[ A_1 = -\int_Q (\psi_{\delta_1}^+)_t \psi_{\delta_2}^+ \hat{S}_{k,m}(v) - \int_Q \psi_{\delta_1}^+ (\psi_{\delta_2}^+)_t \hat{S}_{k,m}(v) + \omega(n) = \omega(n, \delta_1). \]
Next consider \( A_2 \). Notice that \( v_n = T_{2m}(v_n) \) on \( \text{supp}(H_m(v_n)) \). From Proposition 4.10, (iv), the sequence \( \{ A(x, t, \nabla (T_{2m}(v_n) + h_n)) \nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+) \} \) converges to \( F_{2m}.\nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+) \) weakly in \( L^1(Q) \).
Thanks to Remark 5.2 and the convergence of \( \psi_{\delta_1}^+ \psi_{\delta_2}^+ \) in \( X \) to 0 as \( \delta_1 \) tends to 0, we find
\[ A_2 = \int_Q (k - T_k(v)) H_m(v) F_{2m}.\nabla (\psi_{\delta_1}^+ \psi_{\delta_2}^+) + \omega(n) = \omega(n, \delta_1). \]
Then consider \( A_4 \). Then for some \( C = C(p, c_2), \)
\[ |A_4| \leq C \frac{2k}{m} \int_{-2m < v_n \leq -m} \left( |\nabla u_n|^p + |\nabla v_n|^p + |a|^p \right) \psi_{\delta_1}^+ \psi_{\delta_2}^+. \]
Since \( \psi_{\delta_1}^+ \) takes its values in \([0, 1]\), from Lemma 5.3, we get in particular \( A_4 = \omega(n, \delta_1, m, \delta_2) \).
Now estimate $A_5$. The sequence $\{(k - T_k(v_n))H_m(v_n)\psi_{\delta_1}^+\psi_{\delta_2}^+\}$ converges weakly in $X$ to $(k - T_k(v))H_m(v)\psi_{\delta_1}^+\psi_{\delta_2}^+$, and $\{(k - T_k(v_n))H_m(v_n)\}$ converges $*$-weakly in $L^\infty(Q)$ and a.e. in $Q$ to $(k - T_k(v))H_m(v)$. Otherwise $\{f_n\}$ converges to $f$ weakly in $L^1(Q)$ and $\{g_n\}$ converges to $g$ strongly in $(L^p(Q))^N$. Thanks to Remark 5.2 and the convergence of $\psi_{\delta_1}^+\psi_{\delta_2}^+$ to $0$ in $X$ and a.e. in $Q$ as $\delta_1 \to 0$, we deduce that

$$A_5 = -\int_Q (k - T_k(v_n))H_m(v)\psi_{\delta_1}^+\psi_{\delta_2}^+ d\nu_0 + \omega(n) = \omega(n, \delta_1),$$

where $\nu_0 = f - \text{div} g$.

Finally $A_6 \leq 2k \int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+ d\nu_0$; using (5.2) we also find $A_6 \leq \omega(n, \delta_1, m, \delta_2)$. By addition, since $A_3$ does not depend on $m$, we obtain

$$A_3 = \int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+ A(x, t, \nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2).$$

Reasoning as before with $(\psi_{\delta_1}^+\psi_{\delta_2}^+\hat{S}_{k,m})$ as test function in (4.2), where $\hat{S}_{k,m}(r) = -\hat{S}_{k,m}(-r)$, we get in the same way

$$\int_Q \psi_{\delta_1}^+\psi_{\delta_2}^+ A(x, t, \nabla u_n) \nabla T_k(v_n) \leq \omega(n, \delta_1, \delta_2).$$

Then, (5.18) holds.

Next we look at the behaviour far from $E$.

**Lemma 5.5**. Estimate (5.8) holds.

**Proof.** Here we estimate $I_2$: we can write

$$I_2 = \int_{\{|v_n| \leq k\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \nabla (T_k(v_n) - (T_k(v))_\nu).$$

Following the ideas of [51], used also in [49], we define, for any $r \in \mathbb{R}$ and $\ell > 2k > 0$,

$$R_{n,\nu,\ell} = T_{\ell+k} (v_n - (T_k(v))_\nu) - T_{\ell-k} (v_n - T_k(v_n)).$$

Recall that $\|\langle T_k(v)\rangle\|_{\infty, Q} \leq k$, and observe that

$$R_{n,\nu,\ell} = 2k \text{sign}(v_n) \text{ in } \{|v_n| \geq \ell + 2k\}, \quad |R_{n,\nu,\ell}| \leq 4k, \quad R_{n,\nu,\ell} = \omega(n, \nu, \ell) \text{ a.e. in } Q, \quad (5.19)$$

$$\lim_{n \to \infty} R_{n,\nu,\ell} = T_{\ell+k} (v - (T_k(v))_\nu) - T_{\ell-k} (v - T_k(v)), \quad \text{a.e. in } Q, \text{ and weakly in } X. \quad (5.20)$$

Next consider $\xi_{1,n_1} \in C_c^\infty((0, T)), \xi_{2,n_2} \in C_c^\infty((0, T)]$ with values in $[0, 1]$, such that $(\xi_{1,n_1})_t \leq 0$ and $(\xi_{2,n_2})_t \geq 0$; and $\{\xi_{1,n_1}(t)\}$ (resp. $\{\xi_{1,n_2}(t)\}$) converges to $1$, for any $t \in [0, T)$ (resp. $t \in (0, T]$);
and moreover, for any $a \in C([0, T]; L^1(\Omega))$, \(\left\{ \int_\Omega a(\xi_{1,n})_t \right\}\) and \(\int_\Omega a(\xi_{2,n_2})_t\) converge respectively to \(-\int a(T, \cdot)\) and \(\int a(0, \cdot)\). We set

\[
\varphi = \varphi_{n_1,n_2,l_1,l_2}\xi_{1,n_1}(1-\Phi_{\delta_1,\delta_2})[T_{l+k}(v_n-(T_k(v))_\nu)]_{l_1} - \xi_{2,n_2}(1-\Phi_{\delta_1,\delta_2})[T_{l-k}(v_n-T_k(v))]|_{l_2}.
\]

We can see that

\[
\varphi - (1-\Phi_{\delta_1,\delta_2})R_{n_\nu,l} = \omega(l_1,l_2,n_1,n_2).
\]

We can choose \((\varphi, S) = (\varphi_{n_1,n_2,l_1,l_2}, \Phi_{\nu,m})\) as test functions in (4.7) for \(u_n\), where \(\Phi_{\nu,m}\) is defined at (4.14), with \(m > \ell + 2k\). We obtain

\[
A_1 + A_2 + A_3 + A_4 + A_5 = A_6 + A_7,
\]

with

\[
A_1 = \int_\Omega \varphi(T)\Phi_{\nu,m}(v_n(T)) dx,
\]

\[
A_2 = -\int_\Omega \varphi(0)\Phi_{\nu,m}(u_0,n) dx,
\]

\[
A_3 = -\int_\Omega \varphi\Phi_{\nu,m}(v_n),
\]

\[
A_4 = \int_\Omega H_m(v_n)A(x,t,\nabla u_n)\nabla \varphi,
\]

\[
A_5 = \int_\Omega \varphi H'_m(v_n)A(x,t,\nabla u_n)\nabla v_n,
\]

\[
A_6 = \int_\Omega H_m(v_n)\varphi d\lambda_{n,0},
\]

\[
A_7 = \int_\Omega H_m(v_n)\varphi d(\rho_{n,0} - \eta_{n,0}).
\]

**Estimate** of \(A_4\). This term allows to study \(I_2\). Indeed, \(\{H_m(v_n)\}\) converges to 1, \(a.e.\) in \(Q\); thanks to (5.21), (5.19) (5.20), we have

\[
A_4 = \int_\Omega (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n)\nabla R_{n_\nu,\ell} - \int_\Omega R_{n_\nu,\ell}A(x,t,\nabla u_n)\nabla \Phi_{\delta_1,\delta_2} + \omega(l_1,l_2,n_1,n_2,m)
\]

\[
= \int_\Omega (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n)\nabla R_{n_\nu,\ell} + \omega(l_1,l_2,n_1,n_2,m,n,n,\ell)
\]

\[
= I_2 + \int_{\{|v_n| > k\}} (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n)\nabla R_{n_\nu,\ell} + \omega(l_1,l_2,n_1,n_2,m,n,\ell)
\]

\[
= I_2 + B_1 + B_2 + \omega(l_1,l_2,n_1,n_2,m,n,\ell),
\]

where

\[
B_1 = \int_{\{|v_n| > k\}} (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n)\nabla v_n,
\]

\[
B_2 = -\int_{\{|v_n| > k\}} (1-\Phi_{\delta_1,\delta_2})A(x,t,\nabla u_n)\nabla (T_k(v))_\nu.
\]
Now \{A(x, t, \nabla (T_{\ell+2k}(v_n) + h_n)), \nabla (T_k(v))\_\nu\} converges to \(F_{\ell+2k} \nabla (T_k(v))\_\nu\), weakly in \(L^1(Q)\). Otherwise \(\chi_{|v_n| > k} \chi_{|v-n-(T_k(v))\_\nu| \leq \ell+2k}\) converges to \(\chi_{|v| > k} \chi_{|v-(T_k(v))\_\nu| \leq \ell+2k}\), a.e. in \(Q\). And \{\(T_k(v)\)_\nu\} converges to \(T_k(v)\) strongly in \(X\). Thanks to Remark 5.2 we get

\[
B_2 = - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|v| > k} \chi_{|v-(T_k(v))\_\nu| \leq \ell+2k} F_{\ell+2k} \cdot \nabla (T_k(v))\_\nu + \omega(n)
\]

\[
= - \int_Q (1 - \Phi_{\delta_1, \delta_2}) \chi_{|v| > k} \chi_{|v-T_k(v)| \leq \ell+2k} F_{\ell+2k} \cdot \nabla T_k(v) + \omega(n, \nu) = \omega(n, \nu),
\]

since \(\nabla T_k(v) \chi_{|v| > k} = 0\). Besides, we see that, for some \(C = C(p, c_2)\),

\[
|B_1| \leq C \int_{\{\ell-2k \leq |v_n| < \ell+2k\}} (1 - \Phi_{\delta_1, \delta_2}) \left( |\nabla u_n|^p + |\nabla v_n|^p + |\alpha'| \right).
\]

Using (5.3) and (5.4) and applying (5.15) and (5.16) to \(1 - \Phi_{\delta_1, \delta_2}\), we obtain, for \(k > 0\)

\[
\int_{\{m \leq |v_n| < m+4k\}} (|\nabla u_n|^p + |\nabla v_n|^p)(1 - \Phi_{\delta_1, \delta_2}) = \omega(n, m, \delta_1, \delta_2).
\]

Thus, \(B_1 = \omega(n, \nu, \ell, \delta_1, \delta_2)\), hence \(B_1 + B_2 = \omega(n, \nu, \ell, \delta_1, \delta_2)\). Then

\[
A_4 = I_2 + \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2).
\]

**Estimate of \(A_5\).** For \(m > \ell + 2k\), since \(|\phi| \leq 2\ell\), and (5.21) holds, we get, from the dominated convergence Theorem,

\[
A_5 = \int_Q (1 - \Phi_{\delta_1, \delta_2}) R_{n, \nu, \ell} H_m'(v_n) A(x, t, \nabla u_n) \cdot \nabla v_n + \omega(l_1, l_2, n_1, n_2)
\]

\[
= \frac{-2k}{m} \int_{\{m \leq |v_n| < 2m\}} (1 - \Phi_{\delta_1, \delta_2}) A(x, t, \nabla u_n) \cdot \nabla v_n + \omega(l_1, l_2, n_1, n_2);
\]

here, the final equality followed from the relation, since \(m > \ell + 2k\),

\[
R_{n, \nu, \ell} H_m'(v_n) = \frac{-2k}{m} \chi_{m \leq |v_n| \leq 2m}, \ a.e. \ in \ Q.
\]

Next we go to the limit in \(m\), by using (4.3), (4.4) for \(u_n\), with \(\phi = (1 - \Phi_{\delta_1, \delta_2})\). There holds

\[
A_5 = -2k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d \left( (\rho_{n, s} - \eta_{n, s})^+ + (\rho_{n, s} - \eta_{n, s})^- \right) + \omega(l_1, l_2, n_1, n_2, m).
\]

Then, from (5.3) and (5.4), we get \(A_5 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)\).
Estimate of \( A_6 \). Again, from (5.21),
\[
A_6 = \int_Q H_m(v_n) \phi f_n + \int_Q g_n \nabla (H_m(v_n)) \phi
\]
\[
= \int_Q H_m(v_n)(1 - \Phi_{\delta_1, \delta_2}) R_{n,\nu,\ell} f_n + \int_Q g_n \nabla (H_m(v_n)(1 - \Phi_{\delta_1, \delta_2}) R_{n,\nu,\ell}) + \omega(l_1, l_2, n_1, n_2).
\]
Thus we can write \( A_6 = D_1 + D_2 + D_3 + D_4 + \omega(l_1, l_2, n_1, n_2) \), where
\[
D_1 = \int_Q H_m(v_n)(1 - \Phi_{\delta_1, \delta_2}) R_{n,\nu,\ell} f_n, \quad D_2 = \int_Q (1 - \Phi_{\delta_1, \delta_2}) R_{n,\nu,\ell} H'_m(v_n) g_n \nabla v_n,
\]
\[
D_3 = \int_Q H_m(v_n)(1 - \Phi_{\delta_1, \delta_2}) g_n \nabla R_{n,\nu,\ell}, \quad D_4 = - \int_Q H_m(v_n) R_{n,\nu,\ell} g_n \nabla \Phi_{\delta_1, \delta_2}.
\]
Since \( \{f_n\} \) converges to \( f \) weakly in \( L^1(Q) \), and (5.19)-(5.20) hold, we get from Remark 5.2,
\[
D_1 = \int_Q (1 - \Phi_{\delta_1, \delta_2}) (T_{\ell+k} (v - \langle T_k(v) \rangle_\nu) - T_{\ell-k} (v - T_k(v))) f + \omega(m, n) = \omega(m, n, \nu, \ell).
\]
We deduce from (4.10) that \( D_2 = \omega(m) \). Next consider \( D_3 \). Note that \( H_m(v_n) = 1 + \omega(m) \), and (5.20) holds, and \( \{g_n\} \) converges to \( g \) strongly in \( (L^p(Q))^N \), and \( \langle T_k(v) \rangle_\nu \) converges to \( T_k(v) \) strongly in \( X \). Then we obtain successively that
\[
D_3 = \int_Q (1 - \Phi_{\delta_1, \delta_2}) g_n \nabla (T_{\ell+k} (v - \langle T_k(v) \rangle_\nu) - T_{\ell-k} (v - T_k(v))) + \omega(m, n)
\]
\[
= \int_Q (1 - \Phi_{\delta_1, \delta_2}) g_n \nabla (T_{\ell+k} (v - T_k(v)) - T_{\ell-k} (v - T_k(v))) + \omega(m, n, \nu)
\]
\[
= \omega(m, \nu, \ell).
\]
Similarly we also get \( D_4 = \omega(m, \nu, \ell) \). Thus \( A_6 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2) \).

Estimate of \( A_7 \). We have
\[
|A_7| = \left| \int_Q S'_m(v_n) (1 - \Phi_{\delta_1, \delta_2}) R_{n,\nu,\ell} d (\rho_{n,0} - \eta_{n,0}) \right| + \omega(l_1, l_2, n_1, n_2)
\]
\[
\leq 4k \int_Q (1 - \Phi_{\delta_1, \delta_2}) d (\rho_n + \eta_n) + \omega(l_1, l_2, n_1, n_2).
\]
From (5.3) and (5.4) we get \( A_7 = \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2) \).

Estimate of \( A_1 + A_2 + A_3 \). We set
\[
J(r) = T_{\ell-k} (r - T_k(r)), \quad \forall r \in \mathbb{R},
\]
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and use the notations \( J \) and \( J \) of (4.11). From the definitions of \( \xi_{1,n_1}, \xi_{1,n_2} \), we can see that

\[
A_1 + A_2 = -\int_{\Omega} J(v_n(T))H_m(v_n(T)) - \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu})H_m(u_{0,n}) + \omega(l_1, l_2, n_1, n_2)
\]

\[
= -\int_{\Omega} J(v_n(T))v_n(T) - \int_{\Omega} T_{\ell+k}(u_{0,n} - z_{\nu})u_{0,n} + \omega(l_1, l_2, n_1, n_2, m),
\]

(5.25)

where \( z_{\nu} = \langle T_k(v) \rangle_{\nu}(0) \). We can write \( A_3 = F_1 + F_2 \), where

\[
F_1 = -\int_{Q} \left( \xi_{n_1} (1 - \Phi_{\delta_1, \delta_2})[T_{\ell-k} (v_n - \langle T_k(v) \rangle_{\nu})]_{l_1}\right) H_m(v_n),
\]

\[
F_2 = \int_{Q} \left( \xi_{n_2} (1 - \Phi_{\delta_1, \delta_2})[T_{\ell-k} (v_n - T_k(v_n))]_{l_2}\right) H_m(v_n).
\]

**Estimate of** \( F_2 \). We write \( F_2 = G_1 + G_2 + G_3 \), with

\[
G_1 = -\int_{Q} (\Phi_{\delta_1, \delta_2})_{l_1} J(v_n)v_n + \omega(l_1, l_2, n_1, n_2, m),
\]

\[
G_2 = \int_{Q} (1 - \Phi_{\delta_1, \delta_2})(\xi_{n_2})_{l_2} J(v_n)H_m(v_n) + \omega(l_1, l_2) = \int_{\Omega} J(u_{0,n})u_{0,n} + \omega(l_1, l_2, n_1, n_2, m).
\]

Next consider \( G_3 \). Setting \( b = H_m(v_n) \), there holds from (4.13) and (4.12),

\[
(([J(b)]_{-l_2}, b)(\cdot, t) = \frac{b(\cdot, t)}{l_2}(J(b)(\cdot, t) - J(b)(\cdot, t - l_2)).
\]

Hence

\[
([T_{\ell-k} (v_n - T_k(v_n))]_{-l_2}) H_m(v_n) \geq \left( [J(H_m(v_n))]_{-l_2}\right) = ([J(v_n)]_{-l_2}).
\]
since $J$ is constant in $\{|r| \geq m + \ell + 2k\}$. Integrating by parts in $G_3$, we find

$$G_3 \geq \int_Q \xi_{2,n_2}(1 - \Phi_{\delta_1,\delta_2}) (|J(v_n)|)_{l_2}$$

$$= -\int_Q (\xi_{2,n_2}(1 - \Phi_{\delta_1,\delta_2})) [J(v_n)]_{l_2} + \int_{\Omega} \xi_{2,n_2}(T) [J(v_n)]_{l_2}(T)$$

$$= -\int_Q (\xi_{2,n_2})_t (1 - \Phi_{\delta_1,\delta_2}) J(v_n)$$

$$+ \int_Q \xi_{2,n_2}(\Phi_{\delta_1,\delta_2})_t J(v_n) + \int_{\Omega} \xi_{2,n_2}(T) J(v_n(T)) + \omega(l_1, l_2)$$

$$= -\int_{\Omega} J(u_0,n) + \int_Q (\Phi_{\delta_1,\delta_2})_t J(v_n) + \int_{\Omega} J(v_n(T)) + \omega(l_1, l_2, n_1, n_2).$$

Therefore, since $J(v_n) - J(v_n)v_n = -J(v_n)$ and $J(u_0,n) = J(u_0,n)u_0,n - J(u_0,n)$, we obtain

$$F_2 \geq \int_{\Omega} J(u_0,n) - \int_Q (\Phi_{\delta_1,\delta_2})_t J(v_n) + \int_{\Omega} J(v_n(T)) + \omega(l_1, l_2, n_1, n_2, m). \quad (5.26)$$

**Estimate** of $F_1$. Since $m > \ell + 2k$, there holds $T_{\ell+k}(v_n - (T_k(v))) = T_{\ell+k}(\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))$ on $\text{supp} \overline{H_m}(v_n)$. Hence we can write $F_1 = L_1 + L_2$, with

$$L_1 = -\int_Q \left[ \xi_{1,n_1}(1 - \Phi_{\delta_1,\delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu)]_{l_1} (\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu \right.$$

$$L_2 = -\int_Q \left[ \xi_{1,n_1}(1 - \Phi_{\delta_1,\delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu)]_{l_1} (T_k(\overline{H_m}(v)))_\nu \right.$$

Integrating by parts we have, by definition of the Landes-time approximation,

$$L_2 = \int_Q \xi_{1,n_1}(1 - \Phi_{\delta_1,\delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu)]_{l_1} (T_k(\overline{H_m}(v)))_\nu$$

$$+ \int_{\Omega} \xi_{1,n_1}(0) [T_{\ell+k}(\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu)]_{l_1} (0) (T_k(\overline{H_m}(v)))_\nu(0)$$

$$= \nu \int_Q (1 - \Phi_{\delta_1,\delta_2}) T_{\ell+k}(v_n - (T_k(v)))_\nu (T_k(v) - (T_k(v)))_\nu + \int_{\Omega} T_{\ell+k}(u_0,n - z_\nu) z_\nu + \omega(l_1, l_2, n_1, n_2).$$

$$\quad (5.27)$$

We decompose $L_1$ into $L_1 = K_1 + K_2 + K_3$, where

$$K_1 = -\int_Q (\xi_{1,n_1})_t (1 - \Phi_{\delta_1,\delta_2}) [T_{\ell+k}(\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu)]_{l_1} (\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu$$

$$K_2 = -\int_Q \xi_{1,n_1}(\Phi_{\delta_1,\delta_2})_t [T_{\ell+k}(\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu)]_{l_1} (\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu$$

$$K_3 = -\int_Q \xi_{1,n_1}(1 - \Phi_{\delta_1,\delta_2}) (T_{\ell+k}(\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu)]_{l_1} (\overline{H_m}(v_n) - (T_k(\overline{H_m}(v)))_\nu).$$
Then we check easily that

\[ K_1 = \int_{\Omega} T_{\ell+k} (v_n - \langle T_k(v) \rangle_\nu) (T) (v_n - \langle T_k(v) \rangle_\nu) (T) dx + \omega(l_1, l_2, n_1, n_2, m), \]

\[ K_2 = \int_{Q} (\Phi_{\xi_1, \xi_2}) T_{\ell+k} (v_n - \langle T_k(v) \rangle_\nu) (v_n - \langle T_k(v) \rangle_\nu) + \omega(l_1, l_2, n_1, n_2, m). \]

Next consider \( K_3 \). Here we use the function \( T_k \) defined at (4.13). We set \( b = \overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_\nu \). Hence from (4.12),

\[ (([T_{\ell+k}(b)]_{l_1})_t)b(\cdot, t) = \frac{b(\cdot, t)}{l_1} (T_{\ell+k}(b)(\cdot, t + l_1) - T_{\ell+k}(b)(\cdot, t)) \]

\[ \leq \frac{1}{l_1} (T_{\ell+k}(b)(\cdot, t + l_1)) - T_{\ell+k}(b)(\cdot, t) = ([T_{\ell+k}(b)]_{l_1})_t. \]

Thus

\[ \left( [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_\nu)]_{l_1} \right)_t (\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_\nu) \leq \left( [T_{\ell+k}(\overline{H_m}(v_n) - \langle T_k(\overline{H_m}(v)) \rangle_\nu)]_{l_1} \right)_t = (\overline{H_m}(v_n) - \langle T_k(v) \rangle_\nu). \]

Then

\[ K_3 \geq - \int_{Q} \xi_{1, n_1} (1 - \Phi_{\xi_1, \xi_2}) (\overline{T}_{\ell+k} (v_n - \langle T_k(v) \rangle_\nu))_{l_1} \bigg|_{t} \]

\[ = \int_{Q} (\xi_{1, n_1})_{l_1} (1 - \Phi_{\xi_1, \xi_2}) [T_{\ell+k} (v_n - \langle T_k(v) \rangle_\nu)]_{l_1} - \int_{Q} \xi_{1, n_1} (\Phi_{\xi_1, \xi_2})_{l_1} [T_{\ell+k} (v_n - \langle T_k(v) \rangle_\nu)]_{l_1} \]

\[ + \int_{\Omega} \xi_{1, n_1}(0) [T_{\ell+k} (v_n - \langle T_k(v) \rangle_\nu)]_{l_1}(0) \]

\[ = - \int_{\Omega} T_{\ell+k} (v_n(T) - \langle T_k(v) \rangle_\nu(T)) - \int_{Q} (\Phi_{\xi_1, \xi_2})_{l_1} T_{\ell+k} (v_n - \langle T_k(v) \rangle_\nu) \]

\[ + \int_{\Omega} T_{\ell+k} (u_{n, n} - z) + \omega(l_1, l_2, n_1, n_2). \]

We find by addition, since \( T_{\ell+k}(r) - T_{\ell+k}(r) = \overline{T}_{\ell+k}(r) \) for any \( r \in \mathbb{R} \),

\[ L_1 \geq \int_{\Omega} T_{\ell+k} (u_{n, n} - z) + \int_{\Omega} \overline{T}_{\ell+k} (v_n(T) - \langle T_k(v) \rangle_\nu(T)) \]

\[ + \int_{Q} (\Phi_{\xi_1, \xi_2})_{l_1} \overline{T}_{\ell+k} (v_n - \langle T_k(v) \rangle_\nu) + \omega(l_1, l_2, n_1, n_2, m). \]  (5.28)
We deduce from (5.28), (5.27), (5.26),
\[
A_3 \geq \int_{\Omega} J(u_{0,n}) + \int_{\Omega} T_{\ell+k} (u_{0,n} - z_{\nu}) + \int_{\Omega} T_{\ell+k} (u_{0,n} - z_{\nu}) z_{\nu} \\
+ \int_{\Omega} T_{\ell+k} (v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) + \int_{\Omega} J(v_n(T)) + \int_Q (\Phi_{\delta_1, \delta_2})_t (T_{\ell+k} (v_n - \langle T_k(v) \rangle_{\nu}) - \overline{T}(v_n)) \\
+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2})T_{\ell+k} (v_n - \langle T_k(v) \rangle_{\nu}) \nu T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m).
\] (5.29)

Next we add (5.25) and (5.29). Note that \( J(v_n(T)) - J(v_n(T))v_n(T) = -\overline{T}(v_n(T)) \), and also \( T_{\ell+k} (u_{0,n} - z_{\nu}) - T_{\ell+k} (u_{0,n} - z_{\nu})(z_{\nu} - u_{0,n}) = -T_{\ell+k} (u_{0,n} - z_{\nu}) \). Then we find
\[
A_1 + A_2 + A_3 \geq \int_{\Omega} (J(u_{0,n}) - T_{\ell+k} (u_{0,n} - z_{\nu})) + \int_{\Omega} (T_{\ell+k} (v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) - \overline{T}(v_n(T))) \\
+ \int_Q (\Phi_{\delta_1, \delta_2})_t (T_{\ell+k} (v_n - \langle T_k(v) \rangle_{\nu}) - \overline{T}(v_n)) \\
+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2})T_{\ell+k} (v_n - \langle T_k(v) \rangle_{\nu}) \nu T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m).
\]

Notice that \( T_{\ell+k}(r-s) - \overline{T}(r) \geq 0 \) for any \( r, s \in \mathbb{R} \) such that \( |s| \leq k \); thus
\[
\int_{\Omega} (T_{\ell+k} (v_n(T) - \langle T_k(v) \rangle_{\nu}(T)) - \overline{T}(v_n(T))) \geq 0.
\]

And \( \{u_{0,n}\} \) converges to \( u_0 \) in \( L^1(\Omega) \) and \( \{v_n\} \) converges to \( v \) in \( L^1(Q) \) from Proposition 4.10. Thus we obtain
\[
A_1 + A_2 + A_3 \geq \int_{\Omega} (J(u_{0}) - T_{\ell+k} (u_{0} - z_{\nu})) + \int_Q (\Phi_{\delta_1, \delta_2})_t (T_{\ell+k} (v - \langle T_k(v) \rangle_{\nu}) - \overline{T}(v)) \\
+ \nu \int_Q (1 - \Phi_{\delta_1, \delta_2})T_{\ell+k} (v - \langle T_k(v) \rangle_{\nu}) \nu T_k(v) - \langle T_k(v) \rangle_{\nu}) + \omega(l_1, l_2, n_1, n_2, m, n).
\]

Moreover \( T_{\ell+k}(r-s) (T_k(r) - s) \geq 0 \) for any \( r, s \in \mathbb{R} \) such that \( |s| \leq k \), hence
\[
A_1 + A_2 + A_3 \geq \int_{\Omega} (J(u_{0}) - T_{\ell+k} (u_{0} - z_{\nu})) + \int_Q (\Phi_{\delta_1, \delta_2})_t (T_{\ell+k} (v - \langle T_k(v) \rangle_{\nu}) - \overline{T}(v)) \\
+ \omega(l_1, l_2, n_1, n_2, m, n).
\]

As \( \nu \to \infty \), \( \{z_{\nu}\} \) converges to \( T_k(u_0) \), a.e. in \( \Omega \), thus we get
\[
A_1 + A_2 + A_3 \geq \int_{\Omega} (J(u_{0}) - T_{\ell+k} (u_{0} - T_k(u_0))) + \int_Q (\Phi_{\delta_1, \delta_2})_t (T_{\ell+k} (v - T_k(v)) - \overline{T}(v)) \\
+ \omega(l_1, l_2, n_1, n_2, m, n, \nu).
\]

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Finally $|\mathcal{T}_{\ell+k}(r-T_k(r)) - \mathcal{T}(r)| \leq 2k |r| \chi_{\{|r| \geq \ell\}}$ for any $r \in \mathbb{R}$, thus
\[ A_1 + A_2 + A_3 \geq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell). \]
Combining all the estimates, we obtain $I_2 \leq \omega(l_1, l_2, n_1, n_2, m, n, \nu, \ell, \delta_1, \delta_2)$ which implies (5.8), since $I_2$ does not depend on $l_1, l_2, n_1, n_2, m, \ell$.

Next we conclude the proof of Theorem 2.1:

**Lemma 5.6** The function $u$ is a $R$-solution of (1.1).

**Proof.** (i) First show that $u$ satisfies (4.2). Here we proceed as in [49]. Let $\varphi \in X \cap L^\infty(Q)$ such $\varphi \in X' + L^1(Q)$, $\varphi(., T) = 0$, and $S \in W^{2, \infty}(\mathbb{R})$, such that $S'$ has compact support on $\mathbb{R}$, $S(0) = 0$. Let $M > 0$ such that $\text{supp} S' \subset [-M, M]$. Taking successively $(\varphi, S)$ and $(\varphi \psi_\delta^\pm, S)$ as test functions in (4.2) applied to $u_n$, we can write
\[ A_1 + A_2 + A_3 + A_4 = A_5 + A_6 + A_7, \quad A_2, \delta, \pm + A_3, \delta, \pm + A_4, \delta, \pm = A_5, \delta, \pm + A_6, \delta, \pm + A_7, \delta, \pm, \]
where
\[ A_1 = -\int_{\Omega} \varphi(0)S(u_{0,n}), \quad A_2 = -\int_{Q} \varphi_S(v_n), \quad A_2, \delta, \pm = -\int_{Q} (\varphi \psi_\delta^\pm)_S S(v_n), \]
\[ A_3 = \int_{Q} S'(v_n)A(x, t, \nabla u_n).\nabla \varphi, \quad A_3, \delta, \pm = \int_{Q} S'(v_n)A(x, t, \nabla u_n).\nabla (\varphi \psi_\delta^\pm), \]
\[ A_4 = \int_{Q} S'^\prime(v_n)\varphi A(x, t, \nabla u_n).\nabla v_n, \quad A_4, \delta, \pm = \int_{Q} S'^\prime(v_n)\varphi \psi_\delta^\pm A(x, t, \nabla u_n).\nabla v_n, \]
\[ A_5 = \int_{Q} S'(v_n)\varphi d\tilde{\lambda}_{n,0}, \quad A_6 = \int_{Q} S'(v_n)\varphi d\rho_{n,0}, \quad A_7 = -\int_{Q} S'(v_n)\varphi d\eta_{n,0}, \]
\[ A_5, \delta, \pm = \int_{Q} S'(v_n)\varphi \psi_\delta^\pm d\tilde{\lambda}_{n,0}, \quad A_6, \delta, \pm = \int_{Q} S'(v_n)\varphi \psi_\delta^\pm d\rho_{n,0}, \quad A_7, \delta, \pm = -\int_{Q} S'(v_n)\varphi \psi_\delta^\pm d\eta_{n,0}. \]

Since $\{u_{0,n}\}$ converges to $u_0$ in $L^1(\Omega)$, and $\{S(v_n)\}$ converges to $S(v)$ strongly in $X$ and weak* in $L^\infty(Q)$, there holds, from (5.2),
\[ A_1 = -\int_{\Omega} \varphi(0)S(u_0) + \omega(n), \quad A_2 = -\int_{Q} \varphi_S(v) + \omega(n), \quad A_2, \delta, \psi_\delta^\pm = \omega(n, \delta). \]

Moreover $T_M(v_n)$ converges to $T_M(v)$, then $T_M(v_n) + h_n$ converges to $T_k(v) + h$ strongly in $X$, thus
\[ A_3 = \int_{Q} S'(v_n)A(x, t, \nabla (T_M(v_n) + h_n)).\nabla \varphi \]
\[ = \int_{Q} S'(v)A(x, t, \nabla (T_M(v) + h)).\nabla \varphi + \omega(n) \]
\[ = \int_{Q} S'(v)A(x, t, \nabla u).\nabla \varphi + \omega(n); \]
and

\[ A_4 = \int_Q S''(v_n) \varphi A(x,t,\nabla (T_M (v_n) + h_n)).\nabla T_M (v_n) \]
\[ = \int_Q S''(v) \varphi A(x,t,\nabla (T_M (v) + h)).\nabla T_M (v) + \omega(n) \]
\[ = \int_Q S''(v) \varphi A(x,t,\nabla u).\nabla v + \omega(n). \]

In the same way, since \( \psi_\delta^\pm \) converges to 0 in \( X \),

\[ A_{3,\delta,\pm} = \int_Q S'(v)A(x,t,\nabla u).\nabla (\varphi \psi_\delta^\pm) + \omega(n) = \omega(n, \delta), \]
\[ A_{4,\delta,\pm} = \int_Q S''(v)\varphi \psi_\delta^\pm A(x,t,\nabla u).\nabla v + \omega(n) = \omega(n, \delta). \]

And \( \{g_n\} \) converges strongly in \( (L^p'(\Omega))^N \), thus

\[ A_5 = \int_Q S'(v_n)\varphi f_n + \int_Q S'(v_n)g_n.\nabla \varphi + \int_Q S''(v_n)\varphi g_n.\nabla T_M (v_n) \]
\[ = \int_Q S'(v)\varphi f + \int_Q S'(v)g.\nabla \varphi + \int_Q S''(v)\varphi g.\nabla T_M (v) + \omega(n) \]
\[ = \int_Q S'(v)\varphi d\mu_0 + \omega(n). \]

and \( A_{5,\delta,\pm} = \int_Q S'(v)\varphi \psi_\delta^\pm d\lambda_{n,0} + \omega(n) = \omega(n, \delta) \). Then \( A_{6,\delta,\pm} + A_{7,\delta,\pm} = \omega(n, \delta) \). From (5.2) we verify that \( A_{7,\delta,=} = \omega(n, \delta) \) and \( A_{6,\delta,\pm} = \omega(n, \delta) \). Moreover, from (5.6) and (5.2), we find

\[ |A_6 - A_{6,\delta,\pm}| \leq \int_Q |S'(v_n)\varphi| (1 - \psi_\delta^\pm)d\rho_n,0 \leq ||S||_{L^2(\mathbb{R})}||\varphi||_{L^\infty(Q)} \int_Q (1 - \psi_\delta^\pm)d\rho_n = \omega(n, \delta). \]

Similarly we also have \( |A_7 - A_{7,\delta,\pm}| \leq \omega(n, \delta) \). Hence \( A_6 = \omega(n) \) and \( A_7 = \omega(n) \). Therefore, we finally obtain (4.2):

\[- \int_\Omega \varphi(0) S(u_0) - \int_Q \varphi_t S(v) + \int_Q S'(v)A(x,t,\nabla u).\nabla \varphi + \int_Q S''(v)\varphi A(x,t,\nabla u).\nabla v = \int_Q S'(v)\varphi d\mu_0. \]

(5.30)
Clearly, in (5.32), we go to the limit as functions in (5.30), with \( H_m \) as in (4.14). We can write \( D_{1,m} + D_{2,m} = D_{3,m} + D_{4,m} + D_{5,m} \), where

\[
\begin{align*}
D_{1,m} &= -\int_Q ((1 - \psi_\delta) \varphi)_{\delta} H_m(v), \\
D_{2,m} &= \int_Q H_m(v) A(x, t, \nabla u) \cdot \nabla ((1 - \psi_\delta) \varphi), \\
D_{3,m} &= \int_Q H_m(v) (1 - \psi_\delta) \varphi d\mu_0, \\
D_{4,m} &= \frac{1}{m} \int_{m \leq v \leq 2m} (1 - \psi_\delta) \varphi A(x, t, \nabla u). \nabla v, \\
D_{5,m} &= -\frac{1}{m} \int_{-2m \leq v \leq -m} (1 - \psi_\delta) \varphi A(x, t, \nabla u) \cdot \nabla v.
\end{align*}
\]

(5.31)

Taking the same test functions in (4.2) applied to \( u_n \), there holds \( D^n_{1,m} + D^n_{2,m} = D^n_{3,m} + D^n_{4,m} + D^n_{5,m} \), where

\[
\begin{align*}
D^n_{1,m} &= -\int_Q ((1 - \psi_\delta) \varphi)_{\delta} H_m(v_n), \\
D^n_{2,m} &= \int_Q H_m(v_n) A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta) \varphi), \\
D^n_{3,m} &= \int_Q H_m(v_n) (1 - \psi_\delta) \varphi d\lambda_{n,0} + \rho_{n,0} - \eta_{n,0}, \\
D^n_{4,m} &= \frac{1}{m} \int_{m \leq v \leq 2m} (1 - \psi_\delta) \varphi A(x, t, \nabla u_n). \nabla v_n, \\
D^n_{5,m} &= -\frac{1}{m} \int_{-2m \leq v_n \leq -m} (1 - \psi_\delta) \varphi A(x, t, \nabla u_n) \cdot \nabla v_n.
\end{align*}
\]

(5.32)

In (5.32), we go to the limit as \( m \to \infty \). Since \( \{ H_m(v_n) \} \) converges to \( v_n \) and \( \{ H_m(v_n) \} \) converges to 1, a.e. in \( Q \), and \( \{ \nabla H_m(v_n) \} \) converges to 0, weakly in \( (L^p(Q))^N \), we obtain the relation

\[
D^n_1 + D^n_2 = D^n_3 + D^n_4 + D^n_5,
\]

where

\[
\begin{align*}
D^n_1 &= -\int_Q ((1 - \psi_\delta) \varphi)_{\delta} v_n, \\
D^n_2 &= \int_Q A(x, t, \nabla u_n) \cdot \nabla ((1 - \psi_\delta) \varphi), \\
D^n_3 &= \int_Q (1 - \psi_\delta) \varphi d\lambda_{n,0} \\
D^n_4 &= \int_Q (1 - \psi_\delta) \varphi d(\rho_{n,0} - \eta_{n,0}) + \int_Q (1 - \psi_\delta) \varphi d(\rho_{n,s} - \eta_{n,s}) - (\rho_{n,s} - \eta_{n,s}) \}
\]

Clearly, \( D_{i,m} - D^n_i = \omega(n, m) \) for \( i = 1, 2, 3 \). From Lemma (5.3) and (5.2)-(5.4), we obtain \( D_{5,m} = \omega(n, m, \delta) \), and

\[
\begin{align*}
\frac{1}{m} \int_{\{ m \leq v < 2m \}} \psi_\delta \varphi A(x, t, \nabla u). \nabla v &= \omega(n, m, \delta),
\end{align*}
\]

thus,

\[
\begin{align*}
D_{4,m} = \frac{1}{m} \int_{\{ m \leq v < 2m \}} \varphi A(x, t, \nabla u). \nabla v + \omega(n, m, \delta).
\end{align*}
\]
Since \( |\int_Q (1 - \psi_\delta^\pm) \varphi \, d\eta_n| \leq \|\varphi\|_{L^\infty} \int_Q (1 - \psi_\delta^\pm) \, d\eta_n \), it follows that \( \int_Q (1 - \psi_\delta^\pm) \varphi \, d\eta_n = \omega(n, m, \delta) \) from (5.4). And \( |\int_Q \psi_\delta^\pm \varphi \, d\rho_n| \leq \|\varphi\|_{L^\infty} \int_Q \psi_\delta^\pm \, d\rho_n \), thus, from (5.2), \( \int_Q (1 - \psi_\delta^\pm) \varphi \, d\rho_n = \int_Q \varphi \, d\mu_\delta^\pm + \omega(n, m, \delta) \). Then \( D = \int_Q \varphi \, d\mu_\delta^\pm + \omega(n, m, \delta) \). Therefore by substraction, we get

\[
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi \, d\mu_\delta^\pm + \omega(n, m, \delta),
\]

hence

\[
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi \, d\mu_\delta^\pm,
\]

which proves (4.3) when \( \varphi \in C_\infty^\infty(Q) \). Next assume only \( \varphi \in C^\infty(\overline{Q}) \). Then

\[
\lim_{m \to \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi A(x, t, \nabla u) \cdot \nabla v = \int_Q \varphi \, d\mu_\delta^\pm + D,
\]

where,

\[
D = \int_Q \varphi (1 - \psi_\delta^\pm) \, d\mu_\delta^\pm + \lim_{n \to \infty} \frac{1}{m} \int_{\{m \leq v < 2m\}} \varphi (1 - \psi_\delta^\pm) A(x, t, \nabla u) \cdot \nabla v = \omega(\delta).
\]

Therefore, (5.33) still holds for \( \varphi \in C_\infty^\infty(\overline{Q}) \), and we deduce (4.3) by density, and similarly, (4.4). This completes the proof of Theorem 2.1.

As a consequence of Theorem 2.1, we get the following:

**Corollary 5.7** Let \( u_0 \in L^1(\Omega) \) and \( \mu \in M_b(\Omega) \). Then there exists a R-solution \( u \) to the problem 1.1 with data \((\mu, u_0)\). Furthermore, if \( v_0 \in L^1(\Omega) \) and \( \omega \in M_b(\Omega) \) such that \( u_0 \leq v_0 \) and \( \mu \leq \omega \), then one can find R-solution \( v \) to the problem 1.1 with data \((\omega, v_0)\) such that \( u \leq v \).

In particular, if \( a \equiv 0 \) in (1.2), then \( u \) satisfies (4.21) and \( \|v\|_{L^\infty((0,T);L^1(\Omega))} \leq M \) with \( M = \|u_0\|_{L^1(\Omega)} + \|\mu\|_{L^1(\Omega)} \).

### 6 Equations with perturbation terms

Let \( A : Q \times \mathbb{R}^N \to \mathbb{R}^N \) satisfying (1.2), (1.3) with \( a \equiv 0 \). Let \( G : \Omega \times (0,T) \times \mathbb{R} \to \mathbb{R} \) be a Caratheodory function. If \( U \) is a function defined in \( Q \) we define the function \( G(U) \) in \( Q \) by

\[
G(U)(x,t) = G(x,t,U(x,t)) \quad \text{for a.e. } (x,t) \in Q.
\]
We consider the problem (1.5):
\[
\begin{cases}
  u_t - \text{div}(A(x,t,\nabla u)) + G(u) = \mu & \text{in } Q, \\
  u = 0 & \text{in } \partial\Omega \times (0,T), \\
  u(0) = u_0 & \text{in } \Omega,
\end{cases}
\]
where $\mu \in \mathcal{M}_b(Q)$, $u_0 \in L^1(\Omega)$. We say that $u$ is a R-solution of problem (1.5) if $G(u) \in L^1(Q)$ and $u$ is a R-solution of (1.1) with data $(\mu - G(u), u_0)$.

### 6.1 Subcritical type results

For proving Theorem 2.2, we begin by an integration Lemma:

**Lemma 6.1** Let $G$ satisfying (2.3). If a measurable function $V$ in $Q$ satisfies
\[
\text{meas}\{ |V| \geq t \} \leq Mt^{-p_c}, \quad \forall t \geq 1,
\]
for some $M > 0$, then for any $L > 1$,
\[
\int_{\{|V| \geq L\}} G(|V|) \leq p_cM \int_L^{\infty} G(s) s^{-1-p_c} ds. \tag{6.1}
\]

**Proof.** Indeed, setting $G_L(s) = \chi_{[L,\infty)}(s)G(s)$, we have
\[
\int_{\{|V| \geq L\}} G(|V|) dxdt = \int_Q G_L(|V|) dxdt \leq \int_0^{\infty} G_L(|V|^*(s)) ds
\]
where $|V|^*$ is and the rearrangement of $|V|$, defined by
\[
|V|^*(s) = \inf \{ a > 0 : \text{meas}\{ |V| > a \} \leq s \}, \quad \forall s \geq 0.
\]
From the assumption, we get $|V|^*(s) \leq \sup \left( (Ms^{-1})^{p_c-1}, 1 \right)$. Thus, for any $L > 1$,
\[
\int_{\{|V| \geq L\}} G(|V|) dxdt \leq \int_0^{\infty} G_L \left( \sup \left( (Ms^{-1})^{p_c-1}, 1 \right) \right) ds = p_cM \int_L^{\infty} G(s) s^{-1-p_c} ds,
\]
which implies (6.1).

**Proof of Theorem 2.2. Proof of (i)** Let $\mu = \mu_0 + \mu_s \in \mathcal{M}_b(Q)$, with $\mu_0 \in \mathcal{M}_0(Q), \mu_s \in \mathcal{M}_s(Q)$, and $u_0 \in L^1(\Omega)$. Then $\mu^+_0, \mu^-_0$ can be decomposed as $\mu^+_0 = (f_1, g_1, h_1), \mu^-_0 = (f_2, g_2, h_2)$. Let $\mu_{s,i} \in C_c^\infty(Q), \mu_{s,i} \geq 0$, converging respectively to $\mu^+_s, \mu^-_s$ in the narrow topology. By Lemma 3.1, we can find $f_{n,i}, g_{n,i}, h_{n,i} \in C_c^\infty(Q)$ which strongly converge to $f_i, g_i, h_i$ in $L^1(Q)$, $\left( L^p(Q) \right)^N$ and
Let \( u_n = \mu_{n,0,1} + \mu_{n,0,2} + \mu_{n,s,1} + \mu_{n,s,2} \), then \( |\mu_n|(|Q|) \leq |\mu|(Q) \). Consider a sequence \( \{u_{0,n}\} \subset C_c^\infty(\Omega) \) which strongly converges to \( u_0 \) in \( L^1(\Omega) \) and satisfies \( ||u_{0,n}||_{1,\Omega} \leq ||u_0||_{1,\Omega} \).

Let \( u_n \) be a solution of

\[
\begin{cases}
(u_n)_t - \text{div}(A(x,t,\nabla u_n)) + G(u_n) = \mu_n & \text{in } Q, \\
u_n = 0 & \text{on } \partial \Omega \times (0,T), \\
u_n(0) = u_{0,n} & \text{in } \Omega.
\end{cases}
\]

We can choose \( \varphi = \varepsilon^{-1}T_\varepsilon(u_n) \) as test function of above problem. Then we find

\[
\int_Q (\varepsilon^{-1}T_\varepsilon(u_n))_t + \int_Q \varepsilon^{-1}A(x,t,\nabla T_\varepsilon(u_n)) \cdot \nabla T_\varepsilon(u_n) + \int_Q G(x,t,u_n)\varepsilon^{-1}T_\varepsilon(u_n) = \int_Q \varepsilon^{-1}T_\varepsilon(u_n)d\mu_n.
\]

Since

\[
\int_Q (\varepsilon^{-1}T_\varepsilon(u_n))_t = \int_\Omega \varepsilon^{-1}T_\varepsilon(u_n(T))dx - \int_\Omega \varepsilon^{-1}T_\varepsilon(u_{0,n})dx \geq -||u_{0,n}||_{L^1(\Omega)},
\]

there holds

\[
\int_Q G(x,t,u_n)\varepsilon^{-1}T_\varepsilon(u_n) \leq |\mu_n|(Q) + ||u_{0,n}||_{L^1(\Omega)} \leq |\mu|(Q) + ||u_0||_{1,\Omega}.
\]

Letting \( \varepsilon \to 0 \), we obtain

\[
\int_Q |G(x,t,u_n)| \leq |\mu|(Q) + ||u_0||_{1,\Omega}.
\]

Next apply Proposition 4.8 and Remark 4.9 to \( u_n \) with initial data \( u_{0,n} \) and measure data \( \mu_n - G(u_n) \in L^1(\Omega) \), we get

\[
\text{meas} \{|u_n| \geq s\} \leq C(|\mu|(Q) + ||u_0||_{L^1(\Omega)}) \frac{\varepsilon^{\frac{n}{s}}}{s^{-p_c}}, \quad \forall s > 0, \forall n \in \mathbb{N},
\]

for some \( C = C(N,p,c_1,c_2) \). Since \( |G(x,t,u_n)| \leq G(|u_n|) \), we deduce from (6.1) that \( \{G(u_n)\} \) is equi-integrable. Then, thanks to Proposition 4.10, up to a subsequence, \( \{u_{n}\} \) converges to some function \( u \), a.e. in \( Q \), and \( \{G(u_n)\} \) converges to \( G(u) \) in \( L^1(Q) \). Therefore, by Theorem 2.1, \( u \) is a R-solution of (2.4).

**Proof of (ii).** Let \( \{u_n\}_{n \geq 1} \) be defined by induction as nonnegative R-solutions of

\[
\begin{cases}
(u_1)_t - \text{div}(A(x,t,\nabla u_1)) = \mu & \text{in } Q, \\
u_1 = 0 & \text{on } \partial \Omega \times (0,T), \\
u_1(0) = u_0 & \text{in } \Omega,
\end{cases}
\]

\[
\begin{cases}
(u_{n+1})_t - \text{div}(A(x,t,\nabla u_{n+1})) = \mu - \lambda G(u_n) & \text{in } Q, \\
u_{n+1} = 0 & \text{on } \partial \Omega \times (0,T), \\
u_{n+1}(0) = u_n & \text{in } \Omega,
\end{cases}
\]

The proof is similar to that of (i) and is omitted here.

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Thanks to Corollary 5.7 we can assume that \{u_n\} is nondecreasing and satisfies for any \( s > 0 \) and \( n \in \mathbb{N} \)

\[
\text{meas}\{ |u_n| \geq s \} \leq C_1 K_n s^{-p_c}, \tag{6.3}
\]

where \( C_1 \) does not depend on \( s,n, \) and

\[
K_1 = (||u_0||_{L^1(\Omega)} + ||\mu||_{L^1(\Omega)})^{\frac{p_c}{p_c - N}},
\]

\[
K_{n+1} = (||u_0||_{L^1(\Omega)} + ||\mu||_{L^1(\Omega)} + \lambda ||\mathcal{G}(u_n)||_{L^1(\Omega)})^{\frac{p_c}{p_c - N}},
\]

for any \( n \geq 1. \) Take \( \varepsilon = \lambda + ||\mu||_{L^1(\Omega)} \leq 1. \) Denoting by \( C_i \) some constants independent on \( n, \varepsilon, \) there holds

\[
K_{n+1} \leq C_3 \varepsilon (||\mathcal{G}(u_n)||_{L^1(\Omega)}^{1+p_c} + 1).
\]

From (6.1) and (6.3), we find

\[
||\mathcal{G}(u_n)||_{L^1(Q)} \leq |Q| G(2) + \int_{\{u_n \geq 2\}} G(|u_n|) \, dx \, dt \leq |Q| G(2) + C_4 K_n \int_2^\infty G(s) s^{-1-p_c} \, ds.
\]

Thus, \( K_{n+1} \leq C_5 \varepsilon (K_n^{1+p_c} + 1). \) Therefore, if \( \varepsilon \) is small enough, \( \{K_n\} \) is bounded. Then, again from (6.1) and the relation \( |\mathcal{G}(x,t,u_n)| \leq G(|u_n|) \) we verify that \( \{\mathcal{G}(u_n)\} \) converges. Then by Theorem 2.1, up to a subsequence, \( \{u_n\} \) converges to a R-solution \( u \) of (2.5).

\[\square\]

### 6.2 General case with absorption terms

In the sequel we assume that \( A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N \) does not depend on \( t. \) We recall a result obtained in [53],[17] in the elliptic case:

**Theorem 6.2** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N. \) Let \( A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N \) satisfying (1.6),(1.7). Then there exists a constant \( \kappa \) depending on \( N,p,c_3,c_4 \) such that, if \( \omega \in M_b(\Omega) \) and \( u \) is a R-solution of problem

\[
\begin{cases}
- \text{div}(A(x,\nabla u)) = \omega & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

there holds

\[
-\kappa W^{2\text{diam}(\Omega)}_{1,p}[\omega^-] \leq u \leq \kappa W^{2\text{diam}(\Omega)}_{1,p}[\omega^+]. \tag{6.4}
\]

Next we give a general result in case of absorption terms:

**Theorem 6.3** Let \( p < N, \) \( A : \Omega \times \mathbb{R}^N \mapsto \mathbb{R}^N \) satisfying (1.6),(1.7), and \( \mathcal{G} : Q \times \mathbb{R} \mapsto \mathbb{R} \) be a Caratheodory function such that the map \( s \mapsto \mathcal{G}(x,t,s) \) is nondecreasing and odd, for a.e. \( (x,t) \) in \( Q. \)
Let $\mu_1, \mu_2 \in M^+_b(Q)$ such that there exist $\omega_n \in M^+_b(\Omega)$ and nondecreasing sequences $\{\mu_{1,n}\}, \{\mu_{2,n}\}$ in $M^+_b(Q)$ with compact support in $Q$, converging to $\mu_1, \mu_2$, respectively in the narrow topology, and
\[
\mu_{1,n}, \mu_{2,n} \leq \omega_n \otimes \chi(0,T), \quad G((n + \kappa W^{2diam(\Omega)}_{1,p}[\omega_n])) \in L^1(Q),
\]
where the constant $c$ is given at Theorem 6.2. Let $u_0 \in L^1(\Omega)$, and $\mu = \mu_1 - \mu_2$. Then there exists a $R$-solution $u$ of problem (1.5).

Moreover if $u_0 \in L^\infty(\Omega)$, and $\omega_n \leq \gamma$ for any $n \in \mathbb{N}$, for some $\gamma \in M^+_b(\Omega)$, then a.e. in $Q$,
\[
|u(x,t)| \leq \kappa W^{2diam\Omega}_{1,p}\gamma(x) + ||u_0||_{\infty,\Omega}. \quad (6.5)
\]

For proving this result, we need two Lemmas:

**Lemma 6.4** Let $G$ satisfy the assumptions of Theorem 6.3 and $G \in L^\infty(\Omega \times \mathbb{R})$. For $i = 1, 2$, let $u_{0,i} \in L^\infty(\Omega)$ be nonnegative, and $\lambda_i = \lambda_{i,0} + \lambda_{i,s} \in M^+_b(Q)$ with compact support in $Q$, $\gamma \in M^+_b(\Omega)$ with compact support in $\Omega$ such that $\lambda_i \leq \gamma \otimes \chi(0,T)$. Let $\lambda_{i,0} = (f_i, g_i, h_i)$ be a decomposition of $\lambda_{i,0}$ into functions with compact support in $Q$. Then, there exist $R$-solutions $u, u_{1,2}$, to problems
\[
u_i - \text{div}(A(x,\nabla u_i)) + G(u_i) = \lambda_i - \lambda_2 \quad \text{in } Q, \quad u = 0 \quad \text{on } \partial\Omega \times (0,T), \quad u(0) = u_{0,1} - u_{0,2},
\]
\[
(u_i)_t - \text{div}(A(x,\nabla u_i)) + G(u_i) = \lambda_i \quad \text{in } Q, \quad u_i = 0 \quad \text{on } \partial\Omega \times (0,T), \quad u_i(0) = u_{0,i},
\]
relative to decompositions $(f_{1,n} - f_{2,n} - G(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n})$, $(f_{i,n} - G(u_{i,n}), g_{i,n}, h_{i,n})$, such that a.e. in $Q$,
\[
-||u_{0,2}||_{\infty,\Omega} - \kappa W^{2diam\Omega}_{1,p}\gamma(x) \leq -u_2(x,t) \leq u(x,t) \leq u_1(x,t) \leq \kappa W^{2diam\Omega}_{1,p}\gamma(x) + ||u_{0,1}||_{\infty,\Omega},
\]
and
\[
\int_Q |G(u)| \leq \sum_{i=1,2} (\lambda_i(Q) + ||u_{0,i}||_{L^1(\Omega)}), \quad \text{and} \quad \int_Q G(u_i) \leq \lambda_i(Q) + ||u_{0,i}||_{1,\Omega}, \quad i = 1, 2. \quad (6.9)
\]
Furthermore, assume that $\mathcal{H}, \mathcal{K}$ have the same properties as $G$, and $\mathcal{H}(x,t,s) \leq G(x,t,s) \leq \mathcal{K}(x,t,s)$ for any $s \in (0,\infty)$ and a.e. in $Q$. Then, one can find solutions $u_i(\mathcal{H}), u_i(\mathcal{K})$, corresponding to $\mathcal{H}, \mathcal{K}$ with data $\lambda_i$, such that $u_i(\mathcal{H}) \geq u_i(\mathcal{K}), i = 1, 2$.

Assume that $\omega_i, \theta_i$ have the same properties as $\lambda_i$ and $\omega_i \leq \lambda_i \leq \theta_i$, $u_{0,1,1}, u_{0,1,2} \in L^\infty(\Omega)$, $u_{0,1,2} \leq u_{0,1} \leq u_{0,1,1}$. Then one can find solutions $u_i(\omega_i), u_i(\theta_i)$, corresponding to $(\omega_i, u_{0,1,2}),(\theta_i, u_{0,1,1})$, such that $u_i(\omega_i, u_{0,1,2}) \leq u_i \leq u_i(\theta_i, u_{0,1,1})$.

**Proof.** Let $\{\varphi_{1,n}\}, \{\varphi_{2,n}\}$ be sequences of mollifiers in $\mathbb{R}$ and $\mathbb{R}^N$, and $\varphi_n = \varphi_{1,n} \varphi_{2,n}$. Set $\gamma_n = \varphi_{2,n} * \gamma$, and for $i = 1, 2$, $u_{0,1,2} = \varphi_{2,n} * u_{0,i}$,
\[
\lambda_{i,n} = \varphi_n * \lambda_i = f_{i,n} - \text{div}(g_{i,n}) + (h_{i,n})_t + \lambda_{i,s,n},
\]
where $f_{i,n} = \varphi_n * f_i, g_{i,n} = \varphi_n * g_i, h_{i,n} = \varphi_n * h_i, \lambda_{i,s,n} = \varphi_n * \lambda_{i,s}$, and
\[
\lambda_n = \lambda_{1,n} - \lambda_{2,n} = f_n - \text{div}(g_n) + (h_n)_t + \lambda_{s,n},
\]

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where \( f_n = f_{1,n} - f_{2,n} \), \( g_n = g_{1,n} - g_{2,n} \), \( h_n = h_{1,n} - h_{2,n} \), \( \lambda_{s,n} = \lambda_{1,s,n} - \lambda_{2,s,n} \). Then for \( n \) large enough, \( \lambda_{1,n}, \lambda_{2,n}, \lambda_n \in C_c^\infty(Q) \), \( \gamma_n \in C_c^\infty(\Omega) \). Thus there exist unique solutions \( u_n, u_{i,n}, v_{i,n} \), \( i = 1, 2 \), of problems

\[
(u_n) - \text{div}(A(x, \nabla u_n)) + G(u_n) = \lambda_{1,n} - \lambda_{2,n} \quad \text{in } Q, \quad u_n = 0 \quad \text{on } \partial \Omega \times (0, T), \quad u_n(0) = u_{0,1,n} - u_{0,2,n} \quad \text{in } \Omega, \\
(u_{i,n}) - \text{div}(A(x, \nabla u_{i,n})) + G(u_{i,n}) = \lambda_n \quad \text{in } Q, \quad u_{i,n} = 0 \quad \text{on } \partial \Omega \times (0, T), \quad u_{i,n}(0) = u_{0,i,n} \quad \text{in } \Omega, \\
- \text{div}(A(x, \nabla w_n)) = \gamma_n \quad \text{in } \Omega, \quad w_n = 0 \quad \text{on } \partial \Omega,
\]

such that

\[-||u_{0,2}||_{\infty, \Omega} - w_n(x) \leq -u_{2,n}(x, t) \leq u_{1,n}(x, t) \leq w_n(x) + ||u_{0,1}||_{\infty, \Omega}, \quad a.e. \text{ in } Q.\]

Moreover, as in the Proof of Theorem 2.2, (i), there holds

\[
\int_Q |G(u_n)| \leq \sum_{i=1,2} (\lambda_i(Q) + ||u_{0,i,n}||_{1, \Omega}), \quad \text{and} \quad \int_Q G(u_{i,n}) \leq \lambda_i(Q) + ||u_{0,i,n}||_{1, \Omega}, \quad i = 1, 2.
\]

By Proposition 4.10, up to a common subsequence, \( \{u_n, u_{1,n}, u_{2,n}\} \) converge to some \( (u, u_1, u_2) \), \( a.e. \) in \( Q \). Since \( G \) is bounded, in particular, \( \{G(u_n)\} \) converges to \( G(u) \) and \( \{G(u_{i,n})\} \) converges to \( G(u_i) \) in \( L^1(\Omega) \). Thus, (6.9) is satisfied. Moreover \( \{\lambda_{i,n} - G(u_{i,n}), f_{i,n} - G(u_{i,n}), g_{i,n}, h_{i,n}, \lambda_{i,s,n}, u_{0,i,n}\} \) and \( \{\lambda_n - G(u_n), f_n - G(u_n), g_n, h_n, \lambda_{s,n}, u_{0,1,n} - u_{0,2,n}\} \) are approximations of \( (\lambda_i - G(u_i), f_i - G(u_i), g_i, h_i, \lambda_i, u_{0,i}) \) and \( (\lambda - G(u), f - G(u), g, h, \lambda, u_{0,1} - u_{0,2}) \), in the sense of Theorem 2.1. Thus, we can find (different) subsequences converging \( a.e. \) to \( u, u_1, u_2 \), \( R \)-solutions of (6.6) and (6.7). Furthermore, from [47, Corollary 3.4], up to a subsequence, \( \{w_n\} \) converges \( a.e. \) in \( Q \) to a \( R \)-solution

\[-\text{div}(A(x, \nabla w)) = \gamma \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial \Omega,
\]

such that \( w \leq cW_{1,p}^{2diam(\Omega)}, \gamma \) \( a.e. \) in \( \Omega \). Hence, we get the inequality (6.8). The other conclusions follow in the same way. \( \blacksquare \)

**Lemma 6.5** Let \( G \) satisfy the assumptions of Theorem 6.3. For \( i = 1, 2 \), let \( u_{0,i} \in L^\infty(\Omega) \) be nonnegative, \( \lambda_i \in M_b^+(Q) \) with compact support in \( Q \), and \( \gamma \in M_b^+(\Omega) \) with compact support in \( \Omega \), such that

\[
\lambda_i \leq \gamma \otimes \chi_{(0,T)}, \quad G(||u_{0,i}||_{\infty, \Omega} + nW_{1,p}^{2diam(\Omega), \gamma}) \in L^1(\Omega). \tag{6.10}
\]

Then, there exist \( R \)-solutions \( u, u_1, u_2 \) of the problems (6.6) and (6.7), respectively relative to the decompositions \( (f_1 - f_2 - G(u), g_1 - g_2, h_1 - h_2), (f_i - G(u_i), g_i, h_i) \), satisfying (6.8) and (6.9).

Moreover, assume that \( \omega_i, \theta_i \) have the same properties as \( \lambda_i \) and \( \omega_i \leq \lambda_i \leq \theta_i \), \( u_{0,i,1}, u_{0,i,2} \in L^\infty(\Omega) \), \( u_{0,i,2} \leq u_{0,i} \leq u_{0,i,1} \). Then, one can find solutions \( u_i(\omega_i, u_{0,i,2}), u_i(\theta_i, u_{0,i,1}), \) corresponding with \( (\omega_i, u_{0,i,2}), (\theta_i, u_{0,i,1}) \), such that \( u_i(\omega_i, u_{0,i,2}) \leq u_i \leq u_i(\theta_i, u_{0,i,1}) \).
Proof. From Lemma 6.4 there exist R-solutions \( u_n, u_{i,n} \) to problems

\begin{align*}
(u_n)_t &- \text{div}(A(x, \nabla u_n)) + T_n(\mathcal{G}(u_n)) = \lambda_1 - \lambda_2 \quad \text{in } Q, \quad u_n(0) = 0 \quad \text{on } \partial \Omega \times (0,T), \quad u_n(0) = u_{0,1} - u_{0,2}
\end{align*}

\begin{align*}
(u_{i,n})_t &- \text{div}(A(x, \nabla u_{i,n})) + T_n(\mathcal{G}(u_{i,n})) = \lambda_i \quad \text{in } Q, \quad u_{i,n}(0) = 0 \quad \text{on } \partial \Omega \times (0,T), \quad u_{i,n}(0) = u_{0,i},
\end{align*}

relative to the decompositions \((f_1 - f_2 - T_n(\mathcal{G}(u_n), g_1 - g_2, h_1 - h_2), (f_1 - T_n(\mathcal{G}(u_{i,n}), g_i, h_i))\); and they satisfy

\begin{equation}
\int_Q |T_n(\mathcal{G}(u_n))| \leq \sum_{i=1,2} \left| (\lambda_i(Q) + \| u_{0,i} \|_{1,\Omega}) \right|, \quad \text{and} \quad \int_Q T_n(\mathcal{G}(u_{i,n})) \leq \lambda_i(Q) + \| u_{0,i} \|_{1,\Omega}.
\end{equation}

As in Lemma 6.4, up to a common subsequence, \( \{u_n, u_{1,n}, u_{2,n}\} \) converges a.e. in \( Q \) to \( \{u, u_1, u_2\} \) for which (6.8) is satisfied a.e. in \( Q \). From (6.10), (6.11) and the dominated convergence Theorem, we deduce that \( \{T_n(\mathcal{G}(u_n))\} \) converges to \( \mathcal{G}(u) \) and \( \{T_n(\mathcal{G}(u_{i,n}))\} \) converges to \( \mathcal{G}(u_i) \) in \( L^1(\Omega) \).

Thus, from Theorem 2.1, \( u \) and \( u_i \) are respective R-solutions of (6.6) and (6.7) relative to the decompositions \((f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2), (f_i - \mathcal{G}(u_i), g_i, h_i)\), and (6.8) and (6.9 hold. The last statement follows from the same assertion in Lemma 6.4.

Proof of Theorem 6.3. By Proposition 3.2, for \( i = 1, 2 \), there exist \( f_{i,n}, f_i \in L^1(\Omega), g_{i,n}, g_i \in (L^p(\Omega))^N \) and \( h_{i,n}, h_i \in X, \mu_{i,n,s}, \mu_{i,s} \in \mathcal{M}_+^1(\Omega) \) such that

\begin{align*}
\mu_i &= f_i - \text{div} g_i + (h_i)_t + \mu_{i,s}, \quad \mu_{i,n} = f_{i,n} - \text{div} g_{i,n} + (h_{i,n})_t + \mu_{i,n,s},
\end{align*}

and \( \{f_{i,n}\}, \{g_{i,n}\}, \{h_{i,n}\} \) strongly converge to \( f_i, g_i, h_i \) in \( L^1(\Omega), (L^p(\Omega))^N \) and \( X \) respectively, and \( \{\mu_{i,n}\}, \{\mu_{i,s}\} \) converge to \( \mu_i, \mu_{i,s} \) (strongly) in \( \mathcal{M}_b(\Omega) \), and

\begin{equation}
\| f_{i,n} \|_{1,\Omega} + \| g_{i,n} \|_{p',Q} + \| h_{i,n} \|_X + \mu_{i,n,s}(\Omega) \leq 2\mu(Q).
\end{equation}

By Lemma 6.5, there exist R-solutions \( u_n, u_{i,n} \) to problems

\begin{align*}
(u_n)_t &- \text{div}(A(x, \nabla u_n)) + \mathcal{G}(u_n) = \mu_{1,n} - \mu_{2,n} \quad \text{in } Q, \quad u_n(0) = 0 \quad \text{on } \partial \Omega \times (0,T), \quad u_n(0) = T_n(u_0)
\end{align*}

\begin{align*}
(u_{i,n})_t &- \text{div}(A(x, \nabla u_{i,n})) + \mathcal{G}(u_{i,n}) = \mu_{i,n} \quad \text{in } Q, \quad u_{i,n}(0) = 0 \quad \text{on } \partial \Omega \times (0,T), \quad u_{i,n}(0) = T_n(u_{0,i}),
\end{align*}

for \( i = 1, 2 \), relative to the decompositions \((f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}), (f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n})\), such that \( \{u_{i,n}\} \) is nondecreasing and nondecreasing, and \( -u_{2,n} \leq u_n \leq u_{1,n} \); and

\begin{equation}
\int_Q |\mathcal{G}(u_n)| \leq \mu_1(Q) + \mu_2(Q) + \| u_0 \|_{1,\Omega} \quad \text{and} \quad \int_Q \mathcal{G}(u_{i,n}) \leq \mu_i(Q) + \| u_0 \|_{1,\Omega}, \quad i = 1, 2.
\end{equation}

As in the proof of Lemma 6.5, up to a common subsequence \( \{u_n, u_{1,n}, u_{2,n}\} \) converge a.e. in \( Q \) to \( \{u, u_1, u_2\} \). Since \( \{\mathcal{G}(u_{i,n})\} \) is nondecreasing and nonnegative, from the monotone convergence
Theorem and (6.14), we obtain that \( \{\mathcal{G}(u_{i,n})\} \) converges to \( \mathcal{G}(u) \) in \( L^1(Q) \), \( i = 1, 2 \). Finally, \( \{\mathcal{G}(u_n)\} \) converges to \( \mathcal{G}(u) \) in \( L^1(Q) \), since \( |\mathcal{G}(u_n)| \leq \mathcal{G}(u_{1,n}) + \mathcal{G}(u_{2,n}) \). Thus, we can see that

\[
\{\mu_{1,n} - \mu_{2,n} - \mathcal{G}(u_n), f_{1,n} - f_{2,n} - \mathcal{G}(u_n), g_{1,n} - g_{2,n}, h_{1,n} - h_{2,n}, \mu_{1,s,n} - \mu_{2,s,n}, T_n(u_n^+) - T_n(u_n^-)\}
\]
is an approximation of \((\mu_1 - \mu_2 - \mathcal{G}(u), f_1 - f_2 - \mathcal{G}(u), g_1 - g_2, h_1 - h_2, \mu_{1,s} - \mu_{2,s}, u_0)\), in the sense of Theorem 2.1; and

\[
\{\mu_{i,n} - \mathcal{G}(u_{i,n}), f_{i,n} - \mathcal{G}(u_{i,n}), g_{i,n}, h_{i,n}, \mu_{i,s,n}, T_n(u_{i,n}^+)\}
\]
is an approximation of \((\mu_i - \mathcal{G}(u_i), f_i - \mathcal{G}(u_i), g_i, h_i, \mu_{i,s}, u_0^+)\). Therefore, \( u \) is a R-solution of (1.5), and (6.5) holds if \( u_0 \in L^\infty(\Omega) \) and \( \omega_n \leq \gamma \) for any \( n \in \mathbb{N} \) and some \( \gamma \in \mathcal{M}_b(\Omega) \).

As a consequence we prove Theorem 2.3. We use the following result of [17]:

**Proposition 6.6 (see [17])** Let \( q > p - 1, \alpha \in \left(0, \frac{N(p+1)}{pq}\right) \), \( r > 0 \) and \( \nu \in \mathcal{M}_b^+(\Omega) \). If \( \nu \) does not charge the sets of \( C_{\alpha p, p+1-r} \)-capacity zero, there exists a nondecreasing sequence \( \{\nu_n\} \subset \mathcal{M}_b^+(\Omega) \) with compact support in \( \Omega \) which converges to \( \nu \) strongly in \( \mathcal{M}_b(\Omega) \) and such that \( W_{\alpha, p}^r[\nu_n] \in L^q(\mathbb{R}^N) \), for any \( n \in \mathbb{N} \).

**Proof of Theorem 2.3.** Let \( f \in L^1(Q), u_0 \in L^1(\Omega) \), and \( \mu \in \mathcal{M}_b(\Omega) \) such that \( |\mu| \leq \omega \otimes F \), where \( F \in L^1((0,T)) \) and \( \omega \) does not charge the sets of \( C_{p, p+1-r} \)-capacity zero. From Proposition 6.6, there exists a nondecreasing sequence \( \{\omega_n\} \subset \mathcal{M}_b^+(\Omega) \) with compact support in \( \Omega \) which converges to \( \omega \), strongly in \( \mathcal{M}_b(\Omega) \), such that \( W_{1,p}^{2\text{diam} \Omega}[\omega_n] \in L^q(\mathbb{R}^N) \). We can write

\[
f + \mu = \mu_1 - \mu_2, \quad \mu_1 = f^+ + \mu^+, \quad \mu_2 = f^- + \mu^-;
\]
and \( \mu^+, \mu^- \leq \omega \otimes F \). We set

\[
Q_n = \{(x,t) \in \Omega \times \left(0, \frac{1}{n}, T - \frac{1}{n}\right) : d(x, \partial \Omega) > \frac{1}{n}\}, \quad F_n = T_n(\chi_{\left(\frac{1}{n}, T - \frac{1}{n}\right)} F),
\]

\[
\mu_{1,n} = T_n(\chi_{Q_n} f^+) + \inf\{\mu^+, \omega_n \otimes F_n\}, \quad \mu_{2,n} = T_n(\chi_{Q_n} f^-) + \inf\{\mu^-, \omega_n \otimes F_n\}.
\]

Then \( \{\mu_{1,n}\}, \{\mu_{2,n}\} \) are nondecreasing sequences with compact support in \( Q \), and \( \mu_{1,n}, \mu_{2,n} \leq \tilde{\omega}_n \otimes \chi_{(0,T)}, \) with \( \tilde{\omega}_n = n(\chi_{\Omega} + \omega_n) \) and \( (n + \kappa W_{1,p}^{2\text{diam} \Omega}[\omega_n])^q \in L^1(Q) \). Besides, \( \omega_n \otimes F_n \) converges to \( \omega \otimes F \) strongly in \( \mathcal{M}_b(Q) \), indeed we easily check that

\[
||\omega_n \otimes F_n - \omega \otimes F||_{\mathcal{M}_b(Q)} \leq ||F_n||_{L^1((0,T))} ||\omega_n - \omega||_{\mathcal{M}_b(\Omega)} + ||\omega||_{\mathcal{M}_b(\Omega)} ||F_n - F||_{L^1((0,T))}.
\]

Observe that for any measures \( \nu, \theta, \eta \in \mathcal{M}_b(Q) \), there holds

\[
||\inf\{\nu, \theta\} ||_{\mathcal{M}_b(Q)} - ||\inf\{\nu, \eta\} ||_{\mathcal{M}_b(Q)} \leq ||\theta - \eta||_{\mathcal{M}_b(Q)};
\]

hence \( \{\mu_{1,n}\}, \{\mu_{2,n}\} \) converge to \( \mu_1, \mu_2 \) respectively in \( \mathcal{M}_b(Q) \). Therefore, the result follows from Theorem 6.3. \( \blacksquare \)
Remark 6.7 Our result improves the existence results of [50], where \( \mu \in \mathcal{M}_0(Q) \). Indeed, let \( p_c = N(p - 1)/(N - p) \) be the critical exponent for the elliptic problem

\[
-\Delta_p w + |w|^{q-1} w = \omega \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial \Omega.
\]

Notice that \( p_c < p_c, \) since \( p > p_1 \). If \( q \geq p_c, \) there exist measures \( \omega \in \mathcal{M}_0^+(\Omega) \) which do not charge the sets of \( C_{p,q+1}^{-1} \)-capacity zero, such that \( \omega \notin \mathcal{M}_{0,c}(\Omega) \). Then for any \( F \in L^1((0,T)) \), \( F \geq 0, F \neq 0, \) we have \( \omega \otimes F \notin \mathcal{M}_0(Q) \).

Remark 6.8 Let \( A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) satisfying (1.6), (1.7). Let \( G : Q \times \mathbb{R} \rightarrow \mathbb{R} \) be a Caratheodory function such that the map \( s \mapsto G(x,t,s) \) is nondecreasing and odd, for a.e. \((x,t) \in Q\). Assume that \( \omega \in \mathcal{M}_{0,c}(\Omega) \). Thus, we have \( \omega\{x : W^{2}\text{diam}(\Omega)[\omega](x) = \infty}\} = 0 \). As in the proof of Theorem 2.3 with \( \eta_n = \chi_{W^{2}\text{diam}(\Omega)[\omega] \leq n} \), we get that (1.5) has a R-solution.

Remark 6.9 As in [17], from Theorem 6.3, we can extend Theorem 2.3 given for \( G(u) = |u|^{q-1} u \), to the case of a function \( G(x,t,.) \), odd for a.e. \((x,t) \in Q\), such that

\[
|G(x,t,u)| \leq G(|u|), \quad \int_1^{\infty} G(s)s^{-q-1}ds < \infty,
\]

where \( G \) is a nondecreasing continuous, under the condition that \( \omega \) does not charge the sets of zero \( C_{p,q+1}^{-1} \)-capacity, where for any Borel set \( E \subset \mathbb{R}^N \),

\[
C_{p,q+1}^{-1}(E) = \inf\{|\varphi|_{L^{\frac{q}{q-p+1}}(\mathbb{R}^N)} : \varphi \in L^{\frac{q}{q-p+1}}(\mathbb{R}^N), \quad G_p * \varphi \geq \chi_E\}
\]

where \( L^{\frac{q}{q-p+1}}(\mathbb{R}^N) \) is the Lorentz space of order \((q/(q-p+1),1)\).

In case \( G \) is of exponential type, we introduce the notion of maximal fractional operator, defined for any \( \eta \geq 0, \) \( R > 0, \) \( x_0 \in \mathbb{R}^N \) by

\[
M_{p,R}^{\omega}[\omega](x_0) = \sup_{t \in (0,R)} \frac{\omega(B(x_0,t))}{t^{N-p}h_{\eta}(t)}, \quad \text{where } h_{\eta}(t) = \inf((\ln t)^{-\eta},(\ln 2)^{-\eta}).
\]

We obtain the following:

Theorem 6.10 Let \( A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) satisfying (1.6), (1.7). Let \( p < N \) and \( \tau > 0, \beta > 1, \mu \in \mathcal{M}_b(Q) \) and \( u_0 \in L^1(\Omega) \). Assume that \( |\mu| \leq \omega \otimes F, \) with \( \omega \in \mathcal{M}_b^{+}(\Omega) \), \( F \in L^1((0,T)) \) be nonnegative. Assume that one of the following assumptions is satisfied:

(i) \( ||F||_{L^\omega((0,T))} \leq 1 \) and for some \( M_0 = M_0(N,p,\beta,\tau,c_3,c_4,\text{diam}\Omega) \),

\[
||M_{p,2\text{diam}(\Omega)}^{\frac{p-1}{p}}[\omega]||_{L^\infty(\mathbb{R}^N)} < M_0,
\]

(ii) there exists \( \beta_0 > \beta \) such that \( M_{p,2\text{diam}(\Omega)}^{\frac{p}{p}}[\omega] \in L^\infty(\mathbb{R}^N) \).
Thus, from Proposition 6.11 we get
\[
\exp(\varepsilon) = \exp(\varepsilon (1 + \varepsilon))
\]
Suppose Proposition 6.11 (see [17], Theorem 2.4) and conclude from Theorem 6.3.

Case 2: Assume that there exists \( \mu \) in \( \Omega \). Thus, \( \varepsilon > 0 \)
\[
\int_\Omega \exp \left( \frac{\delta}{\|M_{p,2 \text{diam}\Omega}[\nu]\|_{L^{\infty}(\mathbb{R}^N)}} \right) \leq \frac{C}{\delta_0 - \delta}.
\]

Proof of Theorem 6.10. Let \( Q_n \) be defined at (6.16), and \( \omega_n = \omega \chi_{\Omega_n} \), where \( \Omega_n = \{ x \in \Omega : d(x, \partial \Omega) > 1/n \} \). We still consider \( \mu_1, \mu_2, F_n, \mu_{1,n}, \mu_{2,n} \) as in (6.15), (6.17).

Case 1: Assume that \( |F|_{L^\infty((0,T))} \leq 1 \) and (6.18) holds. We have \( \mu_{1,n}, \mu_{2,n} \leq n \chi_{\Omega} + \omega \). For any \( \varepsilon > 0 \), there exists \( c_\varepsilon = c_\varepsilon(\varepsilon, N, p, \beta, \kappa, \text{diam}\Omega) > 0 \) such that
\[
(n + \kappa W_{1,p}^{2 \text{diam}\Omega}[n \chi_{\Omega} + \omega])^\beta \leq c_\varepsilon n^{\frac{\beta p}{p-1}} + (1 + \varepsilon)\kappa^\beta (W_{1,p}^{2 \text{diam}\Omega}[\omega])^\beta
\]
a.e. in \( \Omega \). Thus,
\[
\exp \left( \tau(n + \kappa W_{1,p}^{2 \text{diam}\Omega}[n \chi_{\Omega} + \omega])^\beta \right) \leq \exp \left( \tau c_\varepsilon n^{\frac{\beta p}{p-1}} \right) \exp \left( \tau(1 + \varepsilon)\kappa^\beta (W_{1,p}^{2 \text{diam}\Omega}[\omega])^\beta \right)
\]
If (6.18) holds with \( M_0 = (\delta_0/\tau\kappa^\beta)^{(p-1)/\beta} \) then we can chose \( \varepsilon \) such that
\[
\tau(1 + \varepsilon)\kappa^\beta \|M_{p,2 \text{diam}\Omega}[\nu]\|_{L^\infty(\mathbb{R}^N)} < \delta_0.
\]

From Proposition 6.11, we get \( \exp(\tau(1 + \varepsilon)\kappa^\beta W_{1,p}^{2 \text{diam}\Omega}[\omega])^\beta) \in L^1(\Omega) \), which implies \( \exp(\tau(n + \kappa^\beta W_{1,p}^{2 \text{diam}\Omega}[n \chi_{\Omega} + \omega])^\beta) \in L^1(\Omega) \) for all \( n \). We conclude from Theorem 6.3.

Case 2: Assume that there exists \( \varepsilon > 0 \) such that \( M_{p,2 \text{diam}\Omega}^{(p-1)/\beta(\varepsilon + \nu)}[\omega] \in L^\infty(\mathbb{R}^N) \). Now we use the inequality \( \mu_{1,n}, \mu_{2,n} \leq n(\chi_{\Omega} + \omega) \). For any \( \varepsilon > 0 \) and \( n \in \mathbb{N} \) there exists \( c_{\varepsilon,n} > 0 \) such that
\[
(n + \kappa^\beta W_{1,p}^{2 \text{diam}\Omega}[n(\chi_{\Omega} + \omega)])^\beta \leq c_{\varepsilon,n} + \varepsilon(W_{1,p}^{2 \text{diam}\Omega}[\omega])^\beta_0
\]
Thus, from Proposition 6.11 we get \( \exp(\tau(n + \kappa^\beta W_{1,p}^{2 \text{diam}\Omega}[n(\chi_{\Omega} + \omega)])^\beta) \in L^1(\Omega) \) for all \( n \). We conclude from Theorem 6.3. 

\[\blacksquare\]
6.3 Equations with source term

As a consequence of Theorem 6.3, we get a first result for problem (1.1):

Corollary 6.12 Let \( A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) satisfying (1.6)(1.7). Let \( u_0 \in L^\infty(\Omega) \), and \( \mu \in \mathcal{M}_b(Q) \) such that \( |\mu| \leq \omega \otimes \chi_{(0,T)} \) for some \( \omega \in \mathcal{M}_b^+(\Omega) \). Then there exist a R-solution \( u \) of (1.1), such that

\[
|u(x,t)| \leq \kappa W_{1,p}^{2\text{diam}(\Omega)}[\omega](x) + ||u_0||_{\infty,\Omega}, \quad \text{for a.e.} \ (x,t) \in Q, \tag{6.19}
\]

where \( \kappa \) is defined at Theorem 6.2.

**Proof.** Let \( \{\phi_n\} \) be a nonnegative, nondecreasing sequence in \( C_c^\infty(Q) \) which converges to 1, a.e. in \( Q \). Since \( \{\phi_n\mu^+\}, \{\phi_n\mu^-\} \) are nondecreasing sequences, the result follows from Theorem 6.3.

Our proof of Theorem 2.4 is based on a property of Wolff potentials:

**Theorem 6.13** (see [53]) Let \( q > p - 1, \ 0 < p < N, \ \omega \in \mathcal{M}_b^+(\Omega) \). If for some \( \lambda > 0 \),

\[
\omega(E) \leq \lambda C_{p,\beta}(E) \quad \text{for any compact set} \ E \subset \mathbb{R}^N, \tag{6.20}
\]

then \( (W_{1,p}^{2\text{diam}\Omega}[\omega])^q \in L^1(\Omega) \), and there exists \( M = M(N,p,q,\text{diam}(\Omega)) \) such that, a.e. in \( \Omega \),

\[
W_{1,p}^{2\text{diam}\Omega} \left[ W_{1,p}^{2\text{diam}\Omega}[\omega] \right]^q \leq M \lambda^{\frac{q-1}{p-1}} W_{1,p}^{2\text{diam}\Omega}[\omega] < \infty. \tag{6.21}
\]

We deduce the following:

**Lemma 6.14** Let \( \omega \in \mathcal{M}_b^+(\Omega) \), and \( b \geq 0 \) and \( K > 0 \). Suppose that \( \{u_m\}_{m \geq 1} \) is a sequence of nonnegative functions in \( \Omega \) that satisfies

\[
u_1 \leq K W_{1,p}^{2\text{diam}\Omega}[\omega] + b, \quad u_{m+1} \leq K W_{1,p}^{2\text{diam}\Omega}[u_m + b] \quad \text{for all} \ m \geq 1.
\]

Assume that \( \omega \) satisfies (6.20) for some \( \lambda > 0 \). Then there exist \( \lambda_0 \) and \( b_0 \), depending on \( N, p, q, K \), and \( \text{diam}(\Omega) \), such that, if \( \lambda \leq \lambda_0 \) and \( b \leq b_0 \), then \( W_{1,p}^{2\text{diam}\Omega}[\omega] \in L^q(\Omega) \) and for any \( m \geq 1 \),

\[
u_m \leq 2 \beta_p K W_{1,p}^{2\text{diam}\Omega}[\omega] + 2b, \quad \beta_p = \max(1, \frac{2-p}{p-1}). \tag{6.22}
\]

**Proof.** Clearly, (6.22) holds for \( m = 1 \). Now, assume that it holds at the order \( m \). Then

\[
u^q_m \leq 2^{q-1}(2\beta_p)^q (W_{1,p}^{2\text{diam}\Omega}[\omega])^q + 2^{q-1}(2b)^q.
\]

Using (6.21) we get

\[
u_{m+1} \leq K W_{1,p}^{2\text{diam}\Omega} \left[ 2^{q-1}(2\beta_p)^q (W_{1,p}^{2\text{diam}\Omega}[\omega])^q + 2^{q-1}(2b)^q + \omega \right] + b
\]

\[
\leq \beta_p K \left( A_1 W_{1,p}^{2\text{diam}\Omega} \left[ (W_{1,p}^{2\text{diam}\Omega}[\omega])^q \right] + W_{1,p}^{2\text{diam}\Omega} [2b]^q \right) + b
\]

\[
\leq \beta_p K \left( A_1 M \lambda^{\frac{q-1}{p-1}} + 1 \right) W_{1,p}^{2\text{diam}\Omega}[\omega] + \beta_p K W_{1,p}^{2\text{diam}\Omega} [(2b)^q] + b
\]

\[
= \beta_p K \left( A_1 M \lambda^{\frac{q-1}{p-1}} + 1 \right) W_{1,p}^{2\text{diam}\Omega}[\omega] + A_2 b^{\frac{q}{p-1}} + b,
\]

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where $M$ is as in (6.21) and $A_1 = (2^{p-1}(2\beta_p)^{q(p-1)})^{1/(p-1)}$, $A_2 = \beta_p K^{q/(p-1)} B_1^{1/(p-1)} (p')^{-1}(2\text{diam}\Omega)^p$.

Thus, (6.22) holds for $m = n + 1$ if we prove that

$$A_1 M \lambda^{\frac{p-1}{p-1}} \leq 1$$

and $A_2 b^{\frac{n}{q-p+1}} \leq b$,

which is equivalent to

$$\lambda \leq (A_1 M)^{-\frac{(p-1)^2}{q-p+1}}$$

and

$$b \leq A_2^{\frac{n}{q-p+1}}.$$ 

Therefore, we obtain the result with $\lambda_0 = (A_1 M)^{-\frac{(p-1)^2}{q-p+1}}$ and $b_0 = A_2^{\frac{n}{q-p+1}}$. \hfill \blacksquare

**Proof of Theorem 2.4.** From Corollary 5.7 and 6.12, we can construct a sequence of nonnegative nondecreasing R-solutions $\{u_m\}_{m \geq 1}$ defined in the following way: $u_1$ is a R-solution of (1.1), and $u_{m+1}$ is a nonnegative R-solution of

$$\left\{ \begin{array}{l}
(u_{m+1})_t - \text{div}(A(x, \nabla u_{m+1})) = u_m^q + \mu \\
u_{m+1} = 0 \\
u_{m+1}(0) = u_0
\end{array} \right. \quad \text{in } Q,$$

$$u_{m+1} = 0 \quad \text{on } \partial \Omega \times (0,T),$$

$$u_{m+1}(0) = u_0 \quad \text{in } \Omega.$$

Setting $\overline{u}_m = \sup_{t \in (0,T)} u_m(t)$ for all $m \geq 1$, there holds

$$\overline{u}_1 \leq \kappa \mathcal{W}_{1,p}^{2\text{diam}\Omega} \|\omega\| + \|u_0\|_{\infty,\Omega}, \quad \overline{u}_{m+1} \leq \kappa \mathcal{W}_{1,p}^{2\text{diam}\Omega} [\overline{u}_m + \omega] + \|u_0\|_{\infty,\Omega} \quad \forall m \geq 1.$$

From Lemma 6.14, we can find $\lambda_0 = \lambda_0(N, p, q, \text{diam}\Omega)$ and $b_0 = b_0(N, p, q, \text{diam}\Omega)$ such that if (2.7) is satisfied with $\lambda_0$ and $b_0$, then

$$u_m \leq \overline{u}_m \leq 2\beta_p \kappa \mathcal{W}_{1,p}^{2\text{diam}\Omega} \|\omega\| + 2\|u_0\|_{\infty,\Omega} \quad \forall m \geq 1. \quad (6.23)$$

Thus $\{u_m\}$ converges a.e. in $Q$ and in $L^1(Q)$ to some function $u$, for which (2.9) is satisfied in $\Omega$ with $c = 2\beta_p \kappa$. Finally, one can apply Theorem 2.1 to the sequence of measures $\{u_m^q + \mu\}$, and obtain that $u$ is a R-solution of (2.8). \hfill \blacksquare

Next we consider the exponential case.

**Theorem 6.15** Let $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (1.6),(1.7). Let $\tau > 0, l \in \mathbb{N}$ and $\beta \geq 1$ such that $l\beta > p - 1$. Set

$$E(s) = e^s - \sum_{j=0}^{l-1} \frac{s^j}{j!}, \quad \forall s \in \mathbb{R}. \quad (6.24)$$

Let $\mu \in \mathcal{M}_b^+(Q), \omega \in \mathcal{M}_b^+(\Omega)$ such that $\mu \leq \chi_{(0,T)} \otimes \omega$. Then, there exist $b_0$ and $M_0$ depending on $N, p, \beta, \tau, l$ and $\text{diam}\Omega$, such that if

$$\|M_{p,2\text{diam}\Omega}^{\beta(p-1)} \omega\|_{L^\infty(\mathbb{R}^N)} \leq M_0, \quad \|u_0\|_{\infty,\Omega} \leq b_0,$$

the problem

$$\left\{ \begin{array}{l}
u_t - \text{div}(A(x, \nabla u)) = E(\tau u^\beta) + \mu \\
u = 0 \\
u(0) = u_0
\end{array} \right. \quad \text{in } Q,$$

$$u = 0 \quad \text{on } \partial \Omega \times (0,T),$$

$$u(0) = u_0 \quad \text{in } \Omega \quad (6.25)$$

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admits nonnegative R-solution \( u \), which satisfies, a.e. in \( Q \), for some \( c \), depending on \( N, p, c_3, c_4 \)
\[
u(x,t) \leq cW^{2\text{diam}\Omega}_{1,p} (\omega) + 2b_0.
\] (6.26)

For the proof we first recall an approximation property, which is a consequence of \([47, \text{Theorem 2.5}]\):

**Theorem 6.16** Let \( \tau > 0, b \geq 0, K > 0, l \in \mathbb{N} \) and \( \beta \geq 1 \) such that \( l\beta > p - 1 \). Let \( \mathcal{E} \) be defined by (6.24). Let \( \{v_m\} \) be a sequence of nonnegative functions in \( \Omega \) such that, for some \( K > 0 \),
\[
v_1 \leq KW^{2\text{diam}\Omega}_{1,p} [\mu] + b, \quad v_{m+1} \leq KW^{2\text{diam}\Omega}_{1,p} [\mathcal{E}(\tau u_1^{\beta} + \mu)] + b, \quad \forall m \geq 1.
\]
Then, there exist \( b_0 \) and \( M_0 \), depending on \( N, p, \beta, \tau, l, K \) and \( \text{diam}\Omega \) such that if \( b \leq b_0 \) and
\[
\|M_{p,2\text{diam}\Omega}^{(p-1)\beta-1} \mu\|_{\infty, \mathbb{R}^N} \leq M_0,
\] (6.27)
then, setting \( c_p = 2\max(1, 2^{\frac{2}{p-1}}) \),
\[
\exp(\tau(Kc_pW^{2\text{diam}\Omega}_{1,p} [\mu] + 2b_0)^\beta) \in L^1(\Omega),
\]
\[
v_m \leq Kc_pW^{2\text{diam}\Omega}_{1,p} [\mu] + 2b_0, \quad \forall m \geq 1.
\] (6.28)

**Proof of Theorem 6.15.** From Corollary 5.7 and 6.12 we can construct a sequence of nonnegative nondecreasing R-solutions \( \{u_m\}_{m \geq 1} \) defined in the following way: \( u_1 \) is a R-solution of problem (1.1), and by induction, \( u_{m+1} \) is a R-solution of
\[
\begin{cases}
(u_{m+1})_t - \text{div}(A(x, \nabla u_{m+1})) = \mathcal{E}(\tau u_m^{\beta}) + \mu & \text{in } Q, \\
u^m_{m+1} = 0 & \text{on } \partial \Omega \times (0, T), \\
u^m_{m+1}(0) = u_0 & \text{in } \Omega.
\end{cases}
\]
And, setting \( \overline{u}_m = \sup_{t \in (0, T)} u_m(t) \), there holds
\[
\overline{u}_1 \leq \kappa W^{2\text{diam}\Omega}_{1,p} (\omega) + ||u_0||_{\infty, \Omega}, \quad \overline{u}^m_{m+1} \leq \kappa W^{2\text{diam}\Omega}_{1,p} [\mathcal{E}(\tau \overline{u}_m^{\beta}) + \omega] + ||u_0||_{\infty, \Omega}, \quad \forall m \geq 1.
\]
Thus, from Theorem 6.16, there exist \( b_0 \in (0, 1) \) and \( M_0 > 0 \) depending on \( N, p, \beta, \tau, l \) and \( \text{diam}\Omega \) such that, if (6.27) holds, then (6.28) is satisfied with \( v_m = \overline{u}_m \). As a consequence, \( u_m \) is well defined. Thus, \( \{u_m\} \) converges a.e. in \( Q \) to some function \( u \), for which (6.26) is satisfied in \( \Omega \). Furthermore, \( \mathcal{E}(\tau u_m^{\beta}) \) converges to \( \mathcal{E}(\tau u^\beta) \) in \( L^1(Q) \). Finally, one can apply Theorem 2.1 to the sequence of measures \( \mathcal{E}(\tau u_m^{\beta}) + \mu \), and obtain that \( u \) is a R-solution of (6.25). \( \blacksquare \)
Appendix

Proof of Lemma 4.7. Let $J$ be defined by (4.11). Let $\zeta \in C^1_\zeta([0, T])$ with values in $[0, 1]$, such that $\zeta_t \leq 0$, and $\varphi = \zeta \kappa [j(S(v))]^t$. Clearly, $\varphi \in X \cap L^\infty(Q)$; we choose the pair of functions $(\varphi, S)$ as test function in (4.2). Thanks to convergence properties of Steklov time-averages, we easily will obtain (4.15) if we prove that

$$\lim_{l \to 0, \zeta \to 1} \left( - \int_Q (\zeta \kappa [j(S(v))]^t) \xi S(v) \right) \geq - \int_Q \xi_t J(S(v)).$$

We can write $- \int_Q (\zeta \kappa [j(S(v))]^t) \xi S(v) = F + G$, with

$$F = - \int_Q (\zeta \kappa [j(S(v))]^t) \xi S(v), \quad G = - \int_Q \zeta \kappa [j(S(v))](x, t + l) - j(S(v))(x, t)).$$

Using (4.12) and integrating by parts we have

$$G \geq - \int_Q \zeta \kappa \frac{1}{t} (J(S(v)))(x, t + l) - J(S(v))(x, t))$$

$$= - \int_Q \zeta \kappa \frac{\partial}{\partial t} ([J(S(v))]^t_0) = \int_Q (\zeta \kappa [J(S(v))]^t_0) + \int_\Omega \zeta(0) \xi(0) [J(S(v))]^t(0)$$

$$\geq \int_Q (\zeta \kappa [J(S(v))]^t_0),$$

which achieves the proof.

References


