An LQ sub-optimal stabilizing feedback law for switched linear systems

Pierre Riedinger, Jean-Claude Vivalda

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The aim of this paper is the design of a stabilizing feedback law for continuous time linear switched system based on the optimization of a quadratic criterion. The main result provides a control Lyapunov function and a feedback switching law leading to sub optimal solutions. As the Lyapunov function defines a tight upper bound on the value function of the optimization problem, the sub optimality is guaranteed. Practically, the switching law is easy to apply and the design procedure is effective if there exists at least a controllable convex combination of the subsystems.

1 Introduction

Over the past decade, the design of stabilizing laws for switched systems (in continuous and discrete time) has been the focus of considerable research attention. Several approaches have been used to tackle this problem, one can cite for example [18, 26, 29, 30] for dynamic programming approaches, [1, 2, 27] for variational approaches, or [6, 8, 12] for Lyapunov based approaches. This problem is not easy, even numerically [24, 27] and the design of a stabilizing feedback law based on the optimization of a criterion is a challenging task.

LQ regulators are widely used for the control of linear systems because of their simple design and their robustness properties. These regulators can also been used for the

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design of stabilizing feedback laws for linear switched systems but, as yet, one cannot obtain the exact solution of a switched LQ problem. Moreover, the main drawback of this method is the difficulty to get a good numerical approximation of the solution of the optimal problem; this approximation is difficult to obtain even for small dimensional systems. Another possibility is to use open loop control law, which can be achieved through direct or indirect methods [27, 1], nevertheless, in this case singular solutions [22, 2] cause numerical complications [24].

In [12], the author address two Lyapunov based strategies for stabilization of discrete time linear switched systems. The first one is of open loop nature while the second one is of closed-loop nature and is designed from the solution of the Lyapunov-Metzler inequalities. Their approach uses a family of quadratic Lyapunov functions and an upper bound on the cost is provided but the distance from the optimality of the stabilizing feedback law is not estimated.

In this paper, we consider a linear and controlled switched linear system. Together with this system, we consider a quadratic cost function. Our aim is to design a stabilizing feedback law that approaches the solution of the optimal problem related to this cost function. To the best of our knowledge, the problem of finding a continuous stabilizing law which satisfies the optimal criterion has not yet been solved.

In Section 2, the problem statement is given as well as a relaxed version that takes into account all the convex combinations of the subsystems. We explain why this relaxation is useful to solve the problem. Then, the necessary condition of the Pontryagin Maximum Principle are recalled. We also discuss the numerical difficulties encountered when *singular controls* enter in the solution. To circumvent this, a numerical framework is proposed to solve properly the optimization problem.

In Section 3, assuming that there exists at least a globally asymptotically stable (GAS) convex combination of the subsystems and as the positive definite solution of an algebraic Riccati equation is a continuous function with respect to the convex combination, we are able to build a parametrized family of positive definite function whose parameters belong to a a compact set. A control Lyapunov function is then defined as the point-wise infimum of this familly.

We prove that the Lyapunov function is locally Lipschizt and homogenous of degree two. Then, we show that its directional Dini derivative is well defined along trajectories and we deduce a state feedback leading to a cost value less than the value of the Lyapunov function. In addition, it is proved that the feedback makes the system globally exponentially stable. In section 4, we show that the sampled time version of the state feedback law is also globally exponentially stable. Finally, in Section 5, numerical examples are given which show actually that the optimal cost is finely approached.

2 Problem statement and necessary conditions

We consider the class of continuous time linear switched systems:

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t) \quad x(0) = x_0 \tag{1}$$

where $\sigma : [0, +\infty) \to S = \{1, \cdots, s\}$ denotes the switching law that selects the active mode at time t by choosing among a finite collection of linear systems defined by the pairs $(A_i, B_i) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m_i}$, $i \in S$. Each subsystem is also govern by a control $u_i(t) \in \mathbf{R}^{m_i}$, $0 \le m_i \le n$. Our aim is to design a state feedback switching law (i.e. $x \mapsto (\sigma(x), u_{\sigma(x)}(x))$) for system (1) that approaches the optimal solution of the following optimization problem:

Problem 1: Minimize the switched quadratic criterion:

$$\min_{\sigma(\cdot), u_{\sigma(\cdot)}(\cdot)} \quad \frac{1}{2}\int_0^\infty x^{\mathrm{T}}(t)Q_{\sigma(t)}x(t) + u^{\mathrm{T}}_{\sigma(t)}(t)R_{\sigma(t)}u_{\sigma(t)}(t)dt \tag{2}$$

where $Q_i = Q_i^T \ge 0$, $R_i = R_i^T > 0$, $i \in S$ subject to $\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u_{\sigma(t)}(t)$, $x(0) = x_0$.

A usual framework [24, 2] to solve optimal control problem for switched systems ($\dot{x} = f_i(x)$, $i \in S$) is to solve its relaxed version, replacing the vector field set $(f_i(x))$ by its convex hull ($\dot{x} = \mathrm{co}\{f_i(x)\}$). At least, three reasons justify the convexification of the problem: (i) the solutions are well defined (Fillipov; [9]); (ii) the density of the switched system trajectories into the trajectories of its relaxed version [13]; (iii) the existence of *singular* optimal solutions are taking into account [22, 2].

The relaxed version of Problem 1 is then given as a pure continuous time optimal control problem given by :

$$\dot{x}(t) = \sum_{i=1}^{s} \lambda_i(t)(A_i x(t) + B_i u_i(t)) \quad x(0) = x_0 \tag{3}$$

$$\lambda(t) \in \Lambda = \left\{ \lambda \in \mathbf{R}^s : \sum_{i=1}^{s} \lambda_i = 1 \quad \lambda_i \ge 0 \right\}. \tag{4}$$

and by the convexified cost:

$$\min_{\lambda(\cdot),\ u_i(\cdot)} \frac{1}{2}\int_0^\infty \sum_{i=1}^{s} \lambda_i(t)(x^T(t)Q_i x(t) + u_i^T(t)R_i u_i(t))dt \tag{5}$$

In the sequel we denote by $u = (u_1, u_2, \cdots, u_s)$ and we use the following notation:

$$A(\lambda) = \sum_{i \in S} \lambda_i A_i, \quad B(\lambda) = [\lambda_1 B_1, \lambda_2 B_2, \cdots, \lambda_s B_s]$$

$$Q(\lambda) = \sum_{i \in S} \lambda_i Q_i, \quad R(\lambda) = \mathrm{diag}([\lambda_1 R_1, \lambda_2 R_2, \cdots, \lambda_s R_s])$$

Then the dynamics of the relaxed system can be redefined as:

$$\dot{x} = A(\lambda)x + B(\lambda)u$$

and the cost as

$$\min_{\lambda(\cdot), u(\cdot)} \frac{1}{2} \int_0^\infty x^T(t) Q(\lambda(t)) x(t) + u^T(t) R(\lambda) u(t) dt$$

To apply Pontryagin Maximum Principle (PMP) for Problem 1 or its relaxed version, the Hamiltonian function is defined as follow:

$$\mathcal{H}(x, \lambda, u, p) = \sum_{i=1}^{s} \lambda_i \mathcal{H}_i(x, u_i, p) \tag{6}$$

with $\mathcal{H}_i(x, u_i, p) = p^T(A_i x + B_i u_i) + \frac{1}{2}(x^T Q_i x + u_i^T R_i u_i)$ and where p defines the co-state.

This leads to the following classical necessary conditions for optimality [23]:

Theorem 1. *Suppose that (λ^*, u^*) is optimal with the corresponding state x^*. Then, there exists an absolutely continuous function p^*, called co-state, such that:*

1. $p^* \neq 0$,
2. $\dot{p^*} = \sum_{i=1}^{s} \lambda_i^*(t)(-A_i^T p^* - Q_i x^*)$ *for almost all* $t \in \mathbb{R}^+$,
3. $(\lambda^*(t), u^*(t)) \in \arg\min_{(\lambda \in \Lambda, u)} \mathcal{H}(x^*(t), \lambda, u, p^*(t))$,
4. $\mathcal{H}(x^*(t), \lambda^*(t), u^*, p^*(t)) = 0$.

Theorem 1 can be simplified thanks to the following lemma:

Lemma 2. *The optimal value of the u_i's are given by $u_i^*(t) = -R_i^{-1} B_i^T p^*(t)$ and λ^* satisfies:*

$$\lambda^*(t) \in \arg\min_{\lambda \in \Lambda} \sum_{i=1}^{s} \lambda_i \mathcal{H}_i(x^*, -R_i^{-1} B_i^T p^*, p^*). \tag{7}$$

Proof. From Equation 6, the minimum of $\mathcal{H}$ with respect to the u_i's is clearly independent of the value of λ and the result follows. $\square$

From (7), it is clear that if there exists $i \in S$ at time t such that

$$\mathcal{H}_i(x^*(t), u_i^*, p^*(t)) < \mathcal{H}_j(x^*(t), u_j^*, p^*(t)), \quad \forall j \in S \setminus \{i\},$$

then the optimal control has to satisfy $\lambda_i^*(t) = 1$ and $\lambda_j^*(t) = 0$, $\forall j \in S \setminus \{i\}$.

A switching instant can occur at time t if there exists at least a pair $(i, j) \in S^2$ such that $\mathcal{H}_i = \mathcal{H}_j = 0$. At this time, the value of λ cannot be determined directly. Actually, if we suppose that $0 = \mathcal{H}_i = \mathcal{H}_j < \mathcal{H}_k$, $\forall k \in S \setminus \{i, j\}$ then the values that satisfy the relation, $\lambda_i + \lambda_j = 1$, are potential candidate for optimality. Moreover, a so called *singular control* λ can exist, for which $0 = \mathcal{H}_i = \mathcal{H}_j < \mathcal{H}_k$, $\forall k \in S \setminus \{i, j\}$ on a *non empty time interval* (a, b). This is a well known situation in the literature [25, 5, 4] and second order necessary conditions given by the Generalized Legendre-Clebsh Condition [28, 15] can be necessary to solve the optimal control problem.

Definition 3. We call *singular control*, a control $\lambda(.)$ such that there exist at least two indices i, j, for which $\mathcal{H}_i = \mathcal{H}_j = 0$ on a non zero measure time interval (a, b) and satisfying $\lambda(t) \in \Lambda$, $\lambda_k(t) \neq 1$ $\forall k \in S$, $\forall t \in (a, b)$. The corresponding part of the trajectory is called a singular arc.

A singular control defines a Fillipov solution [7] for the original switched system (1). Hence, it allows to extend properly the notion of optimal solution for switched systems. Roughly speaking when an optimal solution of the relaxed problem possesses singular arcs, these arcs define sliding surfaces for the switched system (1) which lead to chattering if the surface is attractive. It is noteworthy that only suboptimal solutions can be achieved for the switched systems due to the limited switching frequency; see for example [2].

2.1 Numerical resolution

If one attempts to solve numerically an optimal control problem in which singular arcs appear, numerical difficulties will be encountered due to the insensitive of the Hamiltonian with respect to the control. On the one hand, standard indirect numerical methods such as multiple shooting methods are not appropriate to deal with singular arcs without a priori information on the structure of the trajectories (see [10], [20] and [3] to apply multiple shooting methods in this context). This information can be achieved using regularization techniques such as the continuation method used in [21], [14], [19]. On the other hand, when direct methods such as nonlinear programming (NLP) are used to solve (1)-(2), a very bad results are generally obtained on the optimal value for the control due to $\frac{\partial \mathcal{H}}{\partial u} = 0$, $\forall u$ in an open subset of U. In [24], we have proposed to use a mix direct-indirect method.

The idea consists to take implicitly into account the singular arcs using the necessary condition of the PMP and the Hamiltonian systems and then to solve directly an augmented constraint optimization problem.

For the LQ switched problem, this leads to the following constraint optimization problem (here $z = (x, p)$):

Problem 2: Minimize (using NLP):

$$\min_{\lambda(\cdot)} \frac{1}{2} \int_0^\infty \sum_{i=1}^s \lambda_i(t)(x^T(t)Q_i x(t) + p^T(t)B_i R_i^{-1} B_i^T p(t))dt \tag{8}$$

$$\text{subject to } \dot{z}(t) = \sum_{i=1}^s \lambda_i(t) \begin{pmatrix} A_i & -B_i R_i^{-1} B_i^T \\ -Q_i & -A_i^T \end{pmatrix} z(t) \tag{9}$$

$$0 \le \lambda_i \perp \mathcal{H}_i(x, -R_i^{-1} B_i^T p, p) \ge 0, \quad i \in S \tag{10}$$

$$\lambda(t) \in \Lambda, \quad x(0) = x_0 \tag{11}$$

where the sign $x \perp y$ means $xy = 0$.

The complementarity constraints (10) allow λ to be multivalued function when the $\mathcal{H}_i$'s vanish for at least two subscripts. This is a key point since the necessary condition of PMP does not imply the unicity of solution of system (9) from an initial condition. When the admissible values for λ are multiple, this formulation by the minimization of the cost (8) yields the optimal value.

Practically, the complementarity constraint are taken into account with penalization terms $\rho \lambda_i \mathcal{H}_i$ in the cost with weight ρ. Hight order constraints can be also added to improve the result see [24] for more details.

3 Lyapunov based switching law

The aim of this part is to define a Lyapunov function as a tight upper bound on the value function. We means tight in the sense that the two functions may coincide at some points.

We denote by $B(\sqrt{\lambda})$ the matrix $B(\sqrt{\lambda}) = \left[\sqrt{\lambda_1}B_1|\sqrt{\lambda_2}B_2|\dots|\sqrt{\lambda_s}B_s\right]$ and we assume there exists a $\lambda^0 \in \Lambda$ such that the pair $(A(\lambda^0), B(\sqrt{\lambda^0}))$ is controllable. Then, the Riccati equation:

$$A(\lambda)^T P_\lambda + P_\lambda A(\lambda) - P_\lambda B(\sqrt{\lambda})R^{-1}B(\sqrt{\lambda})^{\mathrm{T}} P_\lambda + Q(\lambda) = 0. \tag{12}$$

$R = \mathrm{diag}([R_1, R_2, \cdots, R_s])$, admits a (unique) symmetric positive definite solution P_{λ^0} for $\lambda = \lambda^0$. Notice that the pack-writing term $P_\lambda B(\sqrt{\lambda})R^{-1}B(\sqrt{\lambda})^{\mathrm{T}} P_\lambda$ can be expanded as

$$P_\lambda B(\sqrt{\lambda})R^{-1}B(\sqrt{\lambda})^{\mathrm{T}} P_\lambda = \sum_{i\in S} \lambda_i (P_\lambda B_i R_i^{-1} B_i^{\mathrm{T}} P_\lambda).$$

In fact to ensure the existence of a positive definite solution to the Riccati equation (12), as the Q_i's are positive definite, it is enough to assume that the pair $(A(\lambda^0), B(\sqrt{\lambda^0}))$ is stabilizable (*cf.* Lemma 4). Now, if the pair $(A(\lambda^0), B(\sqrt{\lambda^0}))$ is controllable, the same is true if λ belongs to a sufficiently small neighborhood of λ^0; so the Riccati equation (12) admits a unique (positive definite) solution P_λ for every λ in some neighborhood of λ^0. Moreover, it is well known that the positive definite solution of a Riccati equation is a continuous function of $(A(\lambda), B(\sqrt{\lambda}), R, Q(\lambda))$ [17] and so P_λ is a continuous function of λ. Notice also that the condition for the existence of a symmetric nonnegative solution of the Riccati equation (12) can be weakened: in [16], V. Kučera proved that if the pair $(A(\lambda), B(\sqrt{\lambda}))$ is stabilizable and if the matrix

$$M = \begin{pmatrix} A(\lambda) & -B(\sqrt{\lambda})R^{-1}B(\sqrt{\lambda})^{\mathrm{T}} \\ -Q(\lambda) & -A(\lambda)^{\mathrm{T}} \end{pmatrix}$$

has no purely imaginary eigenvalues, then there exists a symmetric nonnegative solution to equation (12). This result allow us to prove the following lemma about Eq. (12).

Lemma 4. *If the pair $(A(\lambda), B(\sqrt{\lambda}))$ is stabilizable and $Q(\lambda)$ is positive definite, then there exists a positive definite solution to Eq.* (12).

Proof. We denote by $S(\lambda)$ the matrix $B(\sqrt{\lambda})R^{-1}B(\sqrt{\lambda})^{\mathrm{T}}$ and we take $x = (x_1^{\mathrm{T}}, x_2^{\mathrm{T}})^{\mathrm{T}}$ a vector of $\mathbf{C}^{2n}$ such that $Mx = i\alpha x$ (with $\alpha \in \mathbf{R}$); we shall see that $x = 0$. We have

$$A(\lambda)x_1 - S(\lambda)x_2 = i\alpha x_1 \tag{13}$$

$$-Q(\lambda)x_1 - A(\lambda)^{\mathrm{T}} x_2 = i\alpha x_2\,. \tag{14}$$

Multiplying on the left the members of equation (13) (resp. Eq. (14)) by $\bar{x}_2^{\mathrm{T}}$ (resp. by $\bar{x}_1^{\mathrm{T}}$) (the bar denotes the conjugate), we get

$$\bar{x}_2^{\mathrm{T}} A(\lambda)x_1 - \bar{x}_2^{\mathrm{T}} S(\lambda)x_2 = i\alpha x_1 \tag{15}$$

$$-\bar{x}_1^{\mathrm{T}} Q(\lambda)x_1 - \bar{x}_1^{\mathrm{T}} A(\lambda)^{\mathrm{T}} x_2 = i\alpha x_2\,. \tag{16}$$

by adding the conjugate of the members of equation (15) to the members of equation (16), we get

$$-\bar{x}_1^{\mathrm{T}} Q(\lambda) x_1 - \bar{x}_2^{\mathrm{T}} S(\lambda) x_2 = 0.$$

If $x_1 \neq 0$, as $Q(\lambda)$ is positive definite, this last equality implies $\bar{x}_2^{\mathrm{T}} S(\lambda) x_2 < 0$, but this inequality cannot occur because $S(\lambda)$ is nonnegative, so we must have $x_1 = 0$ and reporting this equality in (13) and (14), we get $S(\lambda)x_2 = 0$ and $A(\lambda)^{\mathrm{T}} x_2 = -i\alpha\, x_2$, which in turn implies $A(\lambda)^{\mathrm{T}} x_2 = -i\alpha\, x_2$ and $B(\sqrt{\lambda})^{\mathrm{T}} x_2 = 0$. As the pair $(A(\lambda)^{\mathrm{T}}, B(\sqrt{\lambda})^{\mathrm{T}})$ is detectable, the Hautus lemma implies that $x_2 = 0$.

By applicantion of the above-mentionned result from Kučera, we deduce that there exists a symmetryc nonnegative solution P_λ to Eq. (12). Now this solution is necessarily definite, assume indeed that v is a vector such that $Pv = 0$, left-multiply both sides of (12) by $\bar{v}^{\mathrm{T}}$ and right-multiply by v, we get $\bar{v}^{\mathrm{T}} Q v = 0$ which implies $v = 0$ since Q is assumed to be positive definite. □

This lemma proves that for every $\lambda \in \Lambda$ such that the pair $(A(\lambda), B(\sqrt{\lambda}))$ is stabilizable, there exists a positive definite solution, denoted by P_λ, to the Ricatti equation (12).

We denote by Λ^+ the set

$$\Lambda^+ = \left\{ \lambda \in \Lambda \mid \text{the pair}(A(\lambda), B(\sqrt{\lambda})) \text{ is stabilizable and } \max \operatorname{spec}(P_\lambda) \leq \nu_{\max} \right\}$$

where $\operatorname{spec}(P_\lambda)$ denotes the spectrum of P_λ; this set Λ^+ satisfies the following property.

Lemma 5. *The matrices Q_i being positive definite, if there exists $\lambda^0 \in \Lambda$ such that $(A(\lambda^0), B(\sqrt{\lambda^0}))$ is controllable, then, for every $\nu_{\max}$ large enough, set Λ^+ is compact and its interior is not empty in Λ. Moreover, there exist positive real numbers α_m and α_M defined as*

$$\alpha_m = \min_{\lambda \in \Lambda^+} \min(\operatorname{spec}(P_\lambda)) > 0 \qquad \alpha_M = \max_{\lambda \in \Lambda^+} \max(\operatorname{spec}(P_\lambda)) > 0.$$

Proof. As noticed above, we can find a compact neighborhood $\mathscr{U}$ of λ^0 such that the Riccati equation (12) admits a positive definite solution P_λ for every $\lambda \in \mathscr{U}$. The mapping $\lambda \mapsto P_\lambda$ being continuous and $\mathscr{U}$ being compact, we have $\sup_{\lambda \in \mathscr{U}} \max \operatorname{spec}(P_\lambda) < \infty$; this implies that if $\nu_{\max}$ is chosen large enough, the interior of Λ^+ is non empty.

Now, set Λ^+ is included in Λ, therefore it is bounded; we shall show that it is also closed. Suppose that there exists a sequence $(\lambda^k)_{k \geq 1} \in \Lambda^+$ such that $\lim_{k \to \infty} \lambda^k = \bar{\lambda}$. As Λ is a compact set, $\bar{\lambda} \in \Lambda$. Moreover, the sequence $(P_{\lambda^k})_{k \geq 1}$ is bounded, so we can assume that it converges to a symmetric matrix P. As a limit of a sequence positive definite matrices, this matrix is positive (semi) definite; moreover it is a solution of (12) with $\lambda = \bar{\lambda}$. We claim first that P is definite, assume indeed that v is a vector such that $Pv = 0$, left-multiply both sides of (12) by $\bar{v}^{\mathrm{T}}$ and right-multiply by v, we get $\bar{v}^{\mathrm{T}}Qv = 0$ which implies $v = 0$ since Q is assumed to be positive definite. Moreover the pair $(A(\lambda), B(\sqrt{\lambda})$ is stabilizable, to see this let μ be an eigenvalue of $A(\lambda)^{\mathrm{T}}$ such that $\Re(\mu) \geq 0$ ($\Re(\cdot)$ stands for the real part) and let $v \in \mathbf{R}^n$ be a vector such that $A(\lambda)^{\mathrm{T}}v = \mu\, v$ and $B^{\mathrm{T}}(\sqrt{\lambda})v = 0$, we shall see that $v = 0$ which implies that the rank of the matrix $\left(A(\lambda)^{\mathrm{T}} - \mu \,\mathrm{Id}, B^{\mathrm{T}}(\sqrt{\lambda})\right)^{\mathrm{T}}$ is equal to n for every μ in the closed right half plane and so the result will follow from the Hautus lemma. Matrix P being definite, there exists $x \in \mathbf{R}^n$ such that $Px = v$, left-multiply both sides of (12) by $\bar{x}^{\mathrm{T}}$ and right-multiply by x, we get

$$\mu\, \bar{x}^{\mathrm{T}}v + \bar{\mu}\, \bar{v}^{\mathrm{T}}x + \bar{x}^{\mathrm{T}}Q\, x = 0.$$

If $x \neq 0$, as matrix Q is definite positive, this equality implies that $\Re(\mu\, \bar{x}^{\mathrm{T}}v) < 0$ but $\bar{x}^{\mathrm{T}}v = \bar{x}^{\mathrm{T}}Px$ so $\bar{x}^{\mathrm{T}}v > 0$ since P is positive definite, therefore we must have $\Re(\mu) < 0$. This contradicts the fact that $\Re(\mu) \geq 0$, so, we must have $x = 0$, which implies that $v = 0$. Finally, the existence of α_m and α_M follows from the compactness of Λ^+ and the continuity of the mapping $\lambda \mapsto P_\lambda$. □

For the sake of readability, let us introduce the following notations. We denote by:

$$M(\lambda) := \sum_{i\in S} \lambda_i M_i(\lambda) \tag{17}$$

where

$$M_i(\lambda) := A_i - B_i K_i(\lambda) \tag{18}$$

$$K_i(\lambda) := R_i^{-1} B_i^T P_\lambda \tag{19}$$

and by

$$N(\lambda) := \sum_{i\in S} \lambda_i N_i(\lambda) \tag{20}$$

where

$$N_i(\lambda) := Q_i + K_i(\lambda)^T R_i K_i(\lambda).$$

The Riccati equation (12) can then be rewritten as :

$$M(\lambda)^T P_\lambda + P_\lambda M(\lambda) + N(\lambda) = 0. \tag{21}$$

Lemma 6. *For every $(x, \lambda) \in \left(\mathbf{R}^n \smallsetminus \{0\}\right) \times \Lambda^+$, we have*

$$\min_{i\in S} \left(2x^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x + x^{\mathrm{T}} N_i(\lambda) x\right) \leq 0.$$

Proof. Take $x \in \mathbf{R}^n \smallsetminus \{0\}$ and $\lambda \in \Lambda^+$, then equation (21) admits a solution and we can write

$$\sum_{i\in S} \lambda_i \left(2x^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x + x^{\mathrm{T}} N_i(\lambda) x\right) = 0 \tag{22}$$

so we cannot have

$$\left(2x^{\mathrm{T}}M_i^{\mathrm{T}}(\lambda)P_\lambda x + x^{\mathrm{T}}N_i(\lambda)x\right) > 0 \tag{23}$$

for every $i \in S$ because in this case the left-hand member of equality (22) would be positive. Thus, for every pair (x, λ), there exists an index i which is such that the left-hand member in (23) is non positive. □

Let us now introduce the following Lyapunov function

$$V_m(x) := \inf_{\lambda \in \Lambda^+} x^{\mathrm{T}} P_\lambda x \tag{24}$$

where P_λ denotes the solution of equation (21). Clearly, as every P_λ is positive definite when λ belongs to Λ^+ and as the set $\{P_\lambda \mid \lambda \in \Lambda^+\}$ is compact, V_m is a positive definite function; notice also that V_m is homogeneous of degree 2 and locally Lipschitz.

Proposition 7. *The function defined by* (24) *is locally lipschitzian.*

Proof. We have $z^{\mathrm{T}} P_\lambda z \leq \alpha_M \|z\|^2$ for every $(z, \lambda) \in \mathbf{R}^n \times \Lambda^+$. So, if we take x and y in the ball $B(0, R)$, we obtain easily that, for every $\lambda \in \Lambda^+$,

$$|x^{\mathrm{T}} P_\lambda x - y^{\mathrm{T}} P_\lambda y| = \|(x-y)^{\mathrm{T}} P_\lambda (x+y)| \leq K\,\|x - y\|$$

where $K = 2\alpha_M R$. Thus, the family of functions indexed by $\lambda \in \Lambda^+$ and defined by $x \mapsto x^T P_\lambda x$ is uniformly locally Liptchitz with a Lipschitz constant equals to $K := 2MR$ on the ball $B(0, R)$.

Now, as the function $\lambda \mapsto x^{\mathrm{T}} P_\lambda x$ is continuous and set Λ^+ is compact, there exists a pair $(\lambda_1, \lambda_2) \in (\Lambda^+)^2$ such that:

$$V_m(x) = x^T P_{\lambda^1} x \qquad\qquad V_m(y) = y^T P_{\lambda^2} y.$$

From the definition of V_m, we deduce easily that

$$x^{\mathrm{T}} P_{\lambda^1} x - x^{\mathrm{T}} P_{\lambda^2} x \leq 0 \qquad\qquad y^{\mathrm{T}} P_{\lambda^1} y - y^{\mathrm{T}} P_{\lambda^2} y \geq 0$$

therefore, by continuity, there exists z on the line segment $[x\ y]$ such that $z^T P_{\lambda^1} z = z^T P_{\lambda^2} z$ and it follows:

$$\begin{aligned} |V_m(x) - V_m(y)| &\leq |x^T P_{\lambda_1} x - z^T P_{\lambda_1} z| + |z^T P_{\lambda_2} z - y^T P_{\lambda_2} y| &(25)\\ &\leq K\|x - z\| + K\|z - y\| = K\|x - y\|. &(26)\end{aligned}$$

Finally function V_{min} is proper because it is continuous and $\alpha_m \|z\|^2 \leq V_m(z)$ for every $z \in \mathbf{R}^n$. □

Let f be a function defined on $\mathbf{R}^n$ and d be a vector of $\mathbf{R}^n$, as in [11] we shall denote by $f'(x; d)$ the following limit (if it exists)

$$f'(x; d) := \lim_{\substack{h \to 0 \\ h > 0}} \frac{f(x + h\, d) - f(x)}{h}. \tag{27}$$

For the sake of readability, we let $v_\lambda(x) := x^{\mathrm{T}} P_\lambda x$. In order to compute $V'_m(x; d)$, we use Theorem 6.1 in [11] whose conditions of application are clearly met, thus we have

$$V'_m(x; d) = \inf_{\lambda \in L(x)} v'_\lambda(x; d). \tag{28}$$

Here $L(x)$ denotes the set of $\lambda \in \Lambda^+$ such that $V_m(x) = v_\lambda(x)$ (this set is clearly nonempty and compact because Λ^+ is compact and the function $\lambda \mapsto v_\lambda(x)$ is continuous). As the function $x \mapsto v_\lambda(x)$ is smooth, from (28), we infer that

$$V'_m(x; d) = 2 \inf_{\lambda \in L(x)} d^{\mathrm{T}} P_\lambda x.$$

From this formula and from Lemma 6, we get the following properties.

Lemma 8. *For every $(x, \lambda^0) \in \mathbf{R}^n \times L(x)$, there exist $i(x, \lambda^0)$ such that*

$$V'_m(x; M_{i(x,\lambda^0)}(\lambda^0)x) \le -x^T N_{i(x,\lambda^0)}(\lambda^0)x \tag{29}$$

Proof. For every $(x, \lambda^0) \in \mathbf{R}^n \times L(x)$, Lemma 6 implies that there exists $i(x, \lambda^0)$ such that

$$2x^{\mathrm{T}} M^{\mathrm{T}}_{i(x,\lambda^0)}(\lambda^0) P_{\lambda^0} x + x^T N_{i(x,\lambda^0)}(\lambda^0)x \le 0$$

It follows directly that

$$\begin{aligned} V'_m(x; M_{i(x,\lambda^0)}(\lambda^0)x) &= \inf_{\lambda \in L(x)} 2x^{\mathrm{T}} M^{\mathrm{T}}_{i(x,\lambda^0)}(\lambda^0) P_\lambda x \\ &\le 2x^{\mathrm{T}} M^{\mathrm{T}}_{i(x,\lambda^0)}(\lambda^0) P_{\lambda^0} x \\ &\le -x^T N_{i(x,\lambda^0)}(\lambda^0)x \end{aligned}$$

□

In the following theorem, we shall consider mappings from $\mathbf{R}^n$ to $S \times \Lambda^+$ of the form $x \mapsto (i(x), \lambda(x))$ such that $\lambda(x) \in L(x)$. To such a mapping, we relate the following feedback law for system (1): the mode $\sigma(t)$ is equal to $i(x(t))$ for every $t \ge 0$ and $u_{\sigma(t)}$ is equal to $-K_{i(x)}(\lambda(x))\, x$.

Theorem 9. *We assume that the matrices Q_i are positive definite and there exists at least a $\lambda \in \Lambda$ such that the pair $(A(\lambda), B(\sqrt{\lambda}))$ is controllable. For every $x \in \mathbf{R}^n$, we choose*

$$(i(x), \lambda(x)) \in \operatorname*{arg\,min}_{(i,\lambda) \in S \times L(x)} \Big(2x^{\mathrm{T}} M_i(\lambda) P_\lambda x + x^{\mathrm{T}} N_i(\lambda) x \Big).$$

Then the feedback related to $(i(x), \lambda(x))$ *stabilizes the switched system* (1) *with a cost smaller than* $\frac{1}{2}V_{min}(x_0)$*. Moreover the convergence is exponential with a rate* $\beta = \frac{\eta_0}{\alpha_1}$ *where* η_0 *and* α_1 *are given by:*

$$\eta_0 = \min_{i\in S} \inf_{x\in S^{n-1}} \inf_{\lambda\in L(x)} x^T N_i(\lambda)x, \quad \alpha_1 = \max_{x\in S^{n-1}} V_m(x)$$

Remark. From a practical point of view, Th. 9 remains still valid if the feedback switching law is simplified as follows: for a given x, choose $\lambda(x)$ in $L(x)$ and take $i(x)$ as

$$i(x) \in \arg\min_{(i)\in S}\Big(2x^{\mathrm{T}} M_i(\lambda) P_\lambda x + x^{\mathrm{T}} N_i(\lambda)x\Big).$$

Proof of Th. 9. We shall compute the derivative of V_m along the trajectories of system (1) in closed-loop with the feedback introduced in the theorem. Hereafter, for the sake of readability, we denote by $M_{i(x)}$ the matrix $M_{i(x)}(\lambda(x))$ and $\frac{\mathrm{d}}{\mathrm{d}t}V_m(x(t))$ denotes the Dini derivative (*cf.* (27)) of V_m.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V_m(x(t)) &= V_m'(x; M_{i(x)}x) \\
&= 2 \min_{\lambda\in L(x)} x^{\mathrm{T}} M_{i(x)}^{\mathrm{T}} P_\lambda x \\
&\leq 2x^{\mathrm{T}} M_{i(x)}^{\mathrm{T}} P_{\lambda(x)} x \\
&\leq -x^{\mathrm{T}} N_{i(x)}(\lambda(x))x \qquad\qquad \text{from the definition of } (i(x), \lambda(x)) \\
&\leq -\frac{\eta_0}{\alpha_1} V_m(x).
\end{aligned}$$

This inequality implies that

$$V_m(x(t)) \leq e^{-\beta t} V_m(x_0)$$

with $\beta = \frac{\eta_0}{\alpha_1}$. As V_m is homogeneous of degree 2, this last inequality, implies the global exponential stability.

The upper bound on the cost $(1/2V_m(x_0))$ comes from the fact that

$$\begin{aligned}
x^{\mathrm{T}} Q_{i(x)} x + x^{\mathrm{T}} K_{i(x)}^{\mathrm{T}} R_{i(x)} K_{i(x)} x &= x^{\mathrm{T}} N_{i(x)} x \\
&\leq \frac{\mathrm{d}}{\mathrm{d}t} V_m(x(t)).
\end{aligned} \qquad \square$$

Why do we claim that the Lyapunov function can be a tight upper bound on the value function ? Observe first that in the case where all subsystems (related to the pairs (A_i, B_i)) are stabilizable, then the solution P_i of the Algebraic Riccati Equation exists for each mode i and the Lyapunov function satisfies always the following inequality:

$$\frac{1}{2}V_m(x) \leq \min_{i\in S} \frac{1}{2} x^T P_i x$$

One can also observe that for a given state x_0, the value $\frac{1}{2}V_m(x_0)$ is the best cost related to every constant convex combination that stabilizes the relaxed system. The corresponding control is of the form : $\lambda(t) = \lambda(x_0), \forall t \geq 0,\ u_i(t) = -R_i^{-1}B_i^T P_\lambda x(t)$.

In the general case, when can we say that $\frac{1}{2}V_m(x_0)$ is optimal ? The answer is :'Along the part of trajectories where the optimal control λ^* is constant to reach the origin". At least two cases can be mentioned: if the number of switchings is finite which means that a same mode is used after a time t or if the trajectory is steered to the origin by a constant singular control λ (*cf.* Definition 3) for which $P_\lambda > 0$. Note that singular control in dimension $n = 2$ can be algebraically determined [22] and are constant.

4 Sampled time switched control law

Practically, a sampled time version of the continuous time algorithm is applied. So, we shall show that the sampled time version of the above algorithm stabilizes the system for appropriate choice of sampled period.

Let τ be a given sampling period . The control is now piecewise constant and updated every times $t_k = k\tau$, $k \in \mathbb{N}$ following the state feedback provided by Theorem 9. To be more precise, at time $t_0 = 0$, we start with the initial condition x_0 and we choose the mode $i(x_0)$ and $\lambda(x_0)$ as in Theorem 9. Thus we have

$$2x_0^{\mathrm{T}} M_{i_0(x_0)}^{\mathrm{T}} P_{\lambda(x_0)} x_0 \leq -x_0^{\mathrm{T}} N_{i(x_0)}(\lambda(x_0)) x_0 \leq -\eta_0 \left\|x_0\right\|^2,$$

and, by the way, notice that $v_{\lambda(x_0)}(x_0) = V_m(x_0)$. We apply the feedback law related to $i(x_0), \lambda(x_0)$ to system (1), that is to say, we choose the mode $i(x_0)$ ($\sigma(t) = i(x_0)$) and $u_{\sigma(t)} = -K_{i(x_0)}(\lambda(x_0))$, we do so during a time $\tau > 0$. At time, $t_1 = t_0 + \tau$, we arrive at a point x_1 and we choose a new index $i(x_1)$ and a new $\lambda(x_1)$ as in theorem 9. In this way, we build a sequence of points $(x_k)_{k\geq 0}$ together with a sequence of pairs $(i(x_k), \lambda(x_k))_{k\geq 0}$ chosen as in theorem 9. On each interval $[t_k, t_{k+1})$ ($t_k = t_0 + k\,\tau$), we choose the mode $i(x_k)$ and we apply the feedback $u = -K_{i(x_k)}(\lambda(x_k))$. The switched system (1) in closed loop with this feedback writes

$$\begin{cases} \dot{x} = \Big(A_{i(x_k)} - B_{i(x_k)} K_{i(x_k)}(\lambda(x_k))\Big)x, & t \in [t_k, t_{k+1}) \\ x_k = x(t_k) \end{cases} \tag{30}$$

We state the following theorem about this algorithm; notice that we still assume that the Q_i's are positive definite.

Theorem 10. *If τ is chosen sufficiently small, the sampled time switching law described above stabilizes the switched system* (30) *globally exponentially*

For the proof of the theorem, we shall need the following lemma.

Lemma 11. *Consider the solution $t \mapsto x(t)$ of (30) starting from x_0 at some time $t_0 = 0$ where the mode i is chosen such that*

$$x_0^{\mathrm{T}} P_{\lambda(x_0)} x_0 = V_m(x_0)$$
$$2x_0^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda(x_0)) P_{\lambda(x_0)} x_0 \leq -x_0^T N_i(\lambda(x_0)) x_0 \leq -\eta_0 \|x_0\|^2$$

where η_0 is defined in Theorem 9. We define the time T as

$$T = \inf \left\{ t \geq 0 \mid 2x(t)^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda(x_0)) P_{\lambda(x_0)} x(t) \geq -\eta_0 \gamma \|x(t)\|^2 \right\}$$

where γ is a parameter chosen in the interval $(0, 1)$. Then there exists $\tau_0 > 0$ independent from x_0 such that $T \geq \tau_0$.

Proof. For the sake of readability, we shall denote by i and λ the terms $i(x_0)$ and $\lambda(x_0)$ respectively. We have

$$\begin{aligned} 2x(T)^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x(T) &- 2x_0^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x_0 \\ &= 2 \int_0^T \frac{\mathrm{d}}{\mathrm{d}t} x(t)^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x(t) \, \mathrm{d}t \\ &= 2 \int_0^T \Big(x(t)^{\mathrm{T}} \big(M_i^{\mathrm{T}}(\lambda)\big)^2 P_\lambda x(t) + x(t)^{\mathrm{T}} M_i(\lambda) P_\lambda M_i(\lambda) x(t) \Big) \mathrm{d}t. \end{aligned} \tag{31}$$

Now, for every $t \in [0, T)$, we have $x(t) = e^{t\, M_i(\lambda)} x_0$ and so

$$\|x(t)\| \leq e^{t\, \|M_i(\lambda)\|} \|x_0\|.$$

Notice that the matrix $M_i(\lambda(x_0))$ is bounded because $\lambda(x_0)$ evolves on the compact set Λ^+ and the mapping $\lambda \mapsto M_{i_k}(\lambda)$ is continuous. So there exists $\mu > 0$ such that $\|M_i(\lambda)\| \leq \mu$ for every $(i, \lambda) \in S \times \Lambda^+$; also, from Lemma 5, we know that $\|P_\lambda\| \leq \alpha_M$ for every $\lambda \in \Lambda^+$. So, from equality (31), we deduce that

$$\begin{aligned} 2x(T)^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x(T) &- 2x_0^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x_0 \\ &\leq 4\mu^2 \alpha_M \|x_0\|^2 \int_0^T e^{2t\,\mu} \mathrm{d}t \\ &= 2\mu \alpha_M (e^{2(T\,\mu} - 1) \|x_0\|^2. \end{aligned} \tag{32}$$

Now, since

$$2x_0^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x_0 \leq -\eta_0 \|x_0\|^2$$

and

$$2x(T)^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x(T) = -\eta_0 \gamma \|x(T)\|^2 \geq -\eta_0 \gamma\, e^{2T\mu} \|x_0\|^2,$$

we have

$$2x(T)^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x(T) - 2x_0^{\mathrm{T}} M_i^{\mathrm{T}}(\lambda) P_\lambda x_0 \geq \eta_0 \|x_0\|^2 (1 - \gamma\, e^{2T\mu}).$$

Substituting this inequality in (32), we get

$$2\mu\alpha_M(e^{2(T\mu} - 1)\|x_0\|^2 \geq \eta_0\|x_0\|^2(1 - \gamma\, e^{2T\mu})$$

and so

$$(2\mu\,\alpha_M + \eta_0\gamma)e^{2T\mu}) \geq 2\mu\,\alpha_M + \eta_0$$

which implies that

$$T \geq \tau_0 := \frac{1}{2\mu}\ln\left(\frac{2\mu\,\alpha_M + \eta_0}{2\mu\,\alpha_M + \eta_0\gamma}\right).$$

□

Proof of Theorem 10. We take a sampling time $\tau > 0$ no greater than τ_0, and we shall prove that system (30) is globally exponentially stable about the origin. Recall that, due to the choice of the pair (i_k, λ^k), we have $v_{\lambda^k}(x_k) = V_m(x_k)$ for every index k. Recall also that from Lemma 5, we have the inequality $x^{\mathrm{T}} P_\lambda x \geq \alpha_m\|x\|^2$ for every $\lambda \in \Lambda^+$ and every $x \in \mathbf{R}^n$. The expression $2x(t)^{\mathrm{T}} M_{i_k}^{\mathrm{T}}(\lambda^k) P_{\lambda^k} x$ represents the derivative of v_{λ^k} along the trajectories of system (30) on the time interval $[t_k, t_{k+1})$. So due to the choice of our feedback, this derivative is less or equal to $-\eta_0\gamma\,\|x(t)\|^2$ for every $t \in [t_k, t_{k+1})$, so we have

$$\dot{v}_{\lambda^k}(x) \leq -\eta_0\gamma\,\|x\|^2 \leq -\frac{\eta_0\gamma}{\lambda_M(P_{\lambda^k})} v_{\lambda^k}(x)$$

where $\lambda_M(P_{\lambda^k})$ denotes the greatest eigenvalue of matrix P_{λ^k}. From Lemma 5, we have $\lambda_M(P_{\lambda^k}) \leq \alpha_M$ and so we obtain

$$\dot{v}_{\lambda^k}(x) \leq -\frac{\eta_0\gamma}{\alpha_M} v_{\lambda^k}(x)$$

which implies that

$$v_{\lambda^k}(x_{k+1}) \leq v_{\lambda^k}(x_k) e^{-\frac{\eta_0\gamma}{\alpha_M}\tau} \tag{33}$$

for every index k. Now, we have $V_m(x_{k+1}) \leq v_{\lambda^k}(x_{k+1})$ and $V_m(x_k) = v_{\lambda^k}(x_k)$, so from (33), we deduce

$$V_m(x_{k+1}) \leq \theta\, V_m(x_k) \tag{34}$$

where $\theta = e^{-\frac{\eta_0\gamma}{\alpha_M}\tau} \in (0, 1)$ and so

$$V_m(x_k) \leq \theta^k\, V_m(x_0)$$

which implies , as $V_m(x) \geq \alpha_m\|x\|^2$,

$$\|x_k\|^2 \leq \frac{1}{\alpha_m}\,\theta^k\,\|x_0\|^2. \tag{35}$$

Now, on the interval $[t_k, t_{k+1})$, we have $v_{\lambda^k}(x(t)) \leq v_{\lambda^k}(x_k)$, this inequality together with (35), proves the result. □

5 Illustratives examples

Before presenting some examples, it is important to mention that it is not necessary to ensure a stabilizing switched law to determine all the possible values of the set Λ^+. Only one value is sufficient to guarantee the stability. So, a reasonable finite number of values ensures performances. Practically, a finite number have been used using a discretization of the set Λ.

5.1 Example 1 : a regular case

Consider a two mode switched system with the following design parameters:

$$A_1 = \begin{pmatrix} -2.7 & 3.9 \\ 4.4 & -12.6 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -9.5 & -5.1 \\ -7.5 & -3.3 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0.1 \\ 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 4.6 \\ 0 \end{pmatrix},$$

$Q_1 = Q_2 = \mathrm{Id}$, $R_1 = 1$ and $R_2 = 2$.

For each subsystem, an LQ design can be be performed separately. Thus, in order to make some comparisons, Figure 1 shows the state space trajectories for the switching law given by (30) and the optimal one. The later is obtained by NL programming in a suitable formulation taking into account singular arcs [24]. If not, numerical difficulties in the control determination are often encountered. We can see that the two solutions match well together. In the example, it can be observe that two singular arcs (defined by two lines in the state space) occur in the solutions.

Figure 2 compares the optimal cost with the costs obtained by using the switching law, only mode 1 and only mode 2, respectively. This comparison is made for initial states taken on the unit ball, the x-axis represent the angle θ. We have also added the guarantee on the cost provided by upper bound i.e. $\frac{1}{2}V_{min}(x)$. It can be observed that the cost associated to the switching law coincides with the cost of the optimal numerical solution. Of course, the essential difference is that the numerical solution is an open loop control while the switching law defines a closed loop control. It is also clear that the used of a single mode with no switching leads to lower performances.

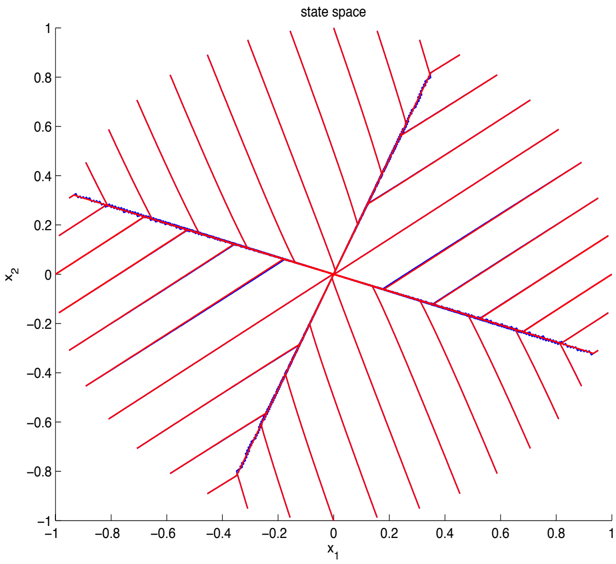

Figure 1: Ex. 1: State space trajectories: (red) optimal solution (NLP); (blue) switching law

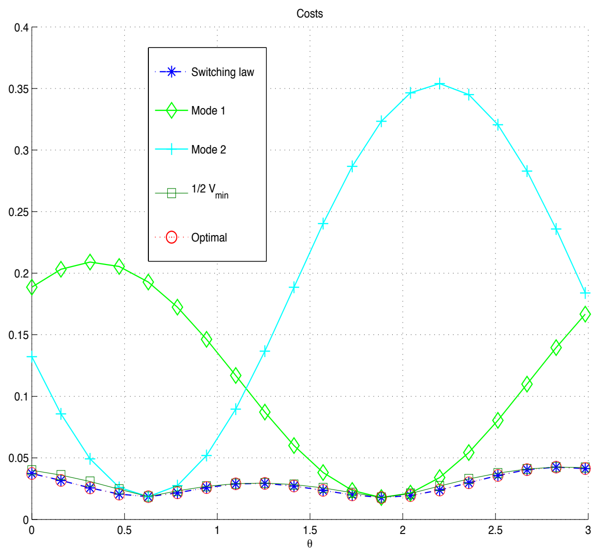

Figure 2: Ex. 1: Cost comparisons for different initial positions taken on the unit ball.

5.2 Example 2 : non stabilizable subsystems

For this second example, we have chosen two non stabilizable subsystems:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, there is no LQ design that can be defined separately for each subsystem. However, in a switched framework and taking $Q_1 = Q_2 = \mathrm{Id}$, $R_1 = 1$ and $R_2 = 2$, as the set Λ^+ is non empty, the switching law presented in this paper can be applied. Once again the optimal solution and the one provided by the switching law are very closed as showed by Figure 3. Figure 4 compares the optimal cost with the costs obtained by using the switching law. The upper bound V_{min} is also plotted.

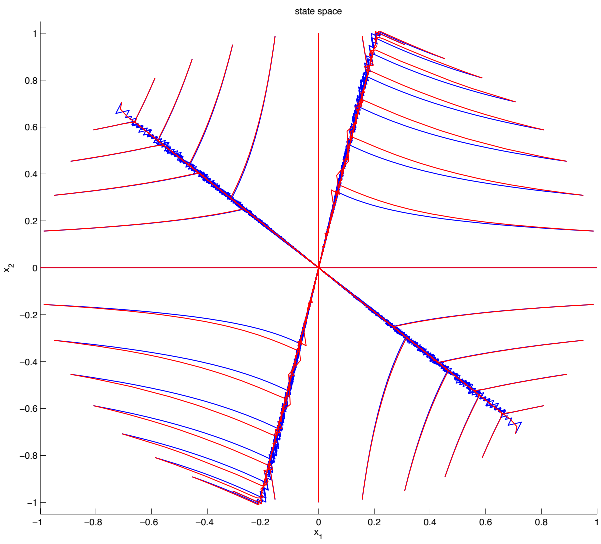

Figure 3: Ex. 2: State space trajectories: (red) optimal solution (NLP); (blue) switching law

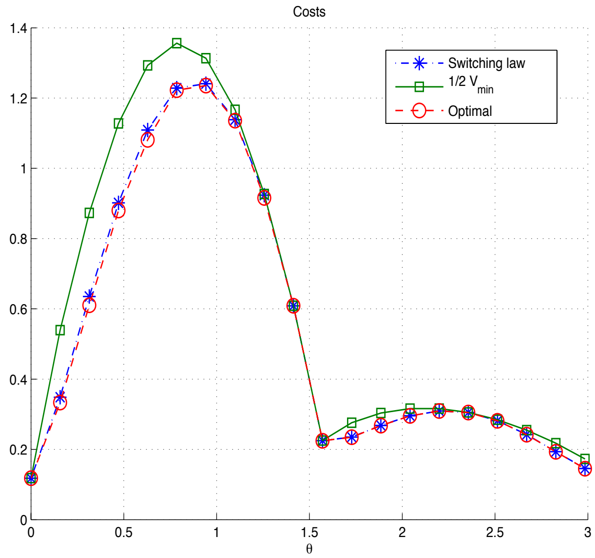

Figure 4: Ex. 2: Cost comparisons for different initial positions taken on the unit ball.

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