Resolution of Singularities of Arithmetical Threefolds I

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Dedicated to Shreeram Shankar Abhyankar, in memoriam.

Laboratoire de Mathématiques LAMA UMR 8100
Université de Versailles
45, avenue des États-Unis
78035 VERSAILLES Cedex (France)

cossart@math.uvsq.fr, piltant@math.uvsq.fr

1 Introduction.

The Resolution of Singularities conjecture has been, and still is, a long standing conjecture. Even since H. Hironaka's celebrated theorem [33] in equal characteristic zero fifty years ago, some new results have bettered our understanding of the problem in equal characteristic zero [8] [55].

In arbitrary characteristic, A.J. de Jong proved a weaker form of the conjecture by using alterations [41], i.e. allowing a finite extension of the function field. Resolution of Singularities in its full birational form is to this date restricted to surfaces: [1] [34] [45] [19], only to mention some contributions. In dimension three, some partial results exist for algebraic varieties over an algebraically closed field k of positive characteristic $p \geq 7$ [4] [25]. These results extend to all characteristics $p > 0$ when $[k : k^p] < +\infty$ [20]

[21] theorem on p. 1839. In mixed characteristics, birational Resolution of Singularities was sofar restricted to surfaces. The first and main purpose of this article is to prove the following theorem:

Theorem 1.1. *Let C be an integral Noetherian curve which is excellent and $\mathcal{X}/C$ be a reduced and separated scheme of finite type and dimension at most three. There exists a proper birational morphism $\pi : \mathcal{X}' \to \mathcal{X}$ with the following properties:*

(i) $\mathcal{X}'$ is everywhere regular;

(ii) π induces an isomorphism $\pi^{-1}(\mathrm{Reg}(\mathcal{X})) \simeq \mathrm{Reg}(\mathcal{X})$;

(iii) $\pi^{-1}(\mathrm{Sing}(\mathcal{X}))$ is a normal crossings divisor on $\mathcal{X}'$.

If furthermore $C = \mathrm{Spec}A$ is affine and $\mathrm{Reg}(\mathcal{X})$ is quasi-projective, one may furthermore take $\mathcal{X}'$ projective.

We emphasize that no assumption is made on the characteristic of C in this theorem. For example, the theorem applies to $C = \mathrm{Spec}\mathcal{O}_K$, K a number field or a complete discretely valued field. An important application of theorem 1.1 is to constructing regular integral models of algebraic surfaces over fields:

Corollary 1.2. *Let $C = \mathrm{Spec}A$ be an irreducible excellent regular Noetherian curve with function field F. Let Σ/F be a reduced algebraic projective surface and $\mathcal{X}$ be a flat projective C-scheme with generic fiber $\mathcal{X}_F = \Sigma$. There exists a projective birational morphism $\pi : \mathcal{X}' \to \mathcal{X}$ with the following properties:*

(i) $\mathcal{X}'$ is everywhere regular.

(ii) π induces an isomorphism $\pi^{-1}(\mathrm{Reg}(\mathcal{X})) \simeq \mathrm{Reg}(\mathcal{X})$.

(iii) $\pi^{-1}(\mathrm{Sing}(\mathcal{X}))$ is a normal crossings divisor on $\mathcal{X}'$.

Similarly, theorem 1.1 extends [21] theorem on p. 1839 to an arbitrary field k of positive characteristic (not necessarily differentially finite over a perfect subfield k_0).

The second purpose of this article is to explore the Resolution of Singularities Conjecture as formulated by A. Grothendieck [29](7.9.6). For this

purpose, we consider finite coverings $\eta : \mathcal{X} \to \mathrm{Spec}S$, where S is an arbitrary excellent regular local ring. A test case for Resolution if S has positive characteristic $p > 0$ is when η is purely inseparable; this was already recognized by O.Zariski [59] p.88 and S. Abhyankar [4] and recently confirmed by M. Temkin's purely inseparable Local Uniformization Theorem [54] theorem 1.3.2, *vid.* remark 1.3.5 (iii). In residue characteristic $p > 0$, we also include Galois coverings of degree p to this test case. The main theorem to be proved in dimension three is:

Theorem 1.3. *Let (S, m_S, k) be an excellent regular local ring of dimension three, quotient field $K := QF(S)$ and residue characteristic* $\mathrm{char}k = p > 0$. *Let*

$$h := X^p + f_1 X^{p-1} + \cdots + f_p \in S[X],\ f_1, \ldots, f_p \in S \tag{1.1}$$

be a reduced polynomial, $\mathcal{X} := \mathrm{Spec}(S[X]/(h))$ and $L := \mathrm{Tot}(S[X]/(h))$ be its total quotient ring. Assume that h satisfies one of the following assumptions:

(i) $\mathcal{X}$ is G-invariant, where $\mathrm{Aut}_K(L) = \mathbb{Z}/p =: G$, or

(ii) $\mathrm{char}K = p$ *and* $f_1 = \cdots = f_{p-1} = 0$.

Let μ be a valuation of L which is centered in m_S. There exists a composition of local Hironaka-permissible blowing ups:

$$(\mathcal{X} =: \mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r), \tag{1.2}$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that $(\mathcal{X}_r, x_r)$ is regular.

We develop an approach to the Resolution of Singularities Conjecture for hypersurface singularities defined by an equation (1.1) in any dimension $n := \dim S \geq 1$. No other assumption on S is required here than excellence of S; in particular, we do not even assume that $[k : k^p] < +\infty$ as suggested by A. Grothendieck *loc.cit.* The main result which is proved here is the existence of a numerical function (definition 2.16)

$$\iota : \mathcal{X} \to \{1, \ldots, p\} \times \mathbb{N} \times \{1, \geq 2\} :\ x \mapsto (m(x), \omega(x), \kappa(x)), \tag{1.3}$$

refining the multiplicity function $x \mapsto m(x)$ at those points $x \in \mathcal{X}$ such that $m(x) = p$. This function has "expected" properties: ι is invariant by regular base change $S \subset \tilde{S}$, $\tilde{S}$ excellent (theorem 2.20) and is constructible on $\mathcal{X}$ (corollary 3.11). A notion of permissible blowing up refining that

of H. Hironaka is developed. Permissible centers $\mathcal{Y} \subset \mathcal{X}$ also extend to permissible centers under regular base change (theorem 3.4). The function ι is nonincreasing with respect to permissible blowing ups (theorem 3.6).

When applied in dimension $n = 3$, this reduces theorem 1.3 to a projection theorem 4.4 which is proved by extending the methods of [21]. This extension is performed in [24], thus giving a complete proof of theorem 1.3 and of theorem 1.1. The sequence (1.2) which is constructed depends in no significant way on the given valuation μ and can be considered as a version of Hironaka's Local Control (Hironaka's A/B Game, in residue characteristic zero) for equations (1.1). Precise statements use the notion of independent sequence (definition 2.18) and are collected in theorem 5.15. The authors hope that theorem 1.3 could be extended to a Resolution of Singularities $\pi : \mathcal{X}' \to \mathcal{X}$, π a composition of Hironaka permissible (global) blowing ups (and with G-invariant centers under assumption (i)).

This article is organized as follows: in section 2, we introduce our main tool which is the Hironaka Characteristic Polyhedron [35]. Our notations are slightly different from Hironaka's because we focus our attention on the variation of the characteristic polyhedron along regular subschemes of Spec$S$. The above assumptions (i) or (ii) provide the structure theorem 2.14 for the initial forms $\text{in}_\alpha h$, $\alpha \in \mathbb{R}^n_{>0}$, of the characteristic polyhedron with respect to its compact faces. This fact is essential because it allows us to reproduce part of the equal characteristic $p > 0$ constructions which were used in [21].

This leads us to the definition 2.16 of the function ι in (1.3). The function ω is a differential version of Hironaka's ϵ function [35] and requires introducing a differential structure (S, h, E) adapted to a normal crossings divisor $E \subset \text{Spec}S$ (section 2.4). A fundamental difference takes place between the Galois case (i) and the purely inseparable case (ii) of the above theorem 1.3: eventually ι is uppersemicontinuous in case (ii) but only constructible in general in case (i), *vid.* example 3.2. When $\omega(x) = 0$ in (1.3) for some $x \in \mathcal{X}$, a simple combinatorial blowing up algorithm (similar to residue characteristic zero) makes the multiplicity function m smaller than p at all points of the blown up space above $\mathcal{X}$ (theorem 2.23). There remains to deal with points $x \in \mathcal{X}$ such that $m(x) = p$, $\omega(x) > 0$.

Section 3 contains the technical bulk of this paper. We develop a notion of permissible blowing up $\pi : \mathcal{X}' \to \mathcal{X}$ which refines that of H. Hironaka.

Roughly speaking, a Hironaka permissible center $\mathcal{Y} \subset \mathcal{X}$ is permissible in our sense if $\mathcal{X}$ is "differentially equimultiple" along $\mathcal{Y}$ (definition 3.1 and definition 3.2). The notion is somewhat subtle but has good properties, the main result being theorem 3.6: ι is nonincreasing along permissible blowing ups. Furthermore, ι decreases except possibly at exceptional points $x' \in \pi^{-1}(x)$ belong to some embedded projective cone

$$PC(x, \mathcal{Y}) \subset \pi^{-1}(x)$$

given in definition 3.3. We also mention:

(1) persistence of permissibility under regular base change (theorem 3.4);

(2) the strict transform $\mathcal{Z}' \subset \mathcal{X}'$ of a permissible center $\mathcal{Z} \subset \mathcal{X}$ by a permissible blowing up π with center $\mathcal{Y} \subset \mathcal{Z}$ is permissible (theorem 3.7);

(3) Hironaka permissible centers are permissible over a dense open subset of their support (theorem 3.10).

We expect these results to be important in order to argue by induction on the dimension of $\mathcal{X}$.

Section 4 is restricted to dimension three and collects together all previous results. A projection number $\kappa(x) \in \{1, 2, 3, 4\}$ is associated to a singular point $x \in \mathcal{X}$ such that $m(x) = p$, $\omega(x) > 0$. This function basically expresses the transverseness or tangency of the initial form $\text{in}_{m_S} h$ of the characteristic polyhedron with respect to the initial face. That $\iota(x)$ in (1.3) can be made smaller by performing Hironaka permissible blowing ups is stated in theorem 4.4 and proved in [24].

Section 5 is an appendix to this article. It consists in adapting some of the equal characteristic $p > 0$ material from [20] to our arbitrary characteristic context. We include adapted proofs of:

(5.1) reduction of theorem 1.1 to its Local Uniformization form along valuations;

(5.2) reduction of Local Uniformization to theorem 1.3.

The section ends with the proof of theorem 1.3, hence of theorem 1.1, assuming theorem 4.4 (proposition 5.10). Section 6 is an excerpt from [24], a special case of which is required in the proof of proposition 5.10.

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2 Adapted structure and primary invariants.

All along this article, we will denote by S a regular local ring of arbitrary dimension $n \geq 1$, and by $(u_1, \ldots, u_n)$ a regular system of parameters (r.s.p. for short) of S. Its maximal ideal is denoted by $m_S := (u_1, \ldots, u_n)$ and its formal completion w.r.t. m_S by $\hat{S}$. The order function ord_{m_S} on S is defined by:

$$\mathrm{ord}_{m_S} f := \sup\{n \in \mathbb{N} : f \in m_S^n\} \in \mathbb{N} \cup \{+\infty\},\ f \in S.$$

This order function extends to a discrete valuation on the quotient field $K := QF(S)$ of S.

We will assume that $\mathrm{char}(S/m_S) > 0$ except for the next three sections. We also assume that S is *excellent* beginning from theorem 2.4 on. The basic reference for excellent rings is [29] 7.8 and 7.9. A useful *compendium* is [46] pp. 255-260; some extensions and examples of non excellent regular local rings can be found in [40] pp. 7-22. Let

$$h := X^m + f_{1,X}X^{m-1} + \cdots + f_{m,X} \in S[X],\ f_{1,X}, \ldots, f_{m,X} \in S \tag{2.1}$$

be a unitary polynomial of degree $m \geq 2$. We denote by

$$\mathcal{X} := \mathrm{Spec}(S[X]/(h)) \text{ and } \eta :\ \mathcal{X} \longrightarrow \mathrm{Spec}S \tag{2.2}$$

respectively the corresponding hypersurface and induced projection.

The total ring of fractions $\mathcal{X}$ is denoted by $L := \mathrm{Tot}(S[X]/(h))$. Given a point $y \in \mathcal{X}$, its residue field is denoted by $k(y)$ and its multiplicity by $m(y)$. Explicitly, we have:

$$m(y) = \mathrm{ord}_{m_{S[X]_y}} h.$$

The singular locus of $\mathcal{X}$ is denoted by :

$$\mathrm{Sing}\mathcal{X} = \{y \in \mathcal{X} : m(y) \geq 2\}.$$

The *locus of multiplicity* m of $\mathcal{X}$ is viewed as an embedded reduced subscheme of $\mathcal{X}$:

$$\mathrm{Sing}_m\mathcal{X} := \{y \in \mathrm{Spec}(S[X]) : \mathrm{ord}_{m_{S[X]_y}} h = m\} \subseteq \mathrm{Sing}\mathcal{X}.$$

Both of $\mathrm{Sing}\mathcal{X}$ and $\mathrm{Sing}_m\mathcal{X}$ are proper closed subsets of $\mathcal{X}$ if S is excellent.

Given a "linear change of" (one also says "translation on") the X-coordinate, say $X' := X - \phi$, $\phi \in \hat{S}$, we still denote by

$$h = X'^m + f_{1,X'}X'^{m-1} + \cdots + f_{m,X'} \in S[X']$$

the corresponding expansion of $h(X' + \phi)$, $f_{1,X'}, \ldots, f_{m,X'} \in \hat{S}$. The explicit formula for this change of coordinate is :

$$f_{i,X'} = \binom{m}{i}\phi^i + \sum_{j=1}^{i}\binom{m-j}{i-j} f_{j,X}\phi^{i-j},\ 1 \leq i \leq m. \tag{2.3}$$

Given $\phi \in S$ and a rational number $d \leq \mathrm{ord}_{m_S}\phi$, we denote by $\mathrm{cl}_d\phi$ the *initial form* of ϕ in $\mathrm{gr}_{m_S}S \simeq S/m_S[U_1, \ldots, U_n]$ (resp. the null form) if $d = \mathrm{ord}_{m_S}\phi$ (resp. otherwise). Similarly, if $I \subseteq S$ and $d \leq \mathrm{ord}_{m_S}I$, we denote

$$\mathrm{cl}_d I := \mathrm{Vect}(\{\mathrm{cl}_d\phi\}_{\phi \in I}) \subseteq S/m_S[U_1, \ldots, U_n]_d.$$

Suppose that a weight vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_{\geq 0}$ is given. Let $\Gamma_\alpha := \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n \subset \mathbb{R}$. For $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n_{\geq 0}$, denote

$$\mid \mathbf{x} \mid_\alpha := \alpha_1 x_1 + \cdots + \alpha_n x_n \in (\Gamma_\alpha)_{\geq 0}.$$

An associated valuation μ_α of K is defined by setting for $f \in S$, $f \neq 0$:

$$\mu_\alpha(f) := \max\{a \in \Gamma_\alpha : f \in I_\alpha(a) := (\{u_1^{x_1} \cdots u_n^{x_n} :\mid \mathbf{x} \mid_\alpha \geq a\})\}.$$

It easily follows from the Noetherianity of S that $\mu_\alpha(f)$ is well defined. One sets

$$\mu_\alpha(f/g) := \mu_\alpha(f) - \mu_\alpha(g) \text{ for } f, g \in S, fg \neq 0.$$

Note that $\mathrm{ord}_{m_S} = \mu_{\mathbf{1}}$, where $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{R}^n_{>0}$. We will systematically use the graded ring $\mathrm{gr}_\alpha S$ of S w.r.t. μ_α:

$$\mathrm{gr}_\alpha S \simeq S/(\{u_i : \alpha_i > 0\})[\{U_i : \alpha_i > 0\}].$$

If $a \in \Gamma_\alpha$ and $\phi \in S$ is given with $a \leq \mu_\alpha(\phi)$, its initial form $\mathrm{cl}_{\alpha,a}\phi \in \mathrm{gr}_\alpha S$ is defined as before. Similarly, if $I \subset S$ and $a \leq \mu_\alpha(I)$, we associate a $(\mathrm{gr}_\alpha S)_0$-module denoted by

$$\mathrm{cl}_{\alpha,a} I := \mathrm{Span}(\{\mathrm{cl}_{\alpha,a}\phi\}_{\phi \in I}) \subseteq (\mathrm{gr}_\alpha S)_a.$$

2.1 Characteristic polyhedron and first invariants.

Given an equation $h \in S[X]$ (2.1) and a r.s.p. $(u_1, \ldots, u_n)$ of S, there is an associated Newton polyhedron w.r.t.the variables $(u_1, \ldots, u_n, X)$:

$$NP(h; u_1, \ldots, u_n; X) \subseteq \mathbb{R}^{n+1}_{\geq 0}.$$

Let $P := (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}_{\geq 0}$, so $P \in \frac{1}{m} NP(h; u_1, \ldots, u_n; X)$, and

$$\mathbf{p} : \mathbb{R}^{n+1} \setminus \{P\} \longrightarrow \mathbb{R}^n$$

be the projection on the $(u_1, \ldots, u_n)$-space. We define a polyhedron by:

$$\Delta(h; u_1, \ldots, u_n; X) := \mathbf{p}\left(\frac{1}{m} NP(h; u_1, \ldots, u_n; X) \setminus \{P\}\right) \subseteq \mathbb{R}^n_{\geq 0}.$$

The *characteristic polyhedron* is introduced in a more general context in [35]. In our setting, it consists in minimizing $\Delta(h; u_1, \ldots, u_n; X')$ over all linear changes of coordinates $X' = X - \phi$, $\phi \in \hat{S}$ (2.3).

In this section, we review and adapt notations to fit our purposes. A fundamental algebraicity result is borrowed from [22] in theorem 2.4 below. Then some of the invariance properties of the characteristic polyhedron under base change are listed.

Let S and $(u_1, \ldots, u_n)$ be fixed as above. Given a subset $J \subseteq \{1, \ldots, n\}$, we denote by

$$I_J := (\{u_j\}_{j \in J}) \subset S \text{ and } \overline{S}^J := S/I_J.$$

We also use the notation $s^J \in \mathrm{Spec} S$ to denote the point $s^J = I_J$, reserving the idealistic notation I_J to commutative algebraic formulæ.

Proposition 2.1. *Let $f \in S$. There exists a unique finite set $\mathbf{S}^J(f) \subset \mathbb{N}^J$ such that the following holds:*

(i) the set of monomials $\{\prod_{j\in J} u_j^{a_j} : \mathbf{a} = (\{a_j\}_{j\in J}) \in \mathbf{S}^J(f)\}$ forms a minimal system of generators of the ideal

$$I(f) := \left(\left\{ \prod_{j\in J} u_j^{a_j} : \mathbf{a} = (\{a_j\}_{j\in J}) \in \mathbf{S}^J(f) \right\} \right);$$

(ii) there is an expansion

$$f = \sum_{\mathbf{a}\in\mathbf{S}^J(f)} \gamma(f,\mathbf{a}) \prod_{j\in J} u_j^{a_j} \in S,\ \gamma(f,\mathbf{a}) \in S \tag{2.4}$$

such that $\gamma(f,\mathbf{a}) \notin I_J$ for every $\mathbf{a} \in \mathbf{S}^J(f)$.

Proof. Let $\widehat{S}^J$ be the formal completion of S *along* I_J. Since $I_J \subseteq m_S$, $\widehat{S}^J$ is faithfully flat over S [46] theorem 8.14(3). Thus $I\widehat{S}^J \cap S = I$ for any ideal $I \subseteq S$, in particular for any monomial ideal in $\{u_j\}_{j\in J}$. One deduces that property (i) and existence of an expansion (2.4) descend from $\widehat{S}^J$ to S.

Suppose that an expansion (2.4) exists for a given $\mathbf{S}^J(f)$ satisfying (i). Each S/I_J^{n+1}, $n \geq 0$ has a structure of free $\overline{S}^J$-module with basis

$$\left\{ \prod_{j\in J} u_j^{a_j} : \mathbf{a} = (\{a_j\}_{j\in J}) \text{ and } \sum_{j\in J} a_j \leq n \right\}.$$

Therefore the class $\gamma(f,\mathbf{a}) + I_J$ is independent of the chosen expansion (2.4) by the minimality property in (i). This proves that the property $\gamma(f,\mathbf{a}) \notin I_J$ in (ii) also descends from $\widehat{S}^J$ to S. In other terms, we may assume that S is I_J-adically complete.

Independent monomial generators in S/I_J^n lift to independent monomial generators in S/I_J^{n+1} for every $n \geq 1$. One easily deduces the existence of an expansion (ii) satisfying (i) for some finite subset $\mathbf{S}^J(f) \subset \mathbb{N}^J$, since S is I_J-adically complete and Noetherian.

Uniqueness of $\mathbf{S}^J(f)$ is also checked by taking images in S/I_J^{n+1} for some $n >> 0$. This concludes the proof.

Definition 2.1. (Associated Polyhedron). Given an equation $h \in S[X]$ (2.1) and $J \subseteq \{1, \ldots, n\}$, we define a rational polyhedron:

$$\Delta_S(h; \{u_j\}_{j\in J}; X) := \mathrm{Conv}\left(\bigcup_{i=1}^{m} \bigcup_{\mathbf{a}\in \mathbf{S}^J(f_{i,X})} \left\{\frac{\mathbf{a}}{i} + \mathbb{R}^J_{\geq 0}\right\}\right) \subseteq \mathbb{R}^J_{\geq 0}.$$

Definition 2.2. (Initial forms). Let $\alpha = (\{\alpha_j\}_{j\in J}) \in \mathbb{R}^J_{>0}$ be a weight vector. We define

$$\delta_\alpha(h; \{u_j\}_{j\in J}; X) := \min\{|\ \mathbf{x}\ |_\alpha \colon \mathbf{x} \in \Delta_S(h; \{u_j\}_{j\in J}; X)\}.$$

The weight vector defines a *compact face* σ_α of $\Delta_S(h; \{u_j\}_{j\in J}; X)$ by:

$$\sigma_\alpha := \{\mathbf{x} \in \Delta_S(h; \{u_j\}_{j\in J}; X) : |\ \mathbf{x}\ |_\alpha = \delta_\alpha(h; \{u_j\}_{j\in J}; X)\}.$$

The *initial form* $\mathrm{in}_\alpha h$ of h w.r.t. α is the polynomial

$$\mathrm{in}_\alpha h := X^m + \sum_{i=1}^{m} F_{i,X,\alpha} X^{m-i} \in (\mathrm{gr}_\alpha S)[X], \tag{2.5}$$

where

$$F_{i,X,\alpha} := \sum_{\mathbf{x}\in\sigma_\alpha} \overline{\gamma}(f_{i,X}, i\mathbf{x}) U^{i\mathbf{x}},$$

and bars denotes images in $(\mathrm{gr}_\alpha S)_0 = \overline{S}^J$, i.e.

$$\overline{\gamma}(f_{i,X}, i\mathbf{x}) := \mathrm{cl}_{\alpha,0}\gamma(f_{i,X}, i\mathbf{x}) \in (\mathrm{gr}_\alpha S)_0 = \overline{S}^J.$$

By convention, we take $\overline{\gamma}(f_{i,X}, i\mathbf{x}) = 0$ in these formulæ whenever $i\mathbf{x} \notin \mathbf{S}^J(f_{i,X})$.

Remark 2.1. Any vertex of $\Delta_S(h; \{u_j\}_{j\in J}; X)$ has coordinates in $\frac{1}{m!}\mathbb{N}$. We have:

$$\Delta_S(h; \{u_j\}_{j\in J}; X) = \emptyset \Leftrightarrow h = X^m.$$

It is worth emphasizing that the polynomial $\mathrm{in}_\alpha h$ only depends on the face σ_α and not on the specific weight vector α defining it. Given h and α, the grading of $\mathrm{gr}_\alpha S$ can be extended to $(\mathrm{gr}_\alpha S)[X]$ by setting:

$$\deg X := \delta_\alpha(h; \{u_j\}_{j\in J}; X).$$

Then $\mathrm{in}_\alpha h$ is a *homogeneous* polynomial of degree $m\delta_\alpha(h; \{u_j\}_{j\in J}; X)$ for this grading.

We now briefly review the behaviour of polyhedra and initial forms under basic operations such as formal completion, localization and projection onto a regular subscheme. The case of regular local morphisms $S \subset \tilde{S}$, $\tilde{S}$ excellent will be considered further on.

With notations as above, let $\alpha \in \mathbb{R}^J_{>0}$ be a weight vector and

$$\sigma_\alpha \subset \Delta_S(h; \{u_j\}_{j\in J}; X),\ \mathrm{in}_\alpha h \in (\mathrm{gr}_\alpha S)[X].$$

Formal Completion: the regular local ring $\hat{S}$ is excellent [29] theorem 7.8.3(iii). Proposition 2.1 and definition 2.1 give an identification

$$\Delta_S(h; \{u_j\}_{j\in J}; X) = \Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X). \tag{2.6}$$

This identification preserves the initial form $\mathrm{in}_\alpha h$ for each weight vector α via the inclusion $\mathrm{gr}_\alpha S \subseteq \mathrm{gr}_\alpha \hat{S} \simeq \mathrm{gr}_\alpha S \otimes_S \hat{S}$.

Localization: the regular local ring S_{s^J} is excellent if S is excellent [29] theorem 7.4.4. Similarly, the identifications

$$\Delta_S(h; \{u_j\}_{j\in J}; X) = \Delta_{S_{s^J}}(h; \{u_j\}_{j\in J}; X) \tag{2.7}$$

also preserve the initial form $\mathrm{in}_\alpha h$ (2.5) via the inclusion

$$\mathrm{gr}_\alpha S \subseteq \mathrm{gr}_\alpha S_{s^J} \simeq (\mathrm{gr}_\alpha S) \otimes_S QF(\overline{S}^J).$$

Projection: let $J \subseteq \{1, \ldots, n\}$ and denote by $J' := \{1, \ldots, n\}\backslash J$ its complement. The regular local ring $\overline{S}^J$ is excellent if S is excellent. A r.s.p. of $\overline{S}^J$ is $(\{\overline{u}_{j'}\}_{j'\in J'})$, where bars denote images in $\overline{S}^J$. With notations as above, we have:

$$\Delta_S(h; \{u_j\}_{j\in J}; X) = \mathrm{pr}^J \Delta_S(h; u_1, \ldots, u_n; X), \tag{2.8}$$

where $\mathrm{pr}^J : \mathbb{R}^n \to \mathbb{R}^J,\ \mathbf{x} \mapsto \mathbf{y} = (\{x_j\}_{j\in J})$ denotes the projection. Let

$$f_{i,X} = \sum_{\mathbf{a}\in \mathbf{S}(f_{i,X})} \gamma(f_{i,X}, \mathbf{a}) u_1^{a_1} \cdots u_n^{a_n} \in S,$$

be an expansion (2.4) (for the subset $\{1, \ldots, n\}$, where $\mathbf{S}(f_{i,X})$ here stands for $\mathbf{S}^{\{1,\ldots,n\}}(f_{i,X})$), $1 \le i \le m$. Then (2.5) is given by

$$F_{i,X,\alpha} := \sum_{\mathbf{y}\in\sigma_\alpha} \left(\sum_{\mathrm{pr}^J(\mathbf{x})=\mathbf{y}} \overline{\gamma}(f_{i,X}, i\mathbf{x}) \prod_{j'\in J'} \overline{u}_{j'}^{ix_{j'}} \right) \prod_{j\in J} U_j^{iy_j}, \tag{2.9}$$

where bars denotes images in $(\mathrm{gr}_\alpha S)_0 = \overline{S}^J$ as before (recall that by convention, we take $\overline{\gamma}(f_{i,X}, i\mathbf{x}) := 0$ in this formula if $i\mathbf{x} \notin \mathbf{S}(f_{i,X})$).

Definition 2.3. (Solvable vertices). Let $\mathbf{x} \in \mathbb{R}^J$ be a vertex of the polyhedron $\Delta_S(h; \{u_j\}_{j\in J}; X)$, that is, a 0-dimensional face $\sigma = \{\mathbf{x}\}$. Denote by

$$\mathrm{in}_{\mathbf{x}} h = X^m + \sum_{i=1}^{m} F_{i,X,\mathbf{x}} X^{m-i} \in (\mathrm{gr}_\alpha S)[X]$$

the initial form polynomial (2.5) w.r.t. any defining weight vector α. We will say that $\mathbf{x}$ is solvable if $\mathbf{x} \in \mathbb{N}^J$ and there exists $\overline{\lambda} \in \overline{S}^J$ such that

$$\mathrm{in}_{\mathbf{x}} h = (X - \overline{\lambda} U^{\mathbf{x}})^m.$$

Explicitly, with notations as in (2.5) *sqq.*, the latter equality means that

$$\overline{\gamma}(f_{i,X}, i\mathbf{x}) = (-1)^i \binom{m}{i} \overline{\lambda}^i \in \overline{S}^J,\ 1 \leq i \leq m.$$

Note that $\binom{m}{i} \in \overline{S}^J$ is not a unit in general when $\mathrm{char}(S/m_S) > 0$.

The following result is a rewriting of Hironaka's vertex preparation lemma and theorem [35] (3.10) and (4.8) in this hypersurface situation.

Proposition 2.2. *(Hironaka). There exists a linear change of the X-coordinate $Z := X - \theta$, with $\theta \in \hat{S}$, such that*

$$\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z) = \min_{X'} \Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X'), \tag{2.10}$$

where the minimum is taken w.r.t. inclusions and over all possible linear changes of coordinates $X' := X - \phi$, $\phi \in \hat{S}$.

Given $X' := X - \phi$, $\phi \in \hat{S}$, $\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X')$ achieves equality in (2.10) if and only if it has no solvable vertex.

If S is excellent, there is an equivalence

$$\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z) = \emptyset \Leftrightarrow \exists g \in S : h = (X - g)^m.$$

Proof. We first recall Hironaka's algorithm: let $\mathbf{x} \in \mathbb{R}^J$ be a solvable vertex of $\Delta_S(h; \{u_j\}_{j\in J}; X)$ with $\mid \mathbf{x} \mid = \sum_{j\in J} x_j$ minimal. By definition 2.3, $\mathbf{x} \in \mathbb{N}^J$ and $\mathrm{in}_{\mathbf{x}} h = (X - \overline{\lambda} U^{\mathbf{x}})^m$ for some $\overline{\lambda} \in \overline{S}^J$. Pick any $\lambda \in S$ whose residue in $\overline{S}^J$ is $\overline{\lambda}$ and let $X_1 := X - \lambda u^{\mathbf{x}}$. By construction

$$\Delta_S(h; \{u_j\}_{j\in J}; X_1) \subseteq \Delta_S(h; \{u_j\}_{j\in J}; X),$$

and equality is strict because $\mathbf{x} \notin \Delta_S(h; \{u_j\}_{j\in J}; X_1)$.

Iterating, we get a decreasing sequence of polyhedra

$$(\Delta_S(h; \{u_j\}_{j\in J}; X_n))_{n\in A},$$

where $A \subseteq \mathbb{N}^J$ indexes solvable vertices $\mathbf{x}_n \in \mathbb{N}^J$. Since for each $a \in \mathbb{N}$, $\mathbb{N}^J \cap \{\mathbf{x} : \mid \mathbf{x} \mid = a\}$ is finite, $\mid \mathbf{x}_n \mid$ goes to infinity with n whenever A is infinite. This proves the existence of $Z := X - \theta$, $\theta \in \hat{S}$, such that $\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z)$ has no solvable vertex (in fact $\theta \in \widehat{S}^J \subseteq \hat{S}$, $\widehat{S}^J$ the formal completion of S along J).

Pick now any $X' := Z - \phi$, $\phi \in \hat{S}$ such that $\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X')$ has no solvable vertex. We claim that

$$\Delta_{\hat{S}}((Z - \phi)^m; \{u_j\}_{j\in J}; Z) \subseteq \Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z). \tag{2.11}$$

Once the claim is proved, one easily gets

$$\Delta(\phi) := \Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X') \subseteq \Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z)$$

from the coordinate change formula (2.3). By symmetry, this proves (2.10) and the second statement in the proposition.

To prove the claim, suppose that (2.11) does not hold. Then there exists a weight vector $\alpha \in \mathbb{R}^J_{>0}$ defining a *vertex* $\mathbf{x}$ of $\Delta(\phi)$ such that

$$\mid \mathbf{x} \mid_\alpha < \min\{\mid \mathbf{x}' \mid_\alpha : \mathbf{x}' \in \Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z)\}.$$

By the coordinate change formula (2.3), we get

$$\mathrm{in}_{\mathbf{x}} h(X') = \mathrm{in}_{\mathbf{x}}(X' + \phi)^m = (X' + \overline{\lambda} U^{\mathbf{x}})^m \in (\mathrm{gr}_\alpha S)[X'],$$

where $\overline{\lambda} \in \overline{S}^J$ is nonzero. This is a contradiction, since it was assumed that $\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; X')$ had no solvable vertex.

We now turn to the last statement in the proposition. The if part is obvious. For the converse, it can be assumed that $J = \{1, \ldots, n\}$ by applying (2.8) to the ring $\hat{S}$. Let $h = P_1(X)^{m_1} \cdots P_s(X)^{m_s}$ be the decomposition of h into monic irreducible factors over $K = QF(S)$.

Since S is excellent, each $\hat{S}[X]/(P_i(X))$ is generically reduced, i.e. the decomposition

$$P_i(X) = \prod_{j=1}^{j_i} \hat{P}_{i,j}^{m_{i,j}}$$

of $P_i(X)$ into monic irreducible factors has $m_{i,j} = 1$, $1 \leq j \leq j_i$, $1 \leq i \leq s$.

The assumption $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z) = \emptyset$ means that $h = (X - g)^m$ for some $g \in \hat{S}$. Hence $s = 1$ and $m_1 = m$, i.e. $g \in K$. Since S is integrally closed, we have $g \in S$ and the conclusion follows.

Definition 2.4. (Characteristic Polyhedron). For $X' := X - \phi$, $\phi \in \hat{S}$, we will say that the polyhedron $\Delta_{\hat{S}}(h; \{u_j\}_{j \in J}; X')$ is minimal if it has no solvable vertex.

With notations and conventions as in (2.1) and (2.2), we have the following result in the case $J = \{1, \ldots, n\}$ and $\alpha = \mathbf{1}$ (so $\mu_{\mathbf{1}} = \mathrm{ord}_{m_S}$) [35] [14]:

Proposition 2.3. *The rational number $\delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z)$ is independent of the r.s.p. $(u_1, \ldots, u_n)$ and $Z = X - \theta$, $\theta \in \hat{S}$ such that $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal.*

If $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal, the following characterizations hold:

(i) $\delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z) > 0 \Leftrightarrow (\eta^{-1}(m_S) = \{x\}$ and $k(x) = S/m_S)$;

(ii) $\delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z) \geq 1 \Leftrightarrow \eta^{-1}(m_S) \cap \mathrm{Sing}_m \mathcal{X} \neq \emptyset$.

Proof. Consider two systems of coordinates $(Z', u'_1, \ldots, u'_n)$ and $(Z, u_1, \ldots, u_n)$ such that both polyhedra $\Delta_{\hat{S}}(h; u'_1, \ldots, u'_n; Z')$ and $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ are minimal. Suppose that $\delta_{\mathbf{1}}(h; u'_1, \ldots, u'_n; Z') > \delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z)$. Then

$$f_{i,Z'}^{m!} \in m_S^{\frac{m!}{i}\delta_{\mathbf{1}}(h; u'_1, \ldots, u'_n; Z')}$$

for each i, $1 \leq i \leq m$, hence

$$\delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z') \geq \delta_{\mathbf{1}}(h; u'_1, \ldots, u'_n; Z') > \delta_{\mathbf{1}}(h; u_1, \ldots, u_n; Z).$$

This contradicts the assumption $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ minimal. The first assertion follows by symmetry.

Let $\overline{h} \in S/m_S[Z]$ be the reduction of h modulo m_S. Since

$$\eta^{-1}(m_S) = \mathrm{Spec}(S/m_S[Z]/(\overline{h})),$$

(i) and the "only if" part in (ii) are immediate from the definitions. We have

$$\mathrm{ord}_x h(Z) \leq \mathrm{ord}_x \overline{h}(Z) \leq m,$$

hence $x \in \mathrm{Sing}_m \mathcal{X}$ implies $\overline{h}(Z) = (Z - \lambda)^m$ for some $\lambda \in S/m_S$. Since $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal, $\mathbf{0} \in \mathbb{R}^n$ is not a solvable vertex and therefore we have $\lambda = 0$. This proves that (i) holds, the "if" part in (ii) being then obvious.

Definition 2.5. Let $s \in \mathrm{Spec}S$, $(v_1, \ldots, v_{n(s)})$ be a r.s.p. of S_s and $y \in \eta^{-1}(s)$. Let $Z := X - \theta$, $\theta \in \widehat{S}_s$ be such that $\Delta_{\widehat{S}_s}(h; v_1, \ldots, v_{n(s)}; Z)$ is minimal, where $\widehat{S}_s$ denotes the formal completion of S_s w.r.t. its maximal ideal. We let:

$$\delta(y) := \delta_{\mathbf{1}}(h; v_1, \ldots, v_{n(s)}; Z) = \min_{1 \leq i \leq m} \left\{ \frac{\mathrm{ord}_{m_{\widehat{S}_s}} f_{i,Z}}{i} \right\} \in \frac{1}{m!}\mathbb{N}.$$

This invariant is classical and appears in e.g. [9], [10] and [6] definition 4.2 and proposition 4.8 in an equal characteristic context. Our main resolution invariants will be defined in terms of coordinates $(u_1, \ldots, u_n)$ and $Z = X - \theta$, $\theta \in \hat{S}$ such that $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal. Since minimizing polyhedra involves in principle choosing *formal* coordinates, an *algebraic* version will be useful for proving the constructibility of our invariants. The following theorem is fundamental for this purpose. When $\mathrm{char}S/m_S = 0$, the first statement in the theorem easily follows from proposition 2.2 by applying the Tschirnhausen transformation (take $\theta = -\frac{1}{m} f_{1,X}$ below).

We assume from this point on that S is excellent.

Theorem 2.4. *[22] Given $h \in S[X]$ (2.1) and a r.s.p. $(u_1, \ldots, u_n)$ of S, there exists $Z := X - \theta$, $\theta \in S$ such that $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal.*

For any such Z, the following holds: for every subset $J \subseteq \{1, \ldots, n\}$, the polyhedron $\Delta_{\widehat{S_{s^J}}}(h; \{u_j\}_{j\in J}; Z)$ is also minimal and is computed by:

$$\Delta_{\widehat{S_{s^J}}}(h; \{u_j\}_{j\in J}; Z) = \mathrm{pr}^J \Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z), \tag{2.12}$$

where $\mathrm{pr}^J : \mathbb{R}^n \to \mathbb{R}^J$, $\mathbf{x} \mapsto \mathbf{y} = (\{x_j\}_{j\in J})$ *denotes the projection. In particular, we have*

$$\delta(y) = \min\left\{ \frac{1}{i} \sum_{j\in J} a_j,\ \mathbf{a} \in \mathbf{S}^{\{1,\ldots,n\}}(f_{i,Z}),\ 1 \le i \le m \right\},\ y \in \eta^{-1}(s^J).$$

Proof. The theorem is trivial if $\mathbf{0} \in \mathbb{R}^n$ is a nonsolvable vertex of the polyhedron $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$, taking $Z := X$. Otherwise it can be assumed that $f_{i,X} \in m_S$, $1 \le i \le m$. Apply [22] theorem II.3 to

$$R := S[X]_{(m_S, X)},\ f := h(X),\ y := X.$$

Since h is monic, it follows from the proof that one may take

$$z = y - \sum_{\mathbf{a}\in\Sigma} \gamma_{\mathbf{a}} u^{\mathbf{a}},\ \gamma_{\mathbf{a}} \in S \text{ a unit},\ \Sigma \text{ finite}.$$

Formula (2.12) follows from (2.6) (2.7) (2.8). Suppose that $\mathbf{y} \in \mathbb{N}^J$ is a solvable vertex of $\Delta_{\widehat{S_{s^J}}}(h; \{u_j\}_{j\in J}; Z)$ defined by some $\alpha \in \mathbb{R}^J_{>0}$. By definition,

$$\exists \overline{\lambda} \in QF(\overline{S}^J) : \mathrm{in}_{\mathbf{y}} h = (Z - \overline{\lambda} U^{\mathbf{y}})^m. \tag{2.13}$$

By (2.9), we have $\overline{\lambda}^m = (-1)^m U^{-m\mathbf{y}} F_{m,Z,\alpha} \in \overline{S}^J$. Hence $\overline{\lambda} \in \overline{S}^J$, since the regular ring $\overline{S}^J$ is integrally closed. By (2.12), there exists a *vertex* $\mathbf{x} \in \Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ such that $\mathbf{y} = \mathrm{pr}^J(\mathbf{x})$. Lifting up, there exists $\beta \in \mathbb{R}^n_{>0}$, $\alpha = \mathrm{pr}^J(\beta)$, defining $\mathbf{x}$ and we let $\alpha' := \mathrm{pr}^{J'}(\beta)$. There is an induced valuation $\mu_{\alpha'}$ on $\overline{S}^J$. The initial form of $\overline{\lambda}$ in $\mathrm{gr}_{\alpha'} \overline{S}^J$ has the form

$$\Lambda = \lambda \prod_{j'\in J'} \overline{U}_{j'}^{x_{j'}},\ \lambda \in S/m_S,\ \lambda \neq 0,\ \{x_{j'}\}_{j'\in J'} \in \mathbb{N}^{J'}.$$

Collecting together (2.9) and (2.13), we get $\mathrm{in}_{\mathbf{x}} h = (Z - \lambda U^{\mathbf{x}})^m$, i.e. $\mathbf{x}$ is a solvable vertex: a contradiction. Therefore $\Delta_{\widehat{S_{s^J}}}(h; \{u_j\}_{j\in J}; Z)$ has no solvable vertex, hence is minimal by the second statement in proposition 2.2.

The last statement is a rewriting of definition 2.5.

Let $S \subseteq \tilde{S}$ be a *local* base change which is *regular*, i.e. flat with geometrically regular fibers [29] definition 6.8.1(iv). In particular $\tilde{S}$ is regular [29] proposition 6.5.1(ii) and faithfully flat. The ring $\tilde{S}$ is not excellent in general, but this certainly holds in the following cases:

(i) $\tilde{S} = \hat{S}$ [29] 7.8.3(iii);

(ii) $\tilde{S}$ is ind-étale over S [40] theorem I.8.1(iv), or

(iii) $\tilde{S}$ is essentially of finite type over S, i.e. smooth over S [29] proposition 7.8.6(i).

An important special case of (ii) is when $\tilde{S}$ is the Henselization or strict Henselization of S. When regular base changes are concerned, we always assume that $\tilde{S}$ is excellent. These conditions are preserved by localizing, i.e. replacing $S \subseteq \tilde{S}$ by $S_s \subseteq \tilde{S}_{\tilde{s}}$, $\tilde{s} \in \mathrm{Spec}\tilde{S}$ and $s \in \mathrm{Spec}S$ its image.

Notation 2.1. Let $S \subseteq \tilde{S}$ be a local base change which is regular, $\tilde{S}$ excellent, $\tilde{s} \in \mathrm{Spec}\tilde{S}$ with image $m_S \in \mathrm{Spec}S$. Any r.s.p. $(u_1, \ldots, u_n)$ of S can be extended to a r.s.p. $(u_1, \ldots, u_{\tilde{n}})$ of $\tilde{S}$. We let $\tilde{h} \in \tilde{S}[X]$ be the image of h and

$$\tilde{\eta}: \ \tilde{\mathcal{X}} = \mathcal{X} \times_S \mathrm{Spec}\tilde{S} \to \mathrm{Spec}\tilde{S}.$$

It follows from definition 2.3 that, if $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ is a nonsolvable vertex of $\Delta_S(h; u_1, \ldots, u_n; Z)$, the vertex

$$(\mathbf{x}, \underbrace{0, \ldots, 0}_{\tilde{n}-n}) \in \Delta_{\tilde{S}}(h; u_1, \ldots, u_{\tilde{n}}; Z) \subseteq \mathbb{R}^{\tilde{n}}_{\geq 0}$$

is nonsolvable provided that $S/m_S \subseteq \tilde{S}/m_{\tilde{S}}$ is inseparably closed. This is of course always satisfied when S/m_S is perfect (e.g. $\mathrm{char}S/m_S = 0$). An obvious consequence of the second statement in proposition 2.2 is:

Proposition 2.5. *Let $S \subseteq \tilde{S}$ be a local base change which is regular, $\tilde{S}$ excellent. Assume that $S/m_S \subseteq \tilde{S}/m_{\tilde{S}}$ is inseparably closed. Let $Z = X - \theta$, $\theta \in S$, be such that $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal. Then*

$$\Delta_{\hat{\tilde{S}}}(h; u_1, \ldots, u_{\tilde{n}}; Z) = \Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z) \times \mathbb{R}^{\tilde{n}-n}_{\geq 0} \subseteq \mathbb{R}^{\tilde{n}}_{\geq 0}$$

and this polyhedron is minimal.

Note that the assumptions of the proposition are satisfied in the above situation (ii): $\tilde{S}$ is ind-étale over S. In situation (iii), i.e. $\tilde{S}$ smooth over S, the following example will make the situation clear:

Example 2.1. Let k be a (nonperfect) field of characteristic $p > 0$ and

$$S := k[u_1]_{(u_1)},\ h := X^p - \lambda u_1^p \in S[X],\ \lambda \in k \backslash k^p.$$

Then $\Delta_{\hat{S}}(h; u_1; X) = [1, +\infty[$ and is minimal. Take $\tilde{S} = S[t]_{(u_1, P(t))}$, where P is a monic polynomial with irreducible residue $\overline{P}(t) \in k[t]$ (resp. $P = 0$). Let $u_2 := P(t)$, so (u_1, u_2) (resp. (u_1)) is a r.s.p. of $\tilde{S}$. Let

$$k(P) := \tilde{S}/m_{\tilde{S}} = k[t]/(\overline{P}(t)) \text{ (resp. } k(0) = \tilde{S}/m_{\tilde{S}} = k(t))$$

be the residue field of $\tilde{S}$. Setting $\{\tilde{x}\} = \tilde{\eta}^{-1}(m_{\tilde{S}})$, we have

$$\begin{cases} \delta(\tilde{x}) = 1 & \text{if} \quad \lambda \notin k(P)^p; \\ \delta(\tilde{x}) = 1 + \frac{1}{p} & \text{if} \quad \lambda \in k(P)^p. \end{cases}$$

Proof: obvious if $\lambda \notin k(P)^p$; if $\lambda \in k(P)^p$, take $Z := X - Q(t)u_1$, $Q(t) \in k[t]$ monic, $\deg Q < \deg P$ and $\overline{Q}(\lambda^{1/p}) = 0$. We have:

$$\Delta_{\tilde{S}}(\tilde{h}; u_1, \tilde{v}; Z) = (1, \frac{1}{p}) + \mathbb{R}^2_{\geq 0}$$

with $\tilde{v} := Q(t)^p - \lambda$. Note that $(u_1, \tilde{v})$ is a r.s.p. of $\tilde{S}$.

In particular, the function on $\mathbb{A}^1_k = \{x\} \times \mathbb{A}^1_k \subset \mathcal{X} \times_k \mathbb{A}^1_k$, $\tilde{x} \mapsto \delta(\tilde{x})$ is not a constructible function.

Theorem 2.4 and proposition 2.5 suggest the following question. An affirmative answer would be very useful in order to build geometrical invariants from characteristic polyhedra. Proposition 2.5 answers in the affirmative when S/m_S is perfect, with $\tilde{S} := S$.

Question 2.1. Let S be an excellent regular local ring with r.s.p. $(u_1, \ldots, u_n)$ and $h \in S[X]$ (2.1). Does there exist a smooth local base change $S \subseteq \tilde{S}$, a r.s.p. $(u_1, \ldots, u_{\tilde{n}})$ of $\tilde{S}$ extending $(u_1, \ldots, u_n)$ and $Z = X - \phi$, $\phi \in \tilde{S}$, such that the following holds:

"for every smooth local base change $\tilde{S} \subseteq S'$ and r.s.p. $(u_1, \ldots, u_{n'})$ of S' extending $(u_1, \ldots, u_{\tilde{n}})$, the polyhedron $\Delta_{\hat{S'}}(h; u_1, \ldots, u_{n'}; Z)$ is minimal"?

Uncovering transformation rules for the characteristic polyhedron under blowing up is a major problem, *vid.* [35] p.254. A good behaviour is known in the special case of a blowing up along a Hironaka permissible subscheme and an exceptional point at the origin of some standard chart.

Proposition 2.6. *With notations as before, let $J \subseteq \{1, \ldots, n\}$, $y \in \eta^{-1}(s^J)$ and assume that $\delta(y) \geq 1$. Fix $j_0 \in J$ and let $S' := S[\{u'_j\}_{j\in J}]_{(u'_1,\ldots,u'_n)}$, where*

$$\begin{cases} u'_j & := & u_j/u_{j_0} & \text{if} & j \in J\backslash\{j_0\}; \\ u'_j & := & u_j & \text{if} & j \in J' \cup \{j_0\}. \end{cases}$$

Let $Z = X - \theta$, $\theta \in S$, with $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ minimal and define:

$$h'(Z') := u_{j_0}^{-m} h(Z) = Z'^m + u_{j_0}^{-1} f_{1,Z} Z'^{m-1} + \cdots + u_{j_0}^{-m} f_{m,Z} \in S'[Z'], \quad (2.14)$$

where $Z' := Z/u_{j_0}$. Then $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ is minimal and the map $l : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ given by

$$\mathbf{x} = (x_1, \ldots, x_n) \mapsto \mathbf{x}' = (x_1, \ldots, x_{j_0-1}, \sum_{j\in J} x_j - 1, x_{j_0+1}, \ldots, x_n) \quad (2.15)$$

gives a one-to-one correspondence between vertices of $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ and vertices of $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$.

Proof. The assumption $\delta(y) \geq 1$ forces $f_{i,Z} \in I_J^i$ by the last statement in theorem 2.4. Therefore (2.14) makes sense, i.e. $h'(Z') \in S'[Z']$. It is obvious from definition 2.1 that

$$l(\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)) = \Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$$

and that l induces a bijection between vertices of these polyhedra.

Let $\mathbf{x}' = l(\mathbf{x})$ be a vertex of $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$. Denote

$$\mathrm{in}_{\mathbf{x}} h = Z^m + \lambda_1 U^{\mathbf{x}} Z^{m-1} + \cdots + \lambda_m U^{m\mathbf{x}}, \ \lambda_1, \ldots, \lambda_m \in S/m_S,$$

with the convention as before that $\lambda_i = 0$ if $i\mathbf{x} \notin \mathbb{N}^n$, $1 \leq i \leq m$. Applying l (2.15), we get

$$\mathrm{in}_{\mathbf{x}'} h = Z'^m + \lambda_1 U'^{\mathbf{x}'} Z'^{m-1} + \cdots + \lambda_m U'^{m\mathbf{x}'}.$$

Since $S'/m_{S'} = S/m_S$, definition 2.3 then shows that $\mathbf{x}'$ is solvable if and only if $\mathbf{x}'$ is solvable. Since $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal, the polyhedron $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ is also minimal by proposition 2.2.

2.2 Normal crossings divisors.

We now introduce a normal crossings divisor $E \subseteq \mathrm{Spec}S$. This section fixes the terminology and notations for blowing ups and base changes with respect to E, then introduces the Hironaka ϵ function on $\mathcal{X}$.

Definition 2.6. A r.s.p. $(u_1, \ldots, u_n)$ of S is said to be adapted to E if $E = \mathrm{div}(u_1 \cdots u_e)$ for some e, $0 \leq e \leq n$.

We emphasize that we allow $e = 0$, i.e. $E = \emptyset$ in this definition. In this context, we use the following notion of Hironaka permissible center:

Definition 2.7. Let $\mathcal{Y} \subset \mathcal{X}$ be an integral closed subscheme with generic point y. We say that $\mathcal{Y}$ is Hironaka-permissible (resp. Hironaka-permissible with respect to E) at $x \in \mathcal{Y}$ if condition (i) (resp. condition (ii)) below is satisfied:

(i) $m(y) = m(x)$ and $\mathcal{Y}$ regular at x;

(ii) $\mathcal{Y} \subseteq \mathrm{Sing}_m\mathcal{X}$ and $W := \eta(\mathcal{Y})$ has normal crossings with E at $s := \eta(x)$.

We remind the reader that an integral closed subscheme $W \subseteq \mathrm{Spec}S$ has normal crossings with $E = \mathrm{div}(u_1 \cdots u_e)$ if the family $(u_1, \ldots, u_e)$ can be extended to a r.s.p. $(u_1, \ldots, u_n)$ of S such that the ideal $I(W)$ of W is of the form $I_J = (\{u_j\}_{j \in J}) \subseteq S$, for some $J \subseteq \{1, \ldots, n\}$.

Note that a Hironaka-permissible center w.r.t. any E (e.g. $E = \emptyset$) is Hironaka-permissible: since $\mathcal{Y} \subseteq \mathrm{Sing}_m\mathcal{X}$, we have $m(y) = m(x) = m$ and $y \in \eta^{-1}(w) \cap \mathrm{Sing}_m\mathcal{X}$, where w is the generic point of W; by proposition 2.3 applied to S_w, the map $\mathcal{Y} \to W$ is birational, hence an isomorphism since W is regular.

Since the notion is local on $\mathcal{X}$, a Hironaka-permissible blowing up (w.r.t. E) is simply the blowing up along a center $\mathcal{Y} \subset \mathcal{X}$ which is Hironaka-permissible (w.r.t. E) at each point of its support. By a *local* Hironaka-permissible blowing up, we simply mean the localization at some point of the exceptional divisor $\pi^{-1}(\mathcal{Y})$ of the blowing up π along a Hironaka-permissible center. The important fact is that Hironaka-permissible blowing ups w.r.t. E preserve our structure:

Proposition 2.7. *Let S, $h \in S[X]$ (2.1), $\mathcal{X}$ and $E = \mathrm{div}(u_1 \cdots u_e)$ be as above. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a Hironaka-permissible blowing up w.r.t. E at*

$x \in \mathcal{X}$. *There exists a commutative diagram*

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\pi} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}S & \xleftarrow{\sigma} & \mathcal{S}' \end{array} \tag{2.16}$$

where $\sigma : \mathcal{S}' \to \mathrm{Spec}S$ *is the blowing up along* W.

For every $s' \in \sigma^{-1}(s)$, $S' := \mathcal{O}_{\mathcal{S}',s'}$, *there exists* $h' \in S'[X']$ *unitary of degree* m *such that* $\mathcal{X}'_{s'} = \mathrm{Spec}(S'[X']/(h'))$.

Furthermore, there exists a r.s.p. $(u'_1, \ldots, u'_n)$ *of* S' *adapted to the stalk* $E'_{s'}$, $E' := \sigma^{-1}(E \cup W)_{\mathrm{red}}$.

Proof. By the above remarks, there exists $J \subseteq \{1, \ldots, n\}$ such that $I(W) = I_J = (\{u_j\}_{j\in J})$. By theorem 2.4, there exists $Z := X - \theta$, $\theta \in S$, such that $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal. Since $x, y \in \mathrm{Sing}_m\mathcal{X}$, we have

$$\eta^{-1}(s) = \{x\},\ \eta^{-1}(W) = \mathcal{Y} \text{ and } \delta(x) \geq 1,\ \delta(y) \geq 1$$

by proposition 2.3. In particular, the ideal of $\mathcal{Y}$ at x is

$$I(\mathcal{Y}) = (Z, \{u_j\}_{j\in J}).$$

Since $\delta(y) \geq 1$, the point at infinity $(1 : 0 : \cdots : 0)$ does not belong to $\mathcal{X}'$ so $(\{u_j\}_{j\in J})\mathcal{O}_{\mathcal{X}'}$ is invertible. By the universal property of blowing up, there is a commutative diagram (2.16).

Let $s' \in \sigma^{-1}(s)$ and $j_0 \in J$ be such that u_{j_0} is a local equation of $\pi_0^{-1}(W)$. We take $X' := Z/u_{j_0}$ and

$$h' := u_{j_0}^{-m} h(Z) = X'^m + u_{j_0}^{-1} f_{1,Z} X'^{m-1} + \cdots + u_{j_0}^{-m} f_{m,Z}. \tag{2.17}$$

Note that $h' \in S'[X']$ follows from the last statement in theorem 2.4. The last statement is obvious because $E' = \sigma^{-1}(E \cup W)_{\mathrm{red}}$ is a normal crossings divisor on $\mathcal{S}'$.

We will stick to these notations when local Hironaka-permissible blowing ups are concerned, or compositions of such local blowing ups. We always refer to the reduced total transform of E on the blown up base Spec$S$.

Suppose a base change is given as considered in the previous section, i.e. formal completion $S \subseteq \hat{S}$, localization at a prime $S \subseteq S_s$ or regular local base change $S \subseteq \tilde{S}$, $\tilde{S}$ excellent.

Notation 2.2. Given $S \subseteq S'$ such a base change, we denote

$$E' := E \times_S \mathrm{Spec}S',\ \eta' :\ \mathcal{X}' = \mathcal{X} \times_S \mathrm{Spec}S' \to \mathrm{Spec}S'.$$

The image of h in $S'[X]$ is denoted $h' \in S'[X]$. This notation is used consistently with notation 2.1.

For instance if $s \in \mathrm{Spec}S$, there exists a r.s.p. $(v_1, \ldots, v_{n(s)})$ of S_s which is adapted to E_s, where E_s is the stalk of E at s. We then have $E_s = \mathrm{div}(v_1 \cdots v_{e(s)})$ and may choose $v_j = u_{\varphi(j)}$ for some injective map $\varphi :\ \{1, \ldots, e(s)\} \to \{1, \ldots, e\}$. It is of course not possible in general to extend a given $(v_1, \ldots, v_{n(s)})$ to a r.s.p. $(u_1, \ldots, u_n)$ of S. We let $h_s \in S_s[X]$ be the image of h.

Definition 2.8. Let $s \in \mathrm{Spec}S$ and $(v_1, \ldots, v_{n(s)})$ be an r.s.p. of S_s which is adapted to E_s, $E_s = \mathrm{div}(v_1 \cdots v_{e(s)})$. We say that coordinates

$$(v_1, \ldots, v_{n(s)}; Z_s),\ Z_s := X - \phi_s,\ \phi_s \in S_s,$$

are well adapted at $y \in \eta^{-1}(s)$ if $\Delta_{\widehat{S_s}}(h; v_1, \ldots, v_{n(s)}; Z_s)$ is minimal.

Definition 2.9. Let $(u_1, \ldots, u_n)$ be a r.s.p. of S which is adapted to E. Let j, $1 \leq j \leq e$, and let $\mathcal{Y}_j \subset \mathcal{X}$ be an irreducible component of $\eta^{-1}(\mathrm{div}(u_j))$ with generic point $y_j \in \mathcal{X}$. We let

$$d_j := \delta(y_j) \in \frac{1}{m!}\mathbb{N}.$$

For any $s \in \mathrm{Spec}S$ and $y \in \eta^{-1}(s)$, we let

$$\epsilon(y) := m\left(\delta(y) - \sum_{\mathrm{div}(u_j) \subseteq E_s} d_j\right) \in \frac{1}{(m-1)!}\mathbb{Z}.$$

Summing up results from the previous section, we have:

Proposition 2.8. *Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$. With notations as above, we have*

$$d_j = \min\left\{\frac{a_j}{i},\ \mathbf{a} \in \mathbf{S}^{\{1,\ldots,n\}}(f_{i,Z}),\ 1 \leq i \leq m\right\},\ 1 \leq j \leq e.$$

For $s \in \mathrm{Spec}S$ and $y \in \eta^{-1}(s)$, we have $\epsilon(y) \geq 0$.

Proof. The first (resp. second) statement follows from the last one in theorem 2.4 applied to S and $J := \{j\}$ (resp. to S_s and each $J := \{j\}$ with $\mathrm{div}(u_j) \subseteq E_s$).

2.3 The Galois or purely inseparable assumption.

In this section, we introduce the assumptions of theorem 1.3. The main result is proposition 2.11 which analyzes the consequence w.r.t. the slopes $\delta_\alpha(h; u_1, \ldots, u_n; Z)$ and initial form polynomials $\mathrm{in}_\alpha h$ from definition 2.2. We assume furthermore that the following property holds:

(G) $m = p$ is a prime number, h is reduced, the ring extension $L|K$ is normal and $\mathcal{X}$ is G-invariant, where $G := \mathrm{Aut}_K(L)$.

*Assumption **(G)** is maintained up to the end of this chapter.*

Since $[L : K] = p$ is a prime number, we have either $G = \mathbb{Z}/p$ ($L|K$ separable, cases (a) and (b) below) or $G = (1)$ ($L|K$ inseparable, case (c) below). Case (a) is included here for the sake of completeness and because residual actions in case (b) may lead to case (a). The three cases to be considered are:

(a) h is totally split (product of p pairwise distinct linear factors) over K;

(b) h is irreducible and Galois over K with group $G = \mathbb{Z}/p$;

(c) h is irreducible, $\mathrm{char}S = p$, $f_{i,X} = 0$, $1 \leq i \leq p-1$.

Assumption **(G)** is also preserved by those base changes considered in the previous sections, i.e. formal completion $S \subseteq \hat{S}$, localization at a prime $S \subseteq S_s$ or regular local base change $S \subseteq \tilde{S}$, $\tilde{S}$ excellent. Note that in any case, h reduced implies respectively h_s, $\hat{h}$ (since S is excellent) and $\tilde{h}$ reduced (notation 2.2). Recall notations and definitions of initial forms from definition 2.2.

Proposition and Definition 2.9. *Assume that* $\mathrm{char}S/m_S = p$. *Let* $(u_1, \ldots, u_n)$ *be a given r.s.p. of* S *and* $\alpha \in \mathbb{R}^n_{>0}$ *be a weight vector. The integer*

$$i_0(\alpha) := \min\{i \in \{1, \ldots p\} : F_{i,Z,\alpha} \neq 0\}$$

does not depend on $Z = X - \theta$, $\theta \in \hat{S}$ *such that* $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ *is minimal. If* $i_0(\alpha) < p$, *the form* $F_{i_0(\alpha),Z,\alpha}$ *is also independent of the choice of* $Z = X - \theta$ *as above.*

In case $\alpha = \mathbf{1}$, *the integer* $i_0(\mathbf{1})$ *(also denoted by* $i_0(x)$ *for* $x \in \eta^{-1}(m_S)$*) and form* $F_{i_0(\mathbf{1}),Z} = F_{i_0(\mathbf{1}),Z,\mathbf{1}}$ *(if* $i_0(\mathbf{1}) < p$*) are also independent of the*

choice of the r.s.p. $(u_1, \ldots, u_n)$ *of* S *and* $Z = X - \theta$, $\theta \in \hat{S}$ *such that* $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ *is minimal.*

Proof. Take $Z' = Z - \phi$ such that both polyhedra $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ and $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z')$ are minimal. By minimality, we have

$$\mu_\alpha(\phi) \geq a := \delta_\alpha(h; u_1, \ldots, u_n; Z).$$

The corresponding initial forms $\mathrm{in}_\alpha h(Z) \in (\mathrm{gr}_\alpha S)[Z]$ and $\mathrm{in}_\alpha h(Z') \in (\mathrm{gr}_\alpha S)[Z']$ are related by

$$\mathrm{in}_\alpha h(Z') = \mathrm{in}_\alpha h(Z - \mathrm{cl}_{\alpha,a}\phi).$$

The first statement follows from the elementary fact that $\mu_\alpha \begin{pmatrix} p \\ i \end{pmatrix} > 0$ for $1 \leq i \leq p-1$, since $p \in m_S$. The second statement then follows from proposition 2.3.

Proposition 2.10. *For* $x \in \mathrm{Sing}\mathcal{X}$, $s := \eta(x)$, *we have:*

$$\eta^{-1}(s) = \{x\},\ k(x) = k(s) \text{ and } \delta(x) > 0. \tag{2.18}$$

Assume that a normal crossings divisor $E = \mathrm{div}(u_1 \cdots u_e) \subset \mathrm{Spec}S$ *is specified and let* $\pi : \mathcal{X}' \to \mathcal{X}$ *be a Hironaka-permissible blowing up w.r.t.* E *at* x. *Then, with notations as in proposition 2.7, for every* $s' \in \sigma^{-1}(s)$, $\mathcal{X}'_{s'}$ *satisfies again* ***(G)****.*

Proof. It can be assumed that $s = m_S$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x and $\overline{h}(Z) \in S/m_S[Z]$ be the reduction of h modulo m_S. By **(G)**, G acts transitively on the fiber $\eta^{-1}(s)$. Then $\overline{h}(Z)$ is either a p^{th}-power or satisfies again **(G)** w.r.t. the zero-dimensional regular local ring S/m_S.

If $\overline{h}(Z)$ satisfies **(G)**, then $(h(Z), u_1, \ldots, u_n)$ is a r.s.p. of the local ring $S[Z]_{m_x}$, so x is a regular point of $\mathcal{X}$.

Assume now that $\overline{h}(Z) = (Z - \overline{\lambda})^p$ for some $\overline{\lambda} \in S/m_S$. Now $(0, \ldots, 0)$ is a solvable vertex of $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ unless $\overline{\lambda} = 0$. Since $(u_1, \ldots, u_n; Z)$ are well adapted coordinates at x, we have $\overline{\lambda} = 0$.

The last statement follows from proposition 2.7 and the fact that x is G-invariant by (2.18).

Proposition 2.11. *Let* $x \in \eta^{-1}(m_S)$ *and* $(u_1, \ldots, u_n; Z)$ *be well adapted coordinates at* x. *For* $\alpha \in \mathbb{R}^n_{>0}$ *a weight vector, the following holds:*

(i) the polynomial $\mathrm{in}_\alpha h \in (\mathrm{gr}_\alpha S)[Z]$ *satisfies again* ***(G)*** *w.r.t. the local ring* $(\mathrm{gr}_\alpha S)_{(U_1,\ldots,U_n)}$*;*

(ii) if ($\mathrm{char}S/m_S = p$ *and* $i_0(\alpha) < p$*), then*

$$\delta_\alpha(h; u_1, \ldots, u_n; Z) \in \Gamma_\alpha = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_n;$$

(iii) if $\mathrm{char}S/m_S = 0$ *or if (*$\mathrm{char}S/m_S = p$ *and* $i_0(\alpha) = p$*), then*

$$\delta_\alpha(h; u_1, \ldots, u_n; Z) \in \frac{1}{p}\Gamma_\alpha.$$

Proof. If $\delta(x) = 0$, we have $\delta_\alpha(h; u_1, \ldots, u_n; Z) = 0$ and $\mathrm{in}_\alpha h = \overline{h}(Z)$ with notations as in the previous proof, so the proposition is trivial. Assume that $\delta(x) > 0$.

By proposition 2.2, we have $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z) \neq \emptyset$ and this polyhedron has no solvable vertex. Therefore $\mathrm{in}_\alpha h$ is not a p^{th}-power. Let $z \in L$ be the image of Z and ν_α be any extension of μ_α to L. Then ν_α is centered at x, since $\mathcal{X}$ is G-invariant and $\eta^{-1}(m_S) = \{x\}$ by proposition 2.3(i). We have:

$$\nu_\alpha(z) = \mu_\alpha(f_{i,Z})/i = \delta_\alpha(h; u_1, \ldots, u_n; Z) \in \Gamma_\alpha \otimes_{\mathbb{Z}} \mathbb{Q} \tag{2.19}$$

for each i, $1 \leq i \leq p$ such that $F_{i,Z,\alpha} \neq 0$. Since $L|K$ is normal of degree p, the reduced ramification index e_0 of $\nu_\alpha|\mu_\alpha$ is $e_0 = 1$ or $e_0 = p$.

Assume that ($\mathrm{char}S/m_S = p$ and $i_0(\alpha) = p$). Then $\mathrm{in}_\alpha h$ is in case (c) of **(G)** and we get (iii) from (2.19).

Assume that $\mathrm{char}S/m_S = 0$ or ($\mathrm{char}S/m_S = p$ and $i_0(\alpha) < p$). Then h is in case (a) or (b). Since $G = \mathbb{Z}/p$ in these cases and $\mathcal{X}$ is G-invariant, G acts transitively on the roots of $\mathrm{in}_\alpha h$. We have:

$$\begin{cases} \mathrm{Tot}((\mathrm{gr}_\alpha S)[Z]/(\mathrm{in}_\alpha h)) & = & \prod_{\nu_\alpha} QF(\mathrm{gr}_\alpha S) & \text{if } \mu_\alpha \text{ splits;} \\ QF((\mathrm{gr}_\alpha S)[Z]/(\mathrm{in}_\alpha h)) & = & QF(\mathrm{gr}_{\nu_\alpha} S) & \text{otherwise,} \end{cases}$$

and this proves (i). Statement (iii) follows from (2.19) if $\mathrm{char}S/m_S = 0$.

Assume finally that ($\mathrm{char}S/m_S = p$ and $i_0(\alpha) < p$). By (2.19), we have

$$p\nu_\alpha(z) = p\mu_\alpha(f_{i_0(\alpha),Z})/i_0(\alpha) \in \Gamma_\alpha.$$

Since $\Gamma_\alpha \simeq \mathbb{Z}^r$ for some $r \geq 1$, this implies

$$\delta_\alpha(h; u_1, \ldots, u_n; Z) = \mu_\alpha(f_{i,Z})/i_0(\alpha) \in \Gamma_\alpha$$

which completes the proof of (ii).

Corollary 2.12. *Assume that a normal crossings divisor*

$$E = \mathrm{div}(u_1 \cdots u_e) \subset \mathrm{Spec}S$$

is specified. We have $pd_j \in \mathbb{N}$, $1 \leq j \leq e$, *and* $\epsilon(y) \in \mathbb{N}$ *for every* $y \in \mathcal{X}$.

Proof. In view of definition 2.9 and proposition 2.8, this follows from proposition 2.11 (ii)(iii) applied to the local rings $S_{(u_j)}$ and S_s, $s := \eta(y)$.

This corollary allows us to define the following invariant:

Definition 2.10. Let $(u_1, \ldots, u_n)$ be a r.s.p. of S which is adapted to the normal crossings divisor $E = \mathrm{div}(u_1 \cdots u_e)$. For $y \in \mathcal{X}$, $s := \eta(y)$, we define a principal ideal:

$$H(y) := \left(\prod_{\mathrm{div}(u_j) \subseteq E_s} u_j^{H_j} \right) \subseteq S,$$

where $H_j := pd_j \in \mathbb{N}$.

2.4 The discriminant assumption.

We now introduce now the critical locus of the map $\eta : \mathcal{X} \to \mathrm{Spec}S$ together with its scheme structure given by the discriminant $D := \mathrm{Disc}_X h \in S$. We are interested in the case where D is a normal crossings divisor. Theorem 2.14 below is basically a reduction to characteristic $p > 0$ as dealt with in [20] [21].

Note that D is by definition independent of the choice of regular parameters of S and invariant by those translations $X' := X - \phi$, $\phi \in \hat{S}$ used in minimizing polyhedra. If (S, h, E) is in case (c) of **(G)**, then $D = 0$. We are interested in the case where D is a normal crossings divisor.

Definition 2.11. Let S, $h \in S[X]$ (2.1), $\mathcal{X}$ and $E = \mathrm{div}(u_1 \cdots u_e)$ be specified. We say that (S, h, E) satisfies assumption **(E)** if $\mathrm{char}(S/m_S) = p > 0$ and one of the following properties hold:

$$\begin{cases} (i) \quad D = 0 \quad \text{and} \quad \eta(\mathrm{Sing}_p\mathcal{X}) \subseteq E, \\ \\ (ii) \quad D \neq 0 \quad \text{and} \quad \mathrm{div}(D)_{\mathrm{red}} \subseteq E \subseteq \mathrm{div}(p)_{\mathrm{red}}. \end{cases} \tag{2.20}$$

*Assumption **(E)** is maintained up to the end of this chapter.*

This assumption implies that $\mathrm{Sing}_p\mathcal{X} \subseteq \eta^{-1}(E) \subset \mathcal{X}$, by definition (i) or because $\eta^{-1}(\mathrm{Spec}S\backslash E)$ is regular since $\mathrm{Spec}S\backslash E$ is (ii). In particular $E \neq \emptyset$ if $\mathrm{Sing}_p\mathcal{X} \neq \emptyset$.

Assumption **(E)** is also preserved by those base changes considered in the previous section: formal completion $S \subseteq \hat{S}$, localization at a prime $S \subseteq S_s$ or regular local base change $S \subseteq \tilde{S}$, $\tilde{S}$ excellent. For Hironaka-permissible blowing ups, we have:

Proposition 2.13. *Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a Hironaka-permissible blowing up w.r.t. E at $x \in \mathcal{X}$. Then, with notations as in proposition 2.7, for every $s' \in \sigma^{-1}(s)$, (S', h', E') satisfies again **(E)**.*

Proof. Any Hironaka-permissible center $\mathcal{Y} \subset \mathcal{X}$ w.r.t. E at x is contained in E by the above remarks. Therefore the proposition is obvious in case (i) of definition 2.11.

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x and $h(Z) \in S[Z]$ be the corresponding expansion. With notations as in proposition 2.7 and (2.17), we have $h'(X') = u_{j_0}^{-p} h(X' u_{j_0})$ for some $u_{j_0} \in I(W)$. We deduce that

$$D' := \mathrm{Disc}_{X'} h' = u_{j_0}^{-p(p-1)} \mathrm{Disc}_Z h = u_{j_0}^{-p(p-1)} D,$$

hence $\mathrm{div}(D')_{\mathrm{red}} \subseteq E' \subseteq \mathrm{div}(p)_{\mathrm{red}}$ as required.

Theorem 2.14. *(Reduction to characteristic p). With assumptions as above, let $x \in \eta^{-1}(m_S)$ be such that $\epsilon(x) > 0$. Then $(\mathcal{X}, x)$ is analytically irreducible.*

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x and $\alpha \in \mathbb{R}^n_{>0}$ be a weight vector. Exactly one of the following properties holds.

(1) $i_0(\alpha) = p$, i.e. $\mathrm{in}_\alpha h = Z^p + F_{p,Z,\alpha}$;

(2) $i_0(\alpha) = p-1$ *i.e.* $\mathrm{in}_\alpha h = Z^p + F_{p-1,Z,\alpha}Z + F_{p,Z,\alpha}$, $F_{p-1,Z,\alpha} \neq 0$. *Furthermore, we have*

$$-f_{p-1,Z} = \gamma_{p-1,Z}\prod_{j=1}^{e} u_j^{A_{p-1,j}} \tag{2.21}$$

with $A_{p-1,j} \in (p-1)\mathbb{N}$, $1 \leq j \leq e$, *and* $\gamma_{p-1,Z} \in S$ *a unit with residue* $\overline{\gamma}_{p-1,Z} \in (S/m_S)^{p-1}$. *In particular,* $-F_{p-1,Z,\alpha} = G^{p-1}$ *for some nonzero* $G \in \mathrm{gr}_\alpha S$, *and we have*

$$\mathrm{cl}_{p(p-1)\delta_\alpha(h;u_1,\ldots,u_n;Z)}(\mathrm{Disc}_Z(h)) =< F_{p-1,Z,\alpha}^p > .$$

Proof. First note that $D = \mathrm{Disc}_Z(h)$ is homogeneous of degree $p(p-1)$ for the grading $\deg f_{i,Z} = i$ on the coefficients of h. In particular, we have

$$\mu_\alpha(D) \geq p(p-1)\delta_\alpha(h; u_1, \ldots, u_n; Z),$$

since $\mu_\alpha(f_{i,Z})/i \geq \delta_\alpha(h; u_1, \ldots, u_n; Z)$ for $1 \leq i \leq p$. We deduce the formula

$$\mathrm{cl}_{\alpha,p(p-1)\delta_\alpha(h;u_1,\ldots,u_n;Z)} D = \mathrm{Disc}_Z(\mathrm{in}_\alpha h). \tag{2.22}$$

On the other hand, $\mathrm{in}_\alpha h$ has a multiple root over an algebraic closure of $QF(\mathrm{gr}_\alpha S)$ if and only if $i_0(\alpha) = p$ by proposition 2.11 (i). When this holds, we are in case (1) of this theorem.

Suppose that h is analytically reducible. By proposition 2.8 and definition 2.5, $\epsilon(x) = \delta(x) - \sum_{i=1}^{e} d_j$ is determined by $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$, thus invariant by base change $S \subseteq \hat{S}$. Therefore it can be assumed w.l.o.g. that $S = \hat{S}$ in order to prove the first statement, i.e. that h is in case (a) of property **(G)**. Since h splits, there is a factorization

$$h = \prod_{i=1}^{p}(Z - \varphi_j) \in S[Z],\ \varphi_1, \ldots, \varphi_p \in S.$$

Let $z \in \mathcal{O}_\mathcal{X}$ be the image of Z and $g \in G = \mathbb{Z}/p$, $g \neq 0$. By property **(G)**, we have $g(z) \in \mathcal{O}_\mathcal{X}$ and $g(z)$ is a root of $h(Z)$. Up to reindexing, it can therefore be assumed that

$$g^i(z) = z - \varphi_{i+1} + \varphi_1 \in S,\ 1 \leq i \leq p-1.$$

In particular, we have $g(z) - z = \varphi_1 - \varphi_2 \in S$ and we deduce that

$$g^i(z) - z = \sum_{k=0}^{i-1} g^k(g(z) - z) = i(g(z) - z),\ 1 \leq i \leq p-1.$$

Since $(p-1)!$ is a unit in S, we get a formula

$$D = \mathrm{Disc}_Z(h) = \gamma_0(\varphi_1 - \varphi_2)^{p(p-1)},\ \gamma_0 \in S,\ \gamma_0 \text{ a unit.}$$

By assumption, $(u_1, \ldots, u_n)$ is adapted to E. Then definition 2.11(ii) implies that

$$\varphi_1 - \varphi_2 = \gamma u^{\mathbf{a}},$$

$\gamma \in S$, γ a unit, and $a_j = 0$, $e+1 \leq j \leq n$. Take an expansion (2.4):

$$\varphi_1 = \sum_{\mathbf{x} \in \mathbf{S}(\varphi_1)} \gamma_{\mathbf{x}} u^{\mathbf{x}},\ \gamma_{\mathbf{x}} \in S,\ \gamma_{\mathbf{x}} \text{ unit}$$

with $\mathbf{S}(\varphi_1) \subset \mathbb{N}^n$ finite. If $x_j < a_j$ for some $\mathbf{x} \in \mathbf{S}(\varphi_1)$ and some j, $1 \leq j \leq e$, then $\mathbf{x}$ is a vertex of $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ with initial form

$$\mathrm{in}_{\mathbf{x}} h = (Z - \lambda U^{\mathbf{x}})^p,\ \lambda \in S/m_S,\ \lambda \neq 0.$$

This is a solvable vertex: a contradiction, since $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal. Therefore $\varphi_1 \in (u^{\mathbf{a}})$ and we get $\epsilon(x) = 0$: a contradiction. Hence $(\mathcal{X}, x)$ is analytically irreducible as stated. It can be assumed that h is in case (b) of property **(G)** from now on.

Assume now that $\mathrm{in}_\alpha h$ is in cases (a) or (b) of property **(G)**, i.e. $i_0(\alpha) < p$ and

$$\mathrm{Disc}_Z(\mathrm{in}_\alpha h) \neq 0. \tag{2.23}$$

We now compute $\mathrm{ord}_{(u_j)} D$ for $1 \leq j \leq e$. Let

$$s_j := (u_j) \in \mathrm{Spec} S,\ S_j := S_{s_j} \text{ and } y_j \in \eta^{-1}(s_j).$$

To begin with, $\Delta_{\widehat{S_j}}(h; u_j, Z)$ is minimal by theorem 2.4. We denote by $G(s_j) = k(s_j)[U_j]$ the graded ring of S_j w.r.t. its valuation $\mu_j := \mathrm{ord}_{(u_j)}$ and by in_j the initial form map w.r.t. μ_j. Let:

$$\gamma_{i,j} U_j^{A_{i,j}} := \mathrm{in}_j f_{i,Z} \in G(s_j),\ 1 \leq i \leq p. \tag{2.24}$$

By definition 2.11(ii), we have $\mathrm{char}S/(u_j) = p$. Therefore proposition 2.9 and (2.22) apply to S_j with $\alpha = 1 \in \mathbb{R}$. The corresponding integer $i_0(1)$ is denoted by $i_0(s_j)$ in order to avoid confusion and we have

$$\mu_j(D) \geq p(p-1)\delta(y_j) = (p-1)H_j. \tag{2.25}$$

Case 1: $i_0(s_j) < p$. Then equality holds in the former formula as remarked right after (2.22).

Case 2: $i_0(s_j) = p$. Then equality is strict in the former formula. Since $\Delta_{\widehat{S_j}}(h; u_j, Z)$ is minimal, we have $\gamma_{p,j}U_j^{A_{p,j}} \notin G(s_j)^p$ and $A_{p,j} = H_j$. Let $z \in L$ be the image of Z. The discrete valuation μ_j of K has a unique extension to L, still denoted by μ_j. There is an embedding $G(s_j) \subset G_j$, where G_j is the graded ring of the valuation ring $\mathcal{O}_j := \{f \in L : \mu_j(f) \geq 0\}$.

Case 2a: $H_j \in p\mathbb{N}$. We have

$$G_j = k(s_j)(\gamma_{p,j}^{\frac{1}{p}})[U_j],\ \mathrm{in}_j z = -\gamma_{p,j}^{\frac{1}{p}}U_j^{\frac{H_j}{p}}; \tag{2.26}$$

Case 2b: $H_j \notin p\mathbb{N}$. We have

$$G_j = k(s_j)[\gamma_{p,j}^{\frac{l_j}{p}}U_j^{\frac{1}{p}}],\ \mathrm{in}_j z = -\gamma_{p,j}^{\frac{1}{p}}U_j^{\frac{H_j}{p}}, \tag{2.27}$$

where l_j satisfies $l_jH_j \equiv 1 \bmod p$, since the element $t := z^{l_j}u_j^{-\frac{l_jH_j-1}{p}}$ is a regular parameter of $\mathcal{O}_j$ with $(\mathrm{in}_j t)^p = -\gamma_{p,j}^{l_j}U_j$.

Let $g \in G = \mathrm{Gal}(L|K)$ be nontrivial. We have

$$g(z)^p - z^p + \sum_{i=1}^{p-1} f_{i,Z}(g(z)^{p-i} - z^{p-i}) = 0. \tag{2.28}$$

Since $\mu_j(g(z) - z) > \mu_j(z)$ and $\mu_j((p-1)!) = 0$, we deduce from (2.24) and (2.26)-(2.27) that

$$\mathrm{in}_j(f_{i,Z}(g(z)^{p-i} - z^{p-i})) = (-1)^{p-i}iT_j\gamma_{i,j}\gamma_{p,j}^{(p-i-1)/p}U_j^{(p-i-1)\frac{H_j}{p}+A_{i,j}} \tag{2.29}$$

for $1 \leq i \leq p-1$, where $T_j := \mathrm{in}_j(g(z)-z)$. On the other hand, we have

$$g(z)^p - z^p = (g(z)-z)^p + \sum_{i=1}^{p-1} \left(\begin{array}{c} p \\ i \end{array} \right) (g(z)-z)^{p-i} z^i. \tag{2.30}$$

Computing $\mu_j(D)$ by the Hilbert formula [60] V.11.(8) gives

$$\mu_j(D) = p(p-1)\mu_j(g(z)-z). \tag{2.31}$$

Since equality is strict in (2.25), we have $\mu_j(H(x)^{-(p-1)}D) > 0$ and we deduce that $\mu_j(g(z)-z) > H_j/p$. Computing initial forms for each term on the right hand side of (2.30), we get for $1 \leq i \leq p-1$:

$$\mathrm{in}_j((g(z)-z)^{p-i}z^i) = (-1)^i T_j^{p-i} \gamma_{p,j}^{\frac{i}{p}} U_j^{i\frac{H_j}{p}}.$$

Since $\mu_j(g(z)-z) > H_j/p$ and $\mu_j(\left(\begin{array}{c} p \\ i \end{array} \right)) = \mu_j(p)$, $1 \leq i \leq p-1$, the unique minimal value term in (2.30) inside the summation symbol is obtained with $i = p-1$. This shows

$$\mathrm{in}_j \left(\sum_{i=1}^{p-1} \left(\begin{array}{c} p \\ i \end{array} \right) (g(z)-z)^{p-i} z^i \right) = \mathrm{in}_j(p) T_j \gamma_{p,j}^{\frac{p-1}{p}} U_j^{(p-1)\frac{H_j}{p}}. \tag{2.32}$$

Case 2a. By (2.26), all terms $\gamma_{p,j}^{(p-i-1)/p}$ for $1 \leq i \leq p-1$ appearing in (2.29) are linearly independent over $k(s_j)$. Since $p \in S_j$, $pu_j^{-\mu_j(p)}$ is a unit in S_j. Let $\gamma \in k(s_j)$ be its residue, so the family $(\gamma\gamma_{p,j}^{\frac{p-1}{p}}, \{\gamma_{p,j}^{\frac{p-i-1}{p}}\}_{1 \leq i \leq p-1})$ is a *basis* of the $k(s_j)$-vector space $k(s_j)(\gamma_{p,j}^{1/p})$. Tracing back to (2.28) an (2.30), the value of $(g(z)-z)^p$ is the value of a sum of terms with linearly independent initial forms in G_j. We deduce the formula

$$\mu_j(g(z)-z)^{p-1} = \min\{\mu_j(p) + (p-1)\frac{H_j}{p}, \min_{1 \leq i \leq p-1}\{(p-i-1)\frac{H_j}{p} + A_{i,j}\}\}. \tag{2.33}$$

Case 2b. By (2.27), all values $(p-i-1)H_j/p$ for $1 \leq i \leq p-1$ appearing in (2.29) are pairwise distinct modulo $\mathbb{Z}$. Since $p \in S_j$, the family

$$(\mu_j(p) + (p-1)\frac{H_j}{p}, \{(p-i-1)\frac{H_j}{p} + A_{i,j}\}_{1 \leq i \leq p-1})$$

represent all cosets of $(1/p)\mathbb{Z}$ modulo $\mathbb{Z}$. The argument is now similar to case 2a above and (2.33) holds as well. Note that the minimum in the right hand side of (2.33) is achieved exactly once in this case 2b.

By (2.31) and (2.33), we conclude in all three cases 1, 2a and 2b that

$$\mu_j(H(x)^{-(p-1)}D) = \min\{p\mu_j(p), \min_{1\leq i\leq p-1}\{pA_{i,j} - iH_j\}\}. \tag{2.34}$$

By (2.24) and definition of $i_0(\alpha)$, we have

$$\sum_{j=1}^{e} A_{i_0(\alpha),j}\alpha_j \leq \mu_\alpha(f_{i_0(\alpha),Z}) = i_0(\alpha)\delta_\alpha(h; u_1, \ldots, u_n; Z). \tag{2.35}$$

Collecting together, since it was assumed in (2.23) that $\mathrm{Disc}_Z(\mathrm{in}_\alpha h) \neq 0$, we have

$$\sum_{j=1}^{e} \mu_j(H(x)^{-(p-1)}D)\alpha_j = (p-1)\left(p\delta_\alpha(h; u_1, \ldots, u_n; Z) - \sum_{j=1}^{e} H_j\alpha_j\right)$$

by (2.22). By (2.34)-(2.35), we deduce

$$(p - 1 - i_0(\alpha))(p\delta_\alpha(h; u_1, \ldots, u_n; Z) - \sum_{j=1}^{e} H_j\alpha_j) \leq 0. \tag{2.36}$$

Suppose that $p\delta_\alpha(h; u_1, \ldots, u_n; Z) - \sum_{j=1}^{e} H_j\alpha_j = 0$. Definition 2.10 implies that $f_{i,Z}^p \in H(x)^i$ for $1 \leq i \leq p$. Definition 2.1 yields the equality

$$\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z) = (\frac{H_1}{p}, \ldots, \frac{H_e}{p}, 0, \ldots, 0) + \mathbb{R}^n_{\geq 0}.$$

This is a contradiction, since it is assumed that $\epsilon(x) > 0$.

We thus have $p\delta_\alpha(h; u_1, \ldots, u_n; Z) - \sum_{j=1}^{e} H_j\alpha_j > 0$. By (2.36), this implies $i_0(\alpha) = p - 1$, since $i_0(\alpha) \leq p - 1$ was assumed in (2.23).

We may now sharpen (2.36) as follows, since it is an equality: equality holds in (2.35) *and* the minimum on the right hand side of (2.34) is achieved with $i = i_0(\alpha) = p - 1$ for each j, $1 \leq j \leq e$. These two properties are equivalent to the existence of an expansion (2.21) with $\gamma_{p-1,Z} \in S$ a unit.

By proposition 2.11(i), $G = \mathbb{Z}/p$ acts on the roots of $\mathrm{in}_\alpha h$. Let

$$z_\alpha \in (\mathrm{gr}_\alpha S)[Z]/(\mathrm{in}_\alpha h)$$

be the image of Z. Then $(g(z_\alpha) - z_\alpha)^{p-1} + F_{p-1,Z,\alpha} = 0$ for $g \in G$ nontrivial, so the polynomial $X^{p-1} + F_{p-1,Z,\alpha}$ is totally split over $\mathrm{gr}_\alpha S$, i.e. $-F_{p-1,Z,\alpha}$ is a $(p-1)^{\text{th}}$ in $\mathrm{gr}_\alpha S$ as required. The last formula in the theorem is obvious.

2.5 Adapted differential structure.

In this section, we introduce the differential structure on the graded algebras $\mathrm{gr}_\alpha S$. We will only consider here the case $\alpha = \mathbf{1} \in \mathbb{R}^J_{>0}$ with notations as in definition 2.2. These algebras appear naturally as blow up algebras of S along regular primes. We will adapt and simplify notations as much as possible in order to fit with the forthcoming computations.

Remark 2.2. This construction uses formal coordinates and Nagata derivatives [46] pp.241-245, and could be considerably simplified when

$$E = \mathrm{Spec}(S/(u_1 \cdots u_e)) \subset \mathrm{Spec}S$$

is essentially of finite type over some field. This extra property is satisfied for example when E is contained in the closed fiber of some previously performed blowing ups. In dimension three, this extra property is easily achieved from embedded resolution theorems in smaller dimensions, *vid.* proposition 5.6.

Notation 2.3. Let $W \subseteq E$ be a regular closed subset of $\mathrm{Spec}S$ having normal crossings with E. We now write

$$I(W) := I_J = (\{u_j\}_{j\in J}) \subset S \text{ for some } J \subseteq \{1,\dots,n\}.$$

Let $J_E := J \cap \{1,\dots,e\}$, $J' := \{1,\dots,n\}\backslash J$, so $(J')_E = \{1,\dots,e\}\backslash J_E$.

Let $S_W := S/I(W)$ and $\overline{u}_{j'} \in S_W$ be the image of $u_{j'}$, $j' \in J'$, so

$$\overline{m}_S := m_{S_W} = (\overline{u}_{j'})_{j'\in J'}).$$

Since $W \subseteq E$, **(E)** implies that $\mathrm{char}G(W) = \mathrm{char}(S/m_S) = p > 0$. The formal completion $\widehat{S_W}$ of S_W can be written as

$$\widehat{S_W} \simeq S/m_S[[\{\overline{u}_{j'}\}_{j'\in J'}]]. \tag{2.37}$$

The algebra $\mathrm{gr}_{\mathbf{1}} S$ of definition 2.2 is denoted by:

$$G(W) := \mathrm{gr}_{I(W)} S \simeq S_W[\{U_j\}_{j\in J}].$$

We also denote $\widehat{G(W)} := G(W) \otimes_{S_W} \widehat{S_W}$. In the special case $W = \{m_S\}$, we thus have $\widehat{G(m_S)} = G(m_S)$.

The initial form $\mathrm{in}_{\mathbf{1}} h$ *(ibid.)* w.r.t. the weight vector $\mathbf{1} \in \mathbb{R}^J_{>0}$ is now denoted

$$\mathrm{in}_W h = X^p + \sum_{i=1}^{p} F_{i,X,W} X^{p-i} \in G(W)[X],$$

with $F_{i,X,W} \in G(W)_{i\delta_{\mathbf{1}}(h;u_1,\ldots,u_n;X)}$, $1 \leq i \leq p$.

Any local equation of E has an initial form in $G(W)$, and we denote by $E(W)$ the associated divisor. Explicitly:

$$E(W) := \mathrm{div}\left(\prod_{j \in J_E} U_j \prod_{j' \in (J')_E} \overline{u}_{j'}\right) \subset \mathrm{Spec} G(W).$$

We include in these definitions the case where $W = \mathrm{div}(u_j)$ is an irreducible component of E. This corresponds to $(J')_E = \{1, \ldots, e\} \backslash \{j\}$ and

$$G(W) = S/(u_j)[U_j],\ E(W) = \mathrm{div}\left(U_j \prod_{j' \in (J')_E} \overline{u}_{j'}\right).$$

Let $(\lambda_l)_{l \in \Lambda_0}$ be an absolute p-basis of S/m_S. For this notion and the rest of this section, we refer to [46] pp.201-205 and pp. 235-245. We allow Λ_0 infinite in these constructions. The corresponding derivations $(\frac{\partial}{\partial \lambda_l})_{l \in \Lambda_0}$ of $\mathrm{Der}(S/m_S)$ act on power series in $\widehat{S_W}$ (2.37) coefficientwise. Those derivations $\frac{\partial}{\partial \lambda_l}$, $l \in \Lambda_0$ will be usually called "derivations w.r.t. to constants".

Let $\mathcal{D}(W) \subset \mathrm{Der}(\widehat{G(W)})$ be the submodule generated by the derivations w.r.t. to constants together with

$$\left(\{U_j \frac{\partial}{\partial U_j}\}_{j \in J_E}, \{\frac{\partial}{\partial U_j}\}_{j \in J \backslash J_E}, \{\overline{u}_{j'} \frac{\partial}{\partial \overline{u}_{j'}}\}_{j' \in (J')_E}, \{\frac{\partial}{\partial \overline{u}_{j'}}\}_{j' \in J' \backslash (J')_E}\right). \quad (2.38)$$

Since S_W is excellent and integrally closed, we have $\widehat{S_W}^p \cap S_W = S_W^p$. Therefore for $F \in G(W)$, there is an equivalence:

$$\forall D \in \mathcal{D}(W),\ D \cdot F = 0 \Leftrightarrow F \in G(W)^p. \quad (2.39)$$

If $F \in G(W)_d$ is a *homogeneous* element, $D \cdot F$ is not homogeneous in general for $D \in \mathcal{D}(W)$ because the derivations $(\frac{\partial}{\partial U_j})_{j \in J \setminus J_E}$ lower degrees by one. We define a homogeneous S_W-submodule of $G(W)_{d-1}$ as follows:

$$\mathcal{V}(F, E, W) := < \{\mathrm{cl}_{d-1} \frac{\partial F}{\partial U_j}\}_{j \in J \setminus J_E} > \subseteq G(W)_{d-1}. \tag{2.40}$$

Let $\mathcal{D}_W \subseteq \mathcal{D}(W)$ be the submodule defined by

$$\mathcal{D}_W := \{D \in \mathcal{D}(W) : D \cdot (I(W)/I(W)^2) \subseteq (I(W)/I(W)^2)\}.$$

If $D \in \mathcal{D}(W)$, we have

$$D \in \mathcal{D}_W \Leftrightarrow \forall j \in J \setminus J_E, \ < dU_j, D > \in (I(W)/I(W)^2)\widehat{G(W)}, \tag{2.41}$$

and there is an equivalence

$$\mathcal{D}_W = \mathcal{D}(W) \Leftrightarrow W \text{ is an intersection of components of } E. \tag{2.42}$$

If $F \in G(W)_d$ is a *homogeneous* element, we define a homogeneous $\widehat{S_W}$-submodule of $\widehat{G(W)}_d$ as follows:

$$\mathcal{J}(F, E, W) := \mathrm{cl}_d(\mathcal{D}_W \cdot F) \subseteq \widehat{G(W)}_d. \tag{2.43}$$

Let H_W be the initial form in $G(W)$ of the monomial ideal $H(x) \subseteq S$ (definition 2.10), where $x \in \eta^{-1}(m_S)$, i.e.

$$H_W := \left(\prod_{j \in J_E} U_j^{H_j} \prod_{j' \in (J')_E} \overline{u}_{j'}^{H_{j'}} \right) \subseteq G(W)_{d_W}, \tag{2.44}$$

where $d_W := \sum_{j \in J_E} H_j$. If $F \in H_W G(W)_{d-d_W}$, it follows from the above definitions that

$$\mathcal{V}(F, E, W) \subseteq H_W G(W)_{d-d_W-1} \text{ and } \mathcal{J}(F, E, W) \subseteq H_W \widehat{G(W)}_{d-d_W}.$$

For such $F \in H_W G(W)_{d-d_W}$, we denote:

$$\begin{cases} V(F, E, W) & := & H_W^{-1}\mathcal{V}(F, E, W) & \subseteq & G(W)_{d-d_W-1}, \\ \\ J(F, E, W) & := & H_W^{-1}\mathcal{J}(F, E, W) & \subseteq & \widehat{G(W)}_{d-d_W}. \end{cases} \tag{2.45}$$

If $F_{p,X,W} \in H_W G(W)_{d-d_W}$, the submodules

$$V(F_{p,X,W}, E, W) \subseteq G(W)_{d-d_W-1} \text{ and } J(F_{p,X,W}, E, W) \subseteq \widehat{G(W)}_{d-d_W}$$

are well-defined by (2.45). We will continually apply this definition when the following properties (i) and (ii) hold:

(i) $(u_1, \ldots, u_n; X)$ are well adapted coordinates at $x \in \eta^{-1}(m_S)$ (definition 2.8), and

(ii) $d - d_W = \epsilon(y)$ with $\eta^{-1}(s) = \{y\}$, s the generic point of W.

Note that $F_{p,X,W} \in H_W G(W)_{d-d_W}$ is then a consequence of definition 2.9 and proposition 2.8.

Some considerations will require localizing S at some point $s \in W$. We then denote by W_s the stalk of W at s. This notation is used jointly with notation 2.2 *sqq.* about the stalk E_s. The restriction of s is denoted by $\overline{s} \in \mathrm{Spec} S_W = G(W)_0$. We have

$$G(W_s) = \mathrm{gr}_{I(W_s)} S_s \simeq (S_W)_{\overline{s}}[\{U_j\}_{j \in J}].$$

Consistently $\mathrm{in}_{W_s} h \in G(W_s)[X]$ denotes the initial form. The above construction thus allows to associate to any *homogeneous* element $F \in G(W_s)_d$ homogeneous submodules

$$\mathcal{V}(F, E_s, W_s) \subseteq G(W_s)_{d-1}, \ \mathcal{J}(F, E_s, W_s) \subseteq \widehat{G(W_s)}_d.$$

2.6 Cones, ridge and directrix.

In this section, we recollect some facts about the directrix and Hilbert-Samuel stratum of a homogeneous ideal. These facts are then applied to extract numerical invariants from the vector spaces

$$V(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)-1} \text{ and } J(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)}$$

defined in the previous section (2.45) when $(u_1, \ldots, u_n; Z)$ are well adapted coordinates at $x \in \eta^{-1}(m_S)$. These considerations are based on elementary linear algebra.

Most difficulties in this section appear only for $n \geq 4$, which will eventually lead us to define our main invariant $\omega(x)$ in a different way than in [21]

chapter 1 (for equicharacteristic S of dimension $n = 3$) in the next section.

Let k be a field, R_1 be a k-vector space of finite dimension $n \geq 1$ and $R := k[R_1]$ be the symmetric algebra. Let $\mathbf{V} := \mathrm{Spec}R$ and I be a homogeneous ideal of R which defines a cone $C = C(I) := \mathrm{Spec}(R/I)$. With these notations, we define:

Definition 2.12. The directrix $\mathrm{VDir}(I)$ of $C = C(I)$ is the smallest k-vector subspace W of R_1 such that $I = (I \cap k[W])R$. We denote

$$\tau(I) := \dim_k \mathrm{VDir}(I),\ \mathrm{Dir}(I) := \mathrm{Spec}(R/(\mathrm{VDir}(I))).$$

Definition 2.13. Let $C = C(F)$ be a hypersurface cone, i.e. $I = (F)$ is a nonzero principal ideal. We define a reduced subcone

$$\mathrm{Max}(F) := \{x \in \mathbf{V} : \mathrm{ord}_x F = \mathrm{ord}_0 F\} \subseteq C(F),$$

where 0 is the origin (so $\mathrm{ord}_0 F = \mathrm{deg}F$).

Given a *fixed* degree $d \geq 1$ and an ideal $I = (F_1, \ldots, F_m) \subset R$ defined by homogeneous polynomials $F_1, \ldots, F_m \in R$, $\mathrm{deg}F_i = d$ for $1 \leq i \leq m$, we let

$$\mathrm{Max}(I) := \{x \in \mathbf{V} : \mathrm{ord}_x F_i = d, 1 \leq i \leq m\} \subseteq C(I).$$

The cone $\mathrm{Max}(I)$ is the closed Hilbert-Samuel stratum of $C(I)$. These two objects and the ridge are considered and connected by H. Hironaka in a more general context. See also [26] [27] for definition and computation of the ridge.

Proposition 2.15. *(Hironaka [36]) Let $C = C(F)$ be a hypersurface cone. There are inclusions*

$$\mathrm{Dir}(I) \subseteq \mathrm{Max}(F) \subseteq C(F).$$

If k is perfect or if $\mathrm{dim}R \leq p + 1$, the left hand side inclusion is an equality.

Remark 2.3. Counterexamples to the last statement exist for nonperfect k and $\mathrm{dim}R > p + 1$. For $\mathrm{dim}R \leq 4$, such counterexamples exist only if $\mathrm{dim}R = 4$ and $p = 2$. For applications to the proof of theorem 1.3, we only

have to deal with this difficulty for the initial form polynomial ($\dim R = 4$) which is of the form

$$\mathrm{in}_{m_S} h = Z^2 - \lambda U_1 Z + F_{2,Z},\ F_{2,Z} \in S/m_S[U_1, U_2, U_3]_2,\ \lambda \in S/m_S.$$

By [36], the polynomial $\mathrm{in}_{m_S} h$ is a counterexample to the last statement of proposition 2.15 if and only if $\lambda = 0$ and, up to a linear change of variables,

$$\mathrm{in}_{m_S} h = Z^2 + \lambda_2 U_1^2 + \lambda_1 U_2^2 + \lambda_1 \lambda_2 U_3^2 \tag{2.46}$$

with λ_1, λ_2 2-independent, i.e. $[(S/m_S)^2(\lambda_1, \lambda_2) : (S/m_S)^2] = 4$. This very special case is dealt with in the proof of proposition **??**.

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$ (definition 2.8). In case $\epsilon(x) > 0$, we have $\eta^{-1}(m_S) = \{x\}$, $k(x) = S/m_S$ (proposition 2.3) and the initial form polynomial has the form

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1} Z + F_{p,Z} \in G(m_S)[Z] = S/m_S[U_1, \ldots, U_n][Z] \tag{2.47}$$

by theorem 2.14 applied to $\alpha = \mathbf{1} \in \mathbb{R}^n_{>0}$. There is an associated integer $i_0(x) = p - 1$ (resp. $i_0(x) = p$) if $G \neq 0$ (resp. if $G = 0$). We denote by $H \subseteq G(m_S)_d$ the initial form vector space of the ideal $H(x)$, $d = \sum_{j=1}^e H_j$ (definition 2.10). If $i_0(x) = p - 1$, we have

$$H^{-1} G^p = < \prod_{j=1}^e U_j^{pB_j} >,\ B_j \in \frac{1}{p}\mathbb{N} \text{ and } \sum_{j=1}^e pB_j = \epsilon(x). \tag{2.48}$$

We can restate previous material as follows:

Proposition 2.16. *Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$ and assume that $\epsilon(x) > 0$. The following holds:*

(i) the vector space $V(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)-1}$ satisfies

$$V(F_{p,Z}, E, m_S) = 0 \Leftrightarrow F_{p,Z} \in S/m_S[U_1, \ldots, U_e][U_{e+1}^p, \ldots U_n^p];$$

(ii) the vector space $J(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)}$ satisfies

$$J(F_{p,Z}, E, m_S) = 0 \Leftrightarrow F_{p,Z} \in (S/m_S[U_1, \ldots, U_n])^p\,;$$

(iii) if $i_0(x) = p$, the vector space $V(F_{p,Z}, E, m_S)$ is independent of the well adapted coordinates $(u_1, \dots, u_n; Z)$; if $i_0(x) = p$ and $V(F_{p,Z}, E, m_S) = 0$, the vector space $J(F_{p,Z}, E, m_S)_{\epsilon(x)}$ is independent of the well adapted coordinates $(u_1, \dots, u_n; Z)$.

Proof. The first statement follows from (2.40) and (2.45), while (ii) follows from (2.39). Assume now that $i_0(x) = p$, i.e. $G = 0$.

To begin with, the situation in (ii) does not occur because the polyhedron $\Delta_{\hat{S}}(h; u_1, \dots, u_n; Z)$ is minimal. If $Z' = Z - \theta$, $\theta \in \hat{S}$ with $\mathrm{ord}_{m_S}\theta \geq \delta(x)/p$, we have $F_{p,Z'} = F_{p,Z} + \Theta^p$ for some $\Theta \in S/m_S[U_1, \dots, U_n]_{\delta(x)/p}$ (so $\Theta = 0$ if $\delta(x) \notin \mathbb{N}$). Hence $D \cdot F_{p,Z'} = D \cdot F_{p,Z}$ for every $D \in \mathrm{Der}(G(m_S))$.

By elementary calculus, the vector space

$$V(F_{p,Z}, E, m_S) =< \left\{ \frac{\partial F_{p,Z}}{\partial U_j} \right\}_{e+1 \leq j \leq n} >$$

is unchanged by adapted coordinate change (more generally by changes stabilizing the vector space $< U_1, \dots, U_e >$) and this proves the first statement in (iii). If $V(F_{p,Z}, E, m_S) = 0$, changes of coordinates fixing each $< U_j >$, $1 \leq j \leq e$, do not affect either

$$J(F_{p,Z}, E, m_S) =< \left\{ U_j \frac{\partial F_{p,Z}}{\partial U_j} \right\}_{1 \leq j \leq e}, \left\{ \frac{\partial F_{p,Z}}{\partial \lambda} \right\}_{\lambda \in \Lambda_0} > .$$

This concludes the proof.

We now turn to the version of proposition 2.16(iii) for $i_0(x) = p - 1$. The problem is elementary, though more technical, and the remaining part of this section is devoted to it.

Let $(\mathbf{e}_j)_{1 \leq j \leq n}$ be the standard basis of $\mathbb{R}^n$ and let

$$\mathbb{E} := \{\mathbf{x} \in \mathbb{R}^n : x_{e+1} = \cdots = x_n = 0\} \simeq \mathbb{R}^e.$$

Given $d \in \frac{1}{p}\mathbb{N}$ and $\mathbf{H} \in \mathbb{N}^n \cap \mathbb{E}$, we denote

$$\Delta_{\mathbf{H}}(d) := \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} :| \mathbf{x} |= d \text{ and } x_j \geq \frac{H_j}{p}, 1 \leq j \leq e\}$$

and

$$\mathcal{V}_{\mathbf{H}}(pd) := (U^{\mathbf{H}}) \cap G(m_S)_{pd} \subseteq G(m_S)_{pd}. \tag{2.49}$$

We fix once and for all $\mathbf{b} \in (\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d)) \cap \mathbb{E}$. Note that $\mathcal{V}_{\mathbf{H}}(pd) \neq (0)$ only if $H_1 + \cdots + H_e \leq pd$ and that such $\mathbf{b}$ as above exists only if $d \in \mathbb{N}$. By convention, we take $\{\mathbf{b}\} = \emptyset$ if $d \notin \mathbb{N}$ in the following formulæ. For applications, we will take $d = \delta(x_0)$, $\mathbf{H}$ as in definition 2.10 and $\mathbf{b}$ will be defined by $< G >=:< U_1^{b_1} \cdots U_e^{b_e} >$.

Notation 2.4. Any homogeneous polynomial $F \in \mathcal{V}_{\mathbf{H}}(pd)$ has a unique expansion of the form

$$F := \sum_{\mathbf{x} \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d)} \lambda(\mathbf{x}) U^{p\mathbf{x}}, \lambda(\mathbf{x}) \in S/m_S.$$

We denote

$$\Delta(F) := \mathrm{Conv}(\{\mathbf{x} \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d) : \lambda(\mathbf{x}) \neq 0\} \cup \{\mathbf{b}\}) \subseteq \Delta_{\mathbf{H}}(d).$$

According to theses conventions, we have $\Delta(0) = \{\mathbf{b}\}$.

Definition 2.14. With notations as above, let $T : \mathcal{V}_{\mathbf{H}}(pd) \to \mathcal{V}_{\mathbf{H}}(pd)$ be the S/m_S-linear truncation operator defined as follows: let

$$A := \{\mathbf{x} \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d) : \mathbf{b} + p(\mathbf{x} - \mathbf{b}) \in \Delta_{\mathbf{H}}(d)\}. \tag{2.50}$$

and

$$TF := \sum_{\mathbf{x} \notin A} \lambda(\mathbf{x}) U^{p\mathbf{x}} \in \mathcal{V}_{\mathbf{H}}(pd). \tag{2.51}$$

For $d \notin \mathbb{N}$, we have $A = \emptyset$ and T is the identity map.

The construction of the previous section associates two vector spaces $V(TF, E, m_S)$ and $J(TF, E, m_S)$. Explicitly, we have:

$$V(TF, E, m_S) = U^{-\mathbf{H}} < \frac{\partial TF}{\partial U_j}, e+1 \leq j \leq n > \subseteq G(m_S)_{pd-1-|\mathbf{H}|}$$

for the former one. If $V(TF, E, m_S) = 0$ (and only in this case), we will use the latter one, given explicitly by and

$$J(TF, E, m_S) = U^{-\mathbf{H}} < \{U_j \frac{\partial TF}{\partial U_j}\}_{1 \leq j \leq e}, \{\frac{\partial TF}{\partial \lambda_l}\}_{l \in \Lambda_0} > \subseteq G(m_S)_{pd-|\mathbf{H}|},$$

with notations as in the previous section. We can now state:

Lemma 2.17. *Assume that $d \in \mathbb{N}$. With notations as above, we have*

$$\mathrm{Ker}T = U^{(p-1)\mathbf{b}}\mathcal{V}_{\lceil \frac{\mathbf{H}}{p} \rceil}(d),$$

where $\lceil \frac{\mathbf{H}}{p} \rceil := (\lceil \frac{H_1}{p} \rceil, \ldots, \lceil \frac{H_e}{p} \rceil, 0, \ldots, 0)$.

Let $G := \mu U^{\mathbf{b}}$, $\mu \in S/m_S$, $\Phi \in \mathcal{V}_{\lceil \frac{\mathbf{H}}{p} \rceil}(d)$ *and* $F \in \mathcal{V}_{\mathbf{H}}(pd)$. *Then*

$$V(T(F + \Phi^p - G^{(p-1)}\Phi), E, m_S) = V(TF, E, m_S).$$

If $V(TF, E, m_S) = 0$, *then*

$$J(T(F + \Phi^p - G^{(p-1)}\Phi), E, m_S) = J(TF, E, m_S),$$

Proof. We analyze the definition of T in (2.51). The kernel of T is generated by those monomials $U^{p\mathbf{x}} \in \mathcal{V}_{\mathbf{H}}(pd)$ such that

$$\mathbf{y} := p\mathbf{x} - (p-1)\mathbf{b} \in \Delta_{\mathbf{H}}(d).$$

Since $\mathbf{x} \in \frac{1}{p}\mathbb{N}^n$, $\mathbf{b} \in \mathbb{N}^n$, we have $\mathbf{y} \in \mathbb{N}^n$ for such $\mathbf{y}$. Therefore $\mathrm{Ker}T$ is generated by

$$\mathrm{Ker}T =< \{U^{(p-1)\mathbf{b}}U^{\mathbf{y}} : \mathbf{y} \in \mathbb{N}^n, \mid \mathbf{y} \mid = d \text{ and } y_j \geq \frac{H_j}{p}, 1 \leq j \leq e\} > .$$

This proves the first statement. For the second part, we have proved that

$$T(F + \Phi^p - G^{(p-1)}\Phi) = TF + T\Phi^p.$$

Hence $D \cdot T(F + \Phi^p - G^{(p-1)}\Phi) = D \cdot TF$ for every $D \in \mathrm{Der}(G(m_S))$ and this concludes the proof.

We now study invariance properties of $V(F, E, m_S)$ and $J(F, E, m_S)$ under changes of adapted coordinates. Given two r.s.p.'s $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{u}' = (u'_1, \ldots, u'_n)$ adapted to E, there exists a matrix $M \in \mathcal{M}(S)$,

$$\mathcal{M}(S) := \{(m_{ij}) \in \mathrm{GL}(n, S) : m_{jj'} = 0, (j, j') \in \{1, \ldots, e\} \times \{1, \ldots, n\}, j \neq j'\}$$

such that $\mathbf{u} = M\mathbf{u}'$. The set $\mathcal{M}(S)$ is the set of S-points of an affine S-scheme $\mathcal{M} \subset \mathrm{GL}(n, S)$. Denote by

$$\mathrm{GL}(n, S) \to \mathrm{GL}(n, S/m_S), \ M \mapsto \overline{M}$$

the canonical surjection. Each such $\overline{M}$ induces a graded S/m_S-automorphism of $\mathrm{gr}_{m_S}(S) \simeq S/m_S[U_1, \ldots, U_n]$. By (2.49), this automorphism restricts to an automorphism of $\mathcal{V}_{\mathbf{H}}(pd)$ for each $d \in \frac{1}{p}\mathbb{N}$ still denoted by $\overline{M}$.

Given a homogeneous polynomial $F \in \mathcal{V}_{\mathbf{H}}(pd)$ as above and a matrix $\overline{M} \in \mathcal{M}(S/m_S)$, we denote for simplicity the transformed equation $U \mapsto \overline{M}U'$ by

$$F' =: \sum_{\mathbf{x}' \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d)} \lambda'(\mathbf{x}') U'^{p\mathbf{x}'}. \tag{2.52}$$

Let $\Delta(F') := \mathrm{Conv}(\{\mathbf{x}' \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d) : \lambda'(\mathbf{x}') \neq 0\} \cup \{\mathbf{b}\}) \subseteq \Delta_{\mathbf{H}}(d)$ be the corresponding polytope and T' be the corresponding operator on $\mathcal{V}_{\mathbf{H}}(pd)$ with variable U'. The linear operator T obviously does not commute with $\overline{M}$ in general (i.e. $(TF)' \neq T'F'$ in general), but the lemma below extracts the relevant invariant data. We refer to definition 2.13 for the notation $\mathrm{Max}(I)$, $I \subset G(m_S)$ generated by homogeneous polynomial of one and the same degree.

Notation 2.5. We denote by

$$B := \{j,\ 1 \leq j \leq e : pb_j - H_j > 0\} \text{ and } U_B := \{U_j, j \in B\}. \tag{2.53}$$

We denote $U_{B'} := \{U_j, j \notin B\}$ and stick to our former conventions, i.e.

$$B' = \{1, \ldots, n\} \backslash B,\ (B')_E = \{1, \ldots, e\} \backslash B.$$

Lemma 2.18. *With notations as above, there is an equality of sets*

$$\mathrm{Max}(V(TF, E, m_S)) \cap \{U_B = 0\} = \mathrm{Max}(V(T'F', E, m_S)) \cap \{U'_B = 0\}. \tag{2.54}$$

If $V(TF, E, m_S) = 0$, *then* $V(T'F', E, m_S) = 0$ *and there is an equality of sets*

$$\mathrm{Max}(J(TF, E, m_S)) \cap \{U_B = 0\} = \mathrm{Max}(J(T'F', E, m_S)) \cap \{U'_B = 0\}. \tag{2.55}$$

Proof. The operator T commutes with $\overline{M}$ when $\overline{M}$ stabilizes the vector space $< U_{e+1}, \ldots, U_n >$. In these cases, we have

$$V(T'F', E, m_S) = V((TF)', E, m_S).$$

If $V(TF, E, m_S) = 0$, then

$$V(T'F', E, m_S) = 0 \text{ and } J(T'F', E, m_S) = J((TF)', E, m_S).$$

So the lemma is trivial in this case and we may therefore assume that

$m_{jj'} = 0, (j, j') \in \{e+1, \ldots, n\} \times \{e+1, \ldots, n\}, j \neq j'$ and $m_{jj} = 1, 1 \leq j \leq n$.

By elementary calculus, this new assumption implies for every $\Phi \in G(m_S)$:

$$\frac{\partial \Phi'}{\partial U'_j} = \left(\frac{\partial \Phi}{\partial U_j}\right)', \ e+1 \leq j \leq n. \tag{2.56}$$

Let $\mathbf{x} \in \frac{1}{p}\mathbb{N}^n \cap \Delta_{\mathbf{H}}(d)$. Since $pb_j = H_j$ for $j \in (B')_E$, we have by (2.50):

$$\mathbf{x} \in A \Leftrightarrow \forall j \in B, px_j \geq (p-1)b_j.$$

Expand $TF = \sum_{\mathbf{y}} U_B^{\mathbf{y}} F_{\mathbf{y}}(U_{B'})$, so we have:

$$V(TF, E, m_S) = U^{-\mathbf{H}} < \{\sum_{\mathbf{y}} U_B^{\mathbf{y}} \frac{\partial F_{\mathbf{y}}(U_{B'})}{\partial U_j}\}_{e+1 \leq j \leq n} > .$$

For $P \in \mathrm{Spec}G(m_S)$ such that $(U_B) \subseteq P$, we get:

$$P \in \mathrm{Max}(V(TF, E, m_S)) \Leftrightarrow P \in \bigcap_{\mathbf{y}} \bigcap_{j=e+1}^{n} \mathrm{Max}(G_{\mathbf{y}}), \tag{2.57}$$

where $G_{\mathbf{y}} := U_{B'}^{-\mathbf{H}'} \frac{\partial F_{\mathbf{y}}(U_{B'})}{\partial U_j}$, $\mathbf{H}' := (H_{j'})_{j' \in (B')_E}$.

Suppose furthermore that $\overline{M}$ stabilizes the vector space $< U_{B'} >$. Then T also commutes with $\overline{M}$ and each term $G_{\mathbf{y}}$ in (2.57) is transformed into

$$(G_{\mathbf{y}})' = U_{B'}^{-\mathbf{H}_{B'}} \frac{\partial F'_{\mathbf{y}}(U'_{B'})}{\partial U'_j}$$

by (2.56) and (2.54) follows. Suppose furthermore that $V(TF, E, m_S) = 0$; then $G_{\mathbf{y}} = 0$ for each $\mathbf{y}$ in (2.56) and we get $V(T'F', E, m_S) = 0$. For $1 \leq j \leq e$ and $l \in \Lambda_0$, we have

$$\left(U_j \frac{\partial TF}{\partial U_j}\right)' = U'_j \frac{\partial T'F'}{\partial U'_j}, \ \left(\frac{\partial TF}{\partial \lambda_l}\right)' = \frac{\partial T'F'}{\partial \lambda_l}, \tag{2.58}$$

and (2.55) also follows. Hence we may furthermore assume that

$$m_{jj'} = 0, (j, j') \in \{e+1, \ldots, n\} \times (B')_E.$$

In this situation, T does not commute any longer with $\overline{M}$. However, for each term $G_{\mathbf{y}}$ as above, we have

$$\mathrm{ord}_P(D \cdot G_{\mathbf{y}}) \geq \mathrm{deg} G_{\mathbf{y}} - a \tag{2.59}$$

for any differential operator D on $S/m_S[U_{B'}]$ of order not greater than a. Let

$$(G_{\mathbf{y}})' = \sum_{|\alpha| \leq \mathrm{deg} G_{\mathbf{y}}} (U'_B)^\alpha (D^{(\alpha)} \cdot G_{\mathbf{y}}),\ D^{(\alpha)} \cdot G_{\mathbf{y}} \in S/m_S[U'_{B'}]_{\mathrm{deg} G_{\mathbf{y}} - |\alpha|}$$

be the (characteristic free) Taylor expansion, where $D^{(\alpha)}$ is a differential operator of order $\mid \alpha \mid$. Take again $P \in \mathrm{Spec} G(m_S)$ such that $(U_B) \subseteq P$. By (2.59), we have

$$P \in \mathrm{Max}(G_{\mathbf{y}}) \Rightarrow P \in \bigcap_\alpha \mathrm{Max}(D^{(\alpha)} \cdot G_{\mathbf{y}}) \Rightarrow P \in \mathrm{Max}((G_{\mathbf{y}})').$$

We deduce from (2.57) that

$$P \in \mathrm{Max}(V(TF, E, m_S)) \Rightarrow P \in \mathrm{Max}(V((TF)', E, m_S)).$$

This proves (2.54). If $V(TF, E, m_S) = 0$, (2.55) follows from (2.58) as above.

This lemma is the key to our version of proposition 2.16(iii) for $i_0(x) = p - 1$:

Proposition 2.19. *Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$ and assume that $\epsilon(x) > 0$ and $i_0(x) = p - 1$. Let*

$$d := \delta(x),\ \mathbf{H} := (H_1, \ldots, H_e, 0, \ldots, 0) \text{ and } < U_1^{b_1} \cdots U_e^{b_e} > := < G >$$

be defined respectively by definition 2.5, definition 2.10 and (2) of theorem 2.14. With notations as above, the following holds:

(i) the set

$$\mathrm{Max}(V(TF_{p,Z}, E, m_S)) \cap \{U_B = 0\} \subseteq \mathrm{Spec} G(m_S)$$

is independent of the well adapted coordinates $(u_1, \ldots, u_n; Z)$;

(ii) the property $V(TF_{p,Z}, E, m_S) = 0$ is independent of the well adapted coordinates $(u_1, \ldots, u_n; Z)$; when it holds, the set

$$\mathrm{Max}(J(TF_{p,Z}, E, m_S)) \cap \{U_B = 0\} \subseteq \mathrm{Spec}G(m_S)$$

is also independent of the well adapted coordinates $(u_1, \ldots, u_n; Z)$.

Proof. For such $(u_1, \ldots, u_n; Z)$, the corresponding initial form is

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1}Z + F_{p,Z} \in G(m_S)[Z].$$

Since $G \neq 0$, we have $d = \delta(x) = \deg G \in \mathbb{N}$. If $(u'_1, \ldots, u'_n)$ is an adapted r.s.p. of S, there exists $M \in \mathcal{M}(S)$ such that $\mathbf{u} = M\mathbf{u}'$. Let $(u'_1, \ldots, u'_n; Z')$ be well adapted coordinates at x. We have $Z' = Z - \phi$ for some $\phi \in S$, with $\mathrm{ord}_{m_S}\phi \geq d$. We deduce that

$$\mathrm{in}_{m_S} h = Z'^p - G^{p-1}Z' + \Phi^p - G^{p-1}\Phi + F_{p,Z} \in G(m_S)[Z']$$

for some $\Phi := \mathrm{cl}_d\phi \in G(m_S)_d$. We deduce the formula

$$F_{p,Z'} = F_{p,Z} + \Phi^p - G^{p-1}\Phi.$$

By lemma 2.17, we have $V(TF_{p,Z'}, E, m_S) = V(TF_{p,Z}, E, m_S)$; if moreover $V(TF_{p,Z}, E, m_S) = 0$, then $J(TF_{p,Z'}, E, m_S) = J(TF_{p,Z}, E, m_S)$. By lemma 2.18, we have an equality of sets

$$\mathrm{Max}(V(TF_{p,Z'}, E, m_S)) \cap \{U_B = 0\} = \mathrm{Max}(V(T'F'_{p,Z'}, E, m_S)) \cap \{U'_B = 0\}$$

and this proves (i). If $V(TF_{p,Z'}, E, m_S) = 0$, then $V(T'F'_{p,Z'}, E, m_S) = 0$ by lemma 2.18 and there is an equality of sets

$$\mathrm{Max}(J(TF_{p,Z'}, E, m_S)) \cap \{U_B = 0\} = \mathrm{Max}(J(T'F'_{p,Z'}, E, m_S)) \cap \{U'_B = 0\}.$$

This concludes the proof.

Remark 2.4. We consider proposition 2.16(iii) as the special case $B = \emptyset$, $T = \mathrm{id}$ of proposition 2.19.

2.7 Main invariants.

Let $s \in \mathrm{Spec}S$ and $y \in \eta^{-1}(s)$. The purpose of this section is to attach to y a resolution complexity

$$\iota(y) = (m(y), \omega(y), \kappa(y)) \in \{1, \dots, p\} \times \mathbb{N} \times \{1, \dots, 4\} \tag{2.60}$$

with certain invariance properties. Auxiliary numbers

$$(\tau(y), \tau'(y)) \in \{1, \dots, n+1\} \times \{1, \dots, n\} \tag{2.61}$$

are similarly attached to y.

The pair $(m(y), \tau(y))$ are the standard multiplicity and Hironaka τ-number of $\mathcal{X}$ at y (definition 2.12). The pair $(\omega(y), \tau'(y))$ play the role of a differential multiplicity and differential τ-number attached to $\eta : \mathcal{X} \to \mathrm{Spec}S$ at y. The behavior of the function ι under blowing up is studied in the next sections. The complete definition of $\kappa(y)$ and its properties is restricted to $n \leq 3$ and performed in chapter 4 below.

In all definitions that follow it can be assumed without loss of generality that $s = m_S$ by localizing S at s, since our assumptions **(G)** and **(E)** are stable when changing (S, h, E) to (S_s, h_s, E_s) (notation 2.2).

Definition 2.15. (Multiplicity). Let $x \in \eta^{-1}(m_S)$. We have already defined

$$m(x) = \mathrm{ord}_{m_{S[X]_x}} h(X) \leq p.$$

Let $M_x \subset S[X]$ be the ideal of x, $G_x := \mathrm{Spec}(\mathrm{gr}_{M_x} S[X]_{M_x})$ and H_x be the initial form of h in $(G_x)_{m(x)}$. From definition 2.12, we let

$$\tau(x) := \tau(H_x).$$

If $m(x) < p$, we let $\iota(x) := (m(x), 0, 1)$.

Note that $m(y) < p$ whenever $s = \eta(y) \notin E$ (definition 2.11 and following comments). If $m(y) = p$, we have

$$s = \eta(y) \in E,\ \eta^{-1}(s) = \{y\} \text{ and } k(y) = k(s)$$

by proposition 2.10.

Applying proposition 2.16(iii) (resp. proposition 2.19(ii)) to S if $i_0(x) = p$ (resp. if $i_0(x) = p-1$) proves that $(\omega(x), \kappa(x))$ is well-defined. We recall that $TF_{p,Z} = F_{p,Z}$ whenever $i_0(x) = p$ (see remark 2.4).

Definition 2.16. (Adapted order). Assume that $m(x) = p$, where $\{x\} = \eta^{-1}(m_S)$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x. We let

$$\omega(x) = \begin{cases} \epsilon(x) - 1 & \text{if} \quad V(TF_{p,Z}, E, m_S) \neq 0; \\ \epsilon(x) & \text{if} \quad V(TF_{p,Z}, E, m_S) = 0. \end{cases}$$

We define:

$$\kappa(x) := 1 \text{ if } (\omega(x) = \epsilon(x) \text{ and } i_0(x) = p - 1).$$

Otherwise, we simply let $\kappa(x) \geq 2$.

Remark 2.5. This definition is different from that used in [21] chapter 1, definition **II.4**. The main difference is that definition 2.16 gives a much better behaviour w.r.t. regular base changes. In [21], $\omega(x)$ was defined in terms of local coordinates, say

$$\omega(x) = \min_{(u_1,\ldots,u_n;Z)} \{\omega(h; u_1, \ldots, u_n; Z)\},$$

where $(u_1, \ldots, u_n; Z)$ were well adapted coordinates at x. This brought over many difficulties (*vid.* [21] chapter 1 **II.3.3.1** and **II.3.3.2**; proof of **II.5.4.2**(iv); theorem **II.5.6**) which could be overcome only for $n = 3$.

In chapter 4, we define the projection number $\kappa(x) \in \{2, 3, 4\}$ when $n = 3$ and state that $\iota(x) = (m(x), \omega(x), \kappa(x))$ can be decreased by blowing ups preserving our structure (projection theorem 4.4 below).

We now turn to the definition of the adapted cone and directrix and the attached invariant $\tau'(x)$. Applying proposition 2.16(iii) (resp. proposition 2.19) if $i_0(x) = p$ (resp. if $i_0(x) = p - 1$) proves that $\mathrm{Max}(x)$, $\mathrm{Dir}(x)$ and $\tau'(x)$ are well defined.

Definition 2.17. (Adapted cone and directrix). Assume that $m(x) = p$ and $\omega(x) > 0$, where $\{x\} = \eta^{-1}(m_S)$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x. We define a reduced subcone $\mathrm{Max}(x) \subseteq \mathrm{Spec}G(m_S)$ by:

$$\mathrm{Max}(x) := \begin{cases} \mathrm{Max}(V(TF_{p,Z}, E, m_S)) \cap \{U_B = 0\} & \text{if} \quad \omega(x) = \epsilon(x) - 1; \\ \mathrm{Max}(J(TF_{p,Z}, E, m_S)) \cap \{U_B = 0\} & \text{if} \quad \omega(x) = \epsilon(x) \quad . \end{cases}$$

We define an affine subspace $\mathrm{Dir}(x) \subseteq \mathrm{Spec}G(m_S)$ by

$$\mathrm{Dir}(x) := \begin{cases} \mathrm{Dir}(V(TF_{p,Z}, E, m_S), U_B) & \text{if} \quad \omega(x) = \epsilon(x) - 1; \\ \mathrm{Dir}(J(TF_{p,Z}, E, m_S), U_B) & \text{if} \quad \omega(x) = \epsilon(x) \quad . \end{cases}$$

We let $\mathrm{VDir}(x)$ to be the underlying vector space of $\mathrm{Dir}(x)$ and

$$\tau'(x) := \dim_{k(x)} \mathrm{VDir}(x).$$

Remark 2.6. We will use the invariants $\mathrm{Dir}(x)$ and $\tau'(x)$ only when $\mathrm{Dir}(x) = \mathrm{Max}(x)$ (last statement in proposition 2.15 and following remark).

Let $S \subseteq \tilde{S}$ be a regular local base change, $\tilde{S}$ excellent. Recall notation 2.1 and notation 2.2. It has been explained when defining conditions **(G)** and **(E)** that they are stable by such base changes and by localization at a prime. Let $\tilde{s} \in \mathrm{Spec}\tilde{S}$ and $\tilde{y} \in \tilde{\eta}^{-1}(\tilde{s})$. In order to relate $\iota(\tilde{y})$ and $\iota(y)$ (2.60), where $y \in \mathcal{X}$ is the image of $\tilde{y}$, we may thus assume that $s = m_S$, $\tilde{s} = m_{\tilde{S}}$.

Let $(u_1, \dots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$. Then $(u_1, \dots, u_n)$ can be completed to a r.s.p. $(u_1, \dots, u_{\tilde{n}})$ of $\tilde{S}$ which is adapted to $\tilde{E}$. There is an inclusion

$$G(m_S) = \frac{S}{m_S}[U_1, \dots, U_n] \subseteq G(m_{\tilde{S}}) = G(m_S) \otimes_{\frac{S}{m_S}} \frac{\tilde{S}}{m_{\tilde{S}}}[\tilde{U}_{n+1}, \dots, \tilde{U}_{\tilde{n}}]. \quad (2.62)$$

Theorem 2.20. *Let $S \subseteq \tilde{S}$ be a local base change which is regular, $\tilde{S}$ excellent. Let $\tilde{x} \in \tilde{\eta}^{-1}(m_{\tilde{S}})$ and $x \in \eta^{-1}(m_S)$ be its image. The following holds.*

(1) we have $(m(\tilde{x}), \omega(\tilde{x})) = (m(x), \omega(x))$;

(2) if $m(x) = p$, then

(i) $H(\tilde{x}) = H(x)\tilde{S}$, $i_0(\tilde{x}) = i_0(x)$, and $(\kappa(\tilde{x}) = 1 \Leftrightarrow \kappa(x) = 1)$;

(ii) we have $\epsilon(\tilde{x}) \geq \epsilon(x)$, and $\epsilon(\tilde{x}) > \epsilon(x)$ if and only if

$$\mathrm{in}_{m_S} h = Z^p + F_{p,Z},\ F_{p,Z} \in (k(\tilde{x})[U_1, \dots, U_n])^p$$

where $(u_1, \dots, u_n; Z)$ are well prepared coordinates at x. When this holds, we have $\tilde{n} > n$, $\epsilon(\tilde{x}) = \epsilon(x) + 1$ and

$$\mathrm{in}_{m_{\tilde{S}}} \tilde{h} = \tilde{Z}^p + \sum_{j=n+1}^{\tilde{n}} U_j \Phi_j(U_1, \dots, U_n) + \Psi(U_1, \dots, U_n) \in G(m_{\tilde{S}})[\tilde{Z}],$$

with $\Phi_j \neq 0$ for some $j \geq n+1$ and $\Phi_j \in k(\tilde{x})[U_1^p, \dots, U_n^p]$ for every $j \geq n+1$, where $(u_1, \dots, u_{\tilde{n}}; \tilde{Z})$ are well prepared coordinates at $\tilde{x}$.

Proof. The theorem is trivial if $m(x) = 1$: then $m(\tilde{x}) = 1$ because $S \subseteq \tilde{S}$ is regular.

Assume that $m(x) \geq 2$ and pick well prepared coordinates $(u_1, \ldots, u_n; Z)$ at x, then complete $(u_1, \ldots, u_n)$ to a r.s.p. $(u_1, \ldots, u_{\tilde{n}})$ of $\tilde{S}$ which is adapted to $\tilde{E}$. We have $\delta(x) > 0$, so $h \in (Z, u_1, \ldots, u_n)$, and $k(x) = S/m_S$ by proposition 2.10. Applying (2.62) to the local base change $S[Z]_{(m_S,Z)} \subseteq T[Z]_{(m_T,Z)}$ which is also regular gives

$$m(x) = \mathrm{ord}_x h(Z) = \mathrm{ord}_{\tilde{x}} \tilde{h}(Z) = m(\tilde{x}).$$

This concludes the proof when $m(x) < p$ and we assume from now on that $m(x) = p$. In particular we have $\{\tilde{x}\} = \tilde{\eta}^{-1}(m_{\tilde{S}})$, $k(\tilde{x}) = \tilde{S}/m_{\tilde{S}}$. Let

$$\mathrm{in}_{m_S} h = Z^p + \sum_{i=1}^{p} F_{i,Z} Z^{p-i} \in G(m_S)[Z],$$

be the corresponding initial form polynomial. Let $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ be a vertex of the polyhedron $\Delta_{\hat{S}}(u_1, \ldots, u_n; Z)$. We denote by

$$\tilde{\mathbf{x}} := (\mathbf{x}, \underbrace{0, \ldots, 0}_{\tilde{n}-n}) \in \Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; Z)$$

the corresponding vertex in $\Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; Z)$. Note that $\tilde{\mathbf{x}}$ *may be a solvable vertex* of the latter polyhedron. We have:

$$\tilde{\mathbf{x}} \text{ solvable} \Leftrightarrow \mathrm{in}_{\tilde{\mathbf{x}}} \tilde{h} \in ((\mathrm{gr}_\alpha \tilde{S})[Z])^p$$

with notations as in definition 2.3. Therefore we have

$$\tilde{\mathbf{x}} \text{ solvable} \Leftrightarrow (\mathrm{in}_{\mathbf{x}} h = Z^p + F_{p,Z,\mathbf{x}}, \mathbf{x} \in \mathbb{N}^n,\ F_{p,Z,\mathbf{x}} = \lambda U^{p\mathbf{x}}, \lambda \in k(\tilde{x})^p).$$

We deduce for the initial form polynomial that

$$\delta(\tilde{x}) > \delta(x) \Leftrightarrow (i_0(x) = p \text{ and } F_{p,Z} \in (k(\tilde{x})[U_1, \ldots, U_n])^p). \tag{2.63}$$

Since the fiber ring $\tilde{S}/m_S\tilde{S}$ is geometrically regular over $k(x)$, the ring $\tilde{S}[Y]/(Y^p - l)$ is regular for every unit $l \in S$ with residue $\bar{l} \notin k(x)^p$. Therefore if $\bar{l} \in k(\tilde{x})^p$, we have

$$\forall \tilde{l} \in \tilde{S}, \tilde{v} := \tilde{l}^p - l \in m_{\tilde{S}} \Longrightarrow \tilde{v} \text{ is a regular parameter in } \tilde{S}.$$

Such $\tilde{v}$ restricts to a regular parameter of $\tilde{S}/m_S\tilde{S}$, so the previous formula is refined to:

$$\tilde{v} \text{ is a regular parameter transverse to } \mathrm{div}(u_1 \cdots u_n) \subset \mathrm{Spec}\tilde{S}. \tag{2.64}$$

This equation implies in particular that $\tilde{n} > n$. Let $\xi \in \mathrm{Spec}(\tilde{S}/m_S\tilde{S})$ be the generic point. Applying the above remarks to the regular local base change $S \subset \tilde{S}_\xi$ shows that $k(\xi)^p \cap k(x) = k(x)^p$.

Let $s_j := (u_j) \in \mathrm{Spec}S$, $1 \leq j \leq e$, and apply this remark to the regular local base change $S_{(u_j)} \subseteq \tilde{S}_{(u_j)}$. This proves that the field inclusion $QF(S/(u_j)) \subseteq QF(\tilde{S}/(u_j))$ is inseparably closed.

The polynomial $\mathrm{in}_{(s_j)}h_{s_j} \in QF(S/(u_j))[U_j][Z]$ is not a p^{th}-power by theorem 2.4. Therefore $\mathrm{in}_{(s_j)}h_{s_j}$ is not a p^{th}-power in $QF(\tilde{S}/(u_j))[U_j][Z]$. Turning back to definition 2.9, we get

$$H(\tilde{x}) = H(x)\tilde{S}. \tag{2.65}$$

Definition 2.9 now shows that $\epsilon(\tilde{x}) \geq \epsilon(x)$ and that

$$\epsilon(\tilde{x}) > \epsilon(x) \Leftrightarrow (i_0(x) = p \text{ and } F_{p,Z} \in (k(\tilde{x})[U_1, \ldots, U_n])^p). \tag{2.66}$$

This proves the first part of (2.ii). To go on with the proof, we consider two cases.

Case 1: assume that $i_0(x) < p$. By (2.66), we have $\epsilon(\tilde{x}) = \epsilon(x)$, so the proof of (2.ii) is already complete. Let $\tilde{\phi} \in \tilde{S}$ be such that $\Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; \tilde{Z})$ is minimal, with $\tilde{Z} := Z - \tilde{\phi}$ and $\mathrm{ord}_{m_{\tilde{S}}}\tilde{\phi} \geq \delta(x)$. We have

$$\mathrm{in}_{m_{\tilde{S}}}\tilde{h} = \tilde{Z}^p + \sum_{i=i_0}^{p} F_{i,\tilde{Z}}\tilde{Z}^{p-i} \in G(m_{\tilde{S}})[\tilde{Z}],$$

with $F_{i_0,\tilde{Z}} = F_{i_0,Z}$ by proposition 2.9. Therefore $i_0(\tilde{x}) = i_0(x)$ and it is sufficient to prove that $\omega(\tilde{x}) = \omega(x)$ in order to complete the proof of (1) and (2.i) in the theorem (still under the assumption $i_0(x) < p$). This is obvious if $\epsilon(x) = 0$, since

$$0 \leq \omega(\tilde{x}) \leq \epsilon(\tilde{x}) = \omega(x) = 0.$$

Assume that $\epsilon(x) > 0$. We have $i_0(x) = p - 1$ and $-F_{p-1,Z} = G^{p-1}$, with $< G >=< U^{\mathbf{b}} >$ for some $\mathbf{b} \in \mathbb{N}^n \cap \mathbb{E}$ by theorem 2.14(2) (in particular

$\delta(x) \in \mathbb{N}$). We have

$$V(TF_{p,Z}, E, m_S) =< \left\{ H^{-1} \frac{\partial TF_{p,Z}}{\partial U_j} \right\}_{e+1 \leq j \leq n} > .$$

Note that the truncation maps T and $\tilde{T}$ associated with the local rings S and $\tilde{S}$ (definition 2.14) commute with the inclusion $G(m_S) \subseteq G(m_{\tilde{S}})$ by (2.65). Since $F_{p,Z} \in G(m_S) = k(x)[U_1, \ldots, U_n]$, we have

$$V(\tilde{T}F_{p,Z}, \tilde{E}, m_{\tilde{S}}) =< \left\{ H^{-1} \frac{\partial \tilde{T}F_{p,Z}}{\partial U_j} \right\}_{j=e+1}^{\tilde{n}} >= V(TF_{p,Z}, E, m_S) \otimes_{k(x)} k(\tilde{x})$$

with obvious notations, taking (2.65) into account. There exists $\tilde{\Theta} \in G(m_{\tilde{S}})$ such that

$$F_{p,\tilde{Z}} = F_{p,Z} + \tilde{\Theta}^p - G^{p-1}\tilde{\Theta}.$$

By lemma 2.17 applied to $F_{p,\tilde{Z}} \in G(m_{\tilde{S}})$, we deduce that

$$V(\tilde{T}F_{p,\tilde{Z}}, \tilde{E}, m_{\tilde{S}}) = V(TF_{p,Z}, E, m_S) \otimes_{k(x)} k(\tilde{x}). \tag{2.67}$$

This completes the proof of the theorem when $\omega(x) = \epsilon(x) - 1$, applying definition 2.16. If $\omega(x) = \epsilon(x)$, (1) and the last statement of (2.i) in the theorem also follow from (2.67) and the proof is complete.

Case 2: assume that $i_0(x) = p$. The proof runs parallel to that of case 1 (with $B = \emptyset$, $\tilde{T} = \mathrm{id}$, *cf.* remark 2.4) *provided that* $\epsilon(\tilde{x}) = \epsilon(x)$. Assume now that $\epsilon(\tilde{x}) > \epsilon(x)$. To complete the proof, we have to show that

$$(i_0(\tilde{x}), \omega(\tilde{x})) = (p, \omega(x)),$$

as well as the last statement in (2.ii). By (2.66), we have $\omega(x) = \epsilon(x)$, $\delta(x) \in \mathbb{N}$ and there is an expansion

$$F_{p,Z} = \sum_{|\mathbf{x}| = \delta(x)} \lambda(\mathbf{x}) U^{p\mathbf{x}} \in (k(\tilde{x})[U_1, \ldots, U_n]_{\delta(x)})^p, \ \lambda(\mathbf{x}) \in k(x).$$

Note that this situation possibly occurs only if $k(x)$ is *not* inseparably closed in $k(\tilde{x})$ (in particular $\tilde{n} > n$). We have $\mathbf{x} \in \mathbb{N}^n$ for every $\mathbf{x}$ such that $\lambda(\mathbf{x}) \neq 0$. Without loss of generality, it can be assumed that $\lambda(\mathbf{x}) \notin k(x)^p$ for every $\mathbf{x}$

such that $\lambda(\mathbf{x}) \neq 0$. Let $l(\mathbf{x}) \in S$ be a preimage of $\lambda(\mathbf{x})$. By (2.64), we may pick for every such $\mathbf{x}$ a unit $\tilde{l}(\mathbf{x}) \in T$ such that $\tilde{v}(\mathbf{x}) := \tilde{l}(\mathbf{x})^p - l(\mathbf{x})$ is a regular parameter of $\tilde{S}$ transverse to $\mathrm{div}(u_1 \cdots u_n)$. Expand

$$h = Z^p + \sum_{i=1}^{p} f_{i,Z} Z^{p-i} \in S[Z],\ \mathrm{ord}_{m_S} f_{i,Z} \geq i\delta(x).$$

For $1 \leq i \leq p-1$, the above inequality is strict, since $i_0(x) = p$. On the other hand, we have $\delta(x) \in \mathbb{N}$, so we deduce that

$$\frac{\mathrm{ord}_{m_S} f_{i,Z}}{i} \geq \delta(x) + \frac{1}{i} > \delta(x) + \frac{1}{p},\ 1 \leq i \leq p-1. \tag{2.68}$$

Let

$$\tilde{Z} := Z + \sum_{|\mathbf{x}|=\delta(x)} \tilde{l}(\mathbf{x}) u^{\mathbf{x}}.$$

By (2.68), there is an expansion

$$f_{p,\tilde{Z}} = -\sum_{|\mathbf{x}|=\delta(x)} \tilde{v}(\mathbf{x}) u^{p\mathbf{x}} + g + \tilde{g}, \tag{2.69}$$

with $g \in S$, $\mathrm{ord}_{m_S} g \geq p\delta(x) + 1$ and $\tilde{g} \in \tilde{S}$, $\mathrm{ord}_{m_{\tilde{S}}} \tilde{g} > p\delta(x) + 1$. We deduce that

$$\delta(h; u_1, \ldots, u_{\tilde{n}}; \tilde{Z}) = \delta(x) + \frac{1}{p}.$$

Since $\delta(x) + \frac{1}{p} \notin \mathbb{N}$, $\Delta_{\tilde{S}}(h; u_1, \ldots, u_{\tilde{n}}; \tilde{Z})$ has no solvable vertex within its initial face $\{\tilde{\mathbf{x}} \in \mathbb{R}^{\tilde{n}}_{\geq 0} :| \tilde{\mathbf{x}} |= \delta(x) + \frac{1}{p}\}$.

Let $(u_1, \ldots, u_{\tilde{n}}; \tilde{Z}_1)$ be well adapted coordinates at $\tilde{x}$. Without loss of generality, it can be assumed that $\tilde{Z}_1 = \tilde{Z} - \tilde{\theta}_1$ with $\mathrm{ord}_{m_{\tilde{S}}} \tilde{\theta}_1 \geq \delta(x) + 1$. By (2.69), we get

$$\mathrm{in}_{m_{\tilde{S}}} \tilde{h} = \tilde{Z}_1^p - \sum_{|\mathbf{x}|=\delta(x)} \tilde{V}(\mathbf{x}) U^{p\mathbf{x}} + G(U_1, \ldots, U_n) \in G(m_{\tilde{S}})[\tilde{Z}_1] \tag{2.70}$$

and (2.ii) is proved. We have $i_0(\tilde{x}) = p$, $\delta(\tilde{x}) = \delta(x) + \frac{1}{p}$ and $\epsilon(\tilde{x}) = \epsilon(x) + 1$. Finally, we have

$$\frac{\partial F_{p,\tilde{Z}_1}}{\partial U_j} = \sum_{|\mathbf{x}|=\delta(x)} \frac{\partial \tilde{V}(\mathbf{x})}{\partial \tilde{V}_j} U^{p\mathbf{x}} \in k(\tilde{x})[U_1, \ldots, U_n],\ n+1 \leq j \leq \tilde{n},$$

so $V(F_{p,\tilde{Z}_1}, \tilde{E}, m_{\tilde{S}}) \neq 0$ and $\omega(\tilde{x}) = \epsilon(\tilde{x}) - 1 = \omega(x)$. This concludes the proof.

Remark 2.7. Theorem 2.20 reduces computations of $\omega(x)$ to the case where S is strict Henselian, i.e. Henselian with separably algebraically closed residue field S/m_S by changing S to its strict Henselianization $\tilde{S}$, $\dim\tilde{S} = n = \dim S$.

Applying the theorem to a tower $\tilde{S}$ of smooth local base changes of the form $S \subseteq S[Y]_{(m_S, Y^p - l)}$ with $l \in S$ a unit with residue $\bar{l} \notin (S/m_S)^p$ also reduces computations of $\omega(x)$ to the case of an algebraically closed residue field for some $\tilde{S}$ with $\dim\tilde{S} > n = \dim S$, *vid.* comments before notation 2.1 for the excellent of such $\tilde{S}$.

The cone $\mathrm{Max}(x)$ and directrix $\mathrm{Dir}(x)$ have no such good behavior w.r.t. regular local base changes.

2.8 Resolution when $\omega(x) = 0$.

In this section, we prove that the multiplicity of $\mathcal{X}$ can be reduced at any point x such that $(m(x), \omega(x)) = (p, 0)$. This is achieved by combinatorial blowing ups in a way which is similar to the equal characteristic zero situation. This resolution algorithm does not depend on the choice of a valuation centered at x and we formalize Hironaka's A/B game as follows:

Definition 2.18. Let (S, h, E) be as before, $x \in \mathcal{X}$ and $L = \mathrm{Tot}(S[X]/(h))$. Suppose that for every valuation μ of L centered at x, a composition of local Hironaka-permissible blowing ups (definition 2.7)

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \tag{2.71}$$

is associated, where $x_i \in \mathcal{X}_i$ is the center of μ, $0 \leq i \leq r$. The sequence (2.71) is said to be *independent* if the blowing up center $\mathcal{Y}_i \subset (\mathcal{X}_i, x_i)$ does not depend on the chosen valuation μ *having center in* x_i, $0 \leq i \leq r-1$.

Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$. If $\epsilon(x) > 0$, recall that $\eta^{-1}(m_S) = \{x\}$, $k(x) = S/m_S$, and that

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1}Z + F_{p,Z} \in G(m_S)[Z] = k(x)[U_1, \ldots, U_n][Z]$$

by (2.47). The initial form of $H(x)$ in $G(m_S)$ is denoted H as before.

Lemma 2.21. *Assume that $m(x) = p$ and $\epsilon(x) = 1$, where $\{x\} = \eta^{-1}(m_S)$. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at $x \in \eta^{-1}(m_S)$. If*

$$H^{-1}F_{p,Z} \not\subseteq < U_1, \ldots, U_e >,$$

then $\omega(x) = 0$.

Proof. According to definition 2.16, we must show that $V(TF_{p,Z}, E, m_S) \neq 0$. Expand

$$H^{-1}F_{p,Z} =< \sum_{j=1}^{n} \alpha_j U_j > \subseteq G(m_S)_1,\ \alpha_j \in k(x).$$

By assumption, we have $\alpha_{j_0} \neq 0$ for some j_0, $e+1 \leq j_0 \leq n$, so

$$0 \neq H^{-1}\frac{\partial F_{p,Z}}{\partial U_{j_0}} \subseteq V(F_{p,Z}, E, m_S). \tag{2.72}$$

If $i_0(x) = p$, we have $TF_{p,Z} = F_{p,Z}$. If $i_0(x) = p-1$, then $H^{-1}G^p =< U_{j_1} >$ for some j_1, $1 \leq j_1 \leq e$, by theorem 2.14(2). Comparing with definition 2.14, we have $\mathbf{x} \in A \Longrightarrow px_{j_1} > H_{j_1}$, therefore $F_{p,Z} - TF_{p,Z} \in HU_{j_1}$. So (2.72) implies that $V(TF_{p,Z}, E, m_S) \neq 0$.

Proposition 2.22. *Assume that $(m(x), \omega(x)) = (p, 0)$, where $\{x\} = \eta^{-1}(m_S)$. Let $\mathcal{Y} \subset (\mathcal{X}, x)$ be a Hironaka-permissible center w.r.t. E, $\pi : \mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along $\mathcal{Y}$ and $x' \in \pi^{-1}(x)$.*

If $W := \eta(\mathcal{Y})$ is an intersection of components of E or if $\epsilon(y) = \epsilon(x)$, then $(m(x'), \omega(x') \leq (p, 0)$.

Proof. According to definition 2.16, there are two different cases to consider:

(1) $\epsilon(x) = 0$;

(2) $\epsilon(x) = 1$, $V(TF_{p,Z}, E, m_S) \neq (0)$.

To begin with, we have $\delta(x) \geq 1$ by proposition 2.3(ii). Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x with $I(W) = (\{u_j\}_{j \in J})$ for some subset $J \subseteq \{1, \ldots, n\}$. By definition 2.9, we have:

$$\epsilon(x) = \min_{1 \leq i \leq p} \left\{ \frac{\mathrm{ord}_{m_S}(H(x)^{-i} f_{i,Z}^p)}{i} \right\}. \tag{2.73}$$

Case 1: $\epsilon(x) = 0$. By (2.73), we have

$$\begin{cases} H(x)^{-i} f_{i,Z}^p & \subseteq m_S, \quad 1 \leq i < i_0(x) \\ H(x)^{-i_0(x)} f_{i_0(x),Z}^p & = S, \\ H(x)^{-i} f_{i,Z}^p & \subseteq S, \quad i_0(x) < i \leq p. \end{cases} \tag{2.74}$$

By proposition 2.7, there exists a commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \xleftarrow{\pi} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}S & \xleftarrow{\sigma} & \mathcal{S}' \end{array}$$

where $\sigma : \mathcal{S}' \to \mathrm{Spec}S$ is the blowing up along W. Let

$$\eta' : \mathcal{X}' \to \mathcal{S}', \ s' := \eta'(x'), \ S' := \mathcal{O}_{\mathcal{S}',s'}, \ E' := (\sigma^{-1}(E)_{\mathrm{red}})_{s'}.$$

Since $W \subseteq E$, it can be assumed after possibly reordering coordinates that

$$(J')_E := \{2, \ldots, e_0\}, \ J = \{1, e_0 + 1, \ldots, n_0\}, \ 1 \leq e_0 \leq e \leq n_0.$$

Furthermore, it can be assumed that $s' \in \mathrm{Spec}(S[u_{e_0+1}/u_1, \ldots, u_{n_0}/u_1])$ or that $s' \in \mathrm{Spec}(S[u_1/u_{n_0}, u_{e_0+1}/u_{n_0}, \ldots, u_{n_0-1}/u_{n_0}])$ with $n_0 > e_0$.

We first prove the proposition when $s' \in \mathrm{Spec}(S[u_{e_0+1}/u_1, \ldots, u_{n_0}/u_1])$. Let

$$h' := u_1^{-p} h = Z'^p + f_{1,Z'} Z'^{p-1} + \cdots + f_{p,Z'} \in S'[Z'],$$

where $Z' := Z/u_1, \ f_{i,Z'} := u_1^{-i} f_{i,Z} \in S'$ for $1 \leq i \leq p$. We have

$$E' = \mathrm{div}(u_1 \cdots u_{e_0} \frac{u_{e_0+1}}{u_1} \cdots \frac{u_e}{u_1}) \tag{2.75}$$

and (S', h', E') satisfies both conditions **(G)** and **(E)** by propositions 2.10 and 2.13. There exists an adapted r.s.p. of S' of the form

$$(u_1' := u_1, \ldots, u_{e_0}' := u_{e_0}, u_{e_0+1}', \ldots, u_{n_0'}', u_{n_0+1}' := u_{n_0+1}, \ldots, u_n' := u_n).$$

Since we do not assume that x' is a closed point, we have $e_0 \leq n_0' \leq n_0$ in general, with

$$n' := \mathrm{dim}S' = n - (n_0 - n_0').$$

We emphasize that the number of irreducible components e' of E' satisfies $e_0 \leq e' \leq e$ and that $e' \neq e$ in general because some of the u_j/u_1 in (2.75) may be units. After reordering coordinates, we may also assume that

$$E' = \mathrm{div}(u'_1 \cdots u'_{e'}) \text{ and } u'_j := u_j/u_1,\ e_0 + 1 \leq e' \leq e.$$

Since $\mathcal{Y}$ is Hironaka-permissible at x, we have (see definition 2.10):

$$\mathrm{ord}_W H(x) = p \sum_{j \in J} d_j \geq p.$$

Therefore $I' := u_1^{-p} H(x) \subseteq S'$ and this ideal is monomial in $(u'_1, \ldots, u'_{e'})$, i.e. $I' =: ({u'_1}^{H'_1} \cdots {u'_{e'}}^{H'_{e'}})$. We let:

$$\mathbf{x}' := (H'_1/p, \ldots, H'_{e'}/p, 0, \ldots, 0) \in \frac{1}{p}\mathbb{N}^{n'},$$

where

$$H'_1 = p(\sum_{j \in J} d_j - 1) \text{ and } H'_j = H_j = pd_j,\ 2 \leq j \leq e'. \tag{2.76}$$

Then (2.74) gives:

$$\begin{cases} I'^{-i} f_{i,Z'}^p & \subseteq m_S S' \quad 1 \leq i < i_0(x) \\ I'^{-i_0(x)} f_{i_0(x),Z'}^p & = S' \\ I'^{-i} f_{i,Z'}^p & \subseteq S' \quad i_0(x) < i \leq p. \end{cases} \tag{2.77}$$

This shows that

$$\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z') = \mathbf{x}' + \mathbb{R}^{n'}_{\geq 0}. \tag{2.78}$$

If $i_0(x) < p$, or if $\sum_{j \in J_E} d_j \notin \mathbb{N}$ or if $d_{j'} \notin \mathbb{N}$ for some j', $2 \leq j' \leq e'$, then $\mathbf{x}'$ is not solvable (definition 2.3) by (2.78), hence $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ is minimal. Therefore we may compute $\epsilon(x')$ from (2.78) and get $\epsilon(x') = 0$, so the proposition is proved in this case.

If ($i_0(x) = p$, $\sum_{j \in J_E} d_j \in \mathbb{N}$ and $d_{j'} \in \mathbb{N}$ for all j', $2 \leq j' \leq e'$), write $f_{p,Z} = \gamma u^{p\mathbf{x}}$, $\gamma \in S$ a unit and $\mathbf{x} := (d_1, \ldots, d_e, 0, \ldots, 0) \in \frac{1}{p}\mathbb{N}^n$. We have

$$\mathrm{in}_{\mathbf{x}'} h' = Z'^p + \lambda(\prod_{j=e'+1}^{e} \lambda_j^{H_j}) U'^{p\mathbf{x}'}, \tag{2.79}$$

where $\lambda \in k(x)$ (resp. $\lambda_j \in k(x')$) is the residue of γ (resp. of u_j/u_1). We let:

$$\lambda' := \lambda \prod_{j=e'+1}^{e} \lambda_j^{H_j} \in k(x'),\ \lambda' \neq 0.$$

If $\lambda' \notin k(x')^p$, then $\mathbf{x}'$ is not solvable and we also have $\epsilon(x') = 0$.
If $\lambda' \in k(x')^p$, let

$$C' := \mathrm{Spec}\left(\frac{k(x)[Z, U_1, U_{e_0+1}, \ldots, U_e]}{(\overline{H})}\right),\ \overline{H} := \mathrm{in}_{m_S} h = Z^p + \lambda \prod_{j=e'+1}^{e} U_j^{H_j}.$$

We claim that the affine cone C' is regular away from the torus

$$\mathbb{T} := \mathbb{A}^{e-e_0+2}_{k(x)} \backslash V(Z \prod_{j \in J_E} U_j).$$

To see this, let $(\lambda_l)_{l \in \Lambda_0}$ be an absolute p-basis of $k(x)$. By [46] theorem 30.5, the ideal of the singular locus of C' is:

$$I(\mathrm{Sing} C') = \left(\overline{H}, \{\frac{\partial \overline{H}}{\partial \lambda_l}\}_{l \in \Lambda_0}, \{\frac{\partial \overline{H}}{\partial U_j}\}_{e'+1 \leq j \leq e}\right).$$

If $d_j \notin \mathbb{N}$ for some j, $e'+1 \leq j \leq e$, then $\frac{\partial \overline{H}}{\partial U_j}$ does not vanish on $\mathbb{T}$. Otherwise, we have $\lambda \notin k(x)^p$ because $\mathbf{x}$ is a vertex of $\Delta_{\hat{S}}(u_1, \ldots, u_n; Z)$ and is not solvable. Therefore $\frac{\partial \overline{H}}{\partial \lambda_l}$ does not vanish on $\mathbb{T}$ for any $l \in \Lambda_0$ such that $\frac{\partial \lambda}{\partial \lambda_l} \neq 0$ and the claim is proved. We deduce that there exists a unit $l' \in S'$ such that

$$v' := l'^p + \gamma \prod_{j=e'+1}^{e} \left(\frac{u_j}{u_1}\right)^{H_j}$$

is a regular parameter of S' transverse to

$$E'_1 := \mathrm{div}(u'_1 \cdots u'_{e'} u'_{n_0+1} \cdots u'_{n'}),\ E'_1 \supseteq E'.$$

We may thus take $u'_{e'+1} := v'$ in our r.s.p. of S' adapted to E'. Let $Z'_1 := Z' - l' u'^{p\mathbf{x}'}$, so the polyhedron $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z'_1)$ has a vertex

$$\mathbf{x}'_1 := (H'_1/p, \ldots, H'_{e'}/p, 1/p, 0, \ldots, 0) \in \frac{1}{p}\mathbb{N}^{n'} \tag{2.80}$$

which is not solvable, since $\mathbf{x}'_1 \notin \mathbb{N}^{n'}$. Let $Z'_2 := Z'_1 - \theta'$, $\theta' \in S'$, be such that $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z'_2)$ is minimal. We deduce from (2.77) and (2.80) that

$$H(x') = (u'^{p\mathbf{x}'}),\ \epsilon(x') = 1 \text{ and } H'^{-1}F_{p,Z'_2} \not\subseteq < U'_1, \ldots, U'_{e'} > .$$

We get $m(x') = 1$ if $\mathbf{x}' = \mathbf{0}$, and $(m(x'), \omega(x')) = (p, 0)$ otherwise by lemma 2.21 as required.

If $s' \in \mathrm{Spec}(S[u_1/u_{n_0}, u_{e_0+1}/u_{n_0}, \ldots, u_{n_0-1}/u_{n_0}])$, it can be furthermore assumed that $s' \notin \mathrm{Spec}(S[u_{e_0+1}/u_1, \ldots, u_{n_0}/u_1])$, i.e. u_j/u_{n_0} is *not* a unit in S' for $j \in J_E$. The proof is now a simpler variation of the above one: (2.75) is replaced by

$$E' = \mathrm{div}(\frac{u_1}{u_{n_0}} u_2 \cdots u_{e_0} \frac{u_{e_0+1}}{u_{n_0}} \cdots \frac{u_e}{u_{n_0}} u_{n_0}).$$

The polyhedron $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ in (2.78) is minimal except if ($d_j \in \mathbb{N}$ for each j, $1 \leq j \leq e$, and $\lambda \in k(x')^p$) with notations as above. We have $\epsilon(x') = 0$ (resp. $\epsilon(x') = 1$) in the former (resp. in the latter) situation. This concludes the proof in case 1.

Case 2: $\epsilon(x) = 1$. The proof runs parallel to that in case 1 and we only indicate the necessary changes. By assumption, W is an intersection of components of E (case 2a) or $\epsilon(y) = \epsilon(x) = 1$ (case 2b).

To begin with, let $v \in S$ be such that $H(x)^{-1} f_{p,Z} = (v)$. By assumption, we have $V(TF_{p,Z}, E, m_S) \neq (0)$, so v is transverse to E.

In case 2a, we may assume that $(u_1, \ldots, u_e, v, u_{e+2}, \ldots, u_n)$ is an adapted r.s.p. of S after renumbering variables. Since $\mathbf{x}_0 := (d_1, \ldots, d_e, \frac{1}{p}, \ldots, 0) \notin \mathbb{N}^n$ is the unique vertex of $\Delta_{\hat{S}}(h; u_1, \ldots, u_e, v, u_{e+2}, \ldots, u_n; Z)$ induced by $f_{p,Z}$, this polyhedron has no solvable vertex. In other terms, it can be assumed that $v = u_{e+1}$.

In case 2b, theorem 2.4 implies that $v \in I(W)$, so $(u_1, \ldots, u_e, v)$ can be completed to an adapted r.s.p. of S such that $I(W) = (\{u_j\}_{j \in J})$ for some subset $J \subseteq \{1, \ldots, n\}$. The polyhedron $\Delta_{\hat{S}}(h; u_1, \ldots, u_e, v, u_{e+2}, \ldots, u_n; Z)$ has no solvable vertex either and it can also be assumed that $v = u_{e+1}$.

We remark in both cases 2a and 2b that, if $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ has a vertex distinct from $\mathbf{x}_0$, then it has exactly two vertices: this follows from theorem 2.14(2), the other vertex being then given by

$$\mathbf{x}_1 := (\frac{D_1}{p(p-1)}, \ldots, \frac{D_e}{p(p-1)}, 0, \ldots, 0),\ (\mathrm{Disc}_Z(h)) =: (u_1^{D_1} \cdots u_e^{D_e}). \quad (2.81)$$

After blowing up, we obtain a (S', h', E') again satisfying conditions **(G)** and **(E)**.

In case 2a, there exists an adapted r.s.p. of S' of the form

$$(u'_1 := u_1, \ldots, u'_{e_0} := u_{e_0}, u'_{e_0+1}, \ldots, u'_{e_1}, u'_{e+1} := u_{e+1}, \ldots, u'_n := u_n),$$

with $J = \{1, e_0+1, \ldots, e\}$ and $E' = \mathrm{div}(u'_1 \cdots u'_{e'})$ after reordering variables, $1 \leq e_0 \leq e' \leq e_1 \leq e$. Then $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ has again a vertex

$$\mathbf{x}' := (H'_1/p, \ldots, H'_{e'}/p, 0, \ldots, 0, 1/p, 0, \ldots, 0) \notin \mathbb{N}^{n-(e-e_1)},$$

thus $\mathbf{x}'$ is not solvable. We deduce that $\epsilon(x') \leq 1$ and $\omega(x') = 0$ follows from lemma 2.21 if $(m(x'), \epsilon(x')) = (p, 1)$.

In case 2b, it can be assumed after reordering variables that

$$(J')_E := \{2, \ldots, e_0\},\ J = \{1, e_0+1, \ldots, n_0\},\ 1 \leq e_0 \leq e,\ e+1 \leq n_0.$$

We let $u'_{j'} := u_{j'}$ for $j' \in J'$ and consider three distinct situations depending on x', up to reordering coordinates:

(1) $s' \in \mathrm{Spec}(S[u_{e_0+1}/u_1, \ldots, u_{n_0}/u_1])$ and $u_{e+1}/u_1 \in m_{S'}$. We may complete the family $(\{u_{j'}\}_{j' \in J'})$ to an adapted r.s.p. of S' by adding

$$(u'_1 := u_1, u'_{e_0+1}, \ldots, u'_{e_1}, u'_{e_1+1} := u_{e+1}/u_1),\ n' := \dim S' = n - (n_0 - e_1).$$

Then $\Delta_{\hat{S}'}(h'; u'_1, \ldots, u'_n; Z')$ has a vertex

$$\mathbf{x}' := (H'_1/p, \ldots, H'_{e'}/p, 1/p, 0, \ldots, 0) \notin \mathbb{N}^{n'},$$

thus $\mathbf{x}'$ is not solvable. We conclude that $\epsilon(x') \leq 1$ and that $\omega(x') = 0$ if $(m(x'), \epsilon(x')) = (p, 1)$ by lemma 2.21.

(2) $s' \in \mathrm{Spec}(S[u_1/u_{n_0}, u_{e_0+1}/u_{n_0}, \ldots, u_{n_0-1}/u_{n_0}])$ and $u_{e+1}/u_{n_0} \in m_{S'}$, where $n_0 > e+1$. After dealing with (1), we may assume furthermore that $u_j/u_{n_0} \in m_{S'}$, $j \in J_E$. We complete the family $(\{u_{j'}\}_{j' \in J'})$ to an adapted r.s.p. of S' by adding

$$(u'_{e_0+1} := u_{e_0+1}/u_{n_0}, \ldots, u'_{e+1} := u_{e+1}/u_{n_0}, u'_{n_1}, \ldots, u'_{n_0-1}, u'_{n_0} := u_{n_0}),$$

with $n' := \dim S' = n - (n_1 - e - 2)$. We conclude as in (1).

(3) $I(W)S' = (u_{e+1})$. We complete the family $(\{u_{j'}\}_{j'\in J'})$ to an adapted r.s.p. of S' by adding

$$(u'_1 := u_{e+1}, u'_{e_0+1}, \dots, u'_{n_1}),\ n' := \dim S' = n - (n_0 - n_1).$$

Let $E' =: \mathrm{div}(u'_1 \cdots u'_{e'})$ and consider two situations as in case 1:

If $\frac{1}{p} + \sum_{j\in J_E} d_j \notin \mathbb{N}$ or if $d_{j'} \notin \mathbb{N}$ for some j', $2 \le j' \le e'$, then the polyhedron $\Delta_{\hat{S}'}(h'; u'_1, \dots, u'_n; Z')$ is minimal and we have $\epsilon(x') = 0$.

If $(\frac{1}{p} + \sum_{j\in J_E} d_j \in \mathbb{N}$ and $d_{j'} \in \mathbb{N}$ for every j', $2 \le j' \le e')$, the initial form polynomial $\mathrm{in}_{\mathbf{x}'}h'$ has the form

$$\mathrm{in}_{\mathbf{x}'}h' = Z'^p - \mu^{p-1}U'^{(p-1)\mathbf{x}'}Z' + \lambda(\prod_{j=e'+1}^{e} \lambda_j^{H_j})U'^{p\mathbf{x}'},$$

where $\lambda \in k(x)$ (resp. $\lambda_j \in k(x')$) is the residue of γ (resp. of u_j/u_{e+1}), *vid.* (2.79). We have $\mu \neq 0$ in the above formula precisely if

$$U^{p(\mathbf{x}_1-\mathbf{x}_0)} = U_{j_0}/U_{e+1},\ u_{j_0}/u_{e+1} \in S' \text{ a unit}$$

for some j_0, $e_0 + 1 \le j_0 \le e$ with notations as in (2.81). Then μ^{p-1} is the residue in $k(x')$ of

$$\gamma_{p-1,Z} \prod_{j=e'+1}^{e} \left(\frac{u_j}{u_{e+1}}\right)^{A_{p-1,j}}$$

with notations as in theorem 2.14(2). The end of the proof goes along as in case 1. This completes the proof of (3), hence the proof of the proposition in case 2.

Remark 2.8. This proposition is a lighter version of theorem 3.6 where it is assumed that $\omega(x) > 0$ and that the blowing up centers are permissible of the first or second kind (definitions 3.1 and 3.2 below).

Theorem 2.23. *Assume that* $(m(x), \omega(x)) = (p, 0)$, *where* $\{x\} = \eta^{-1}(m_S)$. *For every valuation* μ *of* $L = \mathrm{Tot}(S[X]/(h))$ *centered at* x, *there exists a finite and independent composition of local Hironaka-permissible blowing ups (2.71) such that* $m(x_r) < p$.

Proof. We will produce a Hironaka-permissible center $\mathcal{Y} \subset (\mathcal{X}, x)$ w.r.t. E satisfying the assumptions of proposition 2.22 and such that the following holds:

(*) let $\pi : \mathcal{X}' \to (\mathcal{X}, x)$ be the blowing up along $\mathcal{Y}$ and $x' \in \pi^{-1}(x)$. Then

$$\delta(x') < \delta(x).$$

Applying proposition 2.22, the center $x_1 \in \mathcal{X}'$ of a given valuation μ again satisfies the assumptions of the theorem if $m(x_1) = p$. Iterating, any finite sequence (2.71) induces a sequence

$$\delta(x_r) < \delta(x_{r-1}) < \cdots < \delta(x)$$

provided that $m(x_i) = p$, $1 \leq i \leq r-1$. Since $\delta(x_i) \in \frac{1}{p}\mathbb{N}$, we have $\delta(x_r) < 1$ for some $r \geq 1$, hence $m(x_r) < p$ by proposition 2.3(2), so the theorem follows from claim (*). In order to construct $\mathcal{Y}$ with the required properties, we consider two cases as in the proof of proposition 2.22.

Case 1: $\epsilon(x) = 0$. We have $\delta(x) = \sum_{j=1}^{e} d_j \geq 1$. Therefore there exists a subset

$$J \subseteq \{1, \ldots, e\},\ \sum_{j \in J} d_j \geq 1,$$

with smaller possible number of elements among all subsets of $\{1, \ldots, e\}$ with this property. Let $W := V(\{u_j\}_{j \in J}) \subset \mathrm{Spec} S$ and remark that

$$\mathrm{ord}_W H(x) = p \sum_{j \in J} d_j \geq p.$$

Hence $\mathcal{Y} := \eta^{-1}(W) = V(Z, \{u_j\}_{j \in J})$ is Hironaka-permissible w.r.t. E and W is an intersection of components of E. By (2.76), we have

$$\mathrm{ord}_{m_{S'}} H(x') \leq p(\delta(x) + \sum_{j \in J \setminus \{j_0\}} d_j - 1), \tag{2.82}$$

where $I(W)S' = (u_{j_0})$. The minimality property required of J implies that

$$\sum_{j \in J \setminus \{j_1\}} d_j < 1 \text{ for every } j_1 \in J \text{ (so } \sum_{j \in J} d_j < 2 \text{ if } \mid J \mid \geq 2). \tag{2.83}$$

If $\epsilon(x') = 0$, we deduce from (2.82) that

$$p\delta(x') = \mathrm{ord}_{m_{S'}} H(x') < p\delta(x)$$

as required in (*). Note that if $\mid J \mid = 1$, we have $\lambda = \lambda'$ in (2.79) and $S = S'$, hence $\lambda' \notin k(x')^p = k(x)^p$. Since $\epsilon(x') = 0$ in this situation, we may now assume that $\mid J \mid \geq 2$.

If $\epsilon(x') = 1$, we are in the situation discussed in (2.80). We may then take $j_0 = 1$, $E' = \mathrm{div}(u'_1 \cdots u'_{e'})$ and have

$$\sum_{j \in J} d_j \in \mathbb{N},\ d_j \in \mathbb{N} \text{ for } 2 \leq j \leq e'.$$

By (2.83), we have $\sum_{j \in J} d_j = 1$, $d_j = 0$ for $2 \leq j \leq e'$, so $H(x') = (1)$ and $m(x') = 1$. This concludes the proof in case 1.

Case 2: $\epsilon(x) = 1$. We have $\delta(x) = \frac{1}{p} + \sum_{j=1}^{e} d_j \geq 1$.

If $\delta(x) > 1$, there exists a subset

$$J \subseteq \{1, \ldots, e\},\ \sum_{j \in J} d_j \geq 1,$$

with smaller possible number of elements among all subsets of $\{1, \ldots, e\}$ with this property as in case 1 and we also let $W := V(\{u_j\}_{j \in J}) \subset \mathrm{Spec} S$. The proof goes along as in case 1, with

$$p\delta(x') - p\delta(x) \leq \mathrm{ord}_{m_{S'}} H(x') - \mathrm{ord}_{m_S} H(x) < 0.$$

If $\delta(x) = 1$, we may assume that $H(x)^{-1} f_{p,Z} = (u_{e+1})$ and that (2.81) holds if $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ has more than one vertex. In this case, this polyhedron has exactly two vertices and we have

$$H(x)^{-(p-1)} f_{p-1,Z}^p = (u_{j_0})^{p-1} \text{ for some } j_0,\ 1 \leq j_0 \leq e$$

by theorem 2.14(2). We deduce that

$$H(x)^{-i} f_{i,Z}^p \subseteq (u_{j_0}, u_{e+1})^i,\ 1 \leq i \leq p \tag{2.84}$$

by definition of $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$. We let $J := \{j : d_j > 0\} \cup \{e+1\}$ and

$$W := V(\{u_j\}_{j \in J}) \subset \mathrm{Spec} S,\ \mathcal{Y} := \eta^{-1}(W) = V(Z, \{u_j\}_{j \in J}).$$

We have $\mathrm{ord}_W H(x) = p$, so $\mathcal{Y}$ is Hironaka-permissible w.r.t. E. Since $H(x)^{-1} f_{p,Z} = (u_{e+1})$, we have $\epsilon(y) = \epsilon(x) = 1$ by (2.84), where $y \in \mathcal{X}$ is the generic point of $\mathcal{Y}$. Thus proposition 2.22 applies and gives $m(x') \leq p-1$ under either assumption (1)(2) or (3) in the proof of proposition 2.22. This concludes the proof.

3 Permissible blowing ups.

3.1 Blowing ups of the first and second kind.

In this section, we introduce a notion of permissible blowing up which is well behaved w.r.t. our main resolution invariant $y \mapsto \iota(y)$ on $\mathcal{X}$. We assume that $m(x) = p$, $\{x\} = \eta^{-1}(m_S)$ and $\omega(x) > 0$ in what follows since theorem 2.23 rules out the case $\omega(x) = 0$.

Definition 3.1. Let $\mathcal{Y} \subset \mathcal{X}$ be an integral closed subscheme with generic point y. We say that $\mathcal{Y}$ is *permissible of the first kind* at x if $m(y) = m(x) = p$ and the following conditions hold:

(i) $\mathcal{Y}$) is Hironaka-permissible w.r.t. E at x(definition 2.7);

(ii) $\epsilon(y) = \epsilon(x)$.

If $y \in \mathcal{X}$ satisfies $m(y) = p$, it follows from the definition that $\mathcal{Y} := \overline{\{y\}}$ is permissible of the first kind at y. It also follows from (ii) that a permissible center of the first kind has codimension at least two in $\mathcal{X}$.

The main result of this chapter (theorem 3.6 below) will require comparing the initial form polynomials $\mathrm{in}_W h$ and $\mathrm{in}_{m_S} h$. We keep notations as in section 2.4: given well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x, we let

$$W := \eta(\mathcal{Y}),\ I(W) = (\{u_j\}_{j \in J}). \tag{3.1}$$

We denote:

$$\mathrm{in}_W h = Z^p + \sum_{i=1}^{p} F_{i,Z,W} Z^{p-i} \in G(W)[Z]$$

and (proposition 2.16(i) since $\epsilon(x) > 0$)

$$\mathrm{in}_{m_S} h = Z^p - G^{p-1} Z + F_{p,Z} \in G(m_S)[Z].$$

There are associated homogeneous submodules

$$H_W \subseteq G(W)_{d_W} \text{ (resp. } H := H_{m_S} \subseteq G(W)_d)$$

by (2.44), with

$$d_W := \sum_{j\in J_E} H_j,\ d = \sum_{j=1}^{e} H_j.$$

A word of caution is required at this point: formula (2.44) *defines* the monomial ideal H_W which is the *initial form* of $H(x)$ in $G(W)$ and is different in general from the ideal $H(\Xi)$ associated to the triple $(G(W)_\Xi, \mathrm{in}_W h, E_W)$, $\Xi := (\{U_j\}_{j\in J}) + m_{S_W}$.

Corresponding to the above choice for H_W (resp. to H), there are associated S_W-submodules

$$V(F_{p,Z,W}, E, W) \subseteq G(W)_{\epsilon(y)-1},\ J(F_{p,Z,W}, E, W) \subseteq \widehat{G(W)}_{\epsilon(y)}$$

(resp. $k(x)$-vector subspaces

$$V(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)-1},\ J(F_{p,Z}, E, m_S) \subseteq G(m_S)_{\epsilon(x)})$$

given by (2.45).

Notation 3.1. We first recall notations and definitions from section 2.4. We denote

$$J_E := J \cap \{1,\dots,e\},\ J' := \{1,\dots,n\}\backslash J \text{ and } (J')_E := \{1,\dots,e\}\backslash J_E.$$

The image $\overline{m}_S$ of m_S in S_W has regular parameters $(\overline{u}_j)_{j\in J'}$, the respective residues of the corresponding parameters of S.

Let now $d \in \mathbb{N}$ be fixed and

$$F = \sum_{|\mathbf{a}|=d} \hat{f}_{\mathbf{a}} U^{\mathbf{a}} \in \widehat{G(W)}_d = \widehat{S_W}[\{U_j\}_{j\in J}]_d.$$

Note that $\mathrm{gr}_{\overline{m}_S}\widehat{G(W)}_d \simeq \mathrm{gr}_{\overline{m}_S} G(W)_d$ and that it has a structure of graded $\mathrm{gr}_{\overline{m}_S} S_W$-module. For any $d_0 \leq \min_{\mathbf{a}}\{\mathrm{ord}_{\overline{m}_S}\hat{f}_{\mathbf{a}}\}$, F has an initial form in $\mathrm{gr}_{\overline{m}_S} G(W)_d$ by taking

$$\overline{F} := \sum_{|\mathbf{a}|=d} (\mathrm{cl}_{d_0}\hat{f}_{\mathbf{a}}) U^{\mathbf{a}} \in (\mathrm{gr}_{\overline{m}_S} G(W)_d)_{d_0}. \tag{3.2}$$

This notation requires specifying d_0 to avoid ambiguity. We extend the notation to homogeneous submodules $M \subseteq \widehat{G(W)}_d$ as follows:

$$\overline{M} :=< \overline{F},\ F \in M >\subseteq (\mathrm{gr}_{\overline{m}_S} G(W)_d)_{d_0}$$

for fixed $d_0 \leq \min\{d_0(F), F \in M\}$ with obvious notations. For fixed d, d_0, there is an inclusion of S/m_S-vector spaces:

$$(\mathrm{gr}_{\overline{m}_S} G(W)_d)_{d_0} \subset \frac{G(m_S)_{d+d_0}}{< (\{U_j\}_{j\in J})^{d+1} \cap G(m_S)_{d+d_0} >}. \tag{3.3}$$

Proposition 3.1. *Let $\mathcal{Y}$ be permissible of the first kind at $x \in \mathcal{Y}$. Then for any well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x such that $I(W) = (\{u_j\}_{j\in J})$, the initial form $\mathrm{in}_{m_S} h \in G(m_S)[Z]$ satisfies*

$$H^{-1} < G^p, F_{p,Z} >\subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)}.$$

Proof. The existence of well adapted coordinates $(u_1, \ldots, u_n; Z)$ such that $I(W) = (\{u_j\}_{j\in J})$ follows from theorem 2.4. This theorem furthermore implies that the polyhedron

$$\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z) = \mathrm{pr}_J(\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)) \text{ is minimal,} \tag{3.4}$$

where $\mathrm{pr}_J : \mathbb{R}^n \to \mathbb{R}^J$ denotes the projection on the $(u_j)_{j\in J}$-space.

By (ii) of definition 3.1, we have $\epsilon(x) = \epsilon(y)$. Therefore

$$H^{-i} F^p_{i,Z} = \mathrm{cl}_0(H_W^{-i} F^p_{i,Z,W}) \subseteq G(m_S)_{i\epsilon(x)} = k(x)[U_1, \ldots, U_n]_{i\epsilon(x)}$$

is simply the reduction of $H_W^{-i} F^p_{i,Z,W}$ modulo $\overline{m}_S$ for $1 \leq i \leq p$, i.e. taking $d_0 = 0$ in notation 3.1, via the inclusion (3.3)

$$k(x)[\{U_j\}_{j\in J}]_{i\epsilon(y)} \simeq (\mathrm{gr}_{\overline{m}_S} G(W)_{i\epsilon(y)})_0 \subset G(m_S)_{i\epsilon(y)} \simeq k(x)[U_1, \ldots, U_n]_{i\epsilon(x)}.$$

We get respectively $(H^{-1}G^p)^{p-1}$, $(H^{-1}F_{p,Z})^p$ for $i = p-1, p$ and this completes the proof.

The following corollary will be required in the proof of the blowing up theorem below. The adapted cone $\mathrm{Max}(x) \subseteq G(m_S)$ is defined in definition 2.17.

Corollary 3.2. *With notations as above, let $\mathcal{Y}$ be permissible of the first kind at x. The defining ideal* $\mathrm{IMax}(x) \subseteq G(m_S)$ *of* $\mathrm{Max}(x)$ *satisfies*

$$\mathrm{IMax}(x) = (\mathrm{IMax}(x) \cap k(x)[\{U_j\}_{j\in J}])G(m_S).$$

Proof. This follows from proposition 3.1 and definition 2.17. Note that the truncation operator T used in the definition of $\mathrm{Max}(x)$ does not affect the conclusion of the corollary since it is obvious from the definitions that:

$$V(F_{p,Z}, E, m_S) \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)-1} \Rightarrow V(TF_{p,Z}, E, m_S) \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)-1}.$$

The same implication holds for $J(F_{p,Z}, E, m_S)$ and $J(TF_{p,Z}, E, m_S)$.

We now define a second kind of permissible blowing up.

Definition 3.2. Let $\mathcal{Y} \subset \mathcal{X}$ be an integral closed subscheme with generic point y. We say that $\mathcal{Y}$ is *permissible of the second kind* at x if $m(y) = m(x) = p$ and the following conditions hold:

(i) $\mathcal{Y}$) is Hironaka-permissible w.r.t. E at x(definition 2.7);

(ii) $\epsilon(y) = \epsilon(x) - 1$ and $i_0(y) \leq i_0(x)$;

(iii) $\overline{J}(F_{p,Z,W}, E, W) := \mathrm{cl}_0 J(F_{p,Z,W}, E, W) \neq 0$.

Proposition 3.3. *Let $\mathcal{Y}$ be permissible of the second kind at x. For any well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x such that $I(W) = (\{u_j\}_{j\in J})$, the initial form* $\mathrm{in}_{m_S} h \in G(m_S)[Z]$ *satisfies*

$$\begin{cases} H^{-1}G^p & \subseteq & U_{j_0} k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)} \text{ for some } j_0 \in (J')_E \\ H^{-1}F_{p,Z} & = & < \sum_{j\in J'} U_{j'}\Phi_{j'}(\{U_j\}_{j\in J}) + \Psi(\{U_j\}_{j\in J}) > \subseteq G(m_S)_{\epsilon(x)}. \end{cases} , \tag{3.5}$$

with $\Phi_{j'} \neq 0$ for some $j' \in J'\backslash(J')_E$. In particular $\epsilon(y) = \omega(x)$.

Proof. We argue as in the proof of proposition 3.1 and build up from (3.4). By (ii) of definition 3.2, we have $\epsilon(x) = \epsilon(y) + 1$. Therefore

$$\mathrm{cl}_0(H_W^{-i} F_{i,Z,W}^p) = 0,\ 1 \leq i \leq p.$$

This shows that $H_W^{-i} F_{i,Z,W}^p \subseteq \overline{m}_S S_W[\{U_j\}_{j\in J_E}]_{i\epsilon(y)}$. We have $\epsilon(y) > 0$, so $F_{i,Z,W} = 0$, $1 \leq i \leq p-2$ by theorem 2.14. For $i = p-1$, we have

$-F_{p-1,Z,W} = G_W^{p-1}$ for some $G_W \in G(W)_{\delta(y)}$ (so $G_W = 0$ if $\delta(y) \notin \mathbb{N}$). We deduce that

$$H_W^{-1}(G_W^p, F_{p,Z,W}) \subseteq \overline{m}_S S_W[\{U_j\}_{j\in J_E}]_{\epsilon(y)}. \tag{3.6}$$

If $i_0(x) = p$, we have $H^{-1}G^p = 0$ so the first part of (3.5) is trivial. If $i_0(x) = p-1$, we have $i_0(y) = p-1$ by definition 3.2(ii), so $G_W \neq 0$. The first part of (3.5) then follows from (3.6), i.e.

$$H^{-1}G^p = \mathrm{cl}_1(H_W^{-1}G_W^p) \subseteq U_{j_0}k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)},$$

for some $j_0 \in (J')_E$.

Going back to the definition of $J(F_{p,Z,W}, E, W)$ in (2.41), we deduce from (3.6) that

$$\overline{J}(F_{p,Z,W}, E, W) =< \mathrm{cl}_0(H_W^{-1}\frac{\partial F_{p,Z,W}}{\partial \overline{u}_{j'}}), j' \in J'\backslash(J')_E >\subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}.$$

Taking classes as in (3.2) with $d_0 = 1$, we get

$$\mathrm{cl}_1(H_W^{-1}F_{p,Z,W}) \subseteq \sum_{j'\in J'} U_{j'}k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}.$$

Since $\mathrm{cl}_1(H_W^{-1}F_{p,Z,W})$ is a homomorphic image of $H^{-1}F_{p,Z} \in G(m_S)_{\epsilon(x)}$ as described in (3.3), there exists an expansion (3.5). For $j' \in J'\backslash(J')_E$, we have

$$H^{-1}\frac{\partial F_{p,Z}}{\partial U_{j'}} = \mathrm{cl}_0(H_W^{-1}\frac{\partial F_{p,Z,W}}{\partial \overline{u}_{j'}}).$$

Collecting together for all $j' \in J'\backslash(J')_E$, we get

$$\overline{J}(F_{p,Z,W}, E, W) =< H^{-1}\frac{\partial F_{p,Z}}{\partial U_{j'}}, j' \in J'\backslash(J')_E >\subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}$$

and the second part of (3.5) follows from definition 3.2(iii).

Note that $\epsilon(y) = \omega(x)$ is an immediate consequence of definition 2.16 if $i_0(m_S) = p$. If $i_0(m_S) = p-1$, we must introduce a truncation operator $T : G(m_S)_{\delta(x)} \to G(m_S)_{\delta(x)}$ in order to compute $\omega(x)$. The first part of (3.5) now shows that there exists $j_0 \in (J')_E$ such that

$$H^{-1}(F_{p,Z} - TF_{p,Z}) \in U_{j_0}k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}.$$

Since $\overline{J}(F_{p,Z,W}, E, W) \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(y)}$, we thus have:

$$H^{-1}\frac{\partial F_{p,Z}}{\partial U_{j'}} = H^{-1}\frac{\partial TF_{p,Z}}{\partial U_{j'}}$$

for every $j' \in J'\backslash(J')_E$. This proves that $\omega(x) = \epsilon(y)$.

Note that it follows from the above proposition that a permissible center of the second kind has codimension at least two in $\mathcal{X}$, since $\epsilon(y) > 0$. We now introduce the adapted cone associated to a permissible blowing up. Recall the definition of B from (2.53) (*cf.* also definition 2.16). We have $B = \emptyset$ if $i_0(m_S) = p$, and

$$B = \{j : U_j \text{ divides } H^{-1}G^p\} \text{ if } i_0(m_S) = p - 1.$$

Definition 3.3. Let $\mathcal{Y} \subset \mathcal{X}$, with generic point y, be a permissible center at x. We define a subcone

$$C(x, \mathcal{Y}) \subset \mathrm{Spec}(k(x)[\{U_j\}_{j\in J}])$$

as follows: if $\mathcal{Y}$ is of the first kind, we let:

$$C(x, \mathcal{Y}) := \mathrm{Spec}\left(\frac{k(x)[\{U_j\}_{j\in J}]}{(\mathrm{IMax}(x) \cap k(x)[\{U_j\}_{j\in J}])}\right);$$

if $\mathcal{Y}$ is of the second kind, we let $B_J := B\backslash\{j_0\}$ with notations as in proposition 3.3 and define:

$$C(x, \mathcal{Y}) := \mathrm{Max}(\overline{J}(F_{p,Z,W}, E, W)) \cap \{U_{B_J} = 0\}.$$

In both cases, we denote the associated projective cone by $PC(x, \mathcal{Y}) \subseteq \mathbb{P}^{|J|-1}_{k(x)}$.

Theorem 3.4. *Let $S \subseteq \tilde{S}$ be a local base change which is regular, $\tilde{S}$ excellent. Let $\tilde{x} \in \tilde{\eta}^{-1}(m_{\tilde{S}})$ and $x \in \eta^{-1}(m_S)$ be its image.*

If $\mathcal{Y} \subset \mathcal{X}$ is a permissible center (of the first or second kind) at x, then

$$\tilde{\mathcal{Y}} := \mathcal{Y} \times_S \mathrm{Spec}\tilde{S} \subseteq \tilde{\mathcal{X}} = \mathcal{X} \times_S \mathrm{Spec}\tilde{S}$$

is permissible (of the first or second kind) at $\tilde{x}$.

Proof. We denote $(\tilde{S}, \tilde{h}, \tilde{E})$ and $(u_1, \ldots, u_{\tilde{n}})$ as in notations 2.1 and 2.2. Since W has normal crossings with E at x, $\tilde{W} := \tilde{\eta}(\tilde{\mathcal{Y}})$ has normal crossings with $\tilde{E}$ at $\tilde{x}$. Since $\mathcal{Y}$ is permissible at x, we have $m(y) = p$. Any generic point $\tilde{y}$ of $\tilde{\mathcal{Y}}$ has $m(\tilde{y}) = p$ by theorem 2.20(1), and $\tilde{\mathcal{Y}}$ itself is irreducible by proposition 2.10. Theorem 2.20(2) applies to $\tilde{y}$ (with $n(y) = \tilde{n}(y)$) and to $\tilde{x}$ and states that

$$\epsilon(\tilde{y}) = \epsilon(y),\ \epsilon(\tilde{x}) \geq \epsilon(x),\ i_0(\tilde{y}) = i_0(y),\ i_0(\tilde{x}) = i_0(x)$$

Cases of inequality $\epsilon(\tilde{x}) > \epsilon(x)$ are classified in *ibid.*(2.ii).

Suppose that $\epsilon(\tilde{x}) > \epsilon(x)$. Then

$$F_{p,Z} \in k(x)[U_1^p, \ldots, U_n^p] \text{ and } i_0(m_S) = i_0(m_{\tilde{S}}) = p.$$

Then $\mathcal{Y}$ is permissible of the first kind since $F_{p,Z} \in k(x)[U_1^p, \ldots, U_n^p]$ is incompatible with the conclusion of proposition 3.3. Note that

$$\epsilon(y) = \epsilon(x) = \epsilon(\tilde{x}) - 1.$$

We claim that $\tilde{\mathcal{Y}}$ is permissible of the second kind at $\tilde{x}$.

To prove the claim, note that definition 3.2(i) and $i_0(\tilde{y}) \leq i_0(\tilde{x}) = p$ are already checked. We have

$$H^{-1}\frac{\partial F_{p,\tilde{Z}}}{\partial U_{j'}} = H^{-1}\Phi_{j'}(U_1, \ldots, U_n) \neq 0, \tag{3.7}$$

with notations as in theorem 2.20(2.ii) for some j', $n+1 \leq j' \leq \tilde{n}$. Since $H(\tilde{x}) = H(x)\tilde{S}$ by theorem 2.20(2.i), and $H^{-1}F_{p,Z} \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)}$ by proposition 3.1, we have

$$H^{-1}F_{p,\tilde{Z}} \subseteq \sum_{j=1}^{\tilde{n}} U_j k(\tilde{x})[\{U_j\}_{j\in J}]_{\epsilon(x)}.$$

This proves that definition 3.2(iii) holds for $\tilde{\mathcal{Y}}$ at $\tilde{x}$. On the other hand this implies that $\epsilon(\tilde{y}) = \epsilon(y)$ because

$$H^{-1}F_{p,\tilde{Z}} \nsubseteq k(\tilde{x})[\{U_j\}_{j\in J}]_{\epsilon(\tilde{x})}$$

follows obviously from (3.7). So definition 3.2(ii) is also checked and $\tilde{\mathcal{Y}}$ is permissible of the second kind at $\tilde{x}$.

Assume now that $\epsilon(\tilde{x}) = \epsilon(x)$. If $\mathcal{Y}$ is permissible of the first kind at x, we have $\epsilon(\tilde{y}) = \epsilon(\tilde{x})$, so $\tilde{\mathcal{Y}}$ is also permissible of the first kind at $\tilde{x}$.

If $\mathcal{Y}$ is permissible of the second kind at x, definition 3.2(ii) is checked. Finally by proposition 3.3, the polyhedron $\Delta_{\tilde{S}}(h; u_1, \ldots, u_n; Z)$ has a vertex $\mathbf{x}$ such that $x_{j'} \notin \mathbb{N}$ for some $j' \in J'\backslash J_E$. The corresponding vertex

$$\tilde{\mathbf{x}} := (\mathbf{x}, \underbrace{0, \ldots, 0}_{\tilde{n}-n}) \in \Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; Z)$$

is thus not solvable. We hence get $\tilde{\mathbf{x}} \in \Delta_{\tilde{S}}(u_1, \ldots, u_{\tilde{n}}; \tilde{Z})$ and definition 3.2(iii) is checked. Hence $\tilde{\mathcal{Y}}$ is permissible of the second kind at $\tilde{x}$ as required, since $H(\tilde{x}) = H(x)T$.

3.2 Blowing up theorem.

Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along a permissible center $\mathcal{Y}$ (of the first or second kind) at $x \in \mathcal{Y}$, $\{x\} = \eta^{-1}(m_S)$. Our objective is to relate $\omega(x')$ to $\omega(x)$ for points $x' \in \pi^{-1}(x)$.

We keep notations as in proposition 2.7 and proposition 2.10. Then $\sigma : \mathcal{S}' \to \mathrm{Spec}S$ denotes the blowing up along W and there is a commutative diagram (2.16). Let

$$\eta' : \mathcal{X}' \to \mathcal{S}',\ s' := \eta'(x') \in \sigma^{-1}(m_S),\ S' := \mathcal{O}_{\mathcal{S}',s'}.$$

We denote by $W' := \sigma^{-1}(W)$ and $E' := \sigma^{-1}(E)_{\mathrm{red}}$. We do not change notations to denote stalks at s', i.e. we will write $\eta' : \mathcal{X}_{s'} \to \mathrm{Spec}S'$ for the stalk at s' of the above map η', and W', E' for the stalks at s' of the corresponding divisors. By proposition 2.10, we have $\eta'^{-1}(s') = \{x'\}$ if x' is not a regular point of X'.

For the purpose of computations, we shall pick well adapted coordinates $(u_1, \ldots, u_n; Z)$ such that

$$I(W) = (\{u_j\}_{j\in J}),\ \mathcal{Y} = V(Z, \{u_j\}_{j\in J}).$$

with notations as in (3.1). We denote by $u \in S'$ a local equation for W', which can be taken to be some u_{j_1}, where $j_1 \in J$ depends on s'. We have $\mathcal{X}' = \mathrm{Spec}(S'[X']/(h'))$, where

$$h' := u^{-p}h = X'^p + f_{1,X'}X'^{p-1} + \cdots + f_{p,X'} \in S'[X'], \tag{3.8}$$

and

$$X' := Z/u,\ f_{i,X'} := u^{-i} f_{i,Z} \in S' \text{ for } 1 \leq i \leq p. \tag{3.9}$$

Since $\mathcal{Y}$ is permissible, we have $\epsilon(y) > 0$ so the initial form $\mathrm{in}_W h$ reduces to :

$$\mathrm{in}_W h = Z^p - G_W^{p-1} Z + F_{p,Z,W} \in G(W)[Z], \tag{3.10}$$

with $G_W \in G(W)_{\delta(y)}$ and $F_{p,Z,W} \in G(W)_{p\delta(y)}$ (in particular $G_W = 0$ if $\delta(y) \notin \mathbb{N}$). Since $\sigma^{-1}(W) = \mathbf{Proj} G(W)$, the restriction map

$$G(W)_d = \Gamma(W', \mathcal{O}_{W'}(d)) \to \Gamma(W' \backslash V(U), \mathcal{O}_{W'}(d))$$

gives an inclusion

$$U^{-d} G(W)_d = S_W[\{U_j/U\}_{j\in J}]_{\leq d} \subset \mathcal{O}_{W',s'} = S'/(u) \tag{3.11}$$

for each $d \geq 0$. There is an identification:

$$U^{-d} G(W')_d = (S_W[\{U_j/U\}_{j\in J}])_{s'} = S'/(u). \tag{3.12}$$

Finally, we note that $\mathcal{D}_{W'} = \mathcal{D}(W')$ by (2.42) since W' is a component of E'. These remarks are essential for stating the blow up formula in proposition 3.5(v) below.

Proposition 3.5. *(Blow up formula) Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along a permissible center $\mathcal{Y}$ at x, $\{x\} = \eta^{-1}(m_S)$ and $x' \in \pi^{-1}(x)$. With notations as above, the following holds:*

(i) there exists a r.s.p. $(u'_1, \ldots, u'_{n'})$ of S' which is adapted to (S', h', E');

(ii) $\mathrm{in}_{W'} h' = X'^p - G_{W'}^{p-1} X' + F_{p,X',W'} \in G(W')[X']$ *and is given by*

$$G_{W'} = U^{-1} G_W \in G(W')_{\delta(y)-1},\ F_{p,X',W'} = U^{-p} F_{p,Z,W} \in G(W')_{p(\delta(y)-1)};$$

(iii) the polyhedron $\Delta_{\hat{S}'}(h'; u; X')$ is minimal;

(iv) we have $H(x') = u^{\epsilon(y)-p} H(x) \subseteq S'$;

(v) there is an equality of ideals of $\hat{S}'/(u)$:

$$\begin{cases} H_{W'}^{-1} G_{W'}^p &= (U^{-\epsilon(y)} H_W^{-1} G_W^p)_{s'} \\ J(F_{p,X',W'}, E', W') &= (U^{-\epsilon(y)} J(F_{p,Z,W}, E, W)) \hat{S}'/(u). \end{cases},$$

Proof. Statement (i) is proved in proposition 2.7. The formula in (ii) is obvious from (3.8), (3.9) and (3.10).

If $i_0(W) = p-1$, i.e. $G_W \neq 0$ in (3.10), we have $G_{W'} \neq 0$ by (ii), so $\Delta_{\widehat{S'}}(h'; u; X') \subseteq \mathbb{R}_{\geq 0}$ is minimal.

If $i_0(W) = p$, then $F_{p,Z,W} \notin G(W)^p$, i.e.

$$\delta(y) \notin p\mathbb{N} \text{ or } U^{-\delta(y)} F_{p,Z,W} \notin k(W')^p.$$

Note that $G(W)^p = (k(W')[U, U^{-1}])^p \cap G(W)$ since $G(W)$ is integrally closed. By (ii), $F_{p,X',W'} = U^{-p} F_{p,Z,W}$ so $F_{p,X',W'} \notin G(W')^p$ and this proves (iii).

To prove (iv), first consider those irreducible components $W_j = \operatorname{div}(u_j)$ of E, $1 \leq j \leq e$, whose strict transform W'_j passes through s'. We may pick a r.s.p. $(u'_1, \ldots, u'_{n'})$ of S' which is adapted to (S', h', E'), containing u and $u'_j := u_j/u$ if $j \in J_E$ (resp. t and $u'_j := u_j$ if $j \notin J_E$) for each such j. Let

$$\operatorname{in}_{W_j} h(Z) = Z^p + F_{1,Z,W_j} Z^{p-1} + \cdots + F_{p,Z,W_j} \in S/(u_j)[U_j][Z].$$

We have $\operatorname{in}_{W'_j} h' = \operatorname{in}_{W_j} u^{-p} h(uX') \in S'/(u'_j)[U'_j][X']$, since u is a unit in $S'_{(u'_j)} = S_{(u_j)}$. Since $\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)$ is minimal, we have

$$\Delta_{\widehat{S_{(u_j)}}}(h; u_j; Z) = \Delta_{\widehat{S'_{(u'_j)}}}(h'; u'_j; X')$$

minimal as well by theorem 2.4, hence $\operatorname{ord}_{(u'_j)} H(x') = \operatorname{ord}_{(u_j)} H(x)$.

By (ii) and (iii), we have $\operatorname{ord}_{(u)} H(x') = p(\delta(y) - 1)$. Therefore

$$\operatorname{ord}_{(u)} H(x') - \operatorname{ord}_{(u)} H(x) = p(\delta(y) - 1) - \operatorname{ord}_W H(x) = \epsilon(y) - p$$

and the conclusion follows.

We now prove (v). The first part of the statement follows immediately from (ii) and (iv). With notations as in (2.43), we have

$$\begin{cases} J(F_{p,Z,W}, E, W) = H_W^{-1} \mathcal{J}(F_{p,Z,W}, E, W) \subseteq \widehat{G(W)}_{\epsilon(y)}, \\ J(F_{p,X',W'}, E', W') = H_{W'}^{-1} \mathcal{J}(F_{p,X',W'}, E', W') \subseteq \widehat{G(W')}_0. \end{cases}$$

Applying (ii) and (iv), we get:

$$F_{p,X',W'} = U^{-p} F_{p,Z,W}, \; H_{W'} = H_W U^{\epsilon(y)-p} G(W').$$

Since $D \cdot U^p = 0$ for every $D \in \mathcal{D}_{W'}$, (v) can be written in the following form:

$$U^{-\deg F_{p,Z,W}} \mathcal{J}(F_{p,Z,W}, E', W') = (U^{-\deg F_{p,Z,W}} \mathcal{J}(F_{p,Z,W}, E, W))\hat{S}'/(u). \quad (3.13)$$

We have $G(W') = G(W)[\{V_j\}_{j \in J\backslash\{j_1\}}]_{s'}$, $V_j := U_j/U \in G(W')_0$, $j \in J\backslash\{j_1\}$. Pick an isomorphism $\widehat{S_W} \simeq k(x)[[\{\overline{u}_{j'}\}_{j' \in J'}]]$ (2.37). By (3.11), there are inclusions

$$k(x)[\{U_j\}_{j\in J}] \subset k(x)[U, \{V_j\}_{j \in J\backslash\{j_1\}}] \subset \hat{S}'/(u, \{\overline{u}_{j'}\}_{j' \in J'})[U] \simeq G(\hat{W}')/(\{\overline{u}_{j'}\}_{j' \in J'}).$$

Let $A := k(x)[\{U_j\}_{j\in J}]$, $A' := k(x)[U, \{V_j\}_{j \in J\backslash\{j_1\}}]$. The A'-module

$$\Omega^1_{A'/\mathbb{F}_p}\left(\log(U \prod_{j \in J\backslash\{j_1\}} V_j)\right)$$

is generated by collecting together dU/U, $\{dV_j/V_j\}_{j\in J\backslash\{j_1\}}$ and the pullback of $\Omega^1_{A/\mathbb{F}_p}$. For $F \in A$, we deduce the following standard formulæ in A' up to linear combinations of the $\frac{\partial F}{\partial \lambda_l}$, $l \in \Lambda_0$:

$$U\frac{\partial F}{\partial U} = \sum_{j\in J} U_j \frac{\partial F}{\partial U_j},\ V_j \frac{\partial F}{\partial V_j} = U_j \frac{\partial F}{\partial U_j}, j \in J\backslash\{j_1\}. \quad (3.14)$$

By (2.41), the $\widehat{G(W)}$-module $\mathcal{D}_W$ is generated by adjoining the family

$$\left(\{U_j \frac{\partial}{\partial U_j}\}_{j\in J_E}, \{U_k \frac{\partial}{\partial U_j}\}_{k\in J, j\in J\backslash J_E}\right) \quad (3.15)$$

together with $(\{\overline{u}_{j'}\frac{\partial}{\partial \overline{u}_{j'}}\}_{j'\in (J')_E}, \{\frac{\partial}{\partial \overline{u}_{j'}}\}_{j'\in J'\backslash (J')_E}, \{\frac{\partial}{\partial \lambda_l}\}_{l\in\Lambda_0})$. Taking $F \in A_d$, $d \in \mathbb{N}$, we have for $j \in J\backslash J_E$,

$$(U^{-d}\{U_k \frac{\partial F}{\partial U_j}\}_{k\in J})A'_{s'} = (U^{-d}U\frac{\partial F}{\partial U_j})A'_{s'}.$$

Collecting together this equation with (3.14) and (3.15), we get

$$U^{-d}\mathcal{J}(F, E', W') = (U^{-d}\mathcal{J}(F, E, W))\hat{S}'/(u)$$

which proves (3.13) as required.

We now state the main theorem of this section. Recall that the function $y \mapsto \omega(y)$ and $\kappa(y) \in \{1, \geq 2\}$ have been defined for given (S, h, E) and $y \in \mathcal{X}$ (definition 2.15 and definition 2.16). By proposition 2.13, (S', h', E') satisfies again conditions **(G)** and **(E)**. The values of $\epsilon(x')$, $\iota(x')$ are computed w.r.t. the adapted structure (S', h', E').

Notation 3.2. *Choice of coordinates:* by proposition 3.5(i), there exists a r.s.p. $(u'_1, \ldots, u'_{n'})$ which is adapted to (S', h', E') for some $n' \leq n$. We take $u'_1 := u$. Let

$$u'_i := \frac{u_{j_i}}{u},\ 2 \leq i \leq e'_0,\ \text{where } \{j_2, \ldots, j_{e'_0}\} := \{j \in J_E : \frac{u_j}{u} \in m_{S'}\}.$$

Let $\{j_{e'_0+1}, \ldots, j_{e'}\} := (J')_E$, $\{j_{e'+1}, \ldots, j_{n'_0}\} =: J' \backslash (J')_E$. We take

$$u'_i := u_{j_i},\ e'_0 + 1 \leq i \leq n'_0.$$

Let

$$u'_i := \frac{u_{j_i}}{u},\ n'_0 + 1 \leq i \leq n'_1,\ \text{where } \{j_{n'_0+1}, \ldots, j_{n'_1}\} := \{j \in J \backslash J_E : \frac{u_j}{u} \in m_{S'}\}$$

and complete $(u'_1, \ldots, u'_{n'_1})$ to a r.s.p. $(u'_1, \ldots, u'_{n'})$ of S'.

Notation 3.3. Let

$$\overline{S'} := \hat{\mathcal{O}}_{\sigma^{-1}(m_S), s'} = \hat{S}'/(u, \{u_{j'}\}_{j' \in J'}) = k(x)[\{\widehat{U_j/U}\}_{j \in J}]_{\overline{m'}},$$

where $\overline{m'}$ denotes the ideal of the restriction of s' to $\sigma^{-1}(m_S)$:

$$\overline{m'} := (\{\overline{u'_i}\}_{i \in F}),\ F := \{2, \ldots, e'_0\} \cup \{n'_0 + 1, \ldots, n'\}.$$

For $I' \subseteq \hat{S}'/(u)$ an ideal, we denote by

$$\mathrm{ord} I' := \mathrm{ord}_{m_{\hat{S}'/(u)}} I' = \min_{\varphi' \in I'} \{\mathrm{ord}_{m_{\hat{S}'/(u)}} \varphi'\},\ \overline{\mathrm{ord}} I' := \mathrm{ord}_{\overline{m'}} I' \overline{S'}.$$

For every $I' \subseteq \hat{S}'/(u)$, we have $\mathrm{ord} I' \leq \overline{\mathrm{ord}} I' \leq +\infty$. If furthermore d' is given, $d' \leq \overline{\mathrm{ord}} I'$, we write

$$\overline{I'} \subseteq \left(\mathrm{gr}_{\overline{m'}} \overline{S'}\right)_{d'} = k(x')[\{U'_i\}_{i \in F}]_{d'}$$

for the initial part of degree d' of the ideal $I' \overline{S'}$.

The cone $C(x,\mathcal{Y}) \subseteq \mathrm{Spec}(k(x)[\{U_j\}_{j\in J}])$ is given by definition 3.3. For the associated projective cone, there is an embedding

$$PC(x,\mathcal{Y}) \hookrightarrow \sigma^{-1}(m_S).$$

Theorem 3.6. *Assume that $m(x) = p$, $\omega(x) > 0$, where $\{x\} = \eta^{-1}(m_S)$. Let $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along a permissible center $\mathcal{Y}$ (of the first kind or second kind) at x, $x' \in \pi^{-1}(x)$ and $\eta' : \mathcal{X}' \to \mathrm{Spec}S'$ be with notations as above, where $s' = \eta'(x')$. Then*

$$(m(x'),\omega(x'),\kappa(x')) \leq (m(x),\omega(x),\kappa(x)). \tag{3.16}$$

If equality holds in (3.16), then $s' \in PC(x,\mathcal{Y})$.

If $\epsilon(x') > \epsilon(x)$, the following holds:

(1) we have $i_0(m_S) = p$, $\epsilon(y) = \epsilon(x) = \omega(x)$, $\delta(y) \in \mathbb{N}$, $H_{j'} \in p\mathbb{N}$ for every $j' \in (J')_E$ and

$$F_{p,Z} \in (k(x')[U_1,\ldots,U_n])^p[\{U_j\}_{j\in J_E\setminus\{j_2,\ldots,j_{e'_0}\}}];$$

(2) let $(u'_1,\ldots,u'_{n'};Z')$ be well adapted coordinates at x'. Then

$$H'^{-1}F_{p,Z'} \not\subseteq k(x')[U'_1,\ldots,U'_{n'_1}]_{\epsilon(x')} \oplus (\{U'_i\}_{i\notin F}) \cap G(m_{S'})_{\epsilon(x')} \tag{3.17}$$

and there exists $\Phi' \in k(x')[U'^p_1,\ldots,U'^p_{n'_1}][U'_{n'_1+1},\ldots,U'_{n'}]_{p\delta(x')}$ such that

$$H'^{-1}(F_{p,Z'} - \Phi') \subseteq (\{U'_i\}_{i\notin F}) \cap G(m_{S'})_{\epsilon(x')}. \tag{3.18}$$

Proof. Since $\mathcal{Y}$ is permissible, $\mathcal{Y}$ is Hironaka-permissible at x and this implies that $m(x') \leq m(x) = p$ in any case. We are done unless equality holds, so assume that $m(x') = p$.

The polyhedron $\Delta_{\widehat{S'}}(h';u'_1,\ldots,u'_{n'};X')$ need not be minimal. We must take $Z' = X' - \theta'$, $\theta' \in S'$ such that the polyhedron $\Delta_{\widehat{S'}}(h';u'_1,\ldots,u'_{n'};Z')$ is minimal in order to read off $\epsilon(x')$ and $\omega(x')$ from $\mathrm{in}_{m_{S'}}h'$.

By proposition 3.5(iii), we have $\mathrm{ord}_{(u)}H(x') = p(\delta(y)-1)$. The initial form $H_{W'}$ of $H(x')$ in $G(W')$ is given by proposition 3.5(iv):

$$H_{W'} =< U^{p(\delta(y)-1)} \prod_{i=2}^{e'} \overline{u'_i}^{H_{j_i}} > . \tag{3.19}$$

We have $\theta'^p \in H(x')$ since $f_{p,X'} \in H(x')$. Let $\Theta' \in G(W')_{\delta(y)-1}$ be the initial form of θ' (in particular $\Theta' = 0$ if $\delta(y) \notin \mathbb{N}$). Then

$$\mathrm{in}_{W'}h' = Z'^p - G_{W'}^{p-1}Z' + F_{p,X',W'} + \Theta'^p - G_{W'}^{p-1}\Theta' \in G(W')[Z'] \qquad (3.20)$$

where $G_{W'} = U^{-1}G_W$, $F_{p,X',W'} = U^{-p}F_{p,Z,W}$ by proposition 3.5(ii). According to our notations, we have:

$$F_{p,Z',W'} = F_{p,X',W'} + \Theta'^p - G_{W'}^{p-1}\Theta'.$$

Note that derivatives in $\mathcal{D}_{W'}$ decrease orders by at most one. Since $H_{W'}$ is the initial form of $H(x')$ in $G(W')$, we have:

$$\epsilon(x') \leq \min\{\mathrm{ord}_{m_{S'/(u)}}(H_{W'}^{-1}G_{W'}^{p}), 1 + \mathrm{ord}_{m_{S'/(u)}}J(F_{p,Z',W'}, E', W')\}. \quad (3.21)$$

Inequality may be strict, since the $H(x')^{-i}f_{i,Z'}^p$, $1 \leq i \leq p$ may acquire terms of lower order not coming from $\mathrm{in}_W h$. Moreover, some derivatives in $\mathcal{D}_{W'}$ do not decrease orders and give a sharper bound in (3.21).

Recall that if $M \subseteq \widehat{G(W)}_d$, $d \in \mathbb{N}$ is a submodule, and d_0 is given, there are associated initial forms

$$\overline{M} \subseteq \left(\mathrm{gr}_{\overline{m}_S}G(W)_d\right)_{d_0} \subset \frac{G(m_S)_{d+d_0}}{< (\{U_j\}_{j\in J})^{d+1} \cap G(m_S)_{d+d_0} >}$$

under the conditions described in (3.2) and (3.3). Note that

$$\left(\mathrm{gr}_{\overline{m}_S}G(W)_d\right)_0 = \Gamma(\sigma^{-1}(m_S), \mathcal{O}_{\sigma^{-1}(m_S)}(d)) = k(x)[\{U_j\}_{j\in J}]_d$$

for $d_0 = 0$.

Since $\theta'^p \in H(x')$, we have $\Theta'^p \in H_{W'}$ in (3.20). We have $\Theta' = 0$ or $\delta(y) \in \mathbb{N}$ and

$$G_{W'}^{p-1}\Theta' \in G_{W'}^{p-1}\left\lceil H_{W'}^{\frac{1}{p}}\right\rceil, \ \left\lceil H_{W'}^{\frac{1}{p}}\right\rceil :=< U^{\delta(y)-1}\prod_{i=2}^{e'}\overline{u'_i}^{\left\lceil \frac{H_{j_i}}{p}\right\rceil} > .$$

Since $D \cdot \Theta'^p = 0$ for every $D \in \mathcal{D}_{W'}$, we deduce from (3.20) that

$$J(F_{p,Z',W'}, E', W') \equiv J(F_{p,X',W'}, E', W') \ \mathrm{mod} H_{W'}^{-1}G_{W'}^{p-1}\left\lceil H_{W'}^{\frac{1}{p}}\right\rceil. \qquad (3.22)$$

Note that if $i_0(m_S) = p$, or if $H_{j'} \notin p\mathbb{N}$ for some $j' \in (J')_E$, we have

$$G_W = 0 \text{ or } \mathrm{ord}_{(u_{j'})}(H_W^{-1}G_W^p) > 0 \text{ for some } j' \in (J')_E \tag{3.23}$$

by applying proposition 2.11(iii) in the latter case. In this case, we obtain the following from proposition 3.5(v) and (3.22):

$$(H_{W'}^{-1}G_{W'}^p)\overline{S'} = 0,\ J(F_{p,Z',W'}, E', W')\overline{S'} = J(F_{p,X',W'}, E', W')\overline{S'}. \tag{3.24}$$

Case 1: $i_0(m_S) = p$ and $\mathcal{Y}$ is of the first kind. In order to get an estimate of $\epsilon(x')$ from (3.21), we take:

$$M = J(F_{p,Z,W}, E, W),\ d = \epsilon(y) = \epsilon(x),\ d_0 = 0.$$

Remark 3.1. By proposition 3.1, there is an equality

$$H^{-1}F_{p,Z} = \mathrm{cl}_{\epsilon(x)}H_W^{-1}F_{p,Z,W} \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)},$$

but we emphasize that the induced inclusion

$$J(F_{p,Z}, E, m_S) \subseteq \mathrm{cl}_{\epsilon(x)}J(F_{p,Z,W}, E, W). \tag{3.25}$$

is strict in general.

By proposition 2.16(ii) and the remark, we have

$$0 \neq J(F_{p,Z}, E, m_S) \subseteq \overline{M} \subseteq k(x)[\{U_j\}_{j\in J}]_{\epsilon(x)}.$$

Let $I' = J(F_{p,X',W'}, E', W') \subseteq \hat{S}'/(u)$, $d' = \mathrm{ord}I'$. By proposition 3.5(v), we have

$$\big(U^{-\epsilon(x)}J(F_{p,Z}, E, m_S)\big)_{\overline{m'}} \subseteq I'\overline{S'}.$$

Since $i_0(m_S) = p$, we obtain from (3.24) that:

$$\big(U^{-\epsilon(x)}J(F_{p,Z}, E, m_S)\big)_{\overline{m'}} \subseteq I'\overline{S'} = J(F_{p,Z',W'}, E_{W'}, W')\overline{S'}. \tag{3.26}$$

If $\omega(x) = \epsilon(x)$, definition 2.17 gives

$$\mathrm{IMax}(x) = (J(F_{p,Z}, E, m_S))G(m_S).$$

We deduce that $\overline{\mathrm{ord}}I' \leq \omega(x)$ and

$$s' \notin PC(x, \mathcal{Y}) \Longrightarrow \overline{\mathrm{ord}}I' < \omega(x). \tag{3.27}$$

If $\omega(x) = \epsilon(x) - 1$, definition 2.17 gives

$$\mathrm{IMax}(x) = (V(F_{p,Z}, E, m_S))G(m_S).$$

Since $U_{j_1}V(F_{p,Z}, E, m_S) \subseteq J(F_{p,Z}, E, m_S)$ (recall that $u = u_{j_1}$), we also deduce that $\overline{\mathrm{ord}}I' \leq \omega(x)$ and (3.27) holds. We have:

$$\epsilon(x') \leq 1 + \mathrm{ord}I' = 1 + d' \leq 1 + \overline{\mathrm{ord}}I',$$

by (3.21). We have proved that

$$\epsilon(x') \leq 1 + \overline{\mathrm{ord}}I' \leq 1 + \omega(x) \tag{3.28}$$

with strict inequality on the right hand side under the assumption of (3.27). The proof is now an easy consequence of the following claim:

$$\epsilon(x') = 1 + \overline{\mathrm{ord}}I' \Longrightarrow \omega(x') = \epsilon(x') - 1.$$

Namely, assuming the claim, we have $\omega(x') \leq \omega(x)$ and this inequality is strict under the assumption of (3.27). The first part of the proof is complete since $i_0(m_S) = p$ implies $\kappa(x) \geq 2$. To prove the claim, let

$$\mathrm{in}_{m_{S'}}h = Z'^p - G'^{p-1}Z' + F_{p,Z'} \in G(m_{S'})[Z']$$

be the initial form polynomial. Since it is assumed that $\epsilon(x') = 1 + \overline{\mathrm{ord}}I'$, we have $\overline{I'} \neq 0$ and:

$$\overline{I'} =< \left\{ H'^{-1}\frac{\partial F_{p,Z'}}{\partial U'_j} \right\}_{j=n'_0+1}^{n'} > \ \mathrm{mod}(\{U'_{j'}\}_{j' \notin F}) \cap G(m_{S'})_{d'}. \tag{3.29}$$

To compute $\omega(x')$, we must introduce a truncation operator

$$T' : G(m_{S'})_{p\delta(x')} \to G(m_{S'})_{p\delta(x')}$$

as in definition 2.16. By (3.19), we have

$$H' := \mathrm{cl}_{p\delta(x')-\epsilon(x')}H(x') =< U^{p(\delta(y)-1)}\prod_{i=2}^{e'} U_i'^{H_{j_i}} >\in G(m_{S'}).$$

Going back to definition 2.14, we have

$$F_{p,Z'} - T'F_{p,Z'} \in < G'^{p-1}U^{\delta(y)-1}\prod_{i=2}^{e'} U_i'^{\left\lceil \frac{H_{j_i}}{p} \right\rceil} > .$$

Since $i_0(m_S) = p$, (3.24) applies and implies that

$$H'^{-1}(F_{p,Z'} - T'F_{p,Z'}) \subseteq (\{U'_i\}_{i\notin F}) \cap G(m_{S'})_{\epsilon(x')}. \tag{3.30}$$

Comparing with (3.29), there exists i, $n'_0 + 1 \leq i \leq n'$ such that

$$H'^{-1}\frac{\partial T'F_{p,Z'}}{\partial U'_i} \neq 0, \tag{3.31}$$

since $\overline{I'} \neq 0$. This proves that $\omega(x') = \epsilon(x') - 1$ as claimed.

To conclude the proof in case 1, assume that $\epsilon(x') > \epsilon(x)$. If some inequality is strict in (3.27), we have $\epsilon(x') \leq \omega(x) \leq \epsilon(x)$: a contradiction. So $\omega(x') = \omega(x)$ and by the above claim, we get

$$\epsilon(x) = \omega(x) = \omega(x') = \epsilon(x') - 1 = \mathrm{ord}I' = \overline{\mathrm{ord}}I'. \tag{3.32}$$

We use notations as in (2.38). Suppose that there exists $j' \in (J')_E$ such that $H_{j'} \notin p\mathbb{N}$. By proposition 3.1, we have

$$H^{-1}U_{j'}\frac{\partial F_{p,Z}}{\partial U_{j'}} \neq 0.$$

Going back to (3.26), we have

$$\phi_{j'} := \left(U^{-\epsilon(x)}H^{-1}U_{j'}\frac{\partial F_{p,Z}}{\partial U_{j'}}\right)_{\overline{m'}} \subseteq J(F_{p,Z',W'}, E', W')\overline{S'}.$$

Applying the transformation rule in proposition 3.5(v), we have

$$\phi_{j'} = (H^{-1}_{W'}\overline{u}_{j'}\frac{\partial F_{p,Z',W'}}{\partial \overline{u}_{j'}})\overline{S'}.$$

Since $\overline{\mathrm{ord}}\phi_{j'} \leq \epsilon(x)$, we deduce that

$$\epsilon(x') \leq \overline{\mathrm{ord}}(H^{-1}_{W'}F_{p,Z',W'}) \leq \overline{\mathrm{ord}}(H^{-1}_{W'}\overline{u}_{j'}\frac{\partial F_{p,Z',W'}}{\partial \overline{u}_{j'}}) \leq \epsilon(x).$$

This is a contradiction with (3.32). Hence $H_{j'} \in p\mathbb{N}$ for every $j' \in (J')_E$.

Suppose that $\delta(y) \notin \mathbb{N}$. Similarly, by proposition 3.1, we have:

$$H^{-1}D \cdot F_{p,Z} \neq 0,\ D := \sum_{j\in J} U_j\frac{\partial}{\partial U_j} \in \mathrm{Der}(G(W)).$$

Note that we have $\Theta' = 0$ in (3.20) since $\delta(y) \notin \mathbb{N}$. We deduce from (3.14) that

$$\phi_D := \left(U^{-\epsilon(x)} H^{-1} D \cdot F_{p,Z}\right) \hat{S}'/(u) = H_{W'}^{-1} U \frac{\partial F_{p,Z',W'}}{\partial U}.$$

Arguing as above, we get a contradiction from:

$$\epsilon(x') \leq \mathrm{ord}(H_{W'}^{-1} F_{p,Z',W'}) \leq \mathrm{ord}(H_{W'}^{-1} U \frac{\partial F_{p,Z',W'}}{\partial U}) \leq \epsilon(x).$$

Let now $i \in \{2, \ldots, e_0'\}$. By (3.26), we have

$$\phi_i := \left(U^{-\epsilon(x)} H^{-1} U_{j_i} \frac{\partial F_{p,Z}}{\partial U_{j_i}}\right)_{\overline{m'}} \subseteq J(F_{p,Z',W'}, E_{W'}, W') \overline{S'}.$$

Applying once again (3.14) and since $\epsilon(x') > \epsilon(x) = \omega(x)$, we get

$$\mathrm{cl}_{\epsilon(x)}(\{H_{W'}^{-1} \overline{u}_i \frac{\partial F_{p,Z,W'}}{\partial \overline{u}_i}\}_{2 \leq i \leq e_0'}) \equiv \mathrm{cl}_{\epsilon(x)}(\{\phi_i\}_{2 \leq i \leq e_0'}) \bmod (\{U_{i'}'\}_{i' \notin F}) \cap G(m_{S'})_{\epsilon(x)}.$$

If $\phi_i \neq 0$ for some i, $2 \leq i \leq e_0'$, we get

$$\epsilon(x') \leq \mathrm{ord}(H_{W'}^{-1} F_{p,Z',W'}) \leq \mathrm{ord}(H_{W'}^{-1} \overline{u}_i \frac{\partial F_{p,Z,W'}}{\partial \overline{u}_i}) \leq \epsilon(x),$$

again a contradiction. Since $\epsilon(x) = \omega(x)$, we have $\frac{\partial F_{p,Z}}{\partial U_j} = 0$ for every $j \in J \backslash J_E$.

Finally, assume that $F_{p,Z} \notin k(x')^p[U_1, \ldots, U_n]$. With notations as in (2.38), we pick a maximal subset $\Lambda_1 \subseteq \Lambda_0$ such that the family of elements $(d\overline{\lambda}_l)_{l \in \Lambda_1}$ in $\Omega^1_{k(x')/\mathbb{F}_p}$ is linearly independent over $k(x')$. Let $(d\overline{\lambda}_{l'})_{l' \in \Lambda_0'}$ be a basis of $\Omega^1_{k(x')/\mathbb{F}_p}$, $\Lambda_1 \subseteq \Lambda_0'$, and pick a preimage $\lambda_{l'} \in \hat{S}'/(u)$ of $\overline{\lambda}_{l'}$ for $l' \in \Lambda_0' \backslash \Lambda_1$.

By assumption, there exists $l \in \Lambda_1$ such that $\frac{\partial F_{p,Z}}{\partial \lambda_l} \neq 0$. Arguing as above, we get

$$\mathrm{cl}_{\epsilon(x)}(H_{W'}^{-1} \frac{\partial F_{p,Z,W'}}{\partial \lambda_l}) \equiv \mathrm{cl}_{\epsilon(x)} \left(U^{-\epsilon(x)} H^{-1} \frac{\partial F_{p,Z}}{\partial \lambda_l}\right)_{\overline{m'}} \bmod (\{U_{i'}'\}_{i' \notin F}) \cap G(m_{S'})_{\epsilon(x)},$$

a contradiction and the proof of (1) in the theorem is complete.

We now proceed to prove (2). By proposition 3.5(i), we have

$$H_{W'}^{-1} F_{p,X',W'} \overline{S'} = (U^{-\epsilon(x)} H_W^{-1} F_{p,Z,W})_{\overline{m'}} = (U^{-\epsilon(x)} H^{-1} F_{p,Z})_{\overline{m'}}.$$

By (1) in the theorem and proposition 3.1, there is an expansion

$$F_{p,Z} = \left(\prod_{i=e'_0+1}^{e'} U_{j_i}^{H_{j_i}} \right) \sum_{\mathbf{a}\in A} F_{p,Z,\mathbf{a}}(\{U_j\}_{j\in J'_1}) \prod_{j\in J_1} U_j^{pa_j},\ A \subset \mathbb{N}^{J_1},$$

with $J_1 := \{j_2, \ldots, j_{e'_0}, j_{n'_0+1}, \ldots, j_{n'_1}\}$, $J'_1 := J\backslash J_1$, $F_{p,Z,\mathbf{a}} \in k(x')^p[\{U_j\}_{j\in J'_1}]$. We deduce that

$$(U^{-\epsilon(x)}H^{-1}F_{p,Z})_{\overline{m'}} = \overline{H'}^{-1} \left(\sum_{\mathbf{a}\in A} F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J'_1}) \prod_{j\in J_1} (\frac{U_j}{U})^{pa_j} \right), \qquad (3.33)$$

with $\overline{H'} := (\prod_{i=2}^{e'_0} \left(\frac{U_{j_i}}{U}\right)^{H_{j_i}}) \subseteq \overline{S'}$. Since $(H_{W'}^{-1}G_{W'}^{p})\overline{S'} = 0$ by (3.24), there exists $\theta' \in S'/(u)$ such that

$$H_{W'}^{-1}F_{p,Z',W'}\overline{S'} = H_{W'}^{-1}(F_{p,X',W'} + \theta'^{p})\overline{S'}. \qquad (3.34)$$

We deduce from (3.33) that there exists a finite subset $A' \subset \mathbb{N}^{J_1}$, $A \subseteq A'$ and elements

$$\theta'_{\mathbf{a}} \in k(x)[\{\frac{U_j}{U}\}_{j\in J'_1}] \text{ for every } \mathbf{a} \in A'$$

such that (letting $F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J'_1}) = 0$ for $\mathbf{a} \in A'\backslash A$) we have:

$$H_{W'}^{-1}F_{p,Z',W'}\overline{S'} = \overline{H'}^{-1} \left(\sum_{\mathbf{a}\in A'} (F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J'_1}) + \theta'^{\,p}_{\mathbf{a}}) \prod_{j\in J_1} (\frac{U_j}{U})^{pa_j} \right).$$

Let $d_{\mathbf{a}} := \epsilon(x') + \sum_{i=2}^{e'_0} H_{j_i} - p \mid \mathbf{a} \mid$ for $\mathbf{a} \in A'$. Since $\overline{\mathrm{ord}}(H_{W'}^{-1}F_{p,Z',W'}) = \epsilon(x')$ we have

$$\overline{\mathrm{ord}}(F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J'_1}) + \theta'^{\,p}_{\mathbf{a}}) \geq d_{\mathbf{a}}$$

for every $\mathbf{a} \in A'$. Taking classes in $G(\overline{m'})$, we define:

$$\Phi'_{\mathbf{a}} := \mathrm{cl}_{d_{\mathbf{a}}}(F_{p,Z,\mathbf{a}}(\{\frac{U_j}{U}\}_{j\in J'_1}) + \theta'^{p}_{\mathbf{a}}) \in k(x')[U'_{n'_1+1}, \ldots, U'_{n'}]_{d_{\mathbf{a}}}.$$

To conclude the proof, let $I_1 := \{2, \ldots, e'_0, n'_0 + 1, \ldots, n'_1\}$. We take

$$\Phi' := U_1'^{p(\delta(y)-1)} \left(\prod_{i=e'_0+1}^{e'} U_i'^{H_{j_i}} \right) \sum_{\mathbf{a}\in A'} \Phi'_{\mathbf{a}} \prod_{i\in I_1} U_i'^{pa_{j_i}}$$

and claim that Φ' satisfies (2) in the theorem. By the above definition and (1) in the theorem, we have $\Phi' \in k(x')[U_1'^p, \ldots, U_{n_1'}'^p][U_{n_1'}', \ldots, U_{n'}']_{p\delta(x')}$. Also (3.18) follows immediately from (3.34).

With notations as in the above proof of (1), we have

$$J(F_{p,Z}, E, m_S) = H^{-1} < \{U_j \frac{\partial F_{p,Z}}{\partial U_j}\}_{j \in J_E \setminus \{j_2, \ldots, j_{e_0'+1}\}}, \{\frac{\partial F_{p,Z}}{\partial \lambda_l}\}_{l \in \Lambda_0 \setminus \Lambda_1}.$$

Applying once more (3.14), we get

$$\mathrm{cl}_{\epsilon(x)}(\{H_{W'}^{-1} \frac{\partial F_{p,Z,W'}}{\partial \overline{u'}_i}\}_{n_1' \le i \le n'}) \equiv \mathrm{cl}_{\epsilon(x)}(U^{-\epsilon(x)} J(F_{p,Z}, E, m_S))_{\overline{m'}} \bmod (\{U_{i'}'\}_{i' \notin F}) \cap G(m_{S'})_{\epsilon(x)}.$$

Since $J(F_{p,Z}, E, m_S) \neq 0$, we obtain that

$$H'^{-1} \frac{\partial F_{p,Z'}}{\partial U_i'} \notin (\{U_{i'}'\}_{i' \notin F}) \cap G(m_{S'})_{\epsilon(x)}$$

for some i, $n_1' \le i \le n'$, and the conclusion follows. This concludes the proof of (2).

Case 2: $i_0(m_S) = p-1$ (*so* $\mathcal{Y}$ *is of the first kind*). We first take $d = \epsilon(y)$ and

$$M := H_W^{-1} G_W^p, \ d_0 = 0.$$

By proposition 3.1, there is an expansion $H^{-1}G^p =< \prod_{j \in J} U_j^{pB_j} >$. With notations as in definition 2.16, we have

$$pb_j - H_j = pB_j, \ j \in J \text{ and } B = \{j \in J : B_j > 0\}. \tag{3.35}$$

We deduce:

$$(0) \neq \overline{M} = (\prod_{j \in B} U_j^{pB_j}) \subseteq k(x)[\{U_j\}_{j \in J}]_{\epsilon(x)}.$$

Let $I_0' = H_{W'}^{-1} G_{W'}^p$, $d_0' = \mathrm{ord} I_0'$. We have:

$$I_0' \overline{S'} = \left(U^{-\epsilon(x)} \prod_{j \in B} U_j^{pB_j} \right)_{\overline{m'}}. \tag{3.36}$$

This proves that $\epsilon(x') \le \overline{\mathrm{ord}} I_0' \le \epsilon(x)$ and equality holds only if

$$s' \in \mathbf{Proj} \left(\frac{k(x)[\{U_j\}_{j \in J}]}{(U_B)} \right). \tag{3.37}$$

Suppose that $\epsilon(x') < \epsilon(x)$. Then :

$$\omega(x') \leq \epsilon(x') \leq \epsilon(x) - 1 \leq \omega(x).$$

If $\omega(x') = \omega(x)$, then $\omega(x) = \epsilon(x) - 1$, so $\kappa(x) \geq 2$. On the other hand, we have $\omega(x') = \epsilon(x')$ and therefore $\kappa(x') = 1$ by definition 2.16. Hence inequality is strict in (3.16). In other terms, it can be assumed from now on that (3.37) holds and that

$$\epsilon(x') = \epsilon(x). \tag{3.38}$$

We now resume the argument used in case 1 by taking

$$M = J(F_{p,X,W}, E_W, W),\ d = \epsilon(y) = \epsilon(x),\ d_0 = 0.$$

To begin with, (3.26) holds whenever (3.24) applies, i.e. if $H_{j'} \notin p\mathbb{N}$ for some $j' \in (J')_E$ or if $\delta(y) \notin \mathbb{N}$. Suppose that $\delta(y) \in \mathbb{N}$ and $H_{j'} \in p\mathbb{N}$ for every $j' \in (J')_E$. In this case, (3.22) reduces to

$$J(F_{p,Z',W'}, E', W') \equiv J(F_{p,X',W'}, E', W') \bmod K' \frac{\hat{S}'}{(u)}, \tag{3.39}$$

$$K' := (\prod_{i=2}^{e'_0} u_i'^{(p-1)b_{j_i} - H_{j_i} + \left\lceil \frac{H_{j_i}}{p} \right\rceil}) \subseteq S'$$

with notations as in (3.35). We let :

$$k' := \sum_{j \in J} \left((p-1)b_j - H_j + \left\lceil \frac{H_j}{p} \right\rceil \right) = \mathrm{ord}_{m_{S'}} K'.$$

Going back to definition 2.16, we have

$$F_{p,Z} - TF_{p,Z} \in (\prod_{j \in J} U_j^{(p-1)b_j + \left\lceil \frac{H_j}{p} \right\rceil} G(m_S))_{p\delta(x)}$$

and we deduce now from (3.39) that

$$J(F_{p,Z',W'}, E_{W'}, W')\overline{S'} \equiv \left(U^{-\epsilon(x)} J(TF_{p,Z}, E, m_S)\right)_{\overline{m'}} \bmod K'\overline{S'}. \tag{3.40}$$

Note that the previous equation remains valid when $H_{j'} \notin p\mathbb{N}$ for some $j' \in (J')_E$ or when $\delta(y) \notin \mathbb{N}$. The proof now goes on as in case 1 and we

deduce that $\overline{\mathrm{ord}}I' \leq \omega(x)$; joining (3.37) and (3.40), we obtain that (3.27) holds, i.e.

$$s' \notin \mathbf{Proj}\left(\frac{k(x)[\{U_j\}_{j\in J}]}{(\mathrm{IMax}(x)\cap k(x)[\{U_j\}_{j\in J}])}\right) \Longrightarrow \overline{\mathrm{ord}}I' < \omega(x).$$

Equation (3.28) now follows, while (3.29) gets replaced by

$$\overline{I'} =< \left\{H'^{-1}\frac{\partial F_{p,Z'}}{\partial U'_j}\right\}_{j=n'_0+1}^{n'} > \ \mathrm{mod}((\{U'_{j'}\}_{j'\notin F}) + (\mathrm{cl}_{k'}K')) \cap G(m_{S'})_{d'}. \tag{3.41}$$

Finally, we obtain that

$$H'^{-1}(F_{p,Z'} - T'F_{p,Z'}) \subseteq ((\{U'_i\}_{i\notin F}) + (\mathrm{cl}_{k'}K')) \cap G(m_{S'})_{\epsilon(x')}$$

and this concludes the proof of the claim, hence of the theorem, as in case 1.

Case 3: $\mathcal{Y}$ is of the second kind. First recall from proposition 3.3 that $\epsilon(x) - 1 = \omega(x)$, so $\kappa(x) \geq 2$ in particular. Let $I'_0 := H'^{-1}_{W'}G^p_{W'}$, $d'_1 = \mathrm{ord}I'_0$.

Suppose that $i_0(m_S) = p-1$. By proposition 3.3, there exists an expansion

$$H^{-1}G^p =< U_{j_1}\prod_{j\in B_J} U_j^{pB_j} >,\ j_1 \in (J')_E,\ B_j > 0 \text{ for } j \in B_J,$$

with notations as in definition 3.3. By proposition 3.5(v), we have:

$$I'_0S'/(u) = \overline{u}_{j_1}\left(U^{-\epsilon(y)}\prod_{j\in B} U_j^{pB_j}\right)_{m_{S'/(u)}}. \tag{3.42}$$

This proves that $\epsilon(x') \leq \mathrm{ord}I'_0 \leq \epsilon(x)$ and equality holds only if

$$s' \in \mathbf{Proj}\left(\frac{k(x)[\{U_j\}_{j\in J}]}{(U_{B_J})}\right). \tag{3.43}$$

Suppose furthermore that $\epsilon(x') < \epsilon(x)$. We have:

$$\omega(x') \leq \epsilon(x') \leq \epsilon(x) - 1 = \omega(x).$$

If $\omega(x') = \omega(x)$, then $\omega(x') = \epsilon(x')$ and therefore $\kappa(x') = 1$ by definition 2.16, so inequality is strict in (3.16). Therefore if $i_0(m_S) = p-1$, it can be

assumed that $\epsilon(x') = \epsilon(x)$ and in particular that (3.43) holds.

Going back to the general situation of case 3, we now take

$$M = J(F_{p,X,W}, E_W, W),\ d = \epsilon(y),\ d_0 = 0.$$

Note that (3.24) is always valid in this case 3: we either have $i_0(m_S) = p$ or (3.23) holds for $j' = j_0$. Applying proposition 3.5(v) gives:

$$J(F_{p,Z',W'}, E_{W'}, W')\overline{S'} = \left(U^{-\epsilon(y)}\overline{J}(F_{p,Z,W}, E_W, W)\right)_{\overline{m'}}.$$

With notations as in proposition 3.3, we have

$$(0) \neq \overline{J}(F_{p,Z,W}, E_W, W) =< \{\Phi_{j'}(\{U_j\}_{j\in J})\}_{j'\in J'\setminus(J')_E} > .$$

We deduce that

$$J(F_{p,Z',W'}, E_{W'}, W')\overline{S'} =< \{\left(U^{-\epsilon(y)}\Phi_{j'}(\{U_j\}_{j\in J})\right)_{\overline{m'}}\}_{j'\in J'\setminus(J')_E} > . \quad (3.44)$$

Since definition 3.3 gives

$$C(x, \mathcal{Y}) := \mathrm{Max}(\overline{J}(F_{p,Z,W}, E, W)) \cap \{U_{B_J} = 0\},$$

we deduce that $\overline{\mathrm{ord}}J(F_{p,Z',W'}, E_{W'}, W') \leq \omega(x)$ and equality holds only if $s' \in PC(x, \mathcal{Y})$. We obtain:

$$\epsilon(x') \leq 1 + \mathrm{ord}J(F_{p,Z',W'}, E_{W'}, W') \leq 1 + \overline{\mathrm{ord}}J(F_{p,Z',W'}, E_{W'}, W') \leq \epsilon(x). \quad (3.45)$$

Suppose that $s' \notin PC(x, \mathcal{Y})$ and $\omega(x') \geq \omega(x)$. Formula (3.45) shows that $\epsilon(x') = \omega(x') = \omega(x)$. If $i_0(m_{S'}) = p - 1$, we get $\kappa(x') = 1$ so inequality is strict in (3.16). If $i_0(m_{S'}) = p$, we may pick $j' = j_i \in J'\setminus(J')_E$, $e'+1 \leq i \leq n'_0$, such that

$$\overline{\mathrm{ord}}\left(U^{-\epsilon(y)}\Phi_{j'}(\{U_j\}_{j\in J})\right)_{\overline{m'}} < \omega(x).$$

By (3.44), we have $H'^{-1}\frac{\partial F_{p,Z'}}{\partial U'_i} \neq 0$. This is a contradiction with the assumption $\epsilon(x') = \omega(x')$. Thus it can be assumed that $s' \in PC(x, \mathcal{Y})$.

We get $\omega(x') \leq \epsilon(x') \leq \omega(x)$ unless all inequalities in (3.45) are equalities. In this case, we claim that $\omega(x') = \epsilon(x') - 1$ and this will conclude the proof. To prove the claim, we may pick $j_i \in J'\setminus(J')_E$, $e' + 1 \leq i \leq n'_0$, such that $\Phi_{j_i}(\{U_j\}_{j\in J}) \neq 0$ by proposition 3.3. Arguing as above, we have

$$H'^{-1}\frac{\partial F_{p,Z'}}{\partial U'_i} \equiv< \mathrm{cl}_{\omega(x)}\left(U^{-\epsilon(y)}\Phi_{j_i}(\{U_j\}_{j\in J})\right)_{\overline{m'}} > \mathrm{mod}((\{U'_{j'}\}_{j'\notin F})\cap G(m_{S'})_{\omega(x)}, \quad (3.46)$$

and this proves that $H'^{-1}\frac{\partial F_{p,Z'}}{\partial U'_i} \neq 0$. If $i_0(m_{S'}) = p$, we get $\omega(x') = \omega(x)$.

If $i_0(m_{S'}) = p-1$, we must introduce a truncation operator

$$T' : G(m_{S'})_{p\delta(x')} \to G(m_{S'})_{p\delta(x')}$$

as in definition 2.16 in order to compute $\omega(x')$. In any case, we have

$$H'^{-1}G'^{p} \subseteq (U'_{i\notin F}) \cap G(m_{S'})_{\epsilon(x')}, \tag{3.47}$$

which follows from the identity $I'_0 S'/(u) = 0$ (resp. from (3.42)) if $i_0(m_S) = p$ (resp. if $i_0(m_S) = p-1$), *cf.* beginning of the proof of case 3.

Going back to definition 2.14, we have

$$H'^{-1}(F_{p,Z'} - T'F_{p,Z'}) \subseteq (\{U'_i\}_{i\notin F}) \cap G(m_{S'})_{\epsilon(x')}.$$

It now follows from (3.46) that

$$H'^{-1}\frac{\partial T'F_{p,Z'}}{\partial U'_i} \equiv < \mathrm{cl}_{\omega(x)}\left(U^{-\epsilon(y)}\Phi_{j_i}(\{U_j\}_{j\in J})\right)_{\overline{m'}} > \mod((\{U'_{j'}\}_{j'\notin F}) \cap G(m_{S'})_{\omega(x)}.$$

This proves at last that $H'^{-1}\frac{\partial T'F_{p,Z'}}{\partial U'_i} \neq 0$, so $\omega(x') = \epsilon(x') - 1$ and this concludes the proof of the claim, hence of the theorem.

3.3 Consequences of the blowing up theorem and constructibility.

In this section, we prove some basic properties of our main invariant

$$y \mapsto (m(y), \omega(y), \kappa(y))$$

and of our notion of permissibility (see introduction). The following theorem expresses the persistence of permissibility under permissible blowing ups.

Theorem 3.7. *Assume that $m(x) = p$, $\omega(x) > 0$, where $\{x\} = \eta^{-1}(m_S)$. Let $\mathcal{Y}_0 \subset \mathcal{Y}_1$ with respective generic point y_0, y_1 be permissible centers at x and $\pi : \mathcal{X}' \to \mathcal{X}$ be the blowing up along $\mathcal{Y}_1$.*

The strict transform $\mathcal{Y}'_0$ of $\mathcal{Y}_0$ is permissible at every $x' \in \pi^{-1}(x)$.

Proof. By definition of permissibility, we have $m(y_0) = m(y_1) = p$. Let $W_i = \eta(\mathcal{Y}_i)$, $i = 0, 1$ be with notations as in the previous theorem. There exist associated subsets $J_0 \subset J_1 \subseteq \{1, \ldots, n\}$ such that $I(W_i) = (\{u_j\}_{j \in J_i})$ for a certain choice of an adapted r.s.p. $(u_1, \ldots, u_n)$ of S. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x. By theorem 2.4, the polyhedron

$$\Delta_{\hat{S}}(h; \{u_j\}_{j \in J_i}; Z) = \mathrm{pr}_{J_i}(\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)) \text{ is minimal,}$$

where $\mathrm{pr}_{J_i} : \mathbb{R}^n \to \mathbb{R}^{J_i}$ denotes the projection on the $(u_j)_{j \in J_i}$-space, $i = 0, 1$. In particular, we have $\mathcal{Y}_i = V(Z, \{u_j\}_{j \in J_i})$, $i = 0, 1$. The strict transform W_0' of W_0 at s' has normal crossings with $E' := \sigma^{-1}(E)_{\mathrm{red}}$. Since $m(x') \geq m(y_0)$ for every $x' \in \mathcal{Y}_0'$, this proves that $\mathcal{Y}_0'$ is Hironaka-permissible w.r.t. E'.

Applying again theorem 2.4, we have

$$\epsilon(y_0) \leq \epsilon(y_1) \leq \epsilon(x), \ \epsilon(y_0) \leq \epsilon(x'). \tag{3.48}$$

On the other hand, theorem 3.6 applied to π gives $\epsilon(x') \leq \epsilon(x) + 1$ while classifying equality cases in (1) and (2). Thus $\mathcal{Y}_0'$ is permissible of the first kind except possibly in the following two cases:

Case 1: $\mathcal{Y}_1$ is of the first kind and $\epsilon(x') = \epsilon(x) + 1$;
Case 2: $\mathcal{Y}_0$ is of the second kind and $\epsilon(x') = \epsilon(x)$.

Since $x' \in \mathcal{Y}_0'$, we have, with notations as in theorem 3.6 (*cf.* notation 3.2):

$$(J_0)_E \subseteq \{j_i, \ 2 \leq i \leq e_0'\}, \ J_0 \backslash (J_0)_E \subseteq \{j_i, \ n_0' + 1 \leq i \leq n_1'\}. \tag{3.49}$$

Also, letting $F_0 := \{2, \ldots, e_0'\} \cup \{n_0' + 1, \ldots, n_1'\}$, we have (*cf.* notation 3.3):

$$J_0 \subseteq F_0 \subseteq F = F_0 \cup \{n_1' + 1, \ldots, n'\}. \tag{3.50}$$

Proof in case 1: an immediate consequence of theorem 3.6(1) is that :

$$i_0(m_S) = p, \ \frac{\partial F_{p,Z}}{\partial U_j} = 0, \ j \in J_0 \text{ or } j \geq e + 1.$$

This is incompatible with definition 3.3(iii) applied to $\mathcal{Y}_0$, so $\mathcal{Y}_0$ is also of the first kind. By proposition 3.1 we deduce that

$$H^{-1}G^p = 0, \ H^{-1}F_{p,Z} \subseteq k(x)[\{U_j\}_{j \in J_0}]_{\epsilon(x)}. \tag{3.51}$$

Since $\epsilon(y_0) = \epsilon(x') - 1$, we also have

$$H'^{-1} < G'^p, F_{p,Z'} > \subseteq (\{U'_i\}_{j_i \in J_0})^{\epsilon(y_0)} \cap G(m_{S'})_{\epsilon(x')}. \tag{3.52}$$

We claim that $\mathcal{Y}'_0$ is permissible of the second kind at x'. To prove the claim, note that (3.51) implies that

$$H_{W_1}^{-1} G_{W_1}^p \subseteq (\overline{u}_{j'}) G(W_1)_{\epsilon(x)} \text{ for some } j' \in (J'_1)_E.$$

Since $\mathcal{Y}_0$ is permissible of the first kind at x, we actually have

$$H_{W_1}^{-1} G_{W_1}^p \subseteq (\overline{u}_{j'}) S / (\{u_j\}_{j \in J_1})[\{U_j\}_{j \in J_0}]_{\epsilon(x)}.$$

Letting $j' =: j_{i'}$, $e'_0 + 1 \leq i' \leq e$, proposition 3.5(ii) then shows that

$$H_{W'_1}^{-1} G_{W'_1}^p \subseteq (\overline{u}_{i'}) S' / (u'_1)[\{U'_i\}_{j_i \in J_0}]_{\epsilon(x)},\ W'_1 := \sigma^{-1}(W_1).$$

In other terms, we have

$$H'^{-1} G'^p \subseteq (U'_1, U_{i'}) k(x')[\{U'_i\}_{j_i \in J_0}],$$

and this proves that $\mathcal{Y}'_0$ satisfies property (ii) of definition 3.2. Finally, applying (3.52) gives an expansion

$$H'^{-1} F_{p,Z'} = < \sum_{i=1}^{n'} U'_i \Phi_i(\{U'_{i'}\}_{j_{i'} \in J_0}) > .$$

Then definition 3.2(iii) is equivalent to:

$$\exists i \in J'_0 \cap \{e' + 1, \ldots, n'\} : \Phi_i \neq 0.$$

By equation (3.17) in theorem 3.6(2), there exists $i \geq n'_1 + 1$ (hence $i \in J'_0$) such that $\Phi_i \neq 0$, since $j_{i'} \in J_0 \Longrightarrow i' \leq n'_1$ by (3.49) and this completes the proof in case 1.

Proof in case 2. Since $\mathcal{Y}_0$ is permissible of the second kind, the initial form $\text{in}_{m_S} h \in G(m_S)[Z]$ satisfies (3.5). The corresponding integer j_0 satisfies $j_0 \notin J'_0$ and the corresponding family $(\Phi_{j'}(\{U_j\}_{j \in J_0}))_{j' \in J'_0}$ is such that $\Phi_{j'} \neq 0$ for some $j' \in J'_0 \backslash (J'_0)_E$. In order to prove that $\mathcal{Y}'_0$ is of the second kind at x', we consider two subcases:

Case 2a: $\mathcal{Y}_1$ is of the second kind at x. Then $j_0 \in J_1'$ and $\Phi_{j'} \neq 0$ for some $j' \in J_1' \backslash (J_1')_E$. By assumption $\epsilon(x') = \epsilon(x)$, and we deduce from (3.42) (resp. from (3.47)) if $i_0(m_S) = p-1$ (resp. if $i_0(m_S) = p$) that the initial form $\mathrm{in}_{m_{S'}} h' \in G(m_{S'})[Z']$ satisfies

$$H'^{-1} G'^p \subseteq U_{j_0'} k(x')[\{U_i'\}_{j_i \in J_0}]_{\epsilon(y_0)} \text{ for some } j_0' \in \{1, e_0'+1, \ldots, e'\} \quad (3.53)$$

and definition 3.2(ii) is checked for $\mathcal{Y}_0'$ at x'. Similarly, definition 3.2(iii) is checked from (3.46): we have $H'^{-1} \frac{\partial F_{p,Z'}}{\partial U_i'} \neq 0$ for any i, $e'+1 \leq i \leq n_0'$ such that $j_i \in J_1' \backslash (J_1')_E$ and $\Phi_{j_i} \neq 0$; take $j_i = j'$ with notations as above.

Case 2b: $\mathcal{Y}_1$ is of the first kind at x. Then $j_0 \in J_1$ and $\Phi_{j'} = 0$ for any $j' \in J_1'$. By proposition 3.3 and our assumption $\epsilon(x') = \epsilon(x)$, we have

$$\omega(x) = \epsilon(y_0) = \epsilon(x) - 1 = \epsilon(x') - 1 \leq \omega(x').$$

Therefore theorem 3.6 implies that $\omega(x') = \omega(x)$. We have $\kappa(x), \kappa(x') \geq 2$ since $\omega(x) = \epsilon(x) - 1$, $\omega(x') = \epsilon(x') - 1$. This is the equality case $(m(x'), \omega(x'), \kappa(x')) = (m(x), \omega(x), \kappa(x))$ discussed in theorem 3.6.

If $i_0(m_S) = p$, we are in the equality case of (3.28). Then (3.53) holds and there exists i, $n_1'+1 \leq i \leq n'$ or ($n_0'+1 \leq i \leq n_1'$ and $\Phi_{j_i} \neq 0$) such that

$$H'^{-1} \frac{\partial F_{p,Z'}}{\partial U_i'} \neq 0 \quad (3.54)$$

by (3.31). We may take here $j_i := j' \in J_0' \backslash (J_0')_E$. This checks definition 3.2(ii) and (iii) respectively.

If $i_0(m_S) = p-1$, the initial form $\mathrm{in}_{m_{S'}} h' \in G(m_{S'})[Z']$ satisfies

$$H'^{-1} G'^p \subseteq U_{i_1}' k(x')[\{U_i'\}_{j_i \in J_0}]_{\epsilon(y_0)},$$

where $j_{i_1} := j_0 \in J_0'$, $2 \leq i_1 \leq e_0'$ and definition 3.2(ii) is checked. Equation (3.54) also remains valid for some i, $n_0'+1 \leq i \leq n'$, in this case: this follows from (3.31) which is still valid (end of the proof of case 2 of theorem 3.6 where (3.41) replaces (3.29). This checks definition 3.2(iii) and the proof is complete.

Remark 3.2. The conclusion of the above theorem fails in general if it is only assumed that $\mathcal{Y}_0 \subset \mathcal{Y}_1$ is such that $\mathcal{Y}_0$ is permissible at x, $\mathcal{Y}_1$ Hironaka-permissible at x w.r.t. E.

A counterexample with $n = 4$ is given for $\mathrm{char}S = p > 0$ by taking:

$$h = Z^p + u_4 u_1^p + u_3 u_2^p,\ E = \mathrm{div}(u_1 u_2 u_3),\ \mathrm{Sing}_p \mathcal{X} = V(Z, u_1, u_2).$$

Then $(u_1, \ldots, u_4; Z)$ are well adapted coordinates. Taking

$$\mathcal{Y}_0 = V(Z, u_1, u_2) \subset \mathcal{Y}_1 = V(Z, u_1, u_2, u_4) \subset \{x\} = V(Z, u_1, u_2, u_3, u_4),$$

we have $\epsilon(y_0) = \epsilon(y_1) = \epsilon(x) - 1 = \omega(x) = p$. Note that $\mathcal{Y}_1$ does not satisfy definition 3.2(iii). There is a unique point

$$x' = (Z', u_1', u_2', u_3, u_4') := (Z/u_4, u_1/u_4, u_2/u_4, u_3, u_4) \in \mathcal{Y}_0' = V(Z', u_1', u_2').$$

A local equation for the strict transform $\mathcal{X}'$ of $\mathcal{X}$ at x is:

$$h' = Z'^p + u_4' u_1'^p + u_3 u_2'^p,\ E' = \mathrm{div}(u_1' u_2' u_3 u_4').$$

Thus $\epsilon(x') = \omega(x') = p + 1 > \omega(x)$ and $\mathcal{Y}_0'$ is not permissible at x' since $\epsilon(y_0) = p < \epsilon(x')$.

It is easily seen that such counterexamples exist only for $\mathcal{Y}_0$ of the second kind and $n \geq 4$.

We now turn to formal arcs on $\mathcal{X}$ and their image. Recall that it is assumed all along this chapter that $m(x) = p$, $\omega(x) > 0$ and $\{x\} = \eta^{-1}(m_S)$. By this, we mean:

Definition 3.4. A formal arc on $(\mathcal{X}, x)$ is a local morphism $\varphi :\ \mathrm{Spec}\mathcal{O} \to (X, x)$, where $(\mathcal{O}, N, l)$ is a complete discrete valuation ring. We denote the closed (resp. generic) point of $\mathrm{Spec}\mathcal{O}$ by O (resp. ξ) and call *support of* φ the subscheme $Z(\varphi) := \overline{\{\varphi(\xi)\}} \subseteq (\mathcal{X}, x)$.

The arc φ is said to be *well parametrized* if the inclusion

$$\mathcal{O}_\xi := \mathcal{O} \cap k(\varphi(\xi)) \subseteq \mathcal{O}$$

induces an isomorphism $\widehat{\mathcal{O}_\xi} \simeq \mathcal{O}$. The arc φ is said to be *nonconstant* if $\varphi(\xi) \neq x = \varphi(O)$.

Given a nonconstant formal arc on $(\mathcal{X}, x)$, and $\pi :\ \mathcal{X}' \to \mathcal{X}$ a blowing up along a permissible center $\mathcal{Y} \subset \mathcal{X}$ at x such that $\mathcal{Y} \subsetneq Z(\varphi)$, there exists a unique lifting $\varphi' :\ \mathrm{Spec}\mathcal{O} \to \mathcal{X}'$. Let

$$x' := \varphi'(O),\ (\mathcal{X}_1, x_1) := (\mathcal{X}', x') \text{ and } \varphi_1 :\ \mathrm{Spec}\mathcal{O} \to (\mathcal{X}_1, x_1)$$

be the induced morphism. The arc φ_1 is again nonconstant, so the process can be iterated. Let

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots \tag{3.55}$$

be a sequence of such local blowing ups and centers with

$$x_r \in \mathcal{Y}_r \subsetneq Z_r(\varphi) := \overline{\{\varphi_r(\xi)\}} \subset \mathcal{X}_r. \tag{3.56}$$

Note that the local ring $\mathcal{O}_{\mathcal{X}_r,\varphi_r(\xi)}$ is independent of $r \geq 0$. In particular, $m(\varphi_r(\xi))$, $\epsilon(\varphi_r(\xi))$ and $\omega(\varphi_r(\xi))$ are independent of $r \geq 0$. An important case of such sequences is when taking $\mathcal{Y}_r = \{x_r\}$ for every $r \geq 0$; then (3.55) is called the *quadratic sequence along* φ.

In any case, given a sequence (3.55), we let

$$d(\varphi) := \min_{r\geq 0}\{\dim \mathcal{O}_{\mathcal{X}_r,x_r}\}.$$

If $m(x) = p$ and $\omega(x) > 0$, theorem 3.6 implies that

$$(m(x_1), \omega(x_1), \kappa(x_1)) \leq (m(x), \omega(x), \kappa(x)).$$

If $m(x_r) = p$ and $\omega(x_r) > 0$ for every $r \geq 0$, we let

$$m(\varphi) := p,\ \omega(\varphi) := \min_{r\geq 0}\{\omega(x_r)\} > 0.$$

Proposition 3.8. *With notations as above, let* $\varphi : \mathrm{Spec}\mathcal{O} \to (\mathcal{X}, x)$ *be a nonconstant well parametrized formal arc on* $(\mathcal{X}, x)$ *whose quadratic sequence is such that* $m(\varphi) = p$ *and* $\omega(\varphi) > 0$. *Then* $l|k(x_r)$ *is algebraic for* $r >> 0$.

Assume that $l|k(x_r)$ *is algebraic with finite inseparable degree for some* $r \geq 0$. *Then there exists* $r_0 \geq 0$ *such that the following holds: the support* $Z_r(\varphi)$ *is Hironaka-permissible at* x_r *and* $\epsilon(x_r) = \epsilon(x_{r_0})$ *for every* $r \geq r_0$*; furthermore exactly one of the following conditions is satisfied:*

(1) $Z_r(\varphi)$ *is permissible of the first kind at* x_r *for every* $r \geq r_0$*;*

(2) there exists a finite sequence (3.55):

$$(\mathcal{X}_{r_0}, x_{r_0}) =: (\mathcal{X}', x') \leftarrow (\mathcal{X}'_1, x'_1) \leftarrow \cdots \leftarrow (\mathcal{X}'_{r_1}, x'_{r_1}) =: (\tilde{\mathcal{X}}, \tilde{x})$$

of local blowing ups with centers contained in and of codimension one in the successive strict transforms of $Z_{r_0}(\varphi)$*, such that the quadratic sequence along* φ*:*

$$(\tilde{\mathcal{X}}, \tilde{x}) =: (\tilde{\mathcal{X}}_0, \tilde{x}_0) \leftarrow (\tilde{\mathcal{X}}_1, \tilde{x}_1) \leftarrow \cdots \leftarrow (\tilde{\mathcal{X}}_r, \tilde{x}_r) \leftarrow \cdots$$

has the following properties for every $r \geq 0$*:*

(a) $\epsilon(\tilde{x}_r) = \epsilon(x_{r_0})$;

(b) $\dim\mathcal{O}_{\tilde{Z}_r(\varphi),\tilde{x}_r} = \dim\mathcal{O}_{Z_{r_0}(\varphi),x_{r_0}} \geq 2$;

(c) $\tilde{Z}_r(\varphi)$ *is permissible of the second kind at* $\tilde{x}_r$ *(resp.* $\omega(\tilde{x}_r) = 0$*) if* $\epsilon(x_{r_0}) \geq 2$ *(resp. if* $\epsilon(x_{r_0}) = 1$*).*

Proof. It can be assumed without loss of generality that

$$d(\varphi) = \dim\mathcal{O}_{\mathcal{X},x},\ m(x) = p \text{ and } \omega(x) = \omega(\varphi) > 0.$$

Since $m(\varphi) = p$ and $\omega(\varphi) > 0$, we let $\eta_r :\ (\mathcal{X}_r, x_r) \to \mathrm{Spec}S_r$ be the corresponding projection, $I_r(\varphi) \subseteq S_r$ be the ideal of $W_r(\varphi) := \eta_r(Z_r(\varphi))$. We drop the reference to φ in what follows in order to avoid cumbersome notations.

For $f \in m_{S_0}$, $f \notin I_0$ we denote by $\overline{f} \in \mathcal{O}$, $\overline{f} \neq 0$ its image by $\varphi^\sharp$. Let v be the discrete valuation associated with $\mathcal{O}$ and let

$$M_r := \{v(\overline{f}),\ f \in S_r \backslash I_r\}$$

be the semigroup of values of S_r w.r.t. v. The group generated by M_r is the value group of the restriction $v_{|\overline{K}}$ to $\overline{K} = QF(S/I_0)$, hence independent of $r \geq 0$, and is denoted by $a\mathbb{Z} \subseteq v(N)\mathbb{Z}$, $a \in \mathbb{N}$.

Suppose that $M_0 \neq a\mathbb{N}$. Let $\alpha \geq 2$, $\beta \in \mathbb{N}\backslash\alpha\mathbb{N}$ be defined by:

$$a\alpha := \min\{M_0\backslash(0)\},\ a\beta := \min\{M_0\backslash a\alpha\mathbb{N}\}. \tag{3.57}$$

We pick $u, w \in m_{S_0}$ such that $v(\overline{u}) = a\alpha$, $v(\overline{w}) = a\beta$. Obviously u is a regular parameter of S and $wu^{-1} \in m_{S_1}$. Suppose $M_1 \neq a\mathbb{N}$. There are associated integers α_1, β_1 as in (3.57) which satisfy $(\alpha_1, \beta_1) < (\alpha, \beta)$ for the lexicographical ordering. This can repeat only finitely many times so we get $M_r = a\mathbb{N}$ for some $r \geq 0$. W.l.o.g. it can be assumed that $M_0 = a\mathbb{N}$.

Let $(u_1, \ldots, u_n)$ be a r.s.p. of $S = S_0$ which is adapted to $E = \mathrm{div}(u_1 \cdots u_e)$. Without loss of generality, it can be assumed that $v(\overline{u}_e) = a$. Up to renumbering coordinates, there exists $e(\varphi)$, $0 \leq e(\varphi) < e$ such that

$$(u_1, \ldots, u_{e(\varphi)}) \subseteq I := I_0,\ u_j \notin I \text{ for } e(\varphi) + 1 \leq j \leq e.$$

For j, $e(\varphi) + 1 \leq j \leq e - 1$, let $v(\overline{u}_j) =: a\alpha_j$, $\alpha_j \geq 1$. Note that $u_j u_e^{-\alpha_j}$ is a unit in S_{α_j}; in other terms, replacing S by $S_{\max\{\alpha_j\}}$, it can be assumed that

$e(\varphi) = e - 1$.

Let $f \in m_{S_0} \backslash I_0$ and write $f = u_e^{\alpha_r(f)} f_r \in S_r$, where u_e does not divide f_r in S_r and note that

$$f_r \in m_{S_r} \Longrightarrow v(\overline{f}) > \alpha_r(f) v(\overline{u}_e) \geq ar.$$

Since $M_0 = a\mathbb{N}$, there exists $r \geq 0$ such that f_r is a unit. This implies that for every ideal $\overline{J} \subseteq S_0/I_0$, $\overline{J}S_r/I_r$ is a principal ideal for $r >> 0$. This is a well known characterization of valuation rings, i.e.

$$\mathcal{O}_{v_{|\overline{K}}} = \bigcup_{r \geq 0} S_r/I_r. \tag{3.58}$$

Let l_0 be the residue field of the valuation $v_{|\overline{K}}$. Then $l|l_0$ is algebraic (of degree at most p) and $l_0|k(x_r)$ is algebraic for $r >> 0$ by (3.58). This proves the first statement in the theorem. We thus may assume from now on, again by (3.58), that

$$l_0|k(x_0) \text{ is separable algebraic.} \tag{3.59}$$

Let S^{sh} be the strict Henselization of S, so $l^{\mathrm{sh}} := S^{\mathrm{sh}}/m_{S^{\mathrm{sh}}}$ is the separable algebraic closure of l. The residue action induces an isomorphism

$$\mathrm{Gal}(S^{\mathrm{sh}}|S^{\mathrm{h}}) \simeq \mathrm{Gal}(l^{\mathrm{sh}}|k(x))$$

where S^{h} is the Henselization of S. Let $\tilde{S}$ be the fixed subring of S^{sh} by the inverse image of $\mathrm{Gal}(l^{\mathrm{sh}}|l_0)$ under the previous group morphism. Then $S \subset \tilde{S}$ is a local ind-étale map such that $l_0 = \tilde{S}/m_{\tilde{S}}$. In particular $S \subset \tilde{S}$ is regular [40] theorem I.8.1(iv). Since $\mathcal{O}$ is Henselian and $l_0 \subseteq l = \mathcal{O}/N$, the morphism φ factors through $\tilde{S}$.

Recall notation 2.1 and notation 2.2 for the regular local base change $S \subset \tilde{S}$. We apply theorem 2.20 with $\tilde{s} := m_{\tilde{S}}$ and get:

$$m(\tilde{x}) = m(x) = p,\ \omega(\tilde{x}) = \omega(\varphi) > 0 \text{ and } \epsilon(\tilde{x}) = \epsilon(x) > 0,$$

the right hand side equality holding because $\tilde{n} = n$. Applying theorem 2.14, $\tilde{\mathcal{X}} = \mathrm{Spec}(\tilde{S}[X]/(\tilde{h}))$ is irreducible, so in the separable case (case (b) of assumption (**G**)), the $G = \mathbb{Z}/p$-action extends uniquely to $\tilde{\mathcal{X}}$ and (**G**) holds for $(\tilde{S}, \tilde{h}, \tilde{E})$. This proves that $(\tilde{S}, \tilde{h}, \tilde{E})$ satisfies the assumption of the proposition, all other assumptions being trivially satisfied.

Now $W_0 \times_{k(x_0)} \mathrm{Spec} l_0$ may be reducible, but $W_r \times_{k(x_r)} \mathrm{Spec} l_0$ is irreducible for $r >> 0$. After possibly changing indices, it can be assumed that $W := W_0 \times_{k(x_0)} \mathrm{Spec} l_0$ is irreducible. Then W has normal crossings with E at x if and only if $\tilde{W} := W \times_S \mathrm{Spec}\tilde{S}$ has normal crossings with $\tilde{E}$ at $\tilde{x}$. Let $\tilde{Z} := Z \times_S \mathrm{Spec}\tilde{S}$ and $\tilde{z}$ be the generic point of a component of $\tilde{Z}$. By theorem 2.20, we have $m(\tilde{z}) = m(z)$, so $\tilde{Z}$ is Hironaka-permissible at $\tilde{x}$ w.r.t. $\tilde{E}$ if and only if Z is Hironaka-permissible at x w.r.t. E. In other terms, we may replace S by $\tilde{S}$ and thus assume that $l_0 = k(x_0)$ in order to prove the second statement.

Let now

$$e_r := \dim_{k(x_r)} \frac{I_r + m_{S_r}^2}{m_{S_r}^2} \geq e - 1,\ t_r := e_r - (e-1) \geq 0$$

for $r \geq 0$. It can be assumed w.l.o.g. that $(u_{e+1}, \ldots, u_{e+t_0}) \subseteq I_0$. We have $e_{r+1} \geq e_r$ for every $r \geq 0$ and let $e_\infty := \max_{r \geq 0}\{e_r\}$. It can be assumed w.l.o.g. that $e_0 = e_\infty$.

Since $l_0 = k(x_r)$ and $M_r = a\mathbb{N}$ for every $r \geq 0$, the ring morphism $S_r \to \widehat{\mathcal{O}_{v_{|K}}}$ factors through $\hat{S}_r$ to a *surjective* morphism

$$\hat{\varphi}_r :\ \hat{S}_r \to \widehat{\mathcal{O}_{v_{|K}}}.$$

Let $\hat{I}_r$ be the kernel of $\hat{\varphi}_r$, so we have

$$I_r \hat{S}_r \subseteq \hat{I}_r \text{ and } I_r = \hat{I}_r \cap S_r. \tag{3.60}$$

After possibly replacing S_0 by S_r for some $r \geq 0$, it can be assumed that the curve $\mathrm{Spec}(\hat{S}_0/\hat{I}_0)$ is transverse to $\hat{E} = \mathrm{div}(u_1 \cdots u_e) \subset \mathrm{Spec}\hat{S}_0$. We claim that

$$I_0 = (u_1, \ldots, u_{e-1}, u_{e+1}, \ldots, u_{e+t_0}). \tag{3.61}$$

To prove the claim, suppose that $I_0 \neq J_0 := (u_1, \ldots, u_{e-1}, u_{e+1}, \ldots, u_{e+t_0})$. We let $\hat{u}_j := u_j$, $1 \leq j \leq e + t_0$ and pick a basis

$$\hat{I}_0 = J_0 + (\hat{u}_{e+t_0+1}, \ldots, \hat{u}_n) \tag{3.62}$$

of $\hat{I}_0$. Since S_0 is excellent, the ring $(\hat{S}_0/I_0)_{\hat{I}_0}$ is regular, hence reduced. By assumption, $I_0 \neq J_0$, so there exists $f \in I_0 \backslash J_0$ such that f restricts to a regular parameter $\overline{f}$ in $\overline{S} := (\hat{S}_0/J_0)_{\hat{I}_0}$:

$$\mathrm{ord}_{\hat{I}_0} f = 1,\ \mathrm{ord}_{m_{\overline{S}}} \overline{f} = 1. \tag{3.63}$$

Let $F \in \mathrm{gr}_{\hat{I}_0}(\hat{S}_0) \simeq \hat{S}_0/\hat{I}_0[\{\hat{U}_j\}_{j\neq e}]$ be the initial form of f. There is an expansion

$$F = \sum_{j\neq e} F_j \hat{U}_j,\ F_j \in \hat{S}_0/\hat{I}_0.$$

By (3.63) we have $F_j \neq 0$ for some j, $1 \leq j \leq e + t_0$. Suppose that

$$\exists j_0,\ 1 \leq j_0 \leq e + t_0 \mid\ m := \min_{j\neq e}\{\mathrm{ord}_{(\overline{u}_e)} F_j\} = \mathrm{ord}_{(\overline{u}_e)} F_{j_0}.$$

Replacing f with $f - \gamma_{j_0} u_{j_0} u_e^m$ for some unit $\gamma_{j_0} \in S_0$ preserves (3.63) while increasing $\mathrm{ord}_{(\overline{u}_e)} F_{j_0}$. Applying finitely many times this procedure, it can be assumed that

$$m := \min_{j\neq e}\{\mathrm{ord}_{(\overline{u}_e)} F_j\} < \min_{j_0 \leq e+t_0}\{\mathrm{ord}_{(\overline{u}_e)} F_{j_0}\}. \tag{3.64}$$

By lemma 3.9 below, there exists $r \geq 1$ and a writing

$$f_r = u_e^{m+r} g_r,\ g_r \notin (u_e) S_r,\ \mathrm{ord}_{m_{S_r}} g_r = 1.$$

Furthermore the last statement in *ibid.* shows that $\mathrm{in}_{\hat{I}_r} g_r \in (\mathrm{gr}_{\hat{I}_r} \hat{S}_r)_1$ is transverse to the initial forms $\overline{u}_e^{-r} U_j$, $1 \leq j \leq e+t_0$, $j \neq e$ by (3.64). Since $g_r \in I_r$, this implies that $e_r > e_0$: a contradiction, so claim (3.61) is proved. Since (3.61) is stable by further blowing ups, this proves that W_r is transverse to the reduced preimage of $\mathrm{div}(u_1 \cdots u_e)$ for every $r >> 0$.

Let $(\hat{u}_1, \ldots, \hat{u}_n; Z)$ be well adapted coordinates at x. There is an associated expansion

$$h = Z^p + f_{1,Z} Z^{p-1} + \cdots + f_{p,Z},\ f_{1,Z}, \ldots, f_{p,Z} \in \hat{S}_0.$$

We factor out $f_{i,Z} = u_e^{m_i} g_{i,Z}$, $1 \leq i \leq p$, with $g_{i,Z} = 0$ or (u_e does not divide $g_{i,Z}$, $m_i \in \mathbb{N}$). The *formal completion* $\hat{S}_1$ of the local blowing up S_1 has a r.s.p. $(\hat{u}'_1, \ldots, \hat{u}'_n)$ given by

$$\hat{u}'_e = \hat{u}_e = u_e \text{ and } \hat{u}'_j = \hat{u}_j/u_e,\ j \neq e.$$

Let $Z' := Z/u_e$, $h' := u_e^{-p} h \in S_1[Z']$ define the strict transform $(\mathcal{X}_1, x_1)$, since $m(\varphi) = p$. We thus have

$$f_{i,Z'} = u_e^{-i} f_{i,Z},\ 1 \leq i \leq p. \tag{3.65}$$

By proposition 2.6, the polyhedron $\Delta_{\hat{S}_1}(h'; \hat{u}'_1, \ldots, \hat{u}'_n; Z')$ is minimal. Applying again lemma 3.9 below, it can be assumed w.l.o.g. that

$$\mathrm{ord}_{m_{\hat{S}_0}} g_{i,Z} = \mathrm{ord}_{\hat{I}_0} g_{i,Z},\ 1 \leq i \leq p. \tag{3.66}$$

Let $\hat{Z}_0 := V(Z', \hat{I}_0) \subset (\hat{\mathcal{X}}_0, \hat{x})$ and $\hat{z}$ be its generic point. Suppose that $\delta(\hat{z}) < 1$ and let i_0 such that $i_0\delta(\hat{z}) = \mathrm{ord}_{\hat{I}_0} f_{i_0,Z} < i_0$. Applying (3.65) gives

$$\mathrm{ord}_{m_{\hat{S}_1}} f_{i_0,Z'} = m_{i_0} + i_0(\delta(\hat{z}) - 1) < m_{i_0}.$$

This can repeat only finitely many times, a contradiction with $m(\varphi) = p$. Hence $\delta(\hat{z}) \geq 1$, i.e. $m(\hat{z}) = p$. By excellence, this implies that $m(z) = p$. Therefore Z_r is Hironaka-permissible at x_r for every $r >> 0$.

Similarly, replacing S_0 by S_r for some $r \geq 0$ and arguing as above, it can be assumed that

$$\epsilon(\hat{z}) = \min_{1 \leq i \leq p} \left\{ \frac{\mathrm{ord}_{\hat{I}_0}(H(x)^{-i} f_{i,Z}^p)}{i} \right\} = \epsilon(\hat{x}).$$

This proves that $\hat{Z}_0$ is permissible of the first kind at $\hat{x}$. Note that this furthermore implies that $\epsilon(x_r) = \epsilon(\hat{z})$ for every $r \geq 0$ and the second statement of the proposition is proved.

In order to prove that alternative (1) in the last statement holds, we may also replace S by $\tilde{S}$ as above and thus assume that $l_0 = k(x_0)$. If $\epsilon(z) = \epsilon(\hat{z})$, then Z_r is permissible of the first kind at x_r (definition 3.1(ii)). This proves that alternative (1) in the proposition is fulfilled or $\epsilon(\hat{z}) > \epsilon(z)$ which we may assume from now on.

By theorem 2.20(2.ii), we have $\dim Z_r \geq 2$ (statement $\tilde{n} > n$ of *ibid.* applied under the assumption $l_0 = k(x_0)$) and

$$\epsilon(\hat{z}) - 1 = \omega(z) = \epsilon(z) = \epsilon(\hat{x}) - 1 = \epsilon(x) - 1,\ i_0(\hat{z}) = i_0(z) = p. \tag{3.67}$$

We pick again well adapted coordinates $(\hat{u}_1, \ldots, \hat{u}_n; \hat{Z})$ at $\hat{x}$. Since $\hat{Z}_0$ is permissible of the first kind at $\hat{x}$, proposition 3.1 (with notations as therein) gives the following property for the initial form $\mathrm{in}_{m_{\hat{S}_0}} h \in G(m_{\hat{S}_0})[\hat{Z}]$:

$$H_0^{-1} G_0^p \in k(\hat{x})[\hat{U}_1, \ldots, \hat{U}_{e-1}, \hat{U}_{e+1}, \ldots \hat{U}_n]_{\epsilon(\hat{x})}.$$

Since $i_0(\hat{z}) = p$, we have $G_0 = 0$, i.e. $i_0(\hat{x}) = p$. This proves that definition 3.1(ii) is satisfied in any case.

To prove that alternative (2) in the proposition is fulfilled, we first assume that $l_0 = k(x_0)$ as before, then push down the result from $\tilde{S}$ to S. Let $(u_1, \ldots, u_n; Z)$ be well adapted coordinates at x and consider the initial form $\mathrm{in}_{W_0} h = Z^p + F_{p,Z,W_0} \in G(W_0)[Z]$. Let

$$J := \{1, \ldots, e-1, e+1, \ldots, e+t_0\}.$$

Since $\epsilon(\hat{z}) > \epsilon(z)$, we have $\delta(z) \in \mathbb{N}$ and

$$G(W_0) = S_0/I_0[\{U_j\}_{j\in J}],\ F_{p,Z,W_0} \in (\hat{S}_0/\hat{I}_0[\{U_j\}_{j\in J}]_{\delta(z)})^p \tag{3.68}$$

by theorem 2.20(2.ii). By theorem 2.4, the polyhedron

$$\Delta_{\hat{S}_0}(h; \{u_j\}_{j\in J}; Z) = \mathrm{pr}_J(\Delta_{\hat{S}}(h; u_1, \ldots, u_n; Z)) \text{ is minimal,}$$

where $\mathrm{pr}_J :\ \mathbb{R}^n \to \mathbb{R}^J$ denotes the projection on the $(u_j)_{j\in J}$-space. Let

$$\Phi_j := H_{W_0}^{-1} \frac{\partial F_{p,Z,W_0}}{\partial \overline{u}_j} \subseteq G(W_0)_{\epsilon(z)},\ \mathrm{cl}_0\Phi_j = 0,\ j \notin J, j \neq e, \tag{3.69}$$

since $\epsilon(x) = \epsilon(z) + 1$. The local blowing up S_1 has a r.s.p. $(u'_1, \ldots, u'_n)$ given by

$$\begin{cases} u'_j &= u_j/u_e \quad \text{if} \quad j \in J \\ u'_e &= u_e \\ u'_j &= u_j/u_e - \delta_j \quad \text{if} \quad j \notin J, j \neq e \end{cases}$$

where $\delta_j \in S_0$ is a unit or zero since we are assuming that $l_0 = k(x_0)$. Let

$$Z' := Z/u_e - \theta,\ \theta \in S_1,\ h' := u_e^{-p} h \in S_1[Z']$$

define the strict transform $(\mathcal{X}_1, x_1)$, with $\Delta_{\hat{S}_1}(h'; u'_1, \ldots, u'_n; Z')$ minimal and consider the initial form

$$\mathrm{in}_{W_1} h = Z'^p + F_{p,Z',W_1} \in G(W_1)[Z'],\ G(W_1) = S_1/I_1[\{U'_j\}_{j\in J}].$$

It is easily derived from (3.68)(3.69) that

$$\Phi'_j := H_{W_1}^{-1} \frac{\partial F_{p,Z',W_1}}{\partial \overline{u}'_j} = \overline{u}_e^{-\epsilon(x)} \Phi_j \subseteq G(W_1)_{\epsilon(z)},\ j \notin J, j \neq e.$$

Applying again lemma 3.9 below, it can be assumed w.l.o.g. that

$$(\Phi_j = \overline{u}_e^{m_j}\Psi_j,\ \mathrm{cl}_0\Psi_j \neq 0) \text{ or } \Phi_j = 0,\ j \notin J, j \neq e. \tag{3.70}$$

This equation is valid when $l_0 = k(x_0)$ and holds for S if and only if it holds for $\tilde{S}$. We may therefore replace S by $\tilde{S}$ as before.

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{N}^n$ be a vertex of $\Delta_{\hat{S}_0}(h; u_1, \ldots, u_n; Z)$ mapping to a vertex of $\Delta_{\hat{S}_0}(h; \{u_j\}_{j\in J}; Z)$ with $\sum_{j\in J} x_j = \delta(y)$. By (3.68) we have $x_j \in \mathbb{N}$ for $j \in J$. Suppose that $x_j \in \mathbb{N}$ for every $j \neq e$. Since $\hat{S}_0/\hat{I}_0 \simeq k(x)[[\overline{u}_e]]$, (3.68) implies that $\mathbf{x}$ is solvable: a contradiction. Taking j such that $x_j \notin \mathbb{N}$, there exists $j \notin J$, $j \neq e$ such that $\Phi_j \neq 0$. This proves that

$$r_1 := \min\{m_j, j \notin J, j \neq e : \Phi_j \neq 0\}$$

is well defined and that we have

$$\Phi_{p,Z,W_0} := \overline{u}_e^{-r_1} H_{W_0}^{-1} F_{p,Z,W_0} \subseteq G(W_0)_{\epsilon(z)},\ \mathrm{cl}_1\Phi_{p,Z,W_0} \notin (\overline{u}_e)G(W_0)_{\epsilon(z)}. \tag{3.71}$$

If $r_1 = 0$, then alternative (2) is fulfilled (definition 3.2(iii)) since

$$\overline{J}(F_{p,Z,W_0}, E, W_0) =< \{\mathrm{cl}_0\Phi_j\}_{j\notin J, j\neq e} >\neq 0.$$

by (3.71). Note that this situation does not occur if $\epsilon(x_{r_0}) = 1$, since $\omega(\varphi) > 0$.

Otherwise, we define $V_0 := V(u_e, I_0)$ and $\mathcal{Y}_0 := \eta_0^{-1}(V_0) \subset Z_0$. Then $\mathcal{Y}_0$ is Hironaka permissible at x_0 and its generic point y_0 has $\epsilon(y_0) = \epsilon(x)$ by (3.71). Let $\tilde{\mathcal{X}}_1$ be the blowing up of $\mathcal{X}_0$ along $\mathcal{Y}_0$ and note that φ lifts to the point $\tilde{x}_1$ on the strict transform $\tilde{Z}_1$ of Z_0. Let $\tilde{h} := u_e^{-p}h \in \tilde{S}_1[\tilde{Z}]$ define the strict transform $(\tilde{\mathcal{X}}_1, \tilde{x}_1)$ of $(\mathcal{X}, x)$, $\tilde{W}_1 := \tilde{\eta}_1(\tilde{Z}_1)$. By proposition 2.6, the initial form

$$\mathrm{in}_{\tilde{W}_1}\tilde{h} = \tilde{Z}^p + F_{p,\tilde{Z},\tilde{W}_1} \in G(\tilde{W}_1)[\tilde{Z}],\ G(\tilde{W}_1) = \tilde{S}_1/\tilde{I}_1[\{\tilde{U}_j\}_{j\in J}]$$

satisfies a relation (3.71) with associated integer $\tilde{r}_1 = r_1 - 1$. Iterating r_1 times this procedure, we get some $(\tilde{\mathcal{X}}_{r_1}, \tilde{x}_{r_1})$ with initial form

$$\mathrm{in}_{\tilde{W}_r}\tilde{h}_r = \tilde{Z}_r^p + F_{p,\tilde{Z}_r,\tilde{W}_r} \in G(\tilde{W}_r)[\tilde{Z}_r],\ G(\tilde{W}_r) = \tilde{S}_r/\tilde{I}_r[\{\tilde{U}_{j,r}\}_{j\in J}]$$

with $\tilde{U}_{j,r} = \overline{u}_e^{-r_1}U_j$, $j \in J$. We have

$$\tilde{\Phi}_r := H_{\tilde{W}_r}^{-1} F_{p,\tilde{Z}_r,\tilde{W}_r}) \subseteq G(\tilde{W}_r)_{\epsilon(z)},\ \mathrm{cl}_1\tilde{\Phi}_r \notin (\overline{u}_e)G(W_0)_{\epsilon(z)}. \tag{3.72}$$

By proposition 3.3, we now have $\omega(\tilde{x}_{r_1}) = \epsilon(z) = \epsilon(x_{r_0}) - 1 \geq 0$. Thus $\omega(\tilde{x}_{r_1}) > 0$ if $\epsilon(x_{r_0}) \geq 2$ and we are done by the former case $r_1 = 0$. Otherwise, $\epsilon(x_{r_0}) = 1$ and $\omega(\tilde{x}_{r_1}) = 0$ and the conclusion follows.

Example 3.1. Take $S = k[u_1, u_2, u_3, u_4]_{(u_1,u_2,u_3,u_4)}$ with k a field of characteristic $p > 0$. We let:

$$h = Z^p + u_2^p u_4 u_3^p + u_3 u_1^p \in S[Z].$$

Then (u_1, u_2, u_3, u_4) are adapted to (S, h, E), $E := \mathrm{div}(u_1 u_2)$ (definition 2.6) and $(u_1, u_2, u_3, u_4; Z)$ are well adapted coordinates at the closed point $x = (Z, u_1, u_2, u_3, u_4)$ of $\mathcal{X} = \mathrm{Spec}(S[Z]/(h))$ (definition 2.8). Indeed, it is easily seen that:

$$\mathrm{Sing}_p\mathcal{X} := \{y \in \mathcal{X} : m(y) = p\} = V(Z, u_1, u_2) \cup V(Z, u_1, u_3),\ \omega(x) = p.$$

Let $\vartheta(t) := \sum_{i\geq 1} \lambda_i t^i \in k[[t]]$ be a power series which is *transcendental* over $k(t)$. We define a nonconstant well-parametrized k-linear formal arc on $(\mathcal{X}, x)$ by:

$$\varphi(Z) = \varphi(u_1) = \varphi(u_3) = 0,\ \varphi(u_2) = u_2,\ \varphi(u_4) = \vartheta(t)^p.$$

The quadratic sequence along φ has well adapted coordinates $(u_1^{(r)}, \ldots, u_4^{(r)}, u_2^{-r} Z)$ at x_r where $u_j^{(r)} := u_j^{(r-1)}/u_2^{(r-1)}$, $j \neq 2$, $u_2^{(r)} := u_2^{(r-1)}$ for every $r \geq 1$, with $u_j^{(0)} := u_j$, $1 \leq j \leq 4$. Let

$$v_4^{(r)} := u_2^{-r}(u_4 - \sum_{ip\leq r} \lambda_i^p u_2^{ip}),\ T_r := u_2^{-r}(Z + (u_3^{(r)})^p \sum_{ip\leq r} \lambda_i^p u_2^{ip}),\ r \geq 0.$$

Then φ factors through

$$(\mathcal{X}_r, x_r) = (\mathrm{Spec}(S_r[T_r]/(h_r), x_r),\ S_r = S[u_1^{(r)}, u_3^{(r)}, u_4^{(r)}]_{(u_1^{(r)},\ldots,u_4^{(r)})},$$

and the strict transform h_r of h is given by

$$h_r := T_r^p + (u_2^{(r)})^r \left((u_2^{(r)})^p v_4^{(r)} (u_3^{(r)})^p + u_3^{(r)} (u_1^{(r)})^p\right).$$

We have $Z_r := V(T_r, u_1^{(r)}, u_3^{(r)})$ for every $r \geq 1$. Note that Z_r is not permissible at x_r. Therefore φ fulfills alternative (2) of proposition 3.8.

Remark 3.3. We do not know if the conclusion of proposition 3.8 is still valid for $n \geq 4$ when removing the assumption "$l|k(x_r)$ is algebraic with finite inseparable degree for some $r \geq 0$". When $n = 3$, it can be proved that the above assumption is actually implied by "$m(\varphi) = p$ and $\omega(\varphi) > 0$". This is a (very) special case of the proof given in [24].

The following lemma is elementary and well-known.

Lemma 3.9. *Let S be a regular local ring (not necessarily excellent) of dimension $n \geq 1$ with r.s.p. $(u_1, \ldots, u_n)$ and*

$$C := V(u_1, \ldots, u_{n-1}) \subset (\mathcal{S}_0, s_0) := \mathrm{Spec} S$$

be a regular curve. Let

$$(\mathcal{S}_0, s_0) \leftarrow (\mathcal{S}_1, s_1) \leftarrow \cdots \leftarrow (\mathcal{S}_i, s_i) \leftarrow \cdots$$

be the composition of local blowing ups such that $\mathcal{S}_i$ is the blowing up of $\mathcal{S}_{i-1}$ along s_{i-1} and $s_i \in \mathcal{S}_i$ is the point on the strict transform C_i of C for $i \geq 1$.

Let $f \in S$, $f \neq 0$ and denote $d := \mathrm{ord}_C f$. There exists $m, i_0 \in \mathbb{N}$ such that for every $i \geq i_0$, there is a decomposition

$$f = u_n^{m+di} g_i, \ g_i \in S_i := \mathcal{O}_{\mathcal{S}_i, s_i} \text{ and } \mathrm{ord}_{C_i} g_i = \mathrm{ord}_{s_i} g_i = d.$$

Furthermore, the initial form $\mathrm{in}_{C_i} g_i \in (\mathrm{gr}_{I_{C_i}} S_i)_d$ is the strict transform of

$$\mathrm{in}_C f \in (\mathrm{gr}_{I_C} S)_d \simeq S/(u_1, \ldots, u_{n-1})[U_1, \ldots, U_{n-1}]_d.$$

Proof. We have $S_i = S_{i-1}[u_1^{(i)}, \ldots, u_{n-1}^{(i)}]_{(u_1^{(i)}, \ldots, u_n^{(i)})}$, where $u_j^{(i)} := u_j^{(i-1)}/u_n^{(i-1)}$, $1 \leq j \leq n-1$, $u_n^{(i)} := u_n^{(i-1)}$ for every $i \geq 1$, with $u_j^{(0)} := u_j$, $1 \leq j \leq n$. Then $C_i = V(u_1^{(i)}, \ldots, u_{n-1}^{(i)})$ with these notations. There is an expansion

$$f = (u_n^{(i-1)})^{m_{i-1}} g_{i-1}, \ g_{i-1} := \sum_{\mathbf{x} \in \mathbf{S}} \gamma(\mathbf{x})^{(i-1)} (u_1^{(i-1)})^{x_1} \cdots (u_n^{(i-1)})^{x_n} \in S_{i-1},$$

where $\gamma(\mathbf{x})^{(i-1)} \in S_{i-1}$ is a unit for each $\mathbf{x} \in \mathbf{S}$, $\mathbf{S} \subset \mathbb{N}^n$ a finite set, $m_{i-1} \in \mathbb{N}$, $g_{i-1} \notin (u_n^{(i-1)})$. Since $\mathrm{ord}_C f = d$, it can be assumed without loss of generality that

$$d = \min_{\mathbf{x} \in \mathbf{S}} \{x_1 + \cdots + x_{n-1}\}.$$

Therefore

$$d = \mathrm{ord}_{C_{i-1}} g_{i-1} \leq d_{i-1} := \mathrm{ord}_{s_{i-1}} g_{i-1} = \min_{\mathbf{x}\in\mathbf{S}}\{|\ \mathbf{x}\ |\}.$$

Note that the initial form $\mathrm{in}_{C_{i-1}} f$ is given by

$$\mathrm{in}_{C_{i-1}} f = \sum_{x_1+\cdots+x_{n-1}=d} \overline{\gamma}(\mathbf{x})^{(i-1)} (\overline{u}_n^{(i-1)})^{x_n} (U_1^{(i-1)})^{x_1} \cdots (U_{n-1}^{(i-1)})^{x_{n-1}},$$

where $\overline{\gamma}(\mathbf{x})^{(i-1)}, \overline{u}_n^{(i-1)} \in S_{i-1}/(u_1^{(i-1)}, \ldots, u_{n-1}^{(i-1)})$ denote the classes of the corresponding elements in S_{i-1}. After blowing up, we get an expansion

$$f = (u_n^{(i)})^{m_{i-1}+d_{i-1}} g_i,\ g_i := \sum_{\mathbf{x}\in\mathbf{S}} \gamma(\mathbf{x})^{(i-1)} (u_1^{(i)})^{x_1} \cdots (u_{n-1}^{(i)})^{x_{n-1}} (u_n^{(i)})^{|\mathbf{x}|-d_{i-1}} \in S_i.$$

Let $A_{i-1} := \{\mathbf{x} \in \mathbf{S} : x_1 + \cdots + x_{n-1} < d_{i-1}\}$. For each $\mathbf{x} \in A_{i-1}$, we have $|\ \mathbf{x}\ | - d_{i-1} < x_n$. We deduce:

$$0 \leq \min_{\mathbf{x}\in A_i}\{x_n\} < \min_{\mathbf{x}\in A_{i-1}}\{x_n\}.$$

This proves that there exists $i_0 \geq 0$ such that $A_i = \emptyset$ for every $i \geq i_0$. Then $d_i = d$ for $i \geq i_0$. This proves the first statement in the lemma, taking $m := m_{i_0} - di_0 \geq 0$. Finally, this construction preserves the initial form $\mathrm{in}_C f$, i.e.

$$\mathrm{in}_{C_i} f = \overline{u}_n^{-(m+di)} (\mathrm{in}_C f) \left(\overline{u}_n^i U_1^{(i)}, \ldots, \overline{u}_n^i U_n^{(i)}\right),$$

and this concludes the proof.

Theorem 3.10. *Let $\mathcal{Y} \subset (\mathcal{X}, x)$ be an integral closed subscheme with generic point y. The set*

$$\Omega(\mathcal{Y}) := \{y' \in \mathcal{Y} : (m(y'), \omega(y'), \kappa(y')) = (m(y), \omega(y), \kappa(y))\} \subseteq \mathcal{Y}$$

contains a nonempty Zariski open subset of $\mathcal{Y}$.

Let furthermore $\mathcal{Z} \supset \mathcal{Y}$ be an integral closed subscheme with generic point z such that $\mathcal{Z}$ is permissible (of the first or second kind) at y. The set

$$\mathrm{Perm}(\mathcal{Y}, \mathcal{Z}) := \{y' \in \mathcal{Y} : \mathcal{Z} \text{ is permissible at } y'\} \subseteq \mathcal{Y}$$

contains a nonempty Zariski open subset of $\mathcal{Y}$.

Proof. Our function (m, ω, κ) refines the multiplicity function m on $\mathcal{X}$, and our notion of permissible blowing up refines the Hironaka-permissibility. We may thus apply the well known openness of these properties. It is therefore sufficient to prove the first statement when $m(y) = p$. For the second statement, we take a nonempty Zariski open set $\mathcal{U}_1 \subseteq \mathcal{Y}$ such that $\mathcal{Z}$ is Hironaka permissible at every $y' \in \mathcal{U}_1$.

Let $W := \eta(\mathcal{Y})$, $s := \eta(y)$, $W_{\mathcal{Z}} := \eta(\mathcal{Z})$ for the second statement. We pick an adapted r.s.p. $(u_1, \ldots, u_{n_s})$ of S_s, where $E_s = \operatorname{div}(u_1 \cdots u_{e_s})$. For every $y' \in \mathcal{U}_1$ there exists an adapted r.s.p. $(u_1, \ldots, u_{n_{y'}})$ of $S_{\eta(y')}$ (i.e. $E_{\eta(y')} = \operatorname{div}(u_1 \cdots u_{e_{y'}})$, $e_{y'} \geq e_s$) such that S_s is the localization of $S_{\eta(y')}$ at some prime

$$I(W_{y'}) = (\{u_j\}_{j \in J_{y'}}),\ J_{y'} \subseteq \{1, \ldots, n_{y'}\}.$$

After possibly shrinking $\mathcal{U}_1 \subseteq \mathcal{Y}$, it can be assumed without loss of generality that $e_{y'} = e_s$ for every $y' \in \mathcal{U}_1$.

We now *choose* any point $y_0 \in \mathcal{U}_1$. Let $(u_1, \ldots, u_{n_0}; Z)$ be well adapted coordinates at y_0, $s_0 := \eta(y_0)$, $S_0 := S_{s_0}$. There is a corresponding expansion

$$h = Z^p + f_{1,Z} Z^{p-1} + \cdots + f_{p,Z} \in S_0[Z],\ f_{1,Z}, \ldots, f_{p,Z} \in S_0.$$

After possibly restricting again $\mathcal{U}_1$, we may assume that the rational functions $u_1, \ldots, u_{n_0}, f_{1,Z}, \ldots, f_{p,Z}$ are regular at $\eta(y')$ for every $y' \in \mathcal{U}_1$. Moreover, we have in $S_{\eta(y')}$

$$I(W) = (\{u_j\}_{j \in J}) \text{ (and } I(W_{\mathcal{Z}}) = (\{u_j\}_{j \in J_{\mathcal{Z}}}) \text{ for the second statement)}$$

with $J_{\mathcal{Z}} \subseteq J = \{1, \ldots, n\}$, $n_{y'} \geq n$, subsets which do not depend on y'. We fix an associated expansion at s_0:

$$f_{i,Z} = \sum_{\mathbf{x} \in \mathbf{S}_i} \gamma(i, \mathbf{x}) u_1^{i x_1} \cdots u_{n_0}^{i x_{n_0}} \in S_0,\ 1 \leq i \leq p,$$

with $\mathbf{S}_i \subset (\frac{1}{i}\mathbb{N})^{n_0}$ finite and $\gamma(i, \mathbf{x}) \in S_0$ a unit for each $\mathbf{x} \in \mathbf{S}_i$. After possibly restricting again $\mathcal{U}_1$, it may also be assumed that each $\gamma(i, \mathbf{x})$ appearing in some $f_{i,Z}$, $1 \leq i \leq p$, is a regular function at $\eta(y')$. By theorem 2.4, the polyhedra

$$\Delta_{\hat{S}_0}(h; \{u_j\}_{j \in J}; Z) \text{ (and } \Delta_{\hat{S}_0}(h; \{u_j\}_{j \in J_{\mathcal{Z}}}; Z)) \text{ are minimal.} \tag{3.73}$$

We define $A_i \subset (\frac{1}{i}\mathbb{N})^J$ (and $A_{i,\mathcal{Z}} \subset (\frac{1}{i}\mathbb{N})^{J_{\mathcal{Z}}}$ for the second statement) to be the respective images of $\mathbf{S}_i$ by the projections $\operatorname{pr}_J : \mathbb{R}^{n_0} \to \mathbb{R}^J$ and

$\mathrm{pr}_{J_{\mathcal{Z}}} : \mathbb{R}^{n_0} \to \mathbb{R}^{J_{\mathcal{Z}}}$. Given $\mathbf{a} \in A_i$, we let:

$$\gamma(i, \mathbf{a}) := \sum_{\mathrm{pr}_J(\mathbf{x})=\mathbf{a}} \gamma(i, \mathbf{x}) \prod_{j \notin J} u_j^{ix_j} \in S_0.$$

By definition of $\epsilon(y)$, we have:

$$\epsilon(y) = p \min_{1 \le i \le p} \min_{\mathbf{a} \in A_i} \{ |\ \mathbf{a}\ | : \gamma(i, \mathbf{a}) \neq 0 \} - \sum_{j=1}^{e_s} H_j. \tag{3.74}$$

Let $B \subset \mathbb{Q}^n$ be the set of $(i, \mathbf{a})$ achieving equality on the right hand side of (3.74). The initial form polynomial $\mathrm{in}_{m_{S_s}} h$ is thus of the form

$$\mathrm{in}_{m_{S_s}} h = Z^p + \sum_{(i,\mathbf{a}) \in B} \overline{\gamma}(i, \mathbf{a}) \prod_{j \in J} U_j^{ia_j} Z^{p-i} \in G(m_{S_s})[Z], \tag{3.75}$$

where $\overline{\gamma}(i, \mathbf{a})$ denotes the image in $k(y)$. Let

$$B_0 := \{(i, \mathbf{a}) \in B : \exists (i, \mathbf{a}) \in B, i \neq p \text{ or } (i = p \text{ and } \mathbf{a} \notin \mathbb{N}^J)\}.$$

Case 1. Suppose that $B_0 \neq \emptyset$. We define:

$$\mathcal{U} := \{y' \in \mathcal{U}_1 : \forall (i, \mathbf{a}) \in B_0, \overline{\gamma}(i, \mathbf{a}) \text{ is a unit in } S_{\eta(y')}\}.$$

Since $\gamma(i, \mathbf{a})$ is nonzero for $(i, \mathbf{a}) \in B$ by (3.74), $\mathcal{U}$ is a nonempty Zariski open subset of $\mathcal{Y}$. To $y' \in \mathcal{U}$, we associate $\mathbf{x} \in \Delta_{\hat{S}_{\eta(y')}}(h; u_1, \ldots, u_{n_{y'}}; Z)$ (depending on $(i, \mathbf{a})$) by

$$\begin{cases} x_j = a_j & \text{if } j \in J \\ x_j = 0 & \text{if } j \notin J \end{cases}$$

Computing initial forms from definition 2.2 with $\alpha_{y'} := (1, \ldots, 1) \in \mathbb{R}^{n_{y'}}$, $\delta_{\alpha_{y'}}(h; u_1, \ldots, u_{n_{y'}}; Z) = \delta(y)$, the corresponding initial form polynomial

$$\mathrm{in}_{\alpha_{y'}} h = Z^p + \sum_{i=1}^{p} F_{i,Z,\alpha_{y'}} Z^{p-i} \in G(m_{S_{\eta(y')}})[Z] \tag{3.76}$$

is such that $F_{i,Z,\alpha_{y'}} \neq 0$ for some $i \neq p$ or $F_{p,Z,\alpha_{y'}} \notin k(y')[U_1^p, \ldots, U_{n_{y'}}^p]$. Therefore $\delta(y') = \delta(y)$ and we deduce that

$$\epsilon(y') = \epsilon(y) \text{ for every } y' \in \mathcal{U}. \tag{3.77}$$

To prove the first statement, note that we are already done by (3.77) if $\epsilon(y) = 0$. Assume now that $\epsilon(y) > 0$. If $i_0(y) = p-1$, there exists some $(p-1, \mathbf{a}_0) \in B_0$ for some $\mathbf{a}_0 \in \mathbb{N}^J$. Let $y' \in \mathcal{U}$ and pick well adapted coordinates $(u_1, \ldots, u_{n_{y'}}; Z_{y'})$ at y'. The corresponding initial form polynomial

$$\mathrm{in}_{m_{S_{\eta(y')}}} h = Z_{y'}^p - G_{y'}^{p-1} Z_{y'} + F_{p,Z_{y'}} \in G(m_{S_{\eta(y')}})[Z_{y'}]$$

is such that $< G_{y'} >=< U^{\mathbf{a}_0} >$ (resp. $G_{y'} = 0$) if $i_0(y) = p-1$ (resp. if $i_0(y) = p$). We have

$$F_{p,Z_{y'}} = \sum_{(p,\mathbf{a}) \in B_0} \lambda_{y'}(p, \mathbf{a}) U^{\mathbf{a}} + \Psi_{y'} \subseteq G(m_{S_{\eta(y')}})_{\epsilon(y)},$$

where $\lambda_{y'}(i, \mathbf{a}) \in k(y')$, $\lambda_{y'}(i, \mathbf{a}) \neq 0$, $\Psi_{y'} \in k(y')[\{U_j^p\}_{j \in J}]$ for every $(p, \mathbf{a}) \in B_0$ and every $y' \in \mathcal{U}$. Comparing with definition 2.16, we have $\omega(y') = \omega(y)$, $\kappa(y') = 1$ if $\kappa(y) = 1$ for $y' \in \mathcal{U}$. This proves the first statement in case 1.

For the second statement, we are also done if $\epsilon(z) = \epsilon(y)$, i.e. if $\mathcal{Z}$ is of the first kind at y. Suppose that $\mathcal{Z}$ is permissible of the second kind at y. In particular, we have $\epsilon(y) > 0$. There exist $j_1(y) \in J \backslash J_{\mathcal{Z}}$ and $j'(y) \in J \backslash J_{\mathcal{Z}}$, $j'(y) \geq e_s + 1$, satisfying the conclusion of proposition 3.3. Let $y' \in \mathcal{U}$ and pick well adapted coordinates $(u_1, \ldots, u_{n_{y'}}; Z_{y'})$ at y'. The corresponding initial form polynomial (3.77) again satisfies

$$H_{y'}^{-1} G_{y'}^p \subseteq U_{j_1(y)} k(y')[U_1, \ldots, U_{n_{y'}}]_{\epsilon(y)}$$

and there is an expansion

$$H_{y'}^{-1} F_{p,Z_{y'}} =< \sum_{j' \in J'} U_{j'} \Phi_{j'}(\{U_j\}_{j \in J}) + \Psi(\{U_j\}_{j \in J}) > \subseteq G(m_{S_{\eta(y')}})_{\epsilon(y)}$$

with $\Phi_{j'(y_0)} \neq 0$, hence $\mathcal{Y}$ is permissible of the second kind at y' and the conclusion follows.

Case 2. Suppose on the contrary that $B_0 = \emptyset$. By (3.75), we have

$$\mathrm{in}_{m_{S_s}} h = Z^p + \sum_{(p,\mathbf{a}) \in B} \overline{\gamma}(p, \mathbf{a}) \prod_{j \in J} U_j^{p a_j} \in G(m_{S_s})[Z] \tag{3.78}$$

and this proves that

$$\delta(y) \in \mathbb{N},\ \omega(y) = \epsilon(y) \text{ and } \kappa(y) \geq 2. \tag{3.79}$$

Since $(\{u_j\}_{j\in J}; Z)$ are well adapted coordinates at y, there exists a vertex $\mathbf{a}_0 \in \Delta_{\hat{S}_s}(h; \{u_j\}_{j\in J}; Z)$, $(p, \mathbf{a}_0) \in B$ which is not solvable, i.e. $\overline{\gamma}(p, \mathbf{a}_0) \notin k(y)^p$. Let $B_1 \subseteq B_0$ be the nonempty subset defined by

$$B_1 := \{(p, \mathbf{a}) \in B : \overline{\gamma}(p, \mathbf{a}) \notin k(y)^p\}.$$

Given $(p, \mathbf{a}) \in B_1$, we define a morphism:

$$\eta_{(p,\mathbf{a})} : \ \mathcal{Y}_{(p,\mathbf{a})} := \mathrm{Spec}\left(\frac{\mathcal{O}_{\mathcal{U}_1}[T]}{(T^p - \overline{\gamma}(p, \mathbf{a}))}\right) \longrightarrow \mathcal{U}_1.$$

Note that $\mathcal{Y}_{(p,\mathbf{a})}$ is integral and $\eta_{(p,\mathbf{a})}$ is finite and purely inseparable. We define:

$$\mathcal{U} := \{y' \in \mathcal{U}_1 : \forall (p, \mathbf{a}) \in B_1, \eta^{-1}_{(p,\mathbf{a})}(y')_{\mathrm{red}} \text{ is a regular point of } \mathcal{Y}_{(p,\mathbf{a})}\}.$$

Since $\mathcal{Y}_{(p,\mathbf{a})}$ is excellent, its regular locus is a nonempty Zariski open set. We deduce that $\mathcal{U}$ is a nonempty Zariski open subset of $\mathcal{Y}$.

For $y' \in \mathcal{U}_1$ and $(p, \mathbf{a}) \in B$, we denote by $\lambda_{y'}(p, \mathbf{a}) \in k(y')$ the residue of $\overline{\gamma}(p, \mathbf{a})$. The property

$$\text{“}\eta^{-1}_{(p,\mathbf{a})}(y')_{\mathrm{red}} \text{ is a regular point of } \mathcal{Y}_{(p,\mathbf{a})}\text{ ”}$$

is equivalently characterized as follows: either (a) $\lambda_{y'}(p, \mathbf{a}) \notin k(y')^p$, or (b) there exists $\delta_{y'}(p, \mathbf{a}) \in \mathcal{O}_{\mathcal{Y},y'}$ such that

$$v_{y'}(p, \mathbf{a}) := \overline{\gamma}(p, \mathbf{a}) - \delta_{y'}(p, \mathbf{a})^p$$

is a regular parameter at y'.

We now prove the first statement. Let $y' \in \mathcal{U}$ and pick well adapted coordinates $(u_1, \ldots, u_{n_{y'}}; Z_{y'})$ at y'. Let

$$B(y') := \{(p, \mathbf{a}) \in B_1 : \text{(a) is satisfied}\}.$$

Suppose that $B(y') \neq \emptyset$. We get $\delta(y') = \delta(y)$, $i_0(y') = p$ and the initial form polynomial $\mathrm{in}_{m_{S_{\eta(y')}}} h \in G(m_{S_{\eta(y')}})[Z_{y'}]$ is

$$\mathrm{in}_{m_{S_{\eta(y')}}} h = Z^p_{y'} + \sum_{(p,\mathbf{a})\in B(y')} \lambda_{y'}(p, \mathbf{a}) U^{\mathbf{a}} + \Psi^p_{y'}$$

where $\lambda_{y'}(p,\mathbf{a}) \notin k(y')^p$ and $\Psi_{y'} \in k(y')[\{U_j^p\}_{j\in J}]$. This shows that

$$\omega(y') = \epsilon(y') = \epsilon(y) = \omega(y),$$

the right hand side equality by (3.79). Moreover $\kappa(y') \geq 2$, so $y' \in \Omega(\mathcal{Y})$.

Suppose on the contrary that $B(y') = \emptyset$. We get

$$\delta(y') = \delta(y) + \frac{1}{p},\ i_0(y') = p \text{ (since } \delta(y') \notin \mathbb{N})$$

and the initial form polynomial $\mathrm{in}_{m_{S_{\eta(y')}}} h \in G(m_{S_{\eta(y')}})[Z_{y'}]$ is

$$\mathrm{in}_{m_{S_{\eta(y')}}} h = Z_{y'}^p + \sum_{(p,\mathbf{a})\in B_1} V_{y'}(p,\mathbf{a})U^{\mathbf{a}} + \Psi_{y'},$$

where $V_{y'}(p,\mathbf{a}) \in < U_1,\ldots,U_{n_{y'}} > \setminus < \{U_j\}_{j\in J} >$, $\Psi_{y'} \in k(y')[\{U_j\}_{j\in J}]_{p\delta(y)+1}$. This shows that $\omega(y') = \epsilon(y')-1 = \epsilon(y) = \omega(y)$, applying again (3.79). Moreover $\kappa(y') \geq 2$, so $y' \in \Omega(\mathcal{Y})$. This concludes the proof of the first statement.

For the second statement, note that $\mathcal{Z}$ is necessarily of the first kind at y in case 2, since (3.78) is not compatible with proposition 3.3. With notations as above, $\mathcal{Z}$ is then permissible of the first kind (resp. of the second kind) at y' if $B(y') \neq \emptyset$ (resp. if $B(y') = \emptyset$). This concludes the proof.

Corollary 3.11. *With notations as above, the function*

$$\iota : \mathcal{X} \to \{1,\ldots,p\} \times \mathbb{N} \times \{0,1,\geq 2\},\ y \mapsto (m(y),\omega(y),\kappa(y))$$

is a constructible function on $\mathcal{X}$. In particular, it takes finitely many distinct values.

Proof. This follows from the previous theorem and Noetherian induction on $\mathcal{X}$.

Remark 3.4. The constructible sets $\mathcal{X}_{p,a} := \{y \in \mathcal{X} : (m(y),\omega(y)) \leq (p,a)\}$, $a \in \mathbb{N}$ are not in general Zariski open (example 3.2 below). See next proposition for closedness of the set $(m(y),\omega(y)) > (p,0)$.

We do not know if the sets $\mathrm{Perm}(\mathcal{Y},\mathcal{Z})$ as in the theorem are constructible subsets of $\mathcal{Y}$. An important issue about permissibility is addressed below in question 3.1.

Theorem 3.10 is sufficient for our applications to Local Uniformization. About a possible extension of our methods to a global Resolution of Singularities statement, we remark the following: let $\mathcal{S}$ be an excellent regular domain,

$$\eta : \ \mathcal{X} \to \mathcal{S}$$

be a finite morphism, $x \in \mathcal{X}$ be such that $(\mathcal{X}, x) \to \mathcal{S}_{\eta(x)}$ satisfies the assumption of theorem 3.10. It is easily seen that its conclusion extends to some affine neighbourhood $\mathcal{U}$ of x on $\mathcal{X}$.

Example 3.2. Let $S = k[[u_1, u_2, u_3]]$, k a (nonperfect) field of characteristic $p > 0$, $\lambda \in k \backslash k^p$. We take:

$$h = Z^p - (u_1^2 u_2)^{p-1} Z + u_1^p (u_3 u_1^{p-1} + \lambda u_3^p) \in S[Z], \ E = \mathrm{div}(u_1 u_2).$$

The coordinates $(u_1, u_2, u_3; Z)$ are well adapted to (S, h, E). Let

$$x := (Z, u_1, u_2, u_3), \ y := (Z, u_1, u_3).$$

We have $H(x) = (u_1^p)$, $m(x) = m(y) = p$, and compute:

$$\mathrm{in}_{m_S} h = Z^p + U_1^p (U_3 U_1^{p-1} + \lambda U_3^p), \ i_0(x) = p, \ \omega(x) = \epsilon(x) - 1 = p - 1.$$

On the other hand, we have:

$$\mathrm{in}_{m_{S_{\eta(y)}}} h = Z^p - (U_1^2 \overline{u}_2)^{p-1} Z + U_1^p (U_3 U_1^{p-1} + \lambda U_3^p), \ i_0(y) = p - 1, \ \epsilon(y) = p.$$

To compute $\omega(y)$, we must introduce a truncation operator

$$T_y : k(y)[U_1, U_3]_{2p} \to k(y)[U_1, U_3]_{2p}$$

as in definition 2.16 and get $T_y F_{p,Z,y} = \lambda U_1^p U_3^p$, so $\omega(y) = p > \omega(x)$. This proves that the set $\mathcal{X}_{(p,p-1)} := \{y \in \mathcal{X} : (m(y), \omega(y)) \leq (p, p-1)\}$ is not Zariski open.

Proposition 3.12. *Let $(\mathcal{X}, x)$ be as in the theorem. The set*

$$\Omega_+(\mathcal{X}) := \{y \in \mathcal{X} : (m(y), \omega(y)) > (p, 0)\} \subseteq \mathcal{X}$$

is Zariski closed and of dimension at most $n - 2$.

Proof. Let $\xi \in \mathcal{X}$ be the generic point of an irreducible component of $\eta^{-1}(E)$. Then $(m(\xi), \epsilon(\xi)) \leq (p, 0)$, so $\xi \notin \Omega_+(\mathcal{X})$. Therefore it is sufficient to prove that $\Omega_+(\mathcal{X})$ is Zariski closed.

We will use the Nagata Criterion to prove openness of $\mathcal{X}\backslash\Omega_+(\mathcal{X})$. By theorem 3.10, it is sufficient to prove that $\Omega_+(\mathcal{X})$ is stable by specialization. Let $y_0 \rightsquigarrow y_1$ be a specialization in $\mathcal{X}$ and assume that $y_1 \notin \Omega_+(\mathcal{X})$. We must prove that $y_0 \notin \Omega_+(\mathcal{X})$, so we are reduced to the case $m(y_0) = p$. Let $\mathcal{Y}_0 := \overline{\{y_0\}}$.

By localizing η at $\eta(y_1)$, it can be furthermore assumed that $y_1 = x$. Arguing by induction on the dimension of $\mathcal{Y}_0$, it can be furthermore assumed that $\mathcal{Y}_0$ is a curve. Let

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots$$

be a sequence of local blowing ups at closed points belonging to the strict transform of $\mathcal{Y}_0$. We have $m(x_r) \geq m(y_0) = p$, so $m(x_r) = p$ for every $r \geq 0$. Since S is excellent, the strict transform of $\mathcal{Y}_0$ in $\mathcal{X}_r$ is Hironaka permissible for $r >> 0$. By construction, these maps induce local isomorphisms at y_0.

We then have $(m(x_r), \omega(x_r)) \leq (p, 0)$ by proposition 2.22, hence $\omega(x_r) = 0$ since $m(x_r) = p$ for every $r \geq 0$. In other words, after possibly replacing $(\mathcal{X}, x)$ by $(\mathcal{X}_r, x_r)$ for some $r \geq 0$, it can be assumed that $\mathcal{Y}_0$ is Hironaka permissible. Then there exist well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x such that

$$I(W_0) = (\{u_j\}_{j\in J_0}), W_0 := \eta(\mathcal{Y}_0)$$

with $J_0 = \{1, \ldots, n\}\backslash\{j'\}$ for some j' (since $\mathcal{Y}_0$ is a curve). We let $s_0 := \eta(y_0)$, $S_0 := S_{s_0}$. By theorem 2.4, the polyhedron $\Delta_{\hat{S}}(h; \{u_j\}_{j\in J}; Z)$ is minimal, so we deduce that $\epsilon(y_0) \leq \epsilon(x)$.

Since $\omega(x) = 0$ by assumption, we have $\omega(y_0) = 0$ except possibly if $\epsilon(y_0) = \epsilon(x) = 1$. Since $\omega(x) = 0$, the initial form polynomial $\mathrm{in}_{W_0} h \in G(m_S)[Z]$ then satisfies

$$H_{W_0}^{-1} F_{p,Z,W_0} =< \sum_{j\in J_0} \gamma_j U_j >\subseteq G(W_0)_1 = S/I(W_0)[\{U_j\}_{j\in J_0}],$$

and there exists $j_0 \in J_0$, $e + 1 \leq j_0 \leq n$ such that γ_{j_0} is a unit in $S/I(W_0)$. This gives $\omega(y_0) = 0$ if $i_0(y) = p$. If $i_0(y) = p - 1$, we must introduce a truncation operator

$$T_0 : \ G(m_{S_0})_{p\delta(y_0)} \to G(m_{S_0})_{p\delta(y_0)},$$

as in definition 2.16 in order to compute $\omega(y_0)$. However, T_0 proceeds from definition 2.14 in the special case $p\delta(y_0) = 1 + \sum_{j\in J_0} H_j$. Lemma 2.17 then implies that

$$H_{W_0}^{-1}\mathrm{Ker}T_0 \subseteq < \{U_j\}_{j\in J_0, j\leq e} > \subset G(m_{S_0})_{p\delta(y_0)}.$$

Since $j_0 \geq e+1$, we thus have $H_{W_0}U_{j_0} \nsubseteq \mathrm{Ker}T_0$ and this proves that $\omega(y_0) = 0$ as required.

A very special case of the following question (for μ a discrete valuation with some extra assumption) has been answered in the affirmative in theorem 3.8 above.

Question 3.1. Let $\mathcal{Y} = \mathcal{Y}_0$ be an integral closed subscheme with generic point y, $m(y) = p$, $\omega(y) > 0$, and let μ be a valuation centered at m_S. Does there exist a finite sequence of permissible local blowing ups along μ:

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r)$$

with centers $\mathcal{Z}_i \subset (\mathcal{Y}_i, x_i)$, $\mathcal{Y}_i$ denoting the strict transform of $\mathcal{Y}$ in $(\mathcal{X}_i, x_i)$, $0 \leq i \leq r$, such that $\mathcal{Y}_r$ is permissible at x_r?

4 Projection number and projection theorem.

Let (S, h, E) satisfy assumptions **(G)** and **(E)**. In this section, we perform induction on the dimension $n+1 = \dim S[Z]$ of the ambient space of $\mathcal{X}$. This step is for now far out of reach in arbitrary dimension and little more than definitions could be stated when $n \geq 4$. Therefore:

We assume from now on that $\dim S \leq 3$.

4.1 The projection number $\kappa(y)$.

For $y \in \mathcal{X}$, $s := \eta(y) \in \mathrm{Spec}S$, the assignment $\kappa(y) \geq 2$ has sofar been used to express $\kappa(y) \neq 1$; we now distinguish $\kappa(y) = 2, 3, 4$ when $\kappa(y) \geq 2$. This completes the definition of the complexity function (2.60):

$$\iota : \mathcal{X} \to \{1, \ldots, p\} \times \mathbb{N} \times \{1, \ldots, 4\}, y \mapsto (m(y), \omega(y), \kappa(y)).$$

The projection number $\kappa(y)$ expresses the transverseness of $\mathrm{VDir}(y)$ w.r.t. E_s. It has no particular invariance property w.r.t. regular local base change $S \subseteq \tilde{S}$ ($\tilde{S}$ excellent and $\dim \tilde{S} \leq 3$) when $\kappa(y) \geq 2$.

In the following definition it can be assumed without loss of generality that $s = m_S$ by localizing S at s, since our assumptions **(G)** and **(E)** are stable when changing (S, h, E) to (S_s, h_s, E_s) (notation 2.2). We write $E = \mathrm{div}(u_1 \cdots u_e)$ as before.

Definition 4.1. Assume that $m(x) = p$, $\omega(x) > 0$ and $\kappa(x) \geq 2$, where $\eta^{-1}(m_S) = \{x\}$. We let

$$\kappa(x) := 4 \text{ if } \mathrm{VDir}(x) \subseteq < U_1, \ldots, U_e > . \tag{4.1}$$

Assume furthermore that $\kappa(x) \neq 4$. We let $\kappa(x) := 3$ if ($\omega(x) = \epsilon(x) - 1$ and one of the following conditions is satisfied).

(1) $E = \mathrm{div}(u_1)$ and there exists well adapted coordinates $(u_1, \ldots, u_n; Z)$ at x (so $n \leq 3$) such that

$$\mathrm{VDir}(x) \subseteq < U_1, U_2 > \text{ and } H^{-1}\frac{\partial F_{p,Z}}{\partial U_3} \subseteq < U_1^{\omega(x)} >;$$

(2) $E = \mathrm{div}(u_1 u_2)$.

Finally, we let $\kappa(x) := 2$ if $\kappa(x) \neq 3, 4$.

Remark 4.1. When $\dim \mathcal{O}_{\mathcal{X},x} = 2$, the definition reduces to: if $E = \mathrm{div}(u_1 u_2)$, let $\kappa(x) := 4$; if $E = \mathrm{div}(u_1)$, let:

$$\kappa(x) := \begin{cases} 2 & \text{if} \quad \omega(x) = \epsilon(x) \text{ and } \mathrm{VDir}(x) \not\subseteq < U_1 >; \\ 3 & \text{if} \quad \omega(x) = \epsilon(x) - 1 \qquad\qquad\qquad\quad ; \\ 4 & \text{if} \quad \omega(x) = \epsilon(x) \text{ and } \mathrm{VDir}(x) \subseteq < U_1 > . \end{cases}$$

We now turn to the statement of the projection theorem. Since it is assumed here that $\omega(x) > 0$, $(\mathcal{X}, x)$ is (analytically) irreducible by theorem 2.14. Assume that a valuation μ of $L = k(\mathcal{X})$ centered at x is given. We will consider finite sequences of local blowing ups along μ:

$$(\mathcal{X}, x) =: (\mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \tag{4.2}$$

with Hironaka-permissible centers $\mathcal{Y}_i \subset (\mathcal{X}_i, x_i)$, where x_i, $0 \leq i \leq r$, denotes the center of μ. We require that our assumptions **(G)** and **(E)** be preserved by such blowing ups and that

$$(m(x_i), \omega(x_i)) \leq (m(x_{i-1}), \omega(x_{i-1})),\ 1 \leq i \leq r.$$

This certainly holds when the blowing up centers are permissible of the first or second kind by proposition 4.3 and theorem 3.6. In [24], we consider further kinds of permissible blowing up having the same property. We recall that all permissibility conditions (definitions 2.7, 3.1 and 3.2) always refer to the reduced total transform E_i of E in S_i, where there are projections

$$\eta_i : (\mathcal{X}_i, x_i) \longrightarrow \mathrm{Spec} S_i,\ 0 \leq i \leq r.$$

Similarly, $\omega(x_i), \epsilon(x_i), \kappa(x_i)$ are always computed w.r.t. E_i.

Finally, we emphasize that we do *not* require any particular behavior about the numbers $\kappa(x_i)$ *along* the process (4.2), i.e. we may have $\kappa(x_i) > \kappa(x)$ for some i, $1 \leq i \leq r$. Our goal is to *eventually* achieve $\kappa(x_r) < \kappa(x)$.

Definition 4.2. Assume that $m(x) = p$ and $\omega(x) > 0$. Given any finite sequence (4.2), we say that x_r is *very near* x if $\iota(x_r) \geq \iota(x)$.

We say that x is *good* if for every valuation μ of $L = k(\mathcal{X})$ centered at x, there exists a finite and independent sequence (4.2) such that $\iota(x_r) < \iota(x)$.

Proposition 4.1. *We denote by $e(x)$ the codimension in $k(x)[Z, U_1, U_2, U_3]$ of the directrix of the tangent cone of h at x, when $e(x) \geq 2$ (i.e. when $\delta(x) = 1$ **Definition** 2.5), x is good.*

Proof. Let us denote by $f(x)$ the codimension of the ridge of the directrix of the tangent cone of h at x [26] p. I-26, [27]1.5p. 203. The invariant we use is

$$(m(x), -f(x), -e(x))_{lex}$$

. A consequence of a Giraud's theorem [26] proposition 4.2 page II-33, $(m(x), -f(x))_{lex}$ does not increase by Hironaka's permissible blowing ups. As a consequence of [36] **Theorem 3** $(m(x), -f(x), -e(x))_{lex}$ does not increase by Hironaka's permissible blowing ups except in the case:

$$p = 2,\ F = \lambda(Z^2 + \lambda_2 U_1^2 + \lambda_1 U_2^2 + \lambda_1\lambda_2 U_3^2)^\alpha,\ [k^2(\lambda_1, \lambda_2) : k^2] = 4. \quad (4.3)$$

When $e(x) = 4$, after the blowing-up centered at x, there is no near point except in the case 4.3.

Since $m(x') = m(x)$, in this particular case, we have

$$x' := V(U_1^2 + \lambda_1 U_3^2, U_2^2 + \lambda_2 U_3^2, Z^2 + \lambda_1\lambda_2 U_3^2)$$

on $\pi'^{-1}(x) = \mathrm{Proj}(k[Z, U_1, U_2, U_3]/(F))$. x' is the point of parameters

$$(Y, v_1, v_2, u_3) \text{ where } Y := \frac{Z}{u_3} + u_1' u_2', v_1 = {u_1'}^2 + \lambda_1, v_2 = {u_2'}^2 + \lambda_2,\ u_1' = \frac{u_1}{u_3}, u_2' = \frac{u_2}{u_3}.$$

In this case, we have $f(x) = 1$. The strict transform of h is

$$h' = Y^2 + v_1 v_2 \bmod (u_3)$$

the ridge h' is contained in two linear subspaces (V_1, V_2) mod U_3 of the tangent cone , we have $f(x') = 3$:

$$(m(x)-, f(x), -e(x)) = (p, -1, -4) >_{lex} (p, -3, -3) \geq_{lex} (m(x')-, f(x'), -e(x')).$$

When $e(x) = 3$, then, if we blow up along x, then $e(x') \geq e(x)$, $f(x') \geq f(x)$. In case $e(x) = 3$, we make only blowing ups at closed points. Either for some n, $m(x_n) < m$, then we get the result; or we have equality for $n \geq 0$. Then, either $e(x_n) = 4$, for some $n \geq 0$, we get the result or x_n is in case 4.3: then as the dimension of the ridge of the tangent cone is not increasing [27], we cannot get a loop, we come back at worse to the case $e(x) = 3$ and codimension of the ridge ≥ 3. We reach the case $e(x_n) = 3$, by an usual argument, the x_n are all on the strict transform of a curve $\mathcal{C}_n$ which, for $n >> 0$ is permissible for (h): we blow it up and there is no near point.

When $e(x) = 2$, we can choose Z, u_3 in the M-adic completion of $S[Z]$ such that

$$\mathrm{VDir}(h) =< U_3, Z >,\ \Delta(h; u_1, u_2; Z, u_3) \text{ is minimal.}$$

$\Delta(h; u_1, u_2; Z, u_3)$ is defined in [35] Definition (1.12). When $\Delta(h; u_1, u_2; Z, u_3) = \emptyset$, $\mathrm{Sing}_m(h) \supset \mathbf{V}(Z, u_3)$, if we blow up along, there is no near point, so $\mathrm{Sing}_m(h) = \mathbf{V}(Z, u_3)$, by excellence, this is a non formal regular center, permissible in Hironaka's sense: x is good.

So the remaining case is $\Delta(h; u_1, u_2; Z, u_3) \neq \emptyset$. This case is solved by [19] **10** in the case $\mathrm{char}(k(x)) \geq 3$. In the case $\mathrm{char}k(x) = 2$, the reader will

realize that the hypothesis is used just to prove that in the algorithm used in the proof, we never reach Hironaka's case which may be intractable when $\dim\mathcal{X}$ is big. But with the hypotheses $\dim\mathcal{X} = 2$, $m(x) = 2$ this case is solved just above.

Definition 4.3. The point x is *combinatoric* if the following algorithm starts and stops with success.
(i) if there exists $\mathrm{div}(u_i) \subset E$ such that $\mathrm{div}(u_i) \cap X$ is Hironaka-permissible, choose one and blow up X along this one,
(ii) if the center $x' \in X'$ of our valuation is not ω-near x: success,
(iii) if $x' \in X'$ ω-near x, and $e(x') \leq 2$: success,
(iv) if $x' \in X'$ ω-near x, and $\bar{e}(x') = 3$ and there exists $\mathrm{div}(u_i) \subset E'$ such that $\mathrm{div}(u_i) \cap X'$ is Hironaka-permissible, go to (i),
(v) else failure.

Proposition 4.2. *A combinatoric point is good.*

Clear by definition and by 4.1

4.2 An extra assumption on the singular locus.

The following extra assumption **(E)'** is used along the proof of the projection theorem 4.4 below in [24]. It is used only as a shortcut in order to ensure that certain exceptional curves on $\mathcal{X}$ are Hironaka-permissible and can be blown up in order to reduce $\omega(x)$ (permissible curves of the third kind). Such blowing up centers are not used in [13]. The authors do not know if such blowing ups are relevant in dimension $n \geq 4$.

Definition 4.4. We say that (S, h, E) satisfies condition **(E)'** if it satisfies condition **(E)** and if for $x \in \eta^{-1}(m_S)$, we have:

$$\iota(x) > (p, p, 2) \Longrightarrow \eta^{-1}(E) = \mathrm{Sing}_p\mathcal{X}.$$

As stated after definition 2.11, we have in any case $\mathrm{Sing}_p\mathcal{X} \subseteq \eta^{-1}(E)$ whenever (S, h, E) satisfies condition **(E)**. If $s \in \mathrm{Spec}S$, then (S_s, h_s, E_s) obviously satisfies condition **(E)'** if (S, h, E) does.

Proposition 4.3. *Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a permissible blowing up (of the first or second kind) at $x \in \eta^{-1}(m_S)$. If (S, h, E) satisfies condition **(E)'**, then, with notations as in proposition 2.7, (S', h', E') satisfies again **(E)'** for every $s' \in \sigma^{-1}(s)$.*

Proof. If $(m(x), \omega(x)) \leq (p, p-1)$, this reduces to proposition 2.13. Assume that $m(x) = p$, $\omega(x) \geq p$. By proposition 3.5(i)(iv), there exists an adapted r.s.p. $(u'_1, \ldots, u'_{n'})$ of S', $E' := \mathrm{div}(u'_1 \cdots u'_{e'})$ and we have

$$H(x') = {u'_1}^{\epsilon(y)-p} H(x),$$

where $y \in \mathrm{Spec}S$ is the generic point of the blowing up center $\mathcal{Y} \subset \mathcal{X}$, $W = \eta(\mathcal{Y}) = V(\{u_j\}_{j \in J})$ and $I(W)S' = (u'_1)$. It is sufficient to prove that:

$$\mathrm{ord}_{(u'_1)} H(x') = \epsilon(y) - p + \mathrm{ord}_W H(x) \geq p. \tag{4.4}$$

By definition 3.1 and proposition 3.3, we have $\epsilon(y) \geq \omega(x) \geq p$. On the other hand, proposition 2.3(ii) and definition 2.10 give:

$$\mathrm{ord}_{(u_j)} H(x) = p d_j \geq p, \ 1 \leq j \leq e$$

since $\eta^{-1}(E) \subseteq \mathrm{Sing}_p \mathcal{X}$. Since $W \subset E$, $J_E \neq \emptyset$ and (4.4) follows from:

$$\mathrm{ord}_W H(x) = p \sum_{j \in J_E} d_j \geq p.$$

4.3 The projection theorem.

The following projection theorem is proved in [24] by extending the methods of [21].

Theorem 4.4. *[24] Assume that $m(x) = p$, $\omega(x) > 0$ and that (S, h, E) satisfies assumption **(E)'**. Then x is good.*

5 Reduction to the projection theorem.

In this chapter and the following one, we deduce theorems 1.1 and 1.3 from theorem 4.4. All results are extensions of [20] and [23]. The proofs are based on the following three characteristic free results which can be found respectively in [2] theorem 3, a special case of [19] theorem 0.3 (with $B = \emptyset$) and [20] proposition 4.2:

Proposition 5.1. *(Abhyankar) Let (R, m) and (R', m') be regular two-dimensional local domains with a common quotient field and such that*

$$R \subseteq R',\ m' \cap R = m.$$

Then R' is an iterated quadratic transform of R.

Proposition 5.2. *(Cossart-Jannsen-Saito) Let $\mathcal{S}$ be a regular Noetherian irreducible scheme of dimension three which is excellent and $X \hookrightarrow \mathcal{S}$ be a reduced subscheme.*

There is a composition of blowing ups along integral regular subschemes $\sigma : \mathcal{S}' \to \mathcal{S}$ such that the strict transform $X' \hookrightarrow \mathcal{S}'$ of X has normal crossings with the reduced exceptional divisor E of σ. Moreover σ restricts to an isomorphism

$$\pi :\ X' \backslash \sigma^{-1}(\mathrm{Sing}X) \simeq X \backslash \mathrm{Sing}X.$$

Proposition 5.3. *(Cossart-Piltant) Let $\mathcal{S}$ be a regular Noetherian irreducible scheme of dimension three which is excellent and $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{S}}$ be a nonzero ideal sheaf. There exists a finite composition of blowing ups*

$$\mathcal{S} =: \mathcal{S}(0) \leftarrow \mathcal{S}(1) \leftarrow \cdots \leftarrow \mathcal{S}(r)$$

with the following properties:

(i) for each j, $0 \leq j \leq r-1$, $\mathcal{S}(j+1)$ is obtained by blowing up a regular integral subscheme $\mathcal{Y}(j) \subset \mathcal{S}(j)$ with

$$\mathcal{Y}(j) \subseteq \{s_j \in \mathcal{S}(j) : \mathcal{I}\mathcal{O}_{\mathcal{S}(j),s_j} \text{ is not locally principal}\}.$$

(ii) $\mathcal{I}\mathcal{O}_{\mathcal{S}(r)}$ is locally principal.

Proof. The assumption "X/k is quasi-projective" is not used in the proof of [20] proposition 4.2. The equicharacteristic assumption is used only via the power series expansions used for defining E and the characteristic polygon "$\Delta(\mathcal{E}; u_1, u_2; y)$ prepared" on pp.1061-1062 of *ibid.*. But this is also characteristic free by [22] theorem II.3.

5.1 Reduction to local uniformization.

In this section, we deduce theorem 1.1 from its local uniformization form (*) below. Let A be a field (resp. an excellent discrete valuation ring) and L be a function field of dimension at most three (resp. of dimension at most two) over A. We assume that the following holds:

(*) for every valuation ring V, $A \subset V \subset L$, $L = QF(V)$, there exists a finitely generated A-algebra B,

$$A \subset B \subset V,\ QF(B) = L$$

such that B_P is regular, where $P := m_V \cap B$.

Assume that (*) holds. We prove that theorem 1.1 holds after a series of reductions. Note that it can be assumed that $\mathcal{X}$ is proper over C by [47]. If $\mathcal{X}\backslash \mathrm{Sing}\mathcal{X}$ is quasi-projective, it can be assumed w.l.o.g. that $\mathcal{X}$ is projective by taking a Chow cover.

Step 1: it can be assumed that $\mathcal{X}$ is irreducible. Let $\mathcal{X}_1, \ldots, \mathcal{X}_c$ be the finitely many pairwise distinct irreducible components of $\mathcal{X}$. We blow up consecutively all scheme theoretic intersections $\widetilde{\mathcal{X}_i} \cap \widetilde{\mathcal{X}_j}$, $i \neq j$, where $\widetilde{\mathcal{X}_k}$ denotes the strict transform of $\mathcal{X}_k$ in the blown up scheme (*vid.* proof of [20] theorem 3.3). Note that $\widetilde{\mathcal{X}_i} \cap \widetilde{\mathcal{X}_j}$ maps to $\mathcal{X}_i \cap \mathcal{X}_j \subseteq \mathrm{Sing}\mathcal{X}$. This constructs a projective and birational morphism

$$\tilde{\pi} : \coprod_{i=1}^{c} \widetilde{\mathcal{X}_i} \to \mathcal{X}$$

isomorphic above $\mathrm{Sing}\mathcal{X}$. The theorem holds for $\mathcal{X}$ if it holds for each $\widetilde{\mathcal{X}_i}$, $1 \leq i \leq c$.

Resolution of singularities is known if $\dim\mathcal{X} \leq 2$ [45], so we may assume that $\mathcal{X}$ is irreducible and $\dim\mathcal{X} = 3$. At this point, the image of the structure morphism $s : \mathcal{X} \to C$ is either a closed point Spec$k$ or C.

In the former case, this is done in [20] proposition 4.8. The assumption "Z/k is quasi-projective" in [20] proposition 4.8 is not used in the proof. We may assume now that $s(\mathcal{X}) = C$.

Step 2: it can be assumed furthermore that $\mathcal{X}$ and C are normal. Since C is excellent, the respective normalizations $\overline{\mathcal{X}}$ and $\overline{C}$ of $\mathcal{X}$ and C are finite [31]

corollary 7.7.3. We may therefore assume that $\overline{\mathcal{X}} = \mathcal{X}$, then $\overline{C} = C$ by the universal property of normalization.

Step 3: construction of a projective birational morphism $\tilde{\pi} : \tilde{\mathcal{X}} \to \mathcal{X}$ with $\tilde{\mathcal{X}}$ regular. To achieve this, it can be assumed that $\mathcal{X}$ itself is projective by taking a Chow cover. The following rephrases [20] proposition 4.7, using the characteristic free proposition 5.3.

Proposition 5.4. *Let $\mathcal{X}$ and $\mathcal{Z}$ be two projective models of L and $T : \mathcal{Z} \cdots \to \mathcal{X}$ be the birational correspondence and $\mathcal{F} \subset \mathcal{Z}$ be its fundamental locus. There exists a composition of blowing ups with regular centers $\pi : \mathcal{Z}' \to \mathcal{Z}$ such that*

(i) π is an isomorphism away from $\mathcal{F}$;

(ii) $\pi \circ T$ is a morphism above $\mathcal{F} \backslash \mathrm{Sing}\mathcal{Z}$.

The quasi-compactness of the Riemann-Zariski space of valuations of L dominating an (unspecified) local ring of C follows from [61] theorem 40 on p.113 and Noetherianity of C. Regularity is an open property for projective models of L because C is excellent. As indicated in [58] on p.539, Zariski's patching theorem only requires propositions 5.1 and 5.4 (here in our characteristic free context) in order to deduce step 3 from (*). This proves (i) of theorem 1.1 with π projective.

Step 4: achieving (ii). Let $\tilde{\pi}$ be as in step 3, $\mathcal{F} \subseteq \mathcal{X}$ be the fundamental locus of $\tilde{\pi}^{-1}$. We define

$$\mathcal{F}_0 := \text{Zariski closure in } \mathcal{X} \text{ of } \mathcal{F} \backslash \mathrm{Sing}\mathcal{X}.$$

Since $\mathcal{X}$ is normal by step 2, $\mathcal{F}_0$ has dimension at most one. Applying the techniques of [16] (see also [49] section 6), there exists a commutative diagram

$$\begin{array}{ccc} \tilde{\mathcal{X}} & \xleftarrow{\tilde{e}} & \tilde{\mathcal{Z}} \\ \downarrow & & \downarrow \\ \mathcal{X} & \xleftarrow{e} & \mathcal{Z} \end{array}$$

such that e (resp. $\tilde{e}$) is a composition of blowing ups with regular centers mapping to $\mathrm{Sing}\mathcal{X}$ (resp. to $\tilde{\pi}^{-1}(\mathrm{Sing}\mathcal{X})$) and the following holds, where $\mathcal{F}_0'$ denotes the strict transform of $\mathcal{F}_0$ in $\mathcal{Z}$:

(i) $\mathcal{F}_0'$ is disjoint from $\mathrm{Sing}\mathcal{Z}$;

(ii) $\mathcal{F}'_0$ is a union of connected components of the fundamental locus of $\tilde{\pi}' : \tilde{\mathcal{Z}} \to \mathcal{Z}$.

By (ii), there exists a nonempty Zariski open set $\mathcal{U} \subset \mathcal{Z}$ with $\mathcal{F}'_0 \subseteq \mathcal{U}$ such that $\mathcal{U}$ and $\tilde{\mathcal{U}} := (\tilde{\pi}')^{-1}(\mathcal{Z} \backslash \mathcal{F}'_0)$ glue together to a proper model $\mathcal{X}'$ of L.

By (i), it can be assumed that $\mathcal{U} \cap \mathrm{Sing}\mathcal{Z} = \emptyset$, hence $\mathcal{X}'$ is regular. Every exceptional point for $\pi : \mathcal{X}' \to \mathcal{X}$ maps to $\mathrm{Sing}\mathcal{X}$ by construction. Hence π satisfies (i) and (ii) of theorem 1.1.

Note that $\mathcal{X}'$ is projective if $\mathcal{X}$ is projective, which can be assumed if $\mathcal{X} \backslash \mathrm{Sing}\mathcal{X}$ is quasi-projective as explained in the beginning of the proof.

Finally (iii) of theorem 1.1 is achieved by applying proposition 5.2 to the the embedding $\pi^{-1}(\mathrm{Sing}\mathcal{X})_{\mathrm{red}} \hookrightarrow \mathcal{X}'$. This concludes the proof.

5.2 Reduction to cyclic coverings.

In this section, we reduce the local uniformization form (*) of the previous section to theorem 1.3. We may assume here that $\mathrm{char}(V/m_V) = p > 0$, the equicharacteristic zero version of theorem 1.1 being known.

Applying lower dimensional results [45], it can also be assumed that L has dimension three (resp. dimension two) over A when A is a field (resp. an excellent discrete valuation ring). Moreover, there exists a finitely generated algebra B as in (*) of dimension at most two except if $m_V \cap A = m_A$ and the residue extension $V/m_V|A/m_A$ is algebraic. Again by [45], it can be assumed that

$$m_V \cap A = m_A,\ V/m_V|A/m_A \text{ is algebraic.}$$

Finally, we may assume that V has rank one applying [48] theorem 1.1 (valid in all dimensions) or using the dimension three techniques in [20] proposition 5.1.

If A is a field, (*) follows from theorem 1.3 by [20] theorem 8.1. Assume now that (A, m_A, k) is a discrete valuation ring. In particular, we have $\mathrm{char}k = \mathrm{char}(V/m_V) = p > 0$. We will prove that the equal characteristic techniques of [20] extend to this situation. The proof of [20] theorem 8.1 extends without change when $\mathrm{char}A = p$ using propositions 5.2 and 5.3 instead of [20] propositions 4.1 and 4.2 respectively. We thus assume that $\mathrm{char}A = 0$.

Let V be a valuation ring as in (*). Assuming theorem 1.3, we will construct $B \subset V$, $QF(B) = QF(V) = L$ such that $B_{m_V \cap B}$ is regular. Pick a transcendence basis (x_1, x_2) of L over $QF(A)$ with $x_1, x_2 \in V$. The ring $A[x_1, x_2]$ is a polynomial ring over A, hence is regular. Let L_0 be the quotient field of $A[x_1, x_2]$, so the field extension $L|L_0$ is algebraic. Let $\overline{L}|L_0$ be a Galois closure. There exists a diagram of fields

$$\begin{array}{ccccccc} L & \subseteq & L^i & \subseteq & L^r & \subseteq & \overline{L} \\ \uparrow & & \uparrow & & \uparrow & & \\ L_0 & \subseteq & L_0^i & \subseteq & L_0^r & & \end{array} \tag{5.1}$$

given by ramification theory of valuations [61] section 12 as in the proof of [20] theorem 8.1. The left-hand side (resp. middle) inclusions in this diagram are unramified (resp. totally ramified Abelian of order prime to p). The extension $L^r|L_0^r$ is a tower of ramified Galois extensions of degree p. In order to connect ramification theory of valuations and ramification theory of A-algebras essentially of finite type, we state the mixed characteristic version of [20] theorem 7.2. For ramification theory of local rings, we refer to [3] (see also [20] section 2 for a quick summary of the required notions and notations).

Definition 5.1. A normal model of $V|A$ is a finitely generated A-algebra B, $A \subset B \subset V$, $QF(B) = L$ such that B_P is normal, where $P := m_V \cap B$.

Let $L'|L$ be a finite Galois extension and V' be a valuation ring such that $QF(V') = L'$ and $V' \cap L = V$. We define a model B' of $V'|A$ by taking the localization of the integral closure $\overline{B}$ of B in L' at the prime $m_{V'} \cap \overline{B}$.

Note that B' is actually a normal model because A, hence B, is excellent. Also note that if B' is a normal model of $V'|A$, then $B' \cap L = B'^{\mathrm{Gal}(L'|L)}$ is a normal model of $V|A$ since $L'|L$ is finite Galois and A excellent.

Proposition 5.5. *Let $L'|L$ be a finite Galois extension and V' be a valuation ring such that $QF(V') = L'$ and $V' \cap L = V$.*

There exists a normal local model B_0 of $V|A$ such that for any normal model B of $V|A$ with $B_0 \subseteq B$, the following holds:

(1) $G^s(V'|V) = G^s(B'|B)$ and $G^i(V'|V) = G^i(B'|B)$;

(2) the normal model $B^r := B'^{G^r(V'|V)}$ of $V^r|A$ satisfies

$$B^r/m_{B^r} = B^i/m_{B^i},$$

where B^i is the inertia ring of B' over B, i.e. $B^i = B'^{G^i(B'|B)}$. Moreover the representation

$$\rho:\ G^i(V'|V)/G^r(V'|V) \to \mathrm{GL}(m_{B^r}/m_{B^r}^2),\ g \mapsto (\overline{x} \mapsto \overline{h.x})$$

is faithful.

We now prove that theorem 1.3 implies (*). To emphasize the dependence on V, we say that $(*V)$ holds if (*) holds for a particular V. Given an extension $\overline{V}$ of V to $\overline{L}$, we denote by $V_0, V_0^i, V^i, V_0^r, V^r$ the respective valuation rings $L^0 \cap \overline{V}, L_0^i \cap \overline{V}, L^i \cap \overline{V}, L_0^r \cap \overline{V}, L^r \cap \overline{V}$. The strategy is to prove successively that $(*V_0^i), (*V_0^r), (*V^r), (*V^i), (*V)$ hold. Note that $(*V_0)$ holds by construction.

Firstly, $(*V_0^i)$ holds follows immediately from proposition 5.5 (1) as in [20] corollary 7.3. Then $(*V_0^r)$ holds because $L_0^r|L_0^i$ is a tower of ramified Galois extensions of prime degree $l \neq p$: the proof relies on proposition 5.5(1) and the Perron algorithm as in [20] proposition 6.3. This is characteristic free.

To prove that $(*V^r)$ holds, we may assume that $L^r|L_0^r$ is a single Galois extension of degree p. Since $\mathrm{char}L = 0$, any such extension is obtained by taking a p^{th}-root of some element $f \in V_0^r$.

Since $(*V_0^r)$ holds, there exists a normal model S of $V_0^r|A$ which is regular. Applying proposition 5.3, it can be assumed furthermore that $f \in S$. We let $h := X^p - f \in S[X]$, so the pair (S, h) satisfies the assumptions of theorem 1.3(i). This theorem then states that $(*V^r)$ holds.

Proving that $(*V^i)$, then $(*V)$ hold are easy adaptations of [20] lemma 9.2 and proposition 9.1 respectively. The former one is characteristic free while the latter one relies on proposition 5.5 (2). This concludes the proof.

5.3 Normal crossings divisors conditions.

In this section, we reduce theorem 1.3 to theorem 4.4. In other terms, it is assumed that assumption **(G)** is satisfied and we must achieve conditions **(E)** and **(E)'** in order to apply theorems 2.23 and 4.4. This is proved in proposition 5.7 below. Theorem 1.3 is then an immediate consequence of theorem 2.23, theorem 4.4 and [23] Main Theorem 1.3. Theorem 5.15 proves that the multiplicity of $\mathcal{X}$ can be made smaller than p independently of the given valuation μ (see introduction).

Assumption **(G)** is not required in an important part of this step and we prove a more general version for arbitrary multiplicity.

Lemma 5.6. *Let S, $h \in S[X]$ (2.1) and $\eta : \mathcal{X} \to \mathrm{Spec}S$ be given. Assume that $\dim S = 3$ and that h is reduced. There exists a composition of Hironaka-permissible blowing ups (2.16) w.r.t. $E = \emptyset$:*

$$\begin{array}{ccc} \mathcal{X} = & \stackrel{\pi}{\longleftarrow} & \mathcal{X}' \\ \downarrow & & \downarrow \\ \mathrm{Spec}S & \stackrel{\sigma}{\longleftarrow} & \mathcal{S}' \end{array}$$

such that $\pi(\mathrm{Sing}_m \mathcal{X}') \subseteq \eta^{-1}(m_S)$.

Proof. This statement means that there exists a diagram

$$\begin{array}{ccccccc} \mathcal{X} =: \mathcal{X}_0 & \stackrel{\pi_0}{\longleftarrow} & \mathcal{X}_1 & \stackrel{\pi_1}{\longleftarrow} & \cdots & \stackrel{\pi_{n-1}}{\longleftarrow} & \mathcal{X}_n =: \mathcal{X}' \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathrm{Spec}S =: \mathcal{S}_0 & \stackrel{\sigma_0}{\longleftarrow} & \mathcal{S}_1 & \stackrel{\sigma_1}{\longleftarrow} & \cdots & \stackrel{\sigma_{n-1}}{\longleftarrow} & \mathcal{S}_n =: \mathcal{S}' \end{array} \tag{5.2}$$

where each morphism π_i, $0 \leq i \leq n-1$, is the blowing up along a Hironaka-permissible center $\mathcal{Y}_i \subset \mathcal{X}_i$ w.r.t. the reduced exceptional divisor E_i of $\pi^{(i)} : \mathcal{X}_i \to \mathcal{X}$. It can be assumed that $\dim(\mathrm{Sing}_m \mathcal{X}) \geq 1$.

Let $y_i \in \mathcal{X}_i$ denote the generic point of such a Hironaka-permissible center $\mathcal{Y}_i \subset \mathcal{X}_i$ w.r.t. E_i. We define:

$$\Delta_i := \{y \in \mathrm{Sing}_m \mathcal{X}_i : \dim \mathcal{O}_{\mathcal{X}_i, y} = \dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y)} = 1\},$$

$$\delta_i := \max\{\delta(y), y \in \Delta_i\},\ N_i := \sharp\{y \in \Delta_i : \delta(y) = \delta_i\}.$$

Let $i \geq 0$. We claim that

$$\begin{cases} (\delta_{i+1}, N_{i+1}) = (\delta_i, N_i) & \text{if } \dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y_i)} \geq 2; \\ (\delta_{i+1}, N_{i+1}) < (\delta_i, N_i) & \text{if } \dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y_i)} = 1. \end{cases} \tag{5.3}$$

Namely, this is an obvious consequence of the definition if $\dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y_i)} \geq 2$. If $\dim \mathcal{O}_{\mathcal{X}, \pi^{(i)}(y_i)} = 1$, let $y \in \mathcal{X}_{i+1}$ with $\pi_i(y) = y_i$. We have

$$(m(y), \delta(y)) \leq (m(y_i), \delta(y_i) - 1)$$

by proposition 2.6 applied for $n = 1$ and the claim follows

Pick $y \in \Delta_i$ with $\delta(y) = \delta_i$ and denote $\mathcal{Y} := \overline{\{y\}} \subset \mathcal{X}_i$. By proposition 5.2, there exists a composition of blowing ups $\mathcal{X}_{i'} \to \mathcal{X}_i$ with regular centers contained in the successive strict transforms of $\mathcal{Y}$ such that $\eta_{i'}(\mathcal{Y}')$ has normal crossings with $E_{i'}$, where $\mathcal{Y}'$ denotes the strict transform of $\mathcal{Y}$ in $\mathcal{X}_{i'}$. Then $\mathcal{Y}'$ itself and each blowing up center in $\mathcal{X}_{i'} \to \mathcal{X}_i$ are Hironaka-permissible w.r.t. $E_{i'}$ because $m(y) = p$.

We have $(\delta_{i'}, N_{i'}) = (\delta_i, N_i)$ by (5.3). Taking as blowing up center $\mathcal{Y}_{i'} := \mathcal{Y}'$ also gives $(\delta_{i'+1}, N_{i'+1}) < (\delta_i, N_i)$ by (5.3). Since Δ_i is a finite set and $\delta_i \in \frac{1}{m}\mathbb{N}$, there exists an index $i_1 > i$ such that $\Delta_{i_1} = \emptyset$ and this is preserved by further Hironaka-permissible blowing ups w.r.t. $E = \emptyset$.

Since $\Delta_{i_1} = \emptyset$, we are done unless $\pi^{(i_1)}(\mathrm{Sing}_m \mathcal{X}_{i_1}) = \mathcal{C}$, where $\mathcal{C}$ has pure dimension one. Let $C \subset \mathrm{Spec}S$ be an irreducible component of $\eta(\mathcal{C})$ and s be its generic point. Note that the stalk $(\mathcal{X}_i)_s$ at s of the S-scheme $\mathcal{X}_i$ is embedded in the regular scheme of dimension three $\mathrm{Spec}S_s[X]$ for $i = 0$ and in an iterated blowing up along regular centers of the former for $i \geq 1$. By proposition 5.2, there exists a composition of Hironaka-permissible blowing ups $\mathcal{X}'_s \to (\mathcal{X}_{i_1})_s$ w.r.t. $(E_{i_1})_s$ such that $\mathrm{Sing}_m \mathcal{X}'_s = \emptyset$.

Let $\mathcal{Y}_s \subseteq (\mathcal{X}_{i_1})_s$ be a Hironaka-permissible center and $\mathcal{Y} \subseteq \mathcal{X}_{i_1}$ be its Zariski closure, so in particular we have $\mathcal{Y} \subseteq \mathrm{Sing}_m \mathcal{X}_{i_1}$. Since $\Delta_{i_1} = \emptyset$, $\mathcal{Y}$ is either (1) a curve mapping onto C, or (2) a surface mapping to some irreducible component of E_{i_1}.

In situation (1), there exists a composition of blowing ups along closed points $\mathcal{X}_{i'_1} \to \mathcal{X}_{i_1}$ such that $\eta_{i'_1}(\mathcal{Y}')$ has normal crossings with $E_{i'_1}$, where $\mathcal{Y}'$ denotes the strict transform of $\mathcal{Y}$ in $\mathcal{X}_{i'_1}$.

In situation (2), $\mathcal{Y}$ itself is Hironaka-permissible w.r.t. E_{i_1} and we let $i'_1 := i_1$.

In both situations, we may blow up $\mathcal{X}_{i'_1}$ along $\mathcal{Y}'$ and iterate: this produces an index $i_2 \geq i_1$ and a composition of Hironaka-permissible blowing ups $\mathcal{X}_{i_2} \to \mathcal{X}_{i_1}$ w.r.t. E_{i_1} such that $\eta^{-1}(s) \cap \pi^{(i_2)}(\mathrm{Sing}_m \mathcal{X}_{i_2}) = \emptyset$. Applying this construction to the finitely many irreducible components of $\eta(\mathcal{C})$ proves the lemma.

Proposition 5.7. *Let $\mathcal{X}'$ satisfy the conclusion of lemma 5.6 and $E' \subset \mathcal{S}'$ be the reduced exceptional divisor of σ. Let $D \subset \mathrm{Spec}S$ be a reduced divisor.*

There exists a composition of Hironaka-permissible blowing ups (2.16)

w.r.t. E'*:*

$$\begin{array}{ccc} \mathcal{X}' & \xleftarrow{\pi'} & \mathcal{X}'' \\ \downarrow & & \downarrow \\ \mathcal{S}' & \xleftarrow{\sigma'} & \mathcal{S}'' \end{array}$$

such that the strict transform D'' *of* D *is empty at each point of* $\eta''(\mathrm{Sing}_m\mathcal{X}'')$*, where* $\eta'' : \mathcal{X}'' \to \mathcal{S}''$ *is the projection.*

Proof. Let $P := \mathrm{I}(D) \subset S$. The problem is to find a sequence (5.2) which monomialize P, i.e. such that $P_n := P\mathcal{O}_{\mathcal{S}_n}$ is a monomial with components at normal crossings with E_n.

Let us write $P_i := H_iQ_i$ where H_i is a monomial whose components are components of E_i. At the beginning, $H = H_0 = 1$. The strategy is to get $P_n = H_n$, $Q_n = 1$ at the end.

We consider the idealistic exponents (h, m) and (Q, b) living in $\mathrm{Spec}S[Z]$, where $b = \mathrm{ord}_{m_S}(Q)$. We make a descending induction on b: the case $b = 0$ means that we get the conclusion of 5.7. Each pair of blowing ups π_i, σ_i is locally centered at some Y_i and $\eta(Y_i)$ respectively, and is Hironaka-permissible for h (resp. Q_i) w.r.t. E_i.

Let $P_{i+1} =: H_{i+1}Q_{i+1}$ where Q_{i+1} is the strict transform of Q_i. This means that (Q_{i+1}, b) is the transform of (Q_i, b). When $\mathrm{ord}_{x_{i+1}}(Q_{i+1}) < b$, we have strictly improved and we go on with the new idealistic exponent (Q_{i+1}, b'), with $b' := \mathrm{ord}_{x_{i+1}}(Q_{i+1})$. To define a sequence of σ_i is a consequence of [19] **Theorem 0.3** (Canonical embedded resolution with boundary), the problem is the sequence of π_i, i.e. to define the pair (σ_i, π_i).

To avoid cumbersome notations, from now on, $x_i, S_i, \mathcal{X}_i$,etc.$_i$ are denoted by $x, S, \mathcal{X}$,etc. and $x_{i+1}, S_{i+1}, \mathcal{X}_{i+1}$,etc.$_{i+1}$ by $x', S', \mathcal{X}'$,etc.$'$. Let us define $\mathrm{VDir}(x, D)$ as $\mathrm{VDir}(h) + \mathrm{VDir}(Q)$. This is a vector space of codimension $\tau(x, D)$ in the Zariski's tangent space of $\mathcal{X}$ at x. Of course, $\tau(x, D) \geq 2$.

Lemma 5.8. *Let* π *be the blowing up along* Y *which is permissible for both* (h, m) *and* (Q, b)*. Let* $x' \in \pi^{-1}(x)$ *be such that* $m(x') = m(x) = m$ *and* $ord_{x'}Q' = b$*. Then* x' *is on* $\mathbf{Proj}(S/\mathrm{IDir}(x, D))$*. In particular,* x' *is on the strict transform of* $\mathrm{div}(Z)$*.*

Proof. By proposition 2.15 and remark 2.3, we have $\mathrm{Dir}(F) = \mathrm{Max}(F)$ except if $p = 2$ and

$$F = \lambda(Z^2 + \lambda_2U_1^2 + \lambda_1U_2^2 + \lambda_1\lambda_2U_3^2)^\alpha,\ [k^2(\lambda_1, \lambda_2) : k^2] = 4 \qquad (5.4)$$

up to a linear change of variables, $\lambda \neq 0$, $\alpha \geq 1$. By 4.1, x is good. Since $m(x') = m(x)$, we have

$$x' := V(U_1^2 + \lambda_1 U_3^2, U_2^2 + \lambda_2 U_3^2, Z^2 + \lambda_1\lambda_2 U_3^2)$$

on $\pi'^{-1}(x) = \mathrm{Proj}(k[Z, U_1, U_2, U_3]/(F))$.

Since $\mathrm{ord}_{x'} Q' = b$, the initial of Q cannot verify (5.4) (only the last three variables occur). Therefore

$$x' \in \mathbf{Proj}(S/\mathrm{IDir}(h)) \cap \mathbf{Proj}(S/\mathrm{IDir}(Q)) = \mathbf{Proj}(S/\mathrm{IDir}(x, D)).$$

Let us come back to the proof of proposition 5.7. We discuss according to the value of $\tau(x, D)$.

When $\tau(x, D) = 4$, the blowing-up centered at x makes b strictly drop.

When $\tau(x, D) = 2$ or 3, then, if we blow up along x, then $\tau(x', D') \geq \tau(x, D)$. In case $\tau(x, D) = 3$, we make only blowing ups at closed points. Either for some n, $(m(x_n), \mathrm{ord}_{x_n}(Q_n)) <_{\mathrm{lex}} (m, b)$, then we stop at this n; or we have equality for $n \geq 0$. Then, $\tau(x_n, D_n) = 3$, $n \geq 0$, by an usual argument, the x_n are all on the strict transform of a curve $\mathcal{C}_n$ which, for $n >> 0$ is permissible for both (h, m) and (Q, b) and $\eta(\mathcal{C}_n)$ is transverse to E_n. Then at step n in (5.2), we blow up along $\mathcal{C}_n$. By lemma 5.8, $(m(x_{n+1}), \mathrm{ord}_{x_{n+1}}(Q_{n+1})) <_{\mathrm{lex}} (m, b)$.

When $\tau(x, D) = 2$, we can choose Z, u_3 such that

$$\mathrm{VDir}(Q) =< U_3 >,\ \mathrm{VDir}(h) \equiv< Z > \mod(U_3).$$

Remark 5.1. If there is a component Y of dimension 2 in $\mathrm{Sing}(h, m) \cap \mathrm{Sing}(Q, b)$, then, we can choose the parameters so that $I(Y) = (Z, u_3)$. Then $Q \in (z, u_3)^b$, i.e. $Q = u_3^b$, up to multiplication by an invertible. Then, if Y has normal crossing with E, we blow up along Y: π is the blowing up along Y and σ is the identity. In fact in S, we just add $\eta(Y) = \mathrm{div}(u_3)$ to E and we get $b = 0$.

We also note that $(h, m) \cap (Q, b) = (hQ, m + b)$. In other words, we have

$$\mathrm{Sing}(h, m) \cap \mathrm{Sing}(Q, b) = \mathrm{Sing}(hQ, m + b)$$

and permissible centers are the same for $(hQ, m + b)$ and for $(h, m) \cap (Q, b)$.

Then we apply those techniques from [19] **10**, **11**, **12**. More precisely, if for some n_0 the number b just strictly drops, we call "old components"

the components of E_{n_0} at x_{n_0} which are components of H and, for $n \geq n_0$, at $x_n, n \geq n_0$ with $b(x_n) = b(x_{n_0})$, the strict transforms of this old components. The first step is to reach the case where x_n is not on the strict transform of this old components: the invariant is $(m, b, o(x))$ where $o(x)$ is the number of these old components. In the language of idealistic exponents, we desingularize $(hQQ_O, mbo(x))$ where Q_O is the equation of the reduced divisor whose components are the old ones. Then we look at the directrix of hQQ_O. When its codimension denoted by $\tau(hQQ_O)$ is 3 or 4, we play the same game that above with $\tau(x, D) = 3$ or 4. We reach the case where $\tau(hQQ_O) = 2$. This means that either $Q_O = 1$ (no old component) or there is one old component which is tangent to Q.

Then we look at the characteristic polyhedron $\Delta(hQQ_0, z, u_3, u_1, u_2)$ as in [19] **Section 7**.

• Case $\Delta(hQQ_0, z, u_3, u_1, u_2) = \emptyset$. This is equivalent to $hQQ_0 \in (z, u_3)^{mbo(x)}$, i.e. this is equivalent to $\dim(\mathrm{Sing}(hQQ_O, mbo(x)) = 2$. So $QQ_O = u_3^{mbo(x)}$, call $Y := \mathbf{V}(z, u_3)$, in fact, at step n_0, as $b(x_0) = b(x)$, Q was a $b(x_0)$ power and, if at x there is one old component, it is a factor of Q: this is impossible, therefore $o(x) = 0$.

So, at x, E is a union of components which are exceptional divisors of the blowing ups σ_n, $n \geq n_0$. By [19]**Theorem 8.3**, they are transverse to u_3: Y is permissible for $(hQQ_O, mbo(x))$ and transverse to E. We apply the first statement of remark 5.1.

• Case where $\dim(\mathrm{Sing}(hQQ_O, mbo(x)) \leq 1$. Then, we apply [19] **Theorem 5.28** which gives the result if $\mathrm{char}k(x) \geq 3$. This hypothesis $p \neq 2$ is used just to get $\mathrm{Dir}(F) = \mathrm{Max}(F)$ at each step, but we showed above in lemma 5.8, that the only case where $\mathrm{Dir}(F) \neq \mathrm{Max}(F)$ stops after blowing up the closed point x.

Proposition 5.9. *Assume furthermore that* $\mathrm{char}S/m_S = p > 0$ *and* (S, h, E) *satisfies condition* ***(G)****. Take*

$$D := \mathrm{div}(p\mathrm{Disc}_X(h)I(E)) \ .$$

Let $\mathcal{X}''$ *satisfy the conclusion of proposition 5.7 and* $E'' := \sigma'^{-1}(E') \subset \mathcal{S}''$*, then we have* ***(G)*** ***(E)*** *at* x" .

Proof. Indeed, E" is the exceptional divisor of σ' and the centers of the blowing ups in 5.8 project on $\eta(x)$: the residual characteristic at the generic

points of the exceptional components of σ' is p, so E"$\cup$ (strict transform of)$p \subset \mathrm{div}(p)$. By 5.7, in S", the strict transforms of $\mathrm{div}(p)$ and of $\mathrm{Disc}_X(h)$ at $\eta(x$") are empty.

Proposition 5.10. *Assume furthermore that* $\mathrm{char}S/m_S = p > 0$ *and* (S, h, E) *satisfies condition* ***(G)***, ***(E)***.

There exists a composition of Hironaka-permissible blowing ups (2.16) w.r.t. E''*:*

$$\begin{array}{ccc} \mathcal{X}'' = & \stackrel{\tilde{\pi}}{\longleftarrow} & \tilde{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathcal{S}'' & \stackrel{\tilde{\sigma}}{\longleftarrow} & \tilde{\mathcal{S}} \end{array}$$

such that for every $\tilde{s} \in \tilde{\eta}(\mathrm{Sing}_p\tilde{\mathcal{X}})$, $\tilde{\mathcal{X}}_{\tilde{s}}$ *satisfies conditions* ***(G)*** *and* ***(E)'***.

Proof. To begin with, $\mathcal{X}''_{s''}$ satisfies conditions **(G)** and **(E)** at each $s'' \in \eta''(\mathrm{Sing}_p\mathcal{X}'')$. By proposition 2.3(i), we may assume

$$\delta(x) \geq 1. \tag{5.5}$$

Let us define the triple $(a(1), a(2), a(3)) \in \mathbb{N}^3$ as follows:

$$H = u_1^{a(1)}u_2^{a(2)}u_3^{a(3)},\ E \subseteq \mathrm{div}(u_1u_2u_3),\ \mathrm{div}(u_i) \subseteq E,\ 1 \leq i \leq e,$$

and $a(i-1) \leq a(i)$, $1 \leq i \leq e$. We define $N(x)$ or N for short as the number of components $\mathrm{div}(u_i)$ of E with $a(i)$ minimal, i.e. $a(i) = a(1)$.

We may assume that $(\omega(x), \kappa(x)) > (p, 2)$. We will prove that, after performing a sequence of permissible blowing ups, we can reach the case $a(1) \geq p$.

Lemma 5.11. *With the notations of proposition 5.10, we blow up the origin* x*. Let* x' *be a point* ω*-near* x *(i.e.* $\omega(x') = \omega(x)$*. Then:*
(i) with obvious notations,

$$(a(1), a(2), a(3), -N(x)) \leq_{lex} (a(1)', a(2)', a(3)', -N(x')), \tag{5.6}$$

$$(a(1), -N(x)) \leq_{lex} (a(1)', -N(x')), \tag{5.7}$$

(ii) $(a(1), -N(x)) = (a(1)', -N(x'))$*, implies that* x' *is on the strict transform of* $div(u_1)$*.*

Proof. Let us blow up the origin, let x' be a point ω-near x.

First chart: $x' \in \mathrm{Spec}S[X', u_1, u'_2, u'_3]$, where

$$(X', u_1, u'_2, u'_3) := (X/u_1, u_1, u_2/u_1, u_3/u_1).$$

We have $H(x') = u_1^{a(1)+a(2)+a(3)+\epsilon(x)-p} {u'_2}^{a(2)} {u'_3}^{a(3)}$. Let us remark that the equality $a(2)+a(3)+\epsilon(x) = p$ implies $\epsilon(x) = \omega(x) = p$, $a(2) = a(3) = 0$, so, by (5.5), $a(1) > 0$: $E = \mathrm{div}(u_1)$. So, if x' is ω-near x, $\mathrm{VDir}(x) \subseteq < U_2, U_3 >$: this implies $\kappa(x) = 2$, which contradicts the hypotheses of 5.10. So

$$a(1) + a(2) + a(3) + \epsilon(x) - p > a(1). \tag{5.8}$$

If $N(x) = 3$, i.e. $a(1) = a(2) = a(3)$, then, $a(1) = a(1)'$ implies that $N(x') < 3$ and $(a(1), a(2), a(3)) < (a(1)', a(2)', a(3)')$. If $N(x) = 2$, i.e. $a(2) = a(1)$, then $a(1) = a(1)'$ implies that x' is on the strict transform of u_2: $N(x') = 1$ and $(a(1), a(2), a(3)) < (a(1)', a(2)', a(3)')$. When $N(x) = 1$, then $a(1) \leq \inf\{a(2), a(1)+a(2)+a(3)+\epsilon(x)-p\} \leq a(1)'$.

Second chart: $x' \in \mathrm{Spec}S[X', u_1, u'_2, u'_3]$, where

$$(X', u'_1, u_2, u'_3) := (X/u_2, u_1/u_2, u_2, u_3/u_2).$$

If x' is not in the first chart, then it is on the strict transform of $\mathrm{V}(X, u_1)$. We have $H(x') = {u'_1}^{a(1)} u_2^{a(1)+a(2)+a(3)+\epsilon(x)-p} {u'_3}^{a(3)}$. When $a(1)+a(3)+\epsilon(x) > p$, it is easy to see that we get (i). Moreover, if $N(x) \geq 2$, then $N(x') < N(x)$, in fact the inequalities are strict in (5.6) and (5.7). Let us remark that the equality $a(1)+a(3)+\epsilon(x) = p$ implies $\epsilon(x) = \omega(x) = p$, $a(1) = a(3) = 0$, so, by (5.5), $a(1) = a(3) = 0$, $a(2) > 0$: $E = \mathrm{div}(u_1 u_2)$, $N(x) = 1$; as $\kappa(x) > 2$, $\mathrm{VDir} \subseteq < U_1, U_2 >$, as x' is ω-near x, $\mathrm{VDir} \subseteq < U_1 >$. In this last case, (i) and (ii) follow easily.

When x' is the point at infinity, i.e. x' is the point with parameters

$$(X', u'_1, u'_2, u_3) := (X/u_3, u_1/u_3, u_2/u_3, u_3),$$

then $H(x') = {u'_1}^{a(1)} {u'_2}^{a(2)} u_3^{a(1)+a(2)+a(3)+\epsilon(x)-p}$. The reader ends the proof.

Lemma 5.12. *We assume $Y := V(X, u_1, u_i)$, $i \neq 1$, is a permissible curve and we blow up along Y.*

(i) When Y is of the second kind, x' is on the strict transform of $div(u_1)$ and we have (5.6) and (5.7) at x', except in the case

$$E = \mathrm{div}(u_1),\ a(1) = 0,\ \omega(x) = p. \tag{5.9}$$

(i)-1 If ($\mathrm{VDir}(x) =< U_1+\lambda U_2 >$, $\mathrm{div}(u_1u_2) = E$, *and either* $(a(1), a(2)) \neq (0,0)$ *or* $\omega(x) \neq p$*), then:* $a(1) < a(1)'$ *or we make* $a(1)$ *strictly jump by the blowing up centered at* x *followed by the blowing up centered at* Y' *the strict transform of* Y*.*

(ii) When Y *is of the first kind,*

(ii)-1 if $< U_1 > \neq \mathrm{Idir}(x)$*, either* $\kappa(x') = 2$ *or* $(a(1), -N(x)) < (a(1)', -N(x'))$*. Furthermore* $a(1) = a(1)'$ *and* $\kappa(x') \geq 3$ *implies that* $i = 3$*,* $a(2) = a(1)$*,* $\mathrm{Idir}(x) =< U_3 >, \mathrm{div}(u_3) \not\subset E$*. and we have (5.6);*

(ii)-2 if $< U_1 >= \mathrm{Idir}(x)$*, then* $a(1) \leq a(1)'$*. When* $a(1) = a(1)'$*, the inequality* $N(x) = 1 < N(x')$ *implies div*$(u_i) \not\subset E$ *and* $\omega(x) = \epsilon(x) = p = \epsilon(x')$*,* $N(x') = 2$ *and* $deg_{U_1'}(\mathrm{in}_{x'}(F_{p,X/u_i}) = p$*, where* $U_1' = (\mathrm{in}_{x'}(u_1/u_i))$*. When* $a(1) = a(1)'$ *and* $N(x) = 2, i = 2$ *then* $N(x') = 1$*.*

In all cases, the equality $a(1) = a(1)'$*,* $N(x) \leq N(x')$ *implies that* x' *is on the strict transform of div*(u_1)*.*

Proof. We first prove (ii)-1.

We have three different cases $\mathrm{Idir}(x) =< U_1, U_i >$ or $(\mathrm{Idir}(x) =< U_1 + \lambda U_i >$, $\mathrm{div}(u_1u_i) \subset E$, $\lambda \in k(x)^*)$ or $\mathrm{Idir}(x) =< U_i >$.

In the case $\mathrm{Idir}(x) =< U_1, U_i >$, there is no x' ω-near x.

In the case $\mathrm{Idir}(x) =< U_1 + \lambda U_i >$, $\mathrm{div}(u_1u_I) \subset E$, there is at most one x' ω-near x, x' is neither on the strict transform of $\mathrm{div}(u_1)$ nor on the strict transform of $\mathrm{div}(u_i)$, let t be the exceptional parameters, $H' = t^{a(1)+a(i)+\epsilon(x)-p}u_j^{a(j)}$, $j \neq 1, i$. When $a(i) + \epsilon(x) - p > 0$, $(a(1), -N(x)) <_{lex} (a(1)', -N(x'))$. In the extreme case $a(i) + \epsilon(x) - p = 0$, we have $\epsilon(x) = p$, $a(i) = 0$, $i = 2$, $j = 3$, $a(3) > 0$: so $E = \mathrm{div}(u_1u_2u_3)$. By [21] **II.5** page 1896, $\mathrm{in}_x(F_{p,x}) = \gamma U_3^{a(3)}(U_1 + \lambda U_2)^p$, $\gamma \in k(x)^*$. Either $a(3) \neq 0 \bmod(p)$, or γ is not a p^{th}-power, as x' is rational over x, this leads to $\omega(x') = p$ and $\kappa(x') = 2$.

In the case $\mathrm{Idir}(x) =< U_i >, \mathrm{div}(u_i) \not\subset E$, then, as $\kappa(x) > 2$, we have $\epsilon(x) = 1+\omega(x)$. Without loss of generality, we suppose $i = 3$, x' is the point of parameters $(X', u_1, u_2, u_3') = (X/u_1, u_1, u_2, u_3/u_1)$, $H(x') = u_1^{a(1)+\epsilon(x)-p}u_2^{a(2)}$, we get

$$a(1)' = \inf\{a(2), a(1) + \epsilon(x) - p\}.$$

We thus have 5.6 and $(a(1), -N(x)) <_{lex} (a(1)', -N(x'))$.

In the case $\mathrm{Idir}(x) =< U_i >, \mathrm{div}(u_i) \subset E$, we suppose $a(i) > a(1)$, else, we can permute u_1, u_i and we are in case (ii)-2 with $N(x) = 2$, and as we

will see below, $(a(1), -N(x)) <_{lex} (a(1)', -N(x'))$. Then

$$a(1)' \geq inf(a(1) + \epsilon(x) + a(i) - p, a(2), a(3)),$$

it is clear that $(a(1), -N(x)) <_{lex} (a(1)', -N(x'))$ and that (5.6) is true at x'.

Proof of (ii)-2. When $\epsilon(x) = \omega(x)$, we get $\mathrm{in}_x(F_{p,X}) = \gamma U_1^{\omega(x)}$, $\gamma \in k(x)^*$. We make the case where $Y = V(X, u_1, u_3)$, x' is the point of parameters $(X', u_1', u_2, u_3) = (X/u_3, u_1/u_3, u_2, u_3/u_2)$, $H(x') = {u_1'}^{a(1)} u_2^{a(2)} u_3^{a(1)+a(3)+\epsilon(x)-p}$. As $\epsilon(x) \geq p$, we get $a(1) = a(1)'$; when $\epsilon(x) > p$, then $N(x') \leq N(x)$; when $\epsilon(x) = p, a(3) = 0$. Then by construction of our triple and because $\delta(x) > 1$, we get $\mathrm{div}(u_3) \not\subseteq E$ and $N(x') \geq 2$ and $F_{p,X'} = \gamma {u_1'}^p \bmod(u_2, u_3)$, this gives the last assertion. The case $Y = V(X, u_1, u_2)$ is left to the reader.

When $\epsilon(x) = 1 + \omega(x)$, we get $\mathrm{in}_x(F_{p,X}) = \gamma U_1^{\omega(x)} U_i + G(U_1, U_i^{\omega(x)})$, $\gamma \in k(x)^*$, G homogeneous of degree $\epsilon(x) = 1 + \omega(x)$. Without loss of generality, we suppose $i = 3$. x' is the point of parameters $(X', u_1', u_2, u_3) = (X/u_3, u_1/u_3, u_2, u_3/u_2)$, x' is on the strict transform of $\mathrm{div}(u_1)$ and we have $H(x') = {u_1'}^{a(1)} u_2^{a(2)} u_3^{a(1)+a(3)+\epsilon(x)-p}$. We get $a(1) = a(1)'$, $N(x') \leq_{lex} N(x)$ and (5.6).

Proof of (i). There is a free variable transverse to Y: we give it the index 3: this means $i = 2$, $\mathrm{div}(u_3) \not\subset E$ and

$$\mathrm{in}_x(F_{p,x}) = U_3 G(U_1, U_2) + K(U_1, U_2),\ G \neq 0,$$

G homogeneous of degree $\omega(x)$, $K = 0$ or K homogeneous of degree $1+\omega(x)$.

We have three different cases ($\mathrm{Idir}(x) =< U_1, U_2 >$ or $\mathrm{Idir}(x) =< U_1 + \lambda U_2 >$, $\mathrm{div}(u_1 u_2) \subset E$, $\lambda \in k(x)^*$ or $\mathrm{Idir}(x) =< U_i >$, $i = 1, 2$).

In the case $\mathrm{Idir}(x) =< U_1, U_2 >$, there is no x' ω-near x.

In the case $\mathrm{Idir}(x) =< U_1 + \lambda U_2 >$, $\mathrm{div}(u_1 u_2) = E$, then $G = \gamma(U_1 + \lambda U_2)^{\omega(x)}$, $\gamma \in k(x)^*$: x' is the point of parameters $(X', u_1, v, u_3) := (X/u_1, 1 + \lambda' u_2/u_1, u_3)$ where λ' is a pull back of λ in R. Then $H(x') = u_1^{a(1)+a(2)+\omega(x)-p}$ it is easy to see that $a(1) < a(1)'$ except in the exceptional case $a(1) = a(2) = 0$, $\omega(x) = p$.

Then it is successful to blow up the origin: x' is a point on the strict transform of $\mathrm{V}(X, u_1 + \lambda' u_2)$. Either x' is the point of parameters $(X', u_1', u_2', u_3) := (X/u_3, u_1/u_3, u_2/u_3, u_3)$, then Y' the strict transform of Y is permissible of first kind, by (iii), the blowing up centered at Y' will give a strict jump

of $a(1)$ or we reach the case $\kappa(x) \leq 2$. Or x' is in the first chart, and, $H(x') = u_1^{a(1)+a(2)+\epsilon(x)-p} = u_1$, then $a(1) = 0 < 1 = a(1)'$.

Case $\mathrm{Idir}(x) =< U_2 >$, i.e. $G = \gamma U_2^{\omega(x)}$, $\gamma \in k(x)^*$: x' is the point of parameters $(X', u_1, u_2', u_3) := (X/u_1, u_2/u_1, u_3)$, $H(x') = u_1^{a(1)+a(2)+\omega(x)-p} {u_2'}^{a(2)}$. Then we have (5.6) and we have $(a(1), -N(x)) \leq_{lex} (a(1)', -N(x'))$ with strict inequality except in the extreme case $a(2) = 0, \omega(x) = p$, $E \subseteq \mathrm{div}(u_1u_2)$ which implies $(a(1) = a(2) = 0$, $E = \mathrm{div}(u_1u_2))$ or $E = \mathrm{div}(u_1)$.

In the subcase $E = \mathrm{div}(u_1)$, then it is successful to blow up the origin: in the first chart, $a(1)' = a(1) + \epsilon(x) - p = a(1) + 1$, there is no ω-near point in the second chart and if the point at infinity x' is ω-near x, then $\kappa(x') \leq 2$.

In the case $a(1) = a(2) = 0, \omega(x) = p$, $E = \mathrm{div}(u_1u_2)$, as $a(1) \leq a(2)$: u_1 and u_2 play symmetric roles. We reach the following case.

Case $\mathrm{Idir}(x) =< U_1 >$, i.e. $G = \gamma U_1^{\omega(x)}$, $\gamma \in k(x)^*$: x' is the point of parameters $(X', u_1', u_2, u_3) := (X/u_2, u_1/u_2, u_2, u_3)$. Then we compute that $H(x') = {u_1'}^{a(1)} u_2^{a(1)+a(2)+\omega(x)-p}$. Then we have $(a(1), a(2), a(3)) \leq_{lex} (a(1)', a(2)', a(3)')$, $N(x') \leq N(x)$ except in the case $E = \mathrm{div}(u_1)$, $a(1) = 0$, $\omega(x) = p$.

Lemma 5.13. *With notations as before, we have:*

(i) in the case $N(x) = 3$, we blow up the origin x: then $(a(1), N(x)) <_{lex} (a(1)', N(x'))$;

(ii) in the case $N(x) = 2$, we blow up the origin, the equality $(a(1), N(x)) = (a(1)', N(x'))$ implies that x' is the point on the strict transform of $V(X, u_1, u_2)$.

Proof. Clear by lemma 5.11.

As a consequence of lemma 5.13, we can reach the case $N(x) = 1$. Indeed, in the case $N(x) = 2$, by 5.13 (ii), after a finite sequence of blowing ups centered at the points above x on the strict transform of $\mathbf{V}(u_1, u_2)$, we reach the case where (X, u_1, u_2) is permissible of the first kind, by lemma 5.12 (ii), if $\kappa(x') \geq 3$, then $N(x') = 1$.

The case $N(x) = 1$ is not stable, the stable case is (5.10) below.

Lemma 5.14. *With notations as before, we assume*

$$N(x) = 1 \text{ or } (N(x) = 2,\ \omega(x) = \epsilon(x) = p \text{ and } deg_{U_1}(\mathrm{in}_x(F_{p,x}) = p). \quad (5.10)$$

We suppose that x is not combinatoric (definition 4.3).

(i) We make a permissible blowing up of the first kind: we blow up either the origin or $\mathbf{V}(X, u_1, u_i)$ $i = 1$ or 2 if it is permissible of first kind. Then If

x' is ω-near x and $a(1) = a(1)'$ and $\kappa(x') > 2$, x' is on the strict transform of $div(u_1)$ and (x', u_1') verifies (5.10).

(ii) We make a permissible blowing up of the second kind. Then if the permissible center is not in the intersection of two components of E, x' is ω-near x and $a(1) = a(1)'$ and $\kappa(x') > 2$, x' is on the strict transform of $div(u_1)$ and (x', u_1') verifies (5.10).

Let $(\omega_0, -a, N_0)$ be the values of $\omega(x)$, $-a(1)$, $N(x)$ at our initial point. As a consequence, $div(u_1)$ has maximal contact for the condition $\mathcal{C}$ defined by
$\mathcal{C}$ (i) if $\omega(x) = p$, any component of dimension 1 of $Sing_p(\mathcal{X} \cap \mathrm{div}(u_1)$ is in $div(u_i) \subset E$, $i = 2, 3$),
$\mathcal{C}$ (ii) (5.10) holds,
$\mathcal{C}$ (iii) $(\omega(x), -a(1)) \geq (\omega_0, -a)$,
$\mathcal{C}$ (iv) x is not combinatoric.

Proof. The case $N(x) = 1$ is a consequence of lemma 5.11. Let us look at the case $N(x) = 2$, $\omega(x) = \epsilon(x) = p$ and $\deg_{U_1}(\mathrm{in}_x(F_{p,x})) = p$. Note that $\mathrm{VDir}(x) \not\subseteq (U_2, U_3)$.

Case of blowing up centered at the origin.

First chart: $x' \in \mathrm{Spec}S[X', u_1, u_2', u_3']$, where

$$(X', u_1, u_2', u_3') := (X/u_1, u_1, u_2/u_1, u_3/u_1).$$

Then

$$H(x') = u_1^{a(1)+a(2)+a(3)+\epsilon(x)-p} {u_2'}^{a(2)} {u_3'}^{a(3)} = u_1^{a(1)+a(2)+a(3)} {u_2'}^{a(2)} {u_3'}^{a(3)} = u_1^{2a(1)+a(3)} {u_2'}^{a(1)} {u_3'}^{a(3)}.$$

When $a(1) = a(2) > 0$, $a(1) + a(2) + a(3) > a(1)$, so $(a(1), N(x)) < (a(1)', N(x'))$. When $a(1) = a(2) = 0$, as $\delta(x) > 1$, then $a(1) + a(2) + a(3) > 0 = a(1)$. So $a(1)' = a(1)$ implies that x' is on the strict transform of $\mathrm{div}(u_2)$. As $\mathrm{VDir}(x) \not\subseteq (U_2, U_3)$ and as x' is not on the strict transform of $\mathrm{div}(u_1)$, $U_1 + \lambda U_3 \in \mathrm{Idir}(x) \mathrm{mod}(U_2)$, for some $\lambda \in k(x)^*$: x' has parameters $(X', u_1, u_2', v := 1 + \lambda u_3)$. Note that $\kappa(x) \neq 2$ implies $\mathrm{div}(u_3) \subset E$. We have $a(3) \neq 0 \bmod(p)$, else x would be combinatoric, so $a(1) + a(2) + a(3) \neq 0 \bmod(p)$, $H(x')$ is not a p^{th}-power up to multiplication by an invertible. We get $\epsilon(x') = p$ and $V \in \mathrm{Idir}(x') \bmod(U_1, U_2')$ which gives $\kappa(x') \leq 2$. There is no x' satisfying the conditions of lemma 5.14 in this chart.

Second chart: $x' \in \mathrm{Spec}S[X', u_1', u_2, u_3']$, where

$$(X', u_1', u_2, u_3') := (X/u_2, u_1/u_2, u_2, u_3/u_2).$$

If x' is not in the first chart, then

$$H(x') = {u'_1}^{a(1)} u_2^{a(1)+a(2)+a(3)+\epsilon(x)-p} {u'_3}^{a(3)} = {u'_1}^{a(1)} u_2^{2a(1)+a(3)} {u'_2}^{a(1)} {u'_3}^{a(3)}.$$

We have $u'_1(x') = 0$. As above, when $a(1) = a(2) > 0$, $a(1) + a(2) + a(3) > a(1) = a(2)$, so $a(1) = a(1)'$, $N(x) = 2 < N(x') = 1$. When $a(1) = a(2) = 0$, then $a(3) \geq 1$, so $a(1) + a(2) + a(3) + \epsilon(x) - p > 0 = a(2)$, $a(1) = a(1)'$, $N(x) = 2 < N(x') = 1$. The reader makes the point at infinity and the cases of blowing ups centered at $V(X, u_1, u_i)$ $i = 1$ or 2 if one is permissible.

Blowing up of the second kind. Then $\epsilon(x) = 1 + \omega(x) > p$, so $N(x) = 1$, there is a free variable transverse to the center, without loss of generality, we suppose it is u_3, so the center is $V(X, u_1, u_2)$. The statement is a consequence of lemma 5.12(i).

End of the proof of proposition 5.10. We start with a point x not combinatoric with $N(x) = 1$, then we make a sequence of blowing ups

There exists a composition of blowing ups centered at closed points

$$\begin{array}{ccc} \mathcal{X}'' = & \xleftarrow{\tilde{\pi}} & \tilde{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathcal{S}'' & \xleftarrow{\tilde{\sigma}} & \tilde{\mathcal{S}} \end{array}$$

such that at the center x of our valuation, we have (5.10), there is locally at most one component C of dimension 1 in $\mathrm{Sing}_p(\tilde{\mathcal{X}})$ which is not in two different components of E, C is permissible at x of first kind. Then we blow up along C: we reach the case where conditions $\mathcal{C}$(i)(ii)(iv) are true at x, then we define $(\omega_0, a) := (\omega(x), a(1))$. The proof ends with theorem 6.11 below.

Theorem 5.15. *Assume that (S, h, E) satisfies assumption **(G)**. For every valuation μ of $L = \mathrm{Tot}(S[X]/(h))$ centered at x, there exists a finite and independent composition of local Hironaka-permissible blowing ups (4.2) such that $m(x_r) < p$.*

Proof. By proposition 5.10 it can be assumed that (S, h, E) satisfy both conditions **(G)** and **(E)**'. Therefore the theorem follows from theorems 2.23 and 4.4 by descending induction on $\iota(x)$, $\{x\} := \eta^{-1}(m_S)$.

6 Maximal contact

We assume in the whole section that (S, h, E) satisfies conditions **(G)** and **(E)**. We complete the proof of proposition 5.10 in this section by proving theorem 6.11. This theorem is given a much more general form than required since it will be used repeatedly in [24] in order to prove special cases of theorem 4.4.

Definition 6.1. We say that $\mathrm{div}(u_1) \subset E \subset \mathcal{X}_n$ has "weak maximal contact" for some condition $\mathcal{C}$ if the sequence of blowing ups along the centers $x_{n+i} \in \mathcal{X}_{n+i}$ of μ,

$$\mathcal{X}_{n+m} \longrightarrow \mathcal{X}_{n+m-1} \longrightarrow ....\mathcal{X}_{n+1} \longrightarrow \mathcal{X}_n, \tag{1}$$

is such that all the x_{n+i} ω-near x_n are on the strict transform of $\mathrm{div}(u_1) \subset E \subset \mathcal{X}_n$ and verify $\mathcal{C}$ or if for some $m \geq 0$, x_{n+m} is good.

We say that $\mathrm{div}(u_1) \subset E \subset \mathcal{X}_n$ has "maximal contact" for some condition $\mathcal{C}$ if for any sequence of ω-permissible blowing ups

$$\mathcal{X}_{n+m} \longrightarrow \mathcal{X}_{n+m-1} \longrightarrow ....\mathcal{X}_{n+1} \longrightarrow \mathcal{X}_n, \tag{2}$$

all the x_{n+i} ω-near x_n are on the strict transform of $\mathrm{div}(u_1) \subset E \subset \mathcal{X}_n$ and verify $\mathcal{C}$ or if for some $m \geq 0$, x_{n+m} is good.

The aim of this section is to prove theorem 6.11 below. The arguments are quite similar to [21] chapter 4 pages 1957 and following.

First we look at what can be achieved by blowing up closed points. We make the infinite sequence of blowing ups centered at $x_i \in \mathcal{X}_i$

$$(\mathcal{X} =: \mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots , \tag{6.1}$$

where $x_i \in \mathcal{X}_i$ is the center of μ.

Proposition 6.1. *We suppose that div(u_1) has weak maximal contact for some condition $\mathcal{C}$, div$(u_1) \subset E \subset$div(u_1u_2), $\omega(x) = \epsilon(x)$ and $U_3 \in \mathrm{Idir}(x)\ \mathrm{mod}(U_1, U_2)$, where* Idir$(x)$ *is the ideal of the adapted directrix 2.17 then*
(1) if μ is archimedean, in the sequence 6.1, for some i_0, one of the following is true:
(i) $\omega(x_{i_0}) < \omega(x)$,
(ii) $\omega(x_{i_0}) = \omega(x)$ and $\mathcal{C}$ is not true at x_{i_0}.

(2) if div(u_1) has maximal contact, there exists a sequence of ω-permissible blowing ups

$$(\mathcal{X} =: \mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots \ , \tag{6.2}$$

such that, for some i_0, (i) or (ii) above is true;
(3) by a finite sequence of blowing ups centered at closed points, we reach the case where (X, u_1, u_3) is permissible of the first kind with the assumptions above.

Proof.

We may suppose $U_3 \in \mathrm{Idir}(x)$.

We blow up along x, by the assumptions, the only possible ω-near point is the point x' of parameters $(X/u_2, u_1/u_2, u_2, u_3/u_2)$ and the assumptions are true at x'. Then, if we repeat, we make a quadratic sequence along a formal arc φ of ideal $(X, u_1, u_3)\mathrm{mod}\ (u_2)$, by 3.8, we get (3).

Then, (1) is an easy consequence of (3).

In case (3), we blow up along $\mathbf{V}(X, u_1, u_3)$, an ω-near point is on the strict transform of u_3 and X and by hypothesis of u_1 also: there is none.

Notation 6.1. We set $m(x)$ the number of components of E going through x. $\mathrm{div}(u_1)$. If $m(x) = 1$, i.e. $E = \mathrm{div}(u_1)$, we set $\gamma := \infty$. In the case $m(x) = 2$, we suppose $\mathrm{div}(u_1u_2) = E$. By definition of maximal contact or weak maximal contact, after an eventual blowing up centered at the origin, we may suppose $m(x) \geq 2$.
(i) Cases 1 and 2.

$$\epsilon(x) = \omega(x).$$

Case 1 is $E = \mathrm{div}(u_1u_2)$, case 2 is $E = \mathrm{div}(u_1u_2u_3)$.

We suppose that (X, u_1, u_2, u_3) are well adapted variables. We look at the characteristic polyhedron $\Delta(h; u_1, u_2, u_3; X) = \Delta(h; u_1, u_2, u_3; X)$.

We translate $\Delta(h; u_1, u_2, u_3; X) \subset \mathbb{R}^3$ of $-(d_1, d_2, d_3)$ (d_j is defined in 2.9), then we make a stereographic projection of the translated polyhedron from $(\omega(x)/p, 0, 0)$ on the plane $x_1 = 0$ where the coordinates in $\mathbb{R}^3$ are denoted (x_1, x_2, x_3), followed by the homothety of center $(0, 0)$ of ratio $\frac{p}{\omega(x)}$. Let $pr(\Delta)$ be the obtained polyhedron.

With notations as above, let us denote
$A_2 := \inf\ \{x_2 | (x_2, x_3) \in pr(\Delta)$, $A_3 := \inf\ \{x_3 | (x_2, x_3) \in pr(\Delta)$;
$B := \inf\ \{x_2 + x_3 \mid (x_2, x_3) \in pr(\Delta)$;
$C := B - A_2 - A_3 \geq 0$;

$\beta := \inf\{x_3 \mid (A_2, x_3) \in pr(\Delta)$.
$\gamma(X, u_1, u_2, u_3) := 1 + \lfloor C \rfloor$ when $\mathrm{div}(u_1u_2u_3) = E$ (case 2),
$\gamma(X, u_1, u_2, u_3) := \sup\{1, \lceil \beta \rceil\}$ if $E = \mathrm{div}(u_1u_2)$ (case 1).

$\lceil \beta \rceil$ means the smallest integer greater or equal to β.

As all the vertices of the translation $\Delta(h; u_1, u_2, u_3; X)$ have module $\geq \frac{\epsilon(x)}{p}$, we have

$$B \geq 1.$$

$B \neq \infty$, else if we denote $h := X^p + \sum_{i=1}^{p} X^{p-i}F_i$, $\mathrm{ord}_{u_1}(F_i) \geq id_1 + i\omega(x) > id_1$, $1 \leq i \leq p$, which contradicts the definition of d_1. By the same argument, $A_i \neq \infty,\ i = 2, 3$.
(ii) Case (ii)

$$\epsilon(x) = 1 + \omega(x).$$

Then $E = \mathrm{div}(u_1u_2)$, we translate $\Delta(h; u_1, u_2, u_3; X) \subset \mathbb{R}^3$ of $-(d_1, d_2, -1/p)$: this translated polyhedron of $\mathbb{R}^3$ may have vertices with negative third coordinate. Then we make a stereographic projection of the translated polyhedron from $(\omega(x)/p, 0, 0)$ on the plane $x_1 = 0$, followed by the homothety of center $(0, 0)$ of ratio $\frac{p}{\omega(x)}$. Let $pr(\Delta)_3$ be the obtained polyhedron.

With notations as above, let us denote
$A_2 := \inf\{x_2 | (x_2, x_3) \in pr(\Delta)_3$, $A_3 := \inf\{x_3 | (x_2, x_3) \in pr(\Delta)_3$;
$\beta_3 := \inf\{x_3 \mid (A_2, x_3) \in pr(\Delta)_3$,
$\gamma := \sup\{1, \lceil \beta_3 \rceil\}$.

Obviously, these definitions may depend on (u_1, u_2, u_3), but, except for β_3, not on X, since $\Delta(h; u_1, u_2, u_3; X)$ is minimal and A_a, B and C, β, γ are computed by evaluating the minimum of a linear function on some (linear) projection of this polyhedron. When there is a risk of confusion, we will make explicit this dependence on (u_1, u_2, u_3) by writing $A_a(u_1, u_2, u_3)$, etc... We also use the notation $A_a(x)$, $A_a(x')$, etc... when dealing with a blowing up $e : X' \to X$ and $x' \in e^{-1}(x)$. In this case, we always compute invariants w.r.t. $E' := (e^{-1}E)_{red}$.

Remark 6.1. Theses numbers B, A_a can be computed directly from the equation h.

In cases 1-2, let (a, b) strictly positive real numbers such that

$$a(d_1 + \frac{\omega(x)}{p}) + b(d_2 + d_3) = 1$$

with the convention $d_3 = 0$ when $\mathrm{div}(u_3) \not\subset E$.

Then

$$B(u_1,u_2,u_3)=\sup\{\frac{a}{b}|v_{(a,b,b)}(h)=p\},$$

the couple (a,b) giving the sup above is said "defining B". See the analogy with [21], theorem **I.4** equation (3) page 1962. As $B\geq 1$, we have $a\geq b$.

In cases 1-2-3, let (a,b) strictly positive real numbers such that

$$a(d_1+\frac{\omega(x)}{p})+bd_2=1,$$

then

$$A_2(u_1,u_2,u_3)=\sup\{\frac{a}{b}|v_{(a,b,0)}(h)=p\},$$

this suitable couple (a,b) is said "defining A_2".

We denote by H_2

$$H_2=\mathrm{in}_{v_{(a,b,0)}}(h)=X^p+\sum_{1\leq i\leq p}X^{p-i}\Phi_i,\Phi_i\in\frac{S}{(u_1,u_2)}[U_1,U_2],$$

where (a,b) "defines A_2". See the analogy with [21], theorem **I.4** valuation μ_1 page 1962. We expend the $\phi_i, 1\leq i\leq p$:

$$\Phi_i=\sum U_1^jU_2^{b(i,j)}\phi_{i,j},\ b(i,j)=\frac{i}{b}-A_2j,$$

where $\frac{1}{b}=1+A_2(d_1+\omega(x))$.

As we will see in 6.4, 6.7, after the blowing up of one closed point, we will get $\Phi_i=0$ for $i\neq p-1,p$. With this hypothesis, we denote:

$$\Phi_p:=U_1^{a(1)}U_2^{a(2)}U_3^{a(3)}(\sum_{i=1}^{\omega(x)}\lambda_iU_2^{iA_2}u_3^{c(i)}),$$

with $\lambda_i\in\frac{S}{(X,u_1,u_2)}$, $\lambda_i=0$ or invertible, $\lambda_i=0$ when $iA_2\notin\mathbb{N}$,

$$\Phi_{p-1}=\lambda U_1^{(p-1)d_1}U_2^{(p-1)d_2}u_3^{(p-1)d_3},\ \lambda\in\frac{S}{(X,u_1,u_2)},$$

$a(j)=pd_j$, $\mathrm{div}(u_j)\subset E$, by convention, $d_3=0$ when $\mathrm{div}(u_3)\not\subset E$. In case 1, we see that $\beta=\inf\{c(i)/i\}$ when $\Phi_{p-1}=0$, $\beta=0$ when $\Phi_{p-1}\neq 0$.

By analogy, in case 3, we define $\beta = \inf\{c(i)/i|\lambda_i \neq 0\}$ when $\Phi_{p-1} = 0$, $\beta = 0$ when $\Phi_{p-1} \neq 0$. We have $\beta_3 = -\frac{1}{p-1}$ when $\Phi_{p-1} \neq 0$, $\beta_3 = \inf\{\frac{c(i)-1}{i}|\lambda_i \neq 0\}$ when $\Phi_{p-1} = 0$. In the case $\Phi_{p-1} = 0$, let i_2 such that $\beta_3 = \frac{c(i_2)-1}{i_2}$ and i_3 such that $\beta = \frac{c(i_3)}{i_3}$. By definitions, $\beta = \frac{c(i_3)}{i_3} \leq \frac{c(i_2)}{i_2}$ and $\beta_3 = \frac{c(i_2)-1}{i_2} \leq \frac{c(i_3)-1}{i_3}$ then, in case 3

$$\beta_3 < \beta, \beta_3 < N \in \mathbb{N} \ \Rightarrow \ \beta \leq N. \tag{6.3}$$

In case 2, $\beta = \inf\{c(i)/i|\lambda_i \neq 0\}$ when $\Phi_{p-1} = 0$, $\beta = 0$ when $\Phi_{p-1} \neq 0$.

In case 2, let (a, b) strictly positive real numbers such that

$$a(d_1 + \frac{\omega(x)}{p}) + bd_3 = 1,$$

then

$$A_3(u_1, u_2, u_3) = \sup\{\frac{a}{b}|v_{(a,0,b)}(h) = p\}.$$

Proposition 6.2. *In cases 1,2, let (a, b) defining B, we define H_B by*

$$H_B := \mathrm{in}_{v_{(a,b,b)}}(h).$$

Then

$$H_B = X^p + \Phi_{p-1}X + \Phi_p \in k(x)[X, U_1, U_2, U_3],$$

with $\Phi_{p-1} = 0$ or $\Phi_{p-1} = -\gamma_{p-1,Z}\prod_{j=1}^{e} u_j^{A_{p-1,j}}$, $\gamma_{p-1,Z} \in k(x)$, with $A_{p-1,j} \in (p-1)\mathbb{N}$, $1 \leq j \leq e$, and $\gamma_{p-1,Z} \in S$ a unit.

This a consequence of 2.14.

Proposition 6.3. *In case 3, let (a, b) strictly positive real numbers such that*

$$a\frac{\omega(x)}{p} + bd_2 = 1$$

we define $B_3(u_1, u_2, u_3) = \sup\{\frac{a}{b}|v_{(a,b,b)}(hu_3^{-1}) = p\}$.

We define H_{B3} by

$$H_{B3} := \mathrm{in}_{v_{(a,b,b)}}(h).$$

Then

$$H_{B3} = X^p + X\Phi_{p-1} + \Phi_p \in k(x)[X, U_1, U_2, U_3]. \tag{6.4}$$

with $\Phi_{p-1} = 0$ or $\Phi_{p-1} = -\gamma_{p-1,Z}\prod_{j=1}^{e} u_j^{A_{p-1,j}}$, $\gamma_{p-1,Z} \in k(x)$, with $A_{p-1,j} \in (p-1)\mathbb{N}$, $1 \leq j \leq e$, and $\gamma_{p-1,Z} \in S$ a unit.

This is a consequence of 2.14.

Now we follow [21] chapter 4.

Proposition 6.4. *cf. [21]**I.4 Theorem** page 1962 With hypotheses and notations of **I.2**, assume x is in case 1-2. We blow-up x and x' is a closed point of the first chart ω-near to x and condition $\mathcal{C}$ is true at x'.*

If $u'_3(x') = 0$, then x' is in case 1-2 and $C(x') \leq C(x)$, $\beta(x') \leq \beta(x)$, $A_2(x') = B(x) - 1$.

From now on, $u'_3(x') \neq 0$, let $d := [k(x') : k(x)]$, we have

$$\gamma(x') \leq \gamma(x),\ \beta(x') < \lfloor \frac{C(x)}{d} \rfloor + 1, \tag{1}$$

and, if x is in case 1,

$$\beta(x') \leq \beta(x) \text{ or } C(x) = 0. \tag{2}$$

If in H_B the term $\Phi_{p-1} \neq 0$, then $\beta(x') = 0$ if x' is in case 1or 3, $\beta_3(x') < 0$ if x' is in case 3.

In the case $m(x) \leq 2$, when $a(1) + \omega(x) \neq 0 \ mod\ (p)$ or $a(2) \neq 0 \ mod\ (p)$ or x' rational over x, then x' is in case 1,

If x' is not rational over x and $\gamma(x) \geq 2$, then $\gamma(x') < \gamma(x)$, except in the following case:

• *$m(x) \leq 2$, $\beta(x) = 2$ then we get $\beta(x') < 2$ and, if x' is in case 3, $\beta 3(x') = 1$, $\beta(x') = 1 + \frac{1}{i_1(x)}$, $a(1) + \omega(x) = 0 \ mod\ (p)$ and $i_1(x) = 0 \ mod\ (p)$ (i_1 defined below). So $\beta(x') \leq 1 + \frac{1}{p}$.*

Furthermore, at x', for suitable parameters

$$H_2(x') = X'^p + \sum_{1 \leq i \leq p} X'^{p-i} \Phi'_i,$$

$\Phi'_i = 0$ *for* $i \neq p, p-1$.

Proof.

$$H_B = X^p + \Phi_{p-1} X + \Phi_p \in k(x)[X, U_1, U_2, U_3],$$

$$\Phi_p =: U_1^{a(1)} U_2^{a(2)} U_3^{a(3)} (\lambda U_1^{\omega(x)} + \sum_{i=1}^{\omega(x)} U_1^{\omega(x)-i} F_i(U_2, U_3) \in k(x)[U_1, U_2, U_3], \tag{6.5}$$

with the convention $a(3) = 0$ when $\mathrm{div}(u_3) \not\subset E$. Then, if $\Phi_{p-1} = 0$, as $B(x) \neq \infty$, the F_i are not all 0, we define $i_1 := \sup\{i | U_1^{\omega(x)-i} F_i \notin (k(x)[U_1, U_2, U_3])^p\}$. As $\Delta(h; u_1, u_2, u_3; X)$ is minimal and $B(u_1, u_2, u_3, X) < \infty$, i_1 does exist.

First we look at the extreme case $\lambda = 0$: $B(x) = 1$ and H_B is the δ-initial form. Then either $U_1 + \mathrm{Idir}(x) =< U_1, U_2, U_3 >$ or $< U_1, U_2 >$ or $< U_1, U_3 >$ or, up to multiply the variables by an invertible, $< U_1, U_2 + U_3 >$. In the first case, there is no x' verifying condition $\mathcal{C}$ and ω-near to x. In the second and third case, x' is at the origin of an usual chart, there is no preparation on the variables to do: the reader verifies the assertion. In the last case, when $E = \mathrm{div}(u_1 u_2)$, up to a change of local parameters, we may suppose $(U_1 + \mathrm{Idir}(x)) = (U_1, U_2)$, we apply proposition 6.1. When $= \mathrm{div}(u_1 u_2 u_3)$, we get

$$H_B = X^p + U_1^{a(1)} U_2^{a(2)} U_3^{a(3)} (\sum_{i=1}^{\omega(x)} U_1^{\omega(x)-i} F_i(U_2, U_3)),$$

with $F_i = 0$ or $\deg_{U_j}(F_i) = i$, $j = 2, 3$: $C(x) = 1$, $\gamma(x) = 2$. x' is the point of parameters $(X', u_1', u_2', v) := (X/U_2, u_1/u_2, 1 + u_2/u_3)$, we get

$$H_2' = X'^p + U_1'^{a(1)} U_2'^{a(1)+a(2)+a(3)+\omega(x)-p} \sum_{i=1}^{\omega(x)} \lambda_i U_1^{\omega(x)-i} V^{c(i)},$$

$$\lambda_i \in \frac{S'}{(X', u_1', u_2')},\ \lambda_i = 0 \text{ or invertible},\ \lambda_{i_1} \neq 0,\ c(i) = i \text{ or } i+1.$$

If any $c(i) = i$, we apply 6.1 at x'. Else, x' is in case 3, we get $\beta_3(x') = 1$: $\gamma(x') = 2$. Or x' is in case 1, then $\beta(x') = \inf\{1 + \frac{1}{i_1(x)} | \lambda_i \neq 0\} \leq 2$: $\gamma(x') = 2$.

Definition 6.2. The general case $\lambda \neq 0$, is denoted by (1*) or (2*).

In case (1*) or (2*), the proof is based on the fact that H_B of x gives H_{A_2} at x'. Indeed, x' is in the chart of origin the point of parameters $(X', u_1', u_2', u_3') := (X/u_2, u_1/u_2, u_2, u_3/u_1)$. By hypotheses, $X'(x') = u_1'(x') = u_2'(x') = 0$. Let (a, b) strictly positive real numbers such that $\frac{a}{b} \geq 1$ and

$$a(d_1 + \frac{\omega(x)}{p}) + b(d_2 + d_3) = 1$$

for any four-uple $(\alpha, \beta, \gamma, \delta) \in \mathbb{N}^4$, we have $X^\alpha u_1^\beta u_2^\gamma u_1^\delta = X'^\alpha u_1'^\beta u_2'^{\alpha+\beta+\gamma+\delta} u_3'^\delta$. With the parameters (X', u_1', u_2', u_3'), the valuation $v(a, b, b)$ becomes $(1 +$

$b')^{-1}v_{(a',b',0)}$, where $v_{(a',b',0)}$ is given by

$$v_{(a',b',0)}(X'^{\alpha}{u'_1}^{\beta}{u'_2}^{\alpha+\beta+\gamma+\delta}{u'_3}^{\delta}) = \alpha + a'\beta = b'(\alpha+\beta+\gamma+\delta),$$

with $(1+b', a'+b', b')$ proportional to $(1, a, b)$, so $\frac{a'}{b'} = \frac{a}{b} - 1$, by taking the sup, we get $A_2(X', u'_1, u'_2, u'_3) = B(x) - 1$. Let h' be the strict transform of h, we get

$$H_{A'_2} = \mathrm{in}_{v_{(a',b',0)}}(h') = X'^{p} + X'\Phi_{p-1} + U_1^{a(1)}U_2^{p(\delta(x)-1)}{u'_3}^{a(3)}\Phi'_p$$

with

$$\Phi'_p = U_1^{a(1)}U_2^{p(\delta(x)-1)}{u'_3}^{a(3)}(\lambda U_1^{\omega(x)} + \sum_{i=1}^{\omega(x)} U_1^{\omega(x)-i}{U'_2}^{i(B(x)-1)}F_i(1, u'_3). \quad (6.6)$$

This ends the proof in the case where x' is the origin of the chart. When $u'_3(x') \neq 0$, we have to make a translation on X' to minimize the characteristic polyhedron, this may just add a p-power to $U_1^{\omega(x)-i_1}{U'_2}^{i_1(B(x)-1)}F_{i_1}(1, u'_3)$.

To end the proof, the reader looks at [21]**I.4 Theorem**.

Corollary 6.5. *([21] **I.4.1 Corollary** p. 1963) With hypotheses and notations of 6.4, we blow up x. If x is in case 1 or 2 and if x' is a point in the first chart very near to x on the strict transform of $div(u_1)$ with $u'_3(x) \neq 0$, either $\Phi_{p-1} \neq 0$ then $\beta(x') = 0$,*
or $\Phi_{p-1} = 0$, then $\beta(x') \leq (1 + \frac{\deg(F_{i_1})}{d})$, where i_1, F_i are defined just above.

Corollary 6.6. *([21] **I.4.2 Corollary**) With hypotheses and notations of 6.4, if x is in case 1 or 2, if $C(x) = 0$, $A_2(x) < 1$ and $A_3(x) < 1$, then if the sequence of blowing ups along the centers $x_{n+i} \in X_{n+i}$ of μ,*

$$X_0 \longrightarrow X_{n+m-1} \longrightarrow ....X_{n+1} \longrightarrow X_n, \quad (1)$$

We make the infinite sequence of blowing ups centered at $x_i \in \mathcal{X}_i$ 6.1, for some i_0, one of the following is true:
(i) $\omega(x_{i_0}) < \omega(x)$,
(ii) $\omega(x_{i_0}) = \omega(x)$ and $\mathcal{C}$ is not true at x_{i_0}.

Definition 6.3. With hypotheses and notations of **I.2**, we suppose x is in case 3. With the notations of 6.2, we expend

$$\Phi_p := U_1^{\omega(x)}(\lambda U_3 + \lambda' U_2) + \sum_{i=1}^{\omega(x)} U_1^{\omega(x)-i} F_i(U_2, U_3),\ \lambda, \lambda' \in k(x), \qquad (6.7)$$

$F_i \in k(x)[U_2, U_3]$ homogeneous of degree $iB3$.

When $\lambda \neq 0$, we say that x is in case 3*.

Proposition 6.7. *[21]* ***I.5 Theorem*** *page 1964.*

With hypotheses and notations of ***I.2****, we suppose x is in case 3. In case 3*, we choose u_3 such that $\lambda' = 0$, the reader sees that it modifies neither H_2, nor β_3, nor $B3$. We blow-up x.*

Let x' be a closed point very near to x verifying condition $\mathcal{C}$ in the chart of origin $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, u'_3 = \frac{u_3}{u_2})$ (first chart).

(i) When x is in case 3 and $u'_3(x') \neq 0$ and ($a_1 + \omega(x) \neq 0$ mod (p) or $a(2) + 1 \neq 0$ mod (p)), x' is in case 1*.*

(ii) If x' is in case 3, then $\beta 3(x') \leq \beta 3(x)$, the inequality is strict if $1 \leq \beta 3(x)$ and x' is not rational over x.

(iii) When $\beta 3(x) = 1$,

(iii)-a if x' is not rational over x, then $\gamma(x') = 1$,

(iii)-b if x' is rational over x, then $\beta(x') \leq 2$ and, if x' is in case 3 and $\gamma(x') = 2$, then $\beta 3(x') = 1$

(iii)-c if x' is rational over x and $\beta(x) \leq 1 + \frac{1}{p}$, then $\beta(x') \leq 1 + \frac{1}{p}$.

(iv) In every case we have

$$\gamma(x') \leq \gamma(x),\ A_2(x') = B3(x) - 1.$$

Furthermore, at x', for suitable parameters

$$H_2(x') = X'^p + \sum_{1 \leq i \leq p} X'^{p-i} \Phi'_i,$$

$\Phi'_i = 0$ *for* $i \neq p, p-1$.

Proof. At x', we choose a R.S.P. (X', u'_1, u'_2, v) with $(X' = \frac{X}{u_2}, u'_1 = \frac{u_1}{u_2}, u'_2 = u_2, u'_3 = \frac{u_3}{u_2})$, $v = u'_3$ when x' is the origin of the chart, $v = u'_3 + \alpha$, α invertible when x' is rational over x and not the origin, $v = P(u'_3) \mathrm{mod}(u_2)$ with $P \in k(x)[U_3]$ irreducible of degree $d \geq 2$ when x' is not rational over x.

The last assertion of (iv) is a consequence of 6.2. In H_{B3},

$$\Phi_p := U_1^{a(1)}U_2^{a(2)}(\lambda U_1^{\omega(x)}U_3 + \sum_{i=i_0>0}^{i_1} U_1^{\omega(x)-i}U_2^{c(i)}\phi_i(U_2,U_3)),$$

$\phi_i = 0$ or is not divisible by U_2 and is homogeneous of degree $i+1-c(i)$, $c(i) \geq iA_2$, $\phi i_0 \neq 0$, it may happen $i_0 = i_1$: then there is only one term in the sum. We have $\beta(x) \geq \frac{\deg(\phi_{i_1})}{i_1}$ and $\beta_3(x) \geq \frac{\deg(\phi_{i_1})-1}{i_1}$. As in cases 1,2, $H_{B3}(x)$ gives $H_2(x')$, up to an eventual translation on $X' := X/u_2$. Indeed

$$H_2(x') = X'^p + \gamma U_1'^{(p-1)d_1}U_2'^{(p-1)(\delta(x)-1}+$$
$$U_1'^{a(1)}U_2'^{a(1)+a(2)+\omega(x)+1-p}(\lambda U_1'^{\omega(x)}u_3' + \sum_{i=i_0>0}^{i_1} U_1'^{\omega(x)-i}U_2'^{i(B3(x)-1)}\phi_i(1,u_3')).$$

(i) (iv) are clear.

In the case $i_1(B3(x)-1) \neq 0 \mod (p)$ or $\omega(x) - i_1 \neq 0 \mod (p)$, a translation on X' cannot spoil $U_1'^{\omega(x)-i_1}U_2'^{i_1(B3(x)-1)}\phi_{i_1}(1,u_3')$ and we get

$$\beta(x') \leq \frac{\deg(\phi_{i_1})}{di_1} \leq \frac{\beta(x)}{d} \leq \frac{1+\lfloor\beta_3(x)\rfloor}{d}.$$

(ii) and (iii)-a-c are clear in this case. Note that $c(i_1)-1 \leq i_1\beta_3(x)$ and that $\beta_3(x') \leq \frac{1}{i_1}\mathrm{ord}_{x'}(\phi_i(1,u_3')) - 1$, this gives (iii)-b in this case.

In the case $i_1(B3(x)-1) = 0 \mod (p)$ and $\omega(x) - i_1 = 0 \mod (p)$, a translation on X' may add a p-power to $U_1'^{\omega(x)-i_1}U_2'^{i_1(B3(x)-1)}\phi_{i_1}(1,u_3')$, then we get

$$i_1\beta(x') \leq \frac{\deg(\phi_{i_1})}{d} + 1,\ \beta(x') \leq \frac{\deg(\phi_{i_1})}{di_1} + \frac{1}{i_1},\ \beta_3(x') \leq \frac{\deg(\phi_{i_1})}{di_1}.$$

This gives (ii) in the case $1 < \beta_3(x)$, the case $1 = \beta_3(x)$ in (ii) is the consequence of (iii)-a whose proof is following.

Proof of (iii). In the case $\beta_3(x) = 1$, we have $\deg(\phi_{i_1}) \leq 1 + i_1\beta_3 = 1 + i_1$, so $\beta(x') \leq \frac{1+i_1}{di_1} + \frac{1}{i_1}$ and $\beta_3(x') \leq \frac{1+i_1}{di_1}$. When $d \geq 2$, we have $\beta(x') \leq 1$ except in the case $i_1 = 1$, $\deg(\phi i_1) = 2$ and $d = 2$. This gives (iii)-a except in this very last case.

Case $i_1 = 1$, $\deg(\phi i_1) = 2$ and $d = 2$. Either $p \neq 2$, then

$$\mathrm{ord}_x((U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-i}U_2^{c(i)})^{-1}\frac{\partial}{\partial U_3}U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-1}U_2^{c(1)}\phi_1(U_2,U_3)) = 1,$$

at x' it has order 0, so there is a derivation $D \in \mathcal{D}(\{x'\})$ (notations of 2.3) such that

$$\mathrm{ord}_{x'}((U_1'^{a(1)}U_2'^{a(1)+a(2)+\omega(x)+1-p}U_1'^{\omega(x)-i}U_2'^{i(B3(x)-1)})^{-1} \\ D((U_1'^{a(1)}U_2'^{a(1)+a(2)+\omega(x)+1-p}U_1'^{\omega(x)-i}U_2'^{B3(x)-1})\phi_1(1,u_3')) = 0$$

so $\mathrm{ord}_{x'}(\phi_i(1,u_3')) + \psi^p) \leq 1$, where $\Psi \in S'$. It means $\beta(x') \leq 1$: this gives (iii)-a.

When $p = 2$, there is a derivation $D \in \mathcal{D}(\{x\})$ such that

$$\mathrm{ord}_x((U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-i}U_2^{c(1)})^{-1}D(U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-1}U_2^{c(1)}\phi_1(U_2,U_3))) \leq 2,$$

at x' it has order ≤ 1, so there is a derivation $D' \in \mathcal{D}(\{x'\})$ such that

$$\mathrm{ord}_{x'}((U_1'^{a(1)}U_2'^{a(1)+a(2)+\omega(x)+1-p}U_1'^{\omega(x)-i}U_2'^{i(B3(x)-1)})^{-1} \\ D((U_1'^{a(1)}U_2'^{a(1)+a(2)+\omega(x)+1-p}U_1'^{\omega(x)-i}U_2'^{i(B3(x)-1)})\phi_i(1,u_3')) \leq 1,$$

as $p = 2$, $\phi_1(1,u_3')) + \psi^p \neq \alpha v^2$, α invertible. So $\mathrm{ord}_{x'}(\phi_1(1,u_3')) + \psi^p) \leq 1$: $\beta(x') \leq 1$, this ends the proof of (iii)-a.

End of the proof of (iii) b-c when $i_1(B3(x)-1) = 0 \bmod (p)$ and $\omega(x) - i_1 = 0 \bmod (p)$. When $\beta_3 < 0$, the proof is clear. When $\deg(\phi_{i_1}) \leq i_1$, the proof is clear. When $\deg(\phi_{i_1}) > i_1$, let i_2 such that i_2 defines $\beta_3(x)$, i.e. $\Phi_{i_2} = u_2^{i_2A_2(x)}\lambda' u_3^{i_2\beta_3(x)+1}\mathrm{mod}(u_2^{i_2A_2(x)+1}, u_1)$, λ' invertible. We get $A_2(x) + \frac{\deg(\phi_{i_1})-1}{i_1} \leq \frac{c(i_1)}{i_1} + \frac{\deg(\phi_{i_1})-1}{i_1} = B3(x) \leq A_2(x) + \beta_3(x) = A_2(x) + 1$, then $\deg(\phi_{i_1}) - 1 \leq i_1$, $\deg(\phi_{i_1}) = 1 + i_1$. Either $1 + i_1 \neq 0\mathrm{mod}(p)$, then

$$\mathrm{ord}_x((U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-i_1}U_2^{c(i_1)})^{-1}\frac{\partial}{\partial U_3}U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-i_1}U_2^{c(i_1)}\phi_1(U_2,U_3)) = i_1,$$

$v = u_3$ or $u_3 + invertible$, by a similar argument as above, we get

$$ord_{x'}(\frac{\partial}{\partial V}\phi_i(1,u_3')) = i_1,$$

$\mathrm{ord}_{x'}(\phi_{i_1}(1,u_3')) + \psi^p) \leq 1 + i_1$: $\beta_3(x') \leq i_1$.

Or, $U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-i_1}U_2^{c(i_1)}$ is a p-power there is a derivation $D \in \mathcal{D}(\{x\})$, derivation "w.r.t. to constant" such that

$$\mathrm{ord}_x((U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-i_1}U_2^{c(i_1)})^{-1}D(U_1^{a(1)}U_2^{a(2)}U_1^{\omega(x)-i_1}U_2^{c(i_1)}\phi_{i_1}(U_2,U_3))) = 1+i_1,$$

we get

$$ord_{x'}(D\phi_{i_1}(1,u_3')) = i_1,$$

$\mathrm{ord}_{x'}(\phi_{i_1}(1,u_3')) + \psi^p) \leq 1 + i_1$: $\beta_3(x') \leq i_1$.

Proposition 6.8. *(cf. [21]**I.5.3 Lemma** page 1966) With hypotheses and notations of **I.2**, we suppose that x is in case 3, $\beta(x) \geq 1$ and we blow up x. Assume $x' = (X', u'_1, u'_2, u'_3) = (\frac{X}{u_3}, \frac{u_1}{u_3}, \frac{u_2}{u_3}, u_3)$ if x' is ω-near x and verifies condition $\mathcal{C}$, then x' is in case 2, $\Delta(h'; u'_1, u'_2, u'_3; X')$ is minimal. We have $(A_2(x'), \beta(x')) = (A_2(x), A_2(x) + \beta_3(x) - 1)$,*

$$A_3(x') = B3(x) - 1,\ C(x') \leq 1 + \beta_3(x')),\ \gamma(x') \leq \gamma(x),$$

with strict inequality when $3 \leq \gamma(x)$. Furthermore
(i) if $B3(x) - A_2(x) \geq 1$, then $\gamma(x') < \gamma(x)$,
(ii) if $\beta_3(x) = 1$, then $\gamma(x') < \gamma(x) = 2$ or $\gamma(x'') = 1$, where x'' is the center of the valuation μ in the blowing up of x',
(iii) if $B3(x) - A_2(x) < 1$ and $\beta_3(x) \neq 1$ and $\gamma(x) = 2$, then the following holds: either (x' is in case 2 and $\gamma(x') < \gamma(x)$) or (x'' is in case 1 and $\beta(x'') < 2$) or (x'' is in case 2 and $\gamma(x') < \gamma(x)$) or (x'' is in case 3 and $\beta_3(x'') \leq 1$), where x'' is the center of μ in the blowing up X'' of X' along x' and x" is ω-near x and verifies condition $\mathcal{C}$.

Proof.

As $\beta(x) \geq 1$, in 6.4, we have $\Phi_{p-1} = 0$.

We are at the origin of the second chart, there is no translation to do on X', etc. In the case (3*), the proof runs along the same lines of [21]. When we have not (3*), then

$$\Phi_p = \sum_{i=1}^{\omega(x)} U_1^{\omega(x)-i} F_i(U_2, U_3),$$

$F_i = 0$ or homogeneous of degree $i+1$ and, for at least one i, $F_i \notin k(x)[U_2]$. At x' we are in case 2 with $C(x') \leq 1$. We have $A_2(x) = A_2(x')$. $C(x') = 1$ iff $A_2(x) = 0$, in this case, $\beta(x') = \beta_3(x) - 1$.

As in [21], we denote by $(\alpha 3_2, \beta 3_2)$ the point of smallest abscissa of the side of slope -1 of $pr(\Delta)_3$ (end of Notations). And, the vertex of smallest abscissa of $pr(\Delta(x'))$ is $(A_2(x), \beta 3(x) + A_1(x) - 1)$, The vertex of smallest abscissa is $(A_2(x), \beta 3(x) + A_1(x) - 1)$. We have $A_3(x') = B3(x) - 1$. We get

$$C(x') \leq \beta 3(x) + A_2(x) - B3(x) = \beta 3(x) - (B3(x) - A_2(x)) \tag{6.8}$$
$$\leq \alpha 3_2 - A_2(x). \tag{6.9}$$

This gives the first assertions of the lemma, (i) and also (ii) when $B3(x) - A_2(x) > 0$. Furthermore, $C(x') < 1 + \lfloor \beta 3(x') \rfloor$ ($\Rightarrow \gamma(x') \leq \gamma(x)$) if $B3(x) - A_2(x) \geq 0$.

Note that $B3(x) = \alpha 3_2 + \beta 3_2$, and $\beta 3_2 \geq -1$: we have $\alpha 3_2 \leq 1 + B3(x)$,

$$C(x') \leq \alpha 3_2 - A_2(x) \leq B3(x) - A_2(x),$$

if $B3(x) - A_2(x) < 0$ or ($B3(x) - A_2(x) = 0$ and $\alpha 3_2 < 1 + B3(x)$), we get $C(x') < 1$. So we get $C(x') < 1 + \lfloor \beta 3(x') \rfloor$ and we get also (ii) when $B3(x) - A_2(x) < 0$ or ($B3(x) - A_2(x) = 0$ and $\alpha 3_2 < 1 + B3(x)$).

The reader reads the last lines of the proof of [21]**I.5.3 Lemma** page 1966.

Proposition 6.9. *We suppose that div(u_1) has weak contact maximal for a condition $\mathcal{C}$. We make the infinite sequence of blowing ups centered at $x_i \in \mathcal{X}_i$ 6.1*

Then for some i_0, one of the following is true:
(i) $\omega(x_{i_0}) < \omega(x)$,
(ii) $\omega(x_{i_0}) = \omega(x)$ and $\mathcal{C}$ is not true at x_{i_0},
(iii) $\omega(x_{i_0}) = \omega(x)$, $\mathcal{C}$ is true at x_{i_0} and $\gamma(x_i) = 1$ for $i \geq i_0$,
(iv) $\omega(x_{i_0}) = \omega(x)$, $\mathcal{C}$ is true at x_{i_0} and $\gamma(x_{i_0}) = \gamma(x_i) \geq 2$, $i \geq i_0$, then all the x_i are on the strict transform of a curve $\mathbf{C}$ included in the locus $\omega > 0$ of $\mathcal{X}$ and $\mathbf{C} \subset \mathrm{div}(u_1)$, $\mathbf{C}$ is contained in no other exceptional component.

Proof. As a consequence of the preceding lemmas and propositions, if we reach neither (i) nor (ii) nor (iii), for $i >> 0$, x_{i+n} is rational over x_i, and in the first chart for all $n \in \mathbb{N}$, this means (iv).

Proposition 6.10. *With the hypotheses given above, suppose $\gamma(x) = 1$ and that there is no curve $\mathbf{C}$ included in the locus $\omega > 0$ of $\mathcal{X}$ and $\mathbf{C} \subset \mathrm{div}(u_1)$, $\mathbf{C}$ is contained in no other exceptional component. There exists a composition of local ω-permissible blowing ups:*

$$(\mathcal{X} =: \mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots \ , \tag{6.10}$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that for some i_0, one of the following is true:
(i) $\omega(x_{i_0}) < \omega(x)$,
(ii) $\omega(x_{i_0}) = \omega(x)$ and $\mathcal{C}$ is not true at x_{i_0}.

Proof.

Suppose

$$\beta \leq 1 \text{ in case 1-2 (resp. } \beta_3 < 1 \text{ in case 3)}, \ A_2 < 1. \tag{6.11}$$

Then we apply 6.1, at each step, A_2 does not increase and strictly drops when $\beta(x) < 1$ (resp. $\beta_3(x) < 1$). If the point x' is at infinity, $\beta(x') = \beta(x) + A_2(x) - 1 < 1$ (resp. $\beta(x') = \beta_3(x) + A_2(x) - 1 < 1$). If the point x' is not rational, then $\beta(x') < 1$ or $\beta_3(x') < 1$.

So, in the sequence 6.1, $A_2(x_{i+1}) \leq A_2(x_i)$, $\beta(x_{i+1}) \leq 1$ (resp. $\beta(x_{i+1}) < 1$) or one of the following is true

(i) $\omega(x_{i+1}) < \omega(x)$,

(ii) $\omega(x_{i+1}) = \omega(x)$ and $\mathcal{C}$ is not true at x_{i_0}.

As $\beta + A_2 \geq 1$ (resp. $\beta_3 + A_2 \geq 1$), if the sequence 6.1 is infinite and if we never get (i)(ii) above, then, for some i_0, $\beta(x_i) = 1$, $\omega(x_i) = \epsilon(x_i)$ for $i \geq i_0$, this means that the x_i are all on the strict transform of a curve going through x_{i_0}, by hypothesis, this curve is contained in two components of E, in fact, this curve is $\mathbf{V}(X, u_1, u_2)$, for i large enough, this curve is permissible of first kind, we blow it up and the reader sees that there is no point ω-near x_i on the strict transform of $\mathrm{div}(u_1)$.

Suppose

$$\beta \leq 1 \text{ in case 1-2 (resp. } \beta_3 < 1 \text{ in case 3)}.. \tag{6.12}$$

We skip the hypothesis $A_2 < 1$. In case 1 (resp. 3), When $A_2 \geq 1$, $Y := \mathbf{V}(X, u_1, u_2)$ is permissible of first kind (resp. second kind): indeed $\delta(x) = a(1)+a(2)+\epsilon(x) \geq p+1$, so $a(1)+a(2)+\omega(x) \geq p$, $\mathbf{V}(X, u_1, u_2)$ is permissible in Hironaka's sense, the inequality $A_2 \geq 1$ implies that $\epsilon(Y) = \omega(x)$. We blow up along (X, u_1, u_2), a small computation shows that the couple (β, A_2) (resp. (β_3, A_2)) becomes $(\beta, A_2 - 1)$ (resp. $(\beta_3, A_2 - 1)$),

When x is in case 2. Then we make the sequence (6.1), either there is some i_0 such that E_{i_0} has only two components, then we get (6.11) at i_0, or for all i, E_i has three components, then by classical computations, we get $C(x_{i_0} = 0$ for some i_0, i.e. $pr(\Delta)$ has only one vertex. When $A_2 \geq 1$ and $a(2) + \omega(x) \geq p$, $\mathbf{V}(X, u_1, u_2)$ is ω-permissible of first kind. We blow it up, then $A_2(x_{i_0+1}) + A_3(x_{i_0+1}) = A_2(x_{i_0}) + A_3(x_{i_0}) - 1$ and $pr(\Delta(x_{i_0+1}))$ has only one vertex. Mutatis mutandis, if $A_3 \geq 1$ and $a(3) + \omega(x) \geq p$. Either by such blowing ups, we reach 6.11, or we reach one of the following:

(a) $a(1)+a(i)+\omega(x) < p$, $i = 2, 3$: then we apply 6.1, at each step $a(2)+a(3)$ strictly drops or we get 6.12 in cases 1-3. The reader sees [21] page 1967 **1.7.1**.

(b) there exists $i = 2, 3$ with $a(1) + a(i) + \omega(x) < p$ and $A_i \geq 1$ and $j = 2, 3$ with $a(1) + a(j) + \omega(x) < p$ and $A_j \geq 1$. We make a descending induction on $(\sup\{A_a, a = 2, 3\}, sup\{a(b), b = 2, 3\})$ for $\leq_{lex}$, see [21] page 1968 **1.7.4**.

We always reach (6.11).

Theorem 6.11. *We suppose that div(u_1) has contact maximal for a condition $\mathcal{C}$. There exists a composition of local ω-permissible blowing ups:*

$$(\mathcal{X} =: \mathcal{X}_0, x_0) \leftarrow (\mathcal{X}_1, x_1) \leftarrow \cdots \leftarrow (\mathcal{X}_r, x_r) \leftarrow \cdots \ , \tag{6.13}$$

where $x_i \in \mathcal{X}_i$ is the center of μ, such that for some i_0, one of the following is true:
(i) $\omega(x_{i_0}) < \omega(x)$,
(ii) $\omega(x_{i_0}) = \omega(x)$ and $\mathcal{C}$ is not true at x_{i_0}.

Proof.

After a sequence of blowing ups centered at closed points, we may suppose that

$$\{\omega > 0\} \cap \mathrm{div}(u_1) \subset \cup_{i \neq 1, \mathrm{div}(u_i) \subset E} \mathrm{div}(u_i) \tag{6.14}$$

$\{\omega > 0\} \cap \mathrm{div}(u_1) \setminus \cup_{i \neq 1, \mathrm{div}(u_i) \subset E} \mathrm{div}(u_i)$ is of dimension 0 or is a permissible curve of first kind. In the second case, we blow up along this curve and we reach the first case 6.14.

Then we make the sequence (6.1), and by the condition (6.14), for some i_0, we get proposition 6.9 (i),(ii) or (iii). In the case (iii), we apply proposition 6.10.

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