The Filtration Induced by a Locally Nilpotent Derivation

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Abstract. We investigate the filtration corresponding to the degree function induced by a non-zero locally nilpotent derivations ∂ and its associated graded algebra. As an application we provide an efficient method to recover the Makar-Limanov invariants, isomorphism classes and automorphism groups of classically known algebras. We also present a new class of examples which can be fully described with this method.

Introduction

Let k be a field of characteristic zero, and let B be a commutative k-domain. A k-derivation $\partial \in \mathrm{Der}_k(B)$ is said to be *locally nilpotent* if for every $a \in B$, there is an integer $n \geq 0$ such that $\partial^n(a) = 0$. An important invariant of k-domains admitting non-trivial locally nilpotent derivations is the so called *Makar-Limanov invariant* which was defined by Makar-Limanov as the intersection $\mathrm{ML}(B) \subset B$ of kernels of all locally nilpotent derivations of B ([2]). This invariant was initially introduced as a tool to distinguish certain k-domains from polynomial rings but it has many other applications for the study of k-algebras and their automorphism groups ([3]). One of the main difficulties in applications is to compute this invariant without a prior knowledge of all locally nilpotent derivations of a given k-domain.

In [4] S. Kaliman and L. Makar-Limanov developed general techniques to determine the ML-invariant for a class of finitely generated k-domains B. The idea is to reduce the problem to the study of homogeneous locally nilpotent derivations on graded algebras $\mathrm{Gr}(B)$ associated to B. For this, one considers appropriate filtrations $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{R}}$ on B generated by so called real-valued weight degree functions in such a way that every non-zero locally nilpotent derivation on B induces a non-zero homogeneous locally nilpotent derivation on the associated graded algebra $\mathrm{Gr}_{\mathcal{F}}(B)$.

In particular, every k-domain B admitting a non-zero locally nilpotent derivation ∂ comes equipped with a natural filtration by the k-sub-vector-spaces $\mathcal{F}_i = ker(\partial^{i+1})$, $i \geq 0$, that we call the *∂-filtration*.

In this article we show that this filtration is convenient for the computation of the ML-invariant, and we give general methods to describe the sub-spaces $ker(\partial^{i+1})$ and their associated graded algebra.

Knowing this filtration gives a very precise understanding of the structure of *semi-rigid* k-domains, that is,k-domains for which every locally nilpotent derivation gives rise to the same filtration. For such rings the study of the ∂-filtration is a very efficient tool to determine isomorphism types and automorphism groups. We illustrate how the computation of ML-invariant of classically known semi-rigid k-domains can be simplified using these types of filtration. We also present a new class of semi-rigid k-domains which can be studied with this method.

1. **Preliminaries**

In this section we briefly recall basic facts on filtered algebra and their relation with derivation in a form appropriate to our needs.

In the sequel, unless otherwise specified B will denote a commutative domain over a field k of characteristic zero. The set $\mathbb{Z}_{\geqslant 0}$ of non-negative integers will be denoted by $\mathbb{N}$.

1.1. **Filtration and associated graded algebra.**

Definition 1.1. An $\mathbb{N}$-*filtration* of B is a collection $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ of k-sub-vector-spaces of B with the following properties:
1- $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ for all $i \in \mathbb{N}$.
2- $B = \cup_{i\in\mathbb{N}}\mathcal{F}_i$.
3- $\mathcal{F}_i.\mathcal{F}_j \subset \mathcal{F}_{i+j}$ for all $i, j \in \mathbb{N}$.
The filtration is called *proper* if the following additional property holds:
4- If $a \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$ and $b \in \mathcal{F}_j \setminus \mathcal{F}_{j-1}$, then $ab \in \mathcal{F}_{i+j} \setminus \mathcal{F}_{i+j-1}$.

There is a one-to-one correspondence between proper $\mathbb{N}$-filtrations and so called $\mathbb{N}$-degree functions:

Definition 1.2. An $\mathbb{N}$-*degree function* on B is a map $\deg : B \longrightarrow \mathbb{N}\cup\{-\infty\}$ such that, for all $a, b \in B$, the following conditions are satisfied:
(1) $\deg(a) = -\infty \Leftrightarrow a = 0$.
(2) $\deg(ab) = \deg(a) + \deg(b)$.
(3) $\deg(a + b) \leq max\{\deg(a), \deg(b)\}$.
If the equality in (2) replaced by the inequality $\deg(ab) \leq \deg(a) + \deg(b)$, we say that deg is an $\mathbb{N}$-*semi-degree function.*

Indeed, for an $\mathbb{N}$-degree function on B, the sub-sets $\mathcal{F}_i = \{b \in B \mid \deg(b) \leq i\}$ are k-subvector spaces of B that give rise to a proper $\mathbb{N}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$. Conversely, every proper $\mathbb{N}$-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$, yields an $\mathbb{N}$-degree function $\omega : B \longrightarrow \mathbb{N}\cup\{-\infty\}$ defined by $\omega(0) = -\infty$ and $\omega(b) = i$ if $b \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$.

Definition 1.3. Given a k-domain $B = \cup_{i\in\mathbb{N}}\mathcal{F}_i$ equipped with a proper $\mathbb{N}$-filtration, the associated graded algebra $\mathrm{Gr}(B)$ is the k-vector space

$$\mathrm{Gr}(B) = \oplus_{i\in\mathbb{N}}\mathcal{F}_i/\mathcal{F}_{i-1}$$

equipped with the unique multiplicative structure for which the product of the elements $a + \mathcal{F}_{i-1} \in \mathcal{F}_i/\mathcal{F}_{i-1}$ and $b + \mathcal{F}_{j-1} \in \mathcal{F}_j/\mathcal{F}_{j-1}$, where $a \in \mathcal{F}_i$ and $b \in \mathcal{F}_j$, is the element

$$(a + \mathcal{F}_{i-1})(b + \mathcal{F}_{j-1}) := ab + \mathcal{F}_{i+j-1} \in \mathcal{F}_{i+j}/\mathcal{F}_{i+j-1}.$$

Property 4 (proper) in Definition 1.1 ensures that $\mathrm{Gr}(B)$ is a commutative k-domain when B is an integral domain. Since for each $a \in B$ the set $\{n \in \mathbb{N} \mid a \in \mathcal{F}_n\}$ has a minimum, there exists i such that $a \in \mathcal{F}_i$ and $a \notin \mathcal{F}_{i-1}$. So we can define a k-linear map $\mathrm{gr} : B \longrightarrow \mathrm{Gr}(B)$ by sending a to its class in $\mathcal{F}_i/\mathcal{F}_{i-1}$, i.e $a \mapsto a + \mathcal{F}_{i-1}$, and $\mathrm{gr}(0) = 0$. We will frequently denote $\mathrm{gr}(a)$ simply by $\overline{a}$. Observe that $\mathrm{gr}(a) = 0$ if and only if $a = 0$.

Denote by deg the $\mathbb{N}$-degree function $\deg : B \longrightarrow \mathbb{N} \cup \{-\infty\}$ corresponding to the proper $\mathbb{N}$-filtration $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$. We have the following properties.

Lemma 1.4. *Given $a, b \in B$ the following holds:*
P1) $\overline{a\,b} = \overline{a}\,\overline{b}$, *i.e.* gr *is a multiplicative map.*
P2) *If* $\deg(a) > \deg(b)$, *then* $\overline{a+b} = \overline{a}$.
P3) *If* $\deg(a) = \deg(b) = \deg(a+b)$, *then* $\overline{a+b} = \overline{a} + \overline{b}$.
P4) *If* $\deg(a) = \deg(b) > \deg(a+b)$, *then* $\overline{a} + \overline{b} = 0$, *in particular* gr *is not an additive map in general.*

Proof. Let us assume that $\deg(a) = i$ and $\deg(b) = j$. By definition, $\deg(ab) = i+j$ means that $ab \in \mathcal{F}_{i+j}$ and $ab \notin \mathcal{F}_{i+j-1}$, so $\overline{ab} = ab + \mathcal{F}_{i+j-1} := (a + \mathcal{F}_{i-1})(b + \mathcal{F}_{j-1}) = \overline{a}\,\overline{b}$. Which gives P1. For P2 we observe that since $\deg(a+b) = \deg(a)$, we have $\overline{a+b} = (a+b) + \mathcal{F}_{i-1} = (a + \mathcal{F}_{i-1}) + (b + \mathcal{F}_{i-1})$, and since $\mathcal{F}_{j-1} \subset \mathcal{F}_j \subseteq \mathcal{F}_{i-1}$ as $i > j$, we get $b + \mathcal{F}_{i-1} = 0$. P3) is immediate, by definition. Finally, assume by contradiction that $\overline{a} + \overline{b} \neq 0$, then $\overline{a} + \overline{b} = (a + \mathcal{F}_{i-1}) + (b + \mathcal{F}_{i-1}) = ((a+b) + \mathcal{F}_{i-1}) \neq 0$, which means that $a + b \notin \mathcal{F}_{i-1}$ and $\deg(a+b) = i$, which is absurd. So P4 follows. □

1.2. **Derivations.**
By a k-*derivation* of B, we mean a k-linear map $D : B \longrightarrow B$ which satisfies the Leibniz rule: For all $a, b \in B$; $D(ab) = aD(b) + bD(a)$. The set of all k-derivations of B is denoted by $\mathrm{Der}_k(B)$.
The *kernel* of a derivation D is the subalgebra $\ker D = \{b \in B; D(b) = 0\}$ of B.
The *plinth ideal* of D is the ideal $\mathrm{pl}(D) = \ker D \cap D(B)$ of $\ker D$, where $D(B)$ denotes the image of B.
An element $s \in B$ such that $D(s) \in \ker(D) \setminus \{0\}$ is called a *local slice* for D.

Definition 1.5. Given a k-algebra $B = \cup_{i \in \mathbb{N}} \mathcal{F}_i$ equipped with a proper $\mathbb{N}$-filtration, a k-derivation D of B is said to respect the filtration if there exists an integer d such that $D(\mathcal{F}_i) \subset \mathcal{F}_{i+d}$ for all $i \in \mathbb{N}$.
If so, we define a k-linear map $\overline{D} : \mathrm{Gr}(B) \longrightarrow \mathrm{Gr}(B)$ as follows: If $D = 0$, then $\overline{D} = 0$ the zero map. Otherwise, if $D \neq 0$ then we let d be the least integer such that $D(\mathcal{F}_i) \subset \mathcal{F}_{i+d}$ for all $i \in \mathbb{N}$ and we define

$$\overline{D} : \mathcal{F}_i / \mathcal{F}_{i-1} \longrightarrow \mathcal{F}_{i+d} / \mathcal{F}_{i+d-1}$$

by the rule $\overline{D}(a + \mathcal{F}_{i-1}) = D(a) + \mathcal{F}_{i+d-1}$. One checks that $\overline{D}$ satisfies the Leibniz rule, therefore it is a k-derivation of the graded algebra $\mathrm{Gr}(B)$. Moreover it is homogeneous of degree d, i.e $\overline{D}$ sends homogeneous elements of degree i to zero or to homogeneous elements of degree $i + d$.

Observe that $\overline{D} = 0$ if and only if $D = 0$. In addition, $\mathrm{gr}(\ker D) \subset \ker \overline{D}$.

2. LND-Filtrations and Associated Graded Algebras

In this section we introduce the ∂-filtration associated with a locally nilpotent derivation ∂. We explain how to compute this filtration and its associated graded algebra in certain situation.

Definition 2.1. A k-derivation $\partial \in \mathrm{Der}_k(B)$ is said to be *locally nilpotent* if for every $a \in B$, there exists $n \in \mathbb{N}$ (depending of a) such that $\partial^n(a) = 0$. The set of all locally nilpotent derivations of B is denoted by $\mathrm{LND}(B)$.

In particular, every locally nilpotent derivation ∂ of B gives rise to a proper $\mathbb{N}$-filtration of B by the sub-spaces $\mathcal{F}_i = \ker\partial^i$, $i \in \mathbb{N}$, that we call the *∂-filtration.* It is straightforward to check (see Prop. 1.9 in [1]) that the ∂-filtration corresponds to the $\mathbb{N}$-degree function $\deg_\partial : B \longrightarrow \mathbb{N} \cup \{-\infty\}$ defined by

$$\deg_\partial(a) := min\{i \in \mathbb{N} \mid \partial^{i+1}(a) = 0\}, \text{ and } \deg_\partial(0) := -\infty.$$

Note that by definition $\mathcal{F}_0 = \ker\partial$ and that $\mathcal{F}_1 \setminus \mathcal{F}_0$ consists of all local slices for ∂. Let $\mathrm{Gr}_\partial(B) = \oplus_{i\in\mathbb{N}}\mathcal{F}_i/\mathcal{F}_{i-1}$ denote the associated graded algebra relative to the ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$. Let $\mathrm{gr}_\partial : B \longrightarrow \mathrm{Gr}_\partial(B)$; $a \overset{\mathrm{gr}_\partial}{\longmapsto} \overline{a}$ be the natural map between B and $\mathrm{Gr}_\partial(B)$ defined in 1.3, where $\overline{a}$ denote $\mathrm{gr}_\partial(a)$.

The next Proposition, which is due to Daigle (Theorem 2.11 in [1]), implies in particular that if B is of finite transcendence degree over k, then every non-zero $D \in \mathrm{LND}(B)$ respects the ∂-filtration and therefore induces a non-zero homogeneous locally nilpotent derivation $\overline{D}$ of $\mathrm{Gr}_\partial(B)$.

Proposition 2.2. (**Daigle**) *Suppose that B is a commutative domain, of finite transcendence degree over k. Then for every pair $D \in \mathrm{Der}_k(B)$ and $\partial \in \mathrm{LND}(B)$, D respects the ∂-filtration. Consequently, $\overline{D}$ is a well defined homogeneous derivation of the integral domain $\mathrm{Gr}_\partial(B)$ relative to this filtration, and it is locally nilpotent if D is locally nilpotent.*

2.1. **Computing the ∂-filtration.**
Here, given a finitely generated k-domain B, we describe a general method which enables the computation of the ∂-filtration for a locally nilpotent derivation ∂ with finitely generated kernel. First we consider a more general situation where the plinth ideal $\mathrm{pl}(\partial)$ is finitely generated as an ideal in $\ker\partial$ then we deal with the case where $\ker\partial$ is itself finitely generated as a k-algebra.

Let $B = k[X_1, \dots, X_n]/I = k[x_1, \dots, x_n]$ be a finitely generated k-domain, and let $\partial \in \mathrm{LND}(B)$ be such that $\mathrm{pl}(\partial)$ is generated by precisely m elements $f_1, \dots, f_m$ as an ideal in $\ker\partial$. Denote by $\mathcal{F} = \{\mathcal{F}_i\}_{i\in\mathbb{N}}$ the ∂-filtration, then:

By definition $\mathcal{F}_0 = \ker\partial$. Furthermore, given elements $s_i \in \mathcal{F}_1$ such that $\partial(s_i) = f_i$ for every $i \in \{1, \dots, m\}$, it is straightforward to check that

$$\mathcal{F}_1 \quad = \quad \mathcal{F}_0 s_1 + \ldots + \mathcal{F}_0 s_m + \mathcal{F}_0.$$

Letting $\deg_\partial(x_i) = d_i$, we denote by H_j the $\mathcal{F}_0$-sub-module in B generated by elements of degree j relative to $\deg_\partial$ of the form $s_1^{u_1} \dots s_m^{u_m} x_1^{v_1} \dots x_n^{v_n}$, i.e.,

$$H_j := \sum_{\sum_j u_j + \sum_i d_i . v_i = j} \mathcal{F}_0\,(s_1^{u_1} \dots s_m^{u_m} x_1^{v_1} \dots x_n^{v_n})$$

where $u_j, v_i \in \mathbb{N}$ for all i and j. The integer $\sum_j u_j + \sum_i d_i v_i$ is nothing but $\deg_\partial(s_1^{u_1}.s_2^{u_2}\dots.s_m^{u_m}.x_1^{v_1}.x_2^{v_2}.\dots.x_n^{v_n})$. Then we define a new $\mathbb{N}$-filration $\mathcal{G} = \{\mathcal{G}_i\}_{i\in\mathbb{N}}$ of B by setting

$$\mathcal{G}_i = \sum_{j\leq i} H_j.$$

By construction $\mathcal{G}_i \subseteq \mathcal{F}_i$ for all $i \in \mathbb{N}$, with equality for $i = 0, 1$. The following result provides a characterization of when these two filtrations coincide:

Lemma 2.3. *The filtrations $\mathcal{F}$ and $\mathcal{G}$ are equal if and only if $\mathcal{G}$ is proper.*

Proof. One direction is clear since $\mathcal{F}$ is proper. Conversely, suppose that $\mathcal{G}$ is proper with the corresponding $\mathbb{N}$-degree function ω on B (see §1.1). Given $f \in \mathcal{F}_i \setminus \mathcal{F}_{i-1}$, $i > 1$, for every local slice $s \in \mathcal{F}_1 \setminus \mathcal{F}_0$, there exist $f_0 \neq 0, a_i \neq 0, a_{i-1}, \ldots, a_0 \in \mathcal{F}_0$ such that $f_0 f = a_i s^i + a_{i-1}s^{i-1} + \cdots + a_0$ (see the proof of Lemma 4 in [7]). Since $\omega(g) = 0$ (resp. $\omega(g) = 1$) for every $g \in \mathcal{F}_0$ (resp. $g \in \mathcal{F}_1 \setminus \mathcal{F}_0$), we obtain

$$\omega(f) = \omega(f_0 f) = \omega(a_i s^i + a_{i-1}s^{i-1} + \cdots + a_0) = \max\{\omega(a_i s^i)\} = i,$$

and so $f \in \mathcal{G}_i$. □

Next, we determine the ∂-filtration, for a locally nilpotent derivation ∂ with finitely generated kernel, by giving an effective criterion to decide when the $\mathbb{N}$-filtration $\mathcal{G}$ defined above is proper.

Hereafter, we assume that $0 \in \mathrm{Spec}(B)$ and that $\ker(\partial)$ is generated by elements $z_i(x_1, \ldots, x_n) \in B$ such that $z_i(0, \ldots, 0) = 0$, $i \in \{1, \ldots, r\}$, which is always possible since $k \subset \ker \partial$. Since $\ker(\partial)$ is finitely generated k-algebra, the plinth ideal $\mathrm{pl}(\partial)$ is finitely generated. So there exist $s_1(x_1, \ldots, x_n)$, $\ldots$, $s_m(x_1, \ldots, x_n) \in \mathcal{F}_1$ such that $\mathcal{F}_1 = \mathcal{F}_0 s_1 + \ldots + \mathcal{F}_0 s_m + \mathcal{F}_0$. We can also assume that $s_i(0, \ldots, 0) = 0$.

Letting $J \subset k^{[r+n+m]} = k[Z_1, \ldots, Z_r][X_1, \ldots, X_n][S_1, \ldots, S_m]$ be the ideal generated by I and the elements $Z_1 = z_1(X_1, \ldots, X_n), \ldots, Z_r = z_r(x_1, \ldots, x_n)$, $S_1 = s_1(X_1, \ldots, X_n), \ldots, S_m = s_m(x_1, \ldots, x_n)$, then we have

$$B = k[Z_1, \ldots, Z_r][X_1, \ldots, X_n][S_1, \ldots, S_m]/J.$$

Note that by construction $\underbrace{(0, \ldots, 0)}_{r+n+m \text{ times}} \in \mathrm{Spec}(k^{[r+n+m]}/J)$.

We define an $\mathbb{N}$-degree function ω on $k^{[r+n+m]}$ by declaring that $\omega(Z_i) = 0 = \deg_\partial(z_i)$ for all $i \in \{1, \ldots, r\}$, $\omega(S_i) = 1 = \deg_\partial(s_i)$ for all $i \in \{1, \ldots, m\}$, and $\omega(X_i) = \deg_\partial(x_i) = d_i$ for all $i \in \{1, \ldots, n\}$. The corresponding proper $\mathbb{N}$-filtration $\mathcal{Q}_i := \{P \in k^{[n]} \mid \omega(P) \leq i\}$, $i \in \mathbb{N}$, on $k^{[r+n+m]}$ has the form $\mathcal{Q}_i = \oplus_{j \leq i}\mathcal{H}_j$ where

$$\mathcal{H}_j := \oplus_{\sum_j u_j + \sum_i d_i v_i = j} k[Z_1, \ldots, Z_r] S_1^{u_1} \ldots S_m^{u_m} X_1^{v_1} \ldots X_n^{v_n}.$$

By construction $\pi\left(\mathcal{Q}_i\right) = \mathcal{G}_i$ where $\pi : k^{[r+n+m]} \longrightarrow B$ denotes the natural projection. Indeed, since

$$\pi\left(\mathcal{Q}_i\right) = \sum_{j \leq i} \pi\left(\mathcal{H}_j\right)$$

and $\pi\left(\mathcal{H}_j\right) = \sum_{\sum_j u_j + \sum_i d_i v_i = j} k[z_1, \ldots, z_r] s_1^{u_1} \ldots . s_m^{u_m} x_1^{v_1} \ldots x_n^{v_n}$, we get

$$\pi\left(\mathcal{H}_j\right) = \sum_{\sum_j u_j + \sum_i d_i v_i = j} (\ker \partial)\, s_1^{u_1} \ldots . s_m^{u_m} x_1^{v_1} \ldots . x_n^{v_n}$$

which means precisely that $\pi\left(\mathcal{Q}_i\right) = \mathcal{G}_i$.

Let $\hat{J} \subset k^{[r+n+m]}$ be the homogeneous ideal generated by the highest homogeneous components relative to ω of all elements in J. Then we have the following result, which is inspired by the technique developed by S. Kaliman and L. Makar-Limanov:

Proposition 2.4. *The $\mathbb{N}$-filration $\mathcal{G}$ is proper if and only if $\hat{J}$ is prime.*

Proof. It is enough to show that $\mathcal{G} = \{\pi\left(\mathcal{Q}_i\right)\}_{i \in \mathbb{N}}$ coincides with the filtration corresponding to the $\mathbb{N}$-semi-degree function ω_B on B defined by $\omega_B(p) := \min_{P \in \pi^{-1}(p)}\{\omega(P)\}$. Indeed, if so, the result will follow from Lemma 3.2 in [4] which asserts in particular that ω_B is an $\mathbb{N}$-degree function on B if and only if $\hat{J}$ is prime. Let $\{\mathcal{G}_i^{'}\}_{i\in\mathbb{N}}$ be the filtration corresponding to ω_B. Given $f \in \mathcal{G}_i^{'}$ there exists $F \in \mathcal{Q}_i$ such that $\pi(F) = f$, which means that $\mathcal{G}_i^{'} \subset \pi(\mathcal{Q}_i)$. Conversely, it is clear that $\omega_B(z_i) = \omega(Z_i) = 0$ for all $i \in \{1, \ldots, r\}$. Furthermore $\omega_B(s_i) = \omega(S_i) = 1$ for all $i \in \{1, \ldots, m\}$, for otherwise $s_i \in \ker \partial$ which is absurd. Finally, if $d_i \neq 0$ and $\omega_B(x_i) < \omega(X_i) = d_i \neq 0$, then $x_i \in \pi(\mathcal{Q}_{d_i-1}) \subset \ker \partial^{d_i-1}$ which implies that $\deg_\partial(x_i) < d_i$, a contradiction. So $\omega_B(x_i) = d_i$. Thus $\omega_B(f) \leq i$ for every $f \in \pi(\mathcal{Q}_i)$ which means that $\pi(\mathcal{Q}_i) \subset \mathcal{G}_i^{'}$. □

The next Proposition, which is a reinterpretation of Prop. 4.1 in [4], describes in particular the associated graded algebra $Gr_\partial(B)$ of the filtered algebra $(B, \mathcal{F})$ in the case where the $\mathbb{N}$-filtration $\mathcal{G}$ is proper:

Proposition 2.5. *If the* $\mathbb{N}$*-filtration* $\mathcal{G}$ *is proper then* $Gr_\partial(B) \simeq k^{[r+n+m]}/\hat{J}$.

Proof. By virtue of (Prop. 4.1 in [4]) the graded algebra associated to the filtered algebra $(B, \mathcal{G})$ is isomorphic to $k^{[r+n+m]}/\hat{J}$. So the assertion follows from Lemma 2.3. □

3. Semi-Rigid and Rigid k-Domains

In [6] D. Finston and S. Maubach considered rings B whose sets of locally nilpotent derivations are "one-dimensional" in the sense that $\mathrm{LND}(B) = \ker(\partial).\partial$ for some non-zero $\partial \in \mathrm{LND}(B)$. They called them almost-rigid rings. Hereafter, we consider the following definition which seems more natural in our context (see Prop. 3.2 below for a comparison between the two notions).

Definition 3.1. A commutative domain B over a field k of characteristic zero is called *semi-rigid* if all non-zero locally nilpotent derivations of B induce the same proper $\mathbb{N}$-filtration (equivalently, the same $\mathbb{N}$-degree function).

Recall that the *Makar-Limanov invariant* of a commutative k-domain B over a field k of characteristic zero is defined by

$$\mathrm{ML}(B) := \cap_{D\in \mathrm{LND}(B)} \ker(D).$$

In particular, B is semi-rigid if and only if $\mathrm{ML}(B) = \ker(\partial)$ for any non-zero $\partial \in \mathrm{LND}(B)$. Indeed, given $D, E \in \mathrm{LND}(B)\setminus\{0\}$ such that $A := \ker(D) = \ker(E)$, there exist non-zero elements $a, b \in A$ such that $aD = bE$ (see [1] Principle 12) which implies that the D-filtration is equal to the E-filtration. So if $\mathrm{ML}(B) = \ker(\partial)$ for any non-zero $\partial \in \mathrm{LND}(B)$ then B is semi-rigid. The other implication is clear by definition.

Recall that $D \in \mathrm{Der}_k(B)$ is *irreducible* if and only if $D(B)$ is contained in no proper principal ideal of B, and that B is said to satisfy the ascending chain condition (ACC) on principal ideals if and only if every infinite chain $(b_1) \subset (b_2) \subset (b_3) \subset \cdots$ of principal ideals of B eventually stabilizes. B is said to be a *highest common factor ring*, or HCF-ring, if and only if the intersection of any two principal ideals of B is again principal.

Proposition 3.2. *Let* B *be a semi-rigid* k*-domain satisfying the ACC on principal ideals. If* $\mathrm{ML}(B)$ *is an HCF-ring, then there exists a unique irreducible* $\partial \in \mathrm{LND}(B)$ *up to multiplication by unit.*

Proof. Existence: since B is satisfies the ACC on principal ideals, then for every non-zero $T \in \mathrm{LND}(B)$, there exists an irreducible $T_0 \in \mathrm{LND}(B)$ and $c \in \ker(T)$ such that $T = cT_0$. (see [1], Prop. 2.2 and Principle 7).

Uniqueness: the following argument is similar to that in [1] Prop. 2.2.b, but with a little difference, that is, in [1] it is assumed that B itself is an HCF-ring while here we only require that $\mathrm{ML}(B)$ is an HCF-ring.

Let $D, E \in \mathrm{LND}(B)$ be irreducible derivations, and denote $A = \mathrm{ML}(B)$. By hypothesis $\ker(D) = \ker(E) = A$, so there exist non-zero $a, b \in A$ such that $aD = bE$ (see [1] Principle 12). Here we can assume that a, b are not units otherwise we are done. Set $T = aD = bE$. Since A is an HCF-ring, there exists $c \in A$ such that $aA \cap bA = cA$. Therefore, $T(B) \subset cB$, and there exists $T_0 \in \mathrm{LND}(B)$ such that $T = cT_0$. Write $c = as = bt$ for $s, t \in B$. Then $cT_0 = asT_0 = aD$ implies $D = sT_0$, and likewise $E = tT_0$. By irreducibility, s and t are units of B, and we are done. □

In particular, for a ring B as in 3.2, every $D \in \mathrm{LND}(B)$ has the form $D = f\partial$ for some irreducible $\partial \in \mathrm{LND}(B)$ and $f \in \ker(\partial)$, and so B is almost rigid in the sense introduced by Finston and Maubach.

Recall that a ring A is called *rigid* if the zero derivation is the only locally nilpotent derivation of A. Equivalently, A is rigid if and only if $\mathrm{ML}(A) = A$. It is well known that the only non rigid k-domains of transcendence degree one are polynomial rings in one variable over algebraic extensions of k ([1] Corollary 1.24 and Corollary 1.29). In particular, we have the following elementary criterion for rigidity that we will use frequently in the sequel:

Lemma 3.3. *A domain B of transcendence degree one over a field k of characteristic zero is rigid if one of the following properties hold:*
(1) *B is not factorial.*
(2) $\mathrm{Spec}(B)$ *has a singular point.*

3.1. **Elementary examples of semi-rigid k-domains.**
The next Proposition, which is due to Makar-Limanov ([7] Lemma 21, also [8] Theorem 3.1), presents some of the simplest examples of semi-rigid k-domains.

Proposition 3.4. (**Makar-Limanov**) *Let A be a rigid domain of finite transcendence degree over a field k of characteristic zero. Then the polynomial ring $A[x]$ is semi-rigid.*

Proof. For the convenience of the reader, we provide an argument formulated in the LND-filtration language. Let ∂ be the locally nilpotent derivation of $A[x]$ defined by $\partial(a) = 0$ for every $a \in A$ and $\partial(x) = 1$. Then the ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is given by $\mathcal{F}_i = Ax^i \oplus \mathcal{F}_{i-1}$ where $\mathcal{F}_0 = \ker(\partial) = A$. So the associated graded algebra is $Gr(A[x]) = \oplus_{i\in\mathbb{N}} A\overline{x}^i$, where $\overline{x} := gr(x)$. By Proposition 2.2, every non-zero $D \in \mathrm{LND}(A[x])$ respects the ∂-filtration and induces a non-zero homogeneous locally nilpotent derivation $\overline{D}$ of $\mathrm{Gr}(A[x])$ of a certain degree d.

Let $f \in \ker(D)$ and assume that $f \notin A$. Then $\overline{x}$ divides $\overline{f}$. Since $\overline{f} \in \ker(\overline{D})$ and $\ker(\overline{D})$ is factorially closed ([1] Principle 1), we have $\overline{x} \in \ker(\overline{D})$. Note that $\overline{D}$ sends homogeneous elements of degree i to zero or to homogeneous elements of degree $i + d$, therefore $\overline{D}$ has the form $\overline{D} = a\overline{x}^d E$, where $a \in A$, $d \in \mathbb{N}$, and $E \in \mathrm{LND}(A)$ see [1] Principle 7. But since A is rigid, $E = 0$. Thus $\overline{D} = 0$, a contradiction. This means $f \in A$ which implies that $\ker(D) \subset A$. Finally, since

$\mathrm{tr.deg}_{\mathrm{k}}(A) = \mathrm{tr.deg}_{\mathrm{k}}(\ker(D))$ and A is algebraically closed in $A[x]$, we get the equality $\ker(D) = A$. Hence $\mathrm{ML}(A[x]) = A$. $\square$

4. Computing the ML-Invariant using LND-Filtration

Here we illustrate the use of the ∂-filtration in the computation of ML-invariants, for classes of already well-studied examples.

4.1. Classical examples of semi-rigid k-domains.

We first consider in 4.1.1, certain surfaces in $\mathbb{A}^{[3]}$ defined by equations $X^nZ - P(Y)$ where $n > 1$ and $\deg_Y P(Y) > 1$, which were first discussed by Makar-Limanov in [7], where he computed their ML-invariants. Later on Poloni [10] used similar methods to compute ML-invariants for a larger class. In the second example 4.1.2, we consider certain threefolds whose invariants were computed by S. Kaliman and L. Makar-Limanov [5] in the context of the linearization problem for $\mathbb{C}^*$-action on $\mathbb{C}^3$.

In these examples, the use of LND-filtrations is more natural and less tedious than other existing approaches.

4.1.1. **Danielewski hypersurfaces**.
Let

$$B_{n,P} = k[X, Y, Z]/\langle X^nZ - P(X, Y)\rangle$$

where

$$P(X, Y) = Y^m + f_{m-1}(X)Y^{m-1} + \cdots + f_0(X),$$

$f_i(X) \in k[X]$, $n \geq 2$, and $m \geq 2$.

Let x, y, z be the images of X, Y, Z in $B_{n,P}$. Define ∂ by $\partial(x) = 0$, $\partial(y) = x^n$, $\partial(z) = \frac{\partial P}{\partial y}$ where

$$\frac{\partial P}{\partial y} = my^{m-1} + (m-1)f_{m-1}(x)y^{m-2} + \ldots + f_1(x)$$

We see that $\partial \in \mathrm{LND}(B_{n,P})$ with $\ker(\partial) = k[x]$, and y is a local slice for ∂. Moreover, we have $\deg_\partial(x) = 0$, $\deg_\partial(y) = 1$, $\deg_\partial(z) = m$. The plinth ideal is $\mathrm{pl}(\partial) = \langle x^n\rangle$. Up to a change of variable of the form $Y \mapsto Y - c$ where $c \in k$, we can always assume that $0 \in \mathrm{Spec}(B)$. A consequence of Lemma 2.3, Proposition 2.4, and Proposition 2.5 (see the Proof of Prop. 5.1 for more details) is that

1- The ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is given by:

$$\mathcal{F}_{mi+j} = k[x]y^jz^i + \mathcal{F}_{mi+j-1}$$

where $i \in \mathbb{N}$ and $j \in \{0, \ldots m-1\}$.

2- The associated graded algebra $\mathrm{Gr}(B_{n,P}) = \oplus_{i\in\mathbb{N}}\overline{B}_i$, where $\overline{B}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$, is generated by $\overline{x} = gr_\partial(x)$, $\overline{y} = gr_\partial(y)$, $\overline{z} = gr_\partial(z)$ as an algebra over k with relation $\overline{x}^n\overline{z} = \overline{y}^m$, i.e. $\mathrm{Gr}(B_{n,P}) = k[\overline{X}, \overline{Y}, \overline{Z}]/\langle \overline{X}^n\overline{Z} - \overline{Y}^m\rangle$. And we have :

$$\overline{B}_{mi+j} = k[\overline{x}]\overline{y}^j\overline{z}^i$$

where $i \in \mathbb{N}$ and $j \in \{0, \ldots m-1\}$.

Proposition 4.1. *With the notation above we have:*
(1) $\mathrm{ML}(B_{n,P}) = k[x]$. *Consequently* $B_{n,P}$ *is semi-rigid.*
(2) *Every* $D \in \mathrm{LND}(B_{n,P})$ *has the form* $D = f(x)\partial$.

Proof. (1) By Proposition 2.2 a non-zero $D \in \mathrm{LND}(B)$ induces a non-zero $\overline{D} \in \mathrm{LND}\left(\mathrm{Gr}(B)\right)$. Let $f \in \ker(D) \setminus k$, then $\overline{f} \in \ker(\overline{D}) \setminus k$. There exists $i \in \mathbb{N}$ such that $\overline{f} \in \overline{B}_i$.

Assume that $\overline{f} \notin k[\overline{x}] = \overline{B}_0$, then one of the elements $\overline{y}, \overline{z}$ must divide $\overline{f}$. Which leads to a contradiction as follows:

If $\overline{y}$ divides $\overline{f}$, then $\overline{y} \in \ker(\overline{D})$ because $\ker(\overline{D})$ is factorially closed in $\mathrm{Gr}(B)$ ([1] Principle 1). For the same reason $\overline{x}, \overline{z} \in \ker(\overline{D})$ because $\overline{x}^n \overline{z} = \overline{y}^m$. So $\overline{D} = 0$, a contradiction.

If $\overline{z}$ divides $\overline{f}$, then $\overline{D}(\overline{z}) = 0$. So $\overline{D}$ extends to a locally nilpotent derivation $\widetilde{D}$ of the ring $\widetilde{B} = k(\overline{z})[\overline{x}, \overline{y}]/\left\langle \overline{x}^n \overline{z} - \overline{y}^m \right\rangle$. Since $0 \in \mathrm{Spec}(B)$ is a singular point when $n \geq 2$ and $m \geq 2$, $\widetilde{B}$ is rigid (Lemma 3.3). Therefore, $\widetilde{D} = 0$ which means $\overline{D} = 0$, a contradiction.

So the only possibility is that $\overline{f} \in k[\overline{x}]$. This means that $\deg_{\partial}(f) = 0$, and hence that $f \in k[x]$. So $\ker(D) \subset k[x]$, and finally $k[x] = \ker(D)$ because $\mathrm{tr.deg_k}(\ker(D)) = 1$ and $k[x]$ is algebraically closed in B. So we get $\mathrm{ML}(B) = k[x]$.

(2) is an immediate consequence of Proposition 3.2. □

4.1.2. **Koras-Russell hypersurfaces of the second type**.

Here we consider hypersurfaces associated with k-algebras of the form:

$$B_{n,e,l,Q} = k[X, Y, Z, T]/\langle Y(X^n + Z^e)^l - Q(X, Z, T\rangle$$

where

$$Q(X, Z, T) = T^m + f_1(X, Z)T^{m-1} + \ldots + f_m(X, Z)$$

$f_i(X, Z) \in k[X, Z]$, $n > 1$, $e > 1$, $l > 1$, and $m > 1$. We may assume without loss of generality that $Q(0, 0, 0) = 0$. A particular case of this family corresponds to the so called Koras-Russell hypersurfaces of the second type considered by S. Kaliman and L. Makar-Limanov ([5]) where they computed their ML-invariants. Here we explain how to apply the LND-filtration method to compute this invariant for all algebras $B_{n,e,l,Q}$.

Let x, y, z, t be the images of X, Y, Z, T in $B_{n,e,l,Q}$. Define ∂ by

$$\partial = \frac{\partial Q}{\partial t}\partial_y + (X^n + Z^e)^l \partial_t$$

We see that $\partial \in \mathrm{LND}(B_{n,e,l,Q})$ with $\ker(\partial) = k[x, z]$, and t is a local slice for ∂. Moreover, we have $\deg_{\partial}(x) = 0$, $\deg_{\partial}(y) = m$, $\deg_{\partial}(z) = 0$, and $\deg_{\partial}(t) = 1$. The plinth ideal is $\mathrm{pl}(\partial) = \langle (X^n + Z^e)^l \rangle$. By Lemma 2.3, Prop. 2.4, and Prop. 2.5 we get the following.

1- The ∂-filtration $\{\mathcal{F}_i\}_{i \in \mathbb{N}}$ is given by:

$$\mathcal{F}_{mi+j} = k[x, z]t^j y^i + \mathcal{F}_{mi+j-1}$$

where $i \in \mathbb{N}$, and $j \in \{0, \ldots, m-1\}$.

2- The associated graded algebra $\mathrm{Gr}(B_{n,e,l,Q}) = \oplus_{i \in \mathbb{N}} \overline{B}_i$, where $\overline{B}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$, is generated by $\overline{x} = gr_{\partial}(x)$, $\overline{y} = gr_{\partial}(y)$, $\overline{z} = gr_{\partial}(z)$, $\overline{t} = gr_{\partial}(t)$ as an algebra over k with the relation $\overline{y}(\overline{x}^n + \overline{z}^e)^l = \overline{t}^m$, i.e. $\mathrm{Gr}(B_{n,e,l,Q}) = k[\overline{X}, \overline{Y}, \overline{Z}, \overline{T}]/\langle \overline{Y}(\overline{X}^n + \overline{Z}^e)^l - \overline{T}^m \rangle$. And we have :

$$\overline{B}_{mi+j} = k[\overline{x}, \overline{z}]\overline{t}^j \overline{y}^i$$

where $i \in \mathbb{N}$, and $j \in \{0, \ldots, m-1\}$.

Proposition 4.2. *With the notation above we have:*
(1) $\mathrm{ML}(B_{n,e,l,Q}) = k[x,z]$. *Consequently B is semi-rigid.*
(2) *Every $D \in \mathrm{LND}(B_{n,e,l,Q})$ has the form $D = f(x,z)\partial$.*

Proof. (1) Given a non-zero $D \in \mathrm{LND}(B)$. By Proposition 2.2 D induces a non-zero $\overline{D} \in \mathrm{LND}\,(\mathrm{Gr}(B))$. Suppose that $f \in \ker(D) \setminus k$, then $\overline{f} \in \ker(\overline{D}) \setminus k$. So there exists $i \in \mathbb{N}$ such that $\overline{f} \in \overline{B}_i$. Assume that $\overline{f} \notin k[\overline{x},\overline{z}] = \overline{B}_0$, then one of the elements $\overline{t}$, $\overline{y}$ must divides $\overline{f}$ (see §4.1.2 above). Which leads to a contradiction as follows:

If $\overline{t}$ divides $\overline{f}$, then $\overline{t} \in \ker(\overline{D})$ as $\ker(\overline{D})$ is factorially closed, and for the same reason $\overline{y}, (\overline{x}^n + \overline{z}^e)^l \in \ker(\overline{D})$ due to the relation $\overline{y}(\overline{x}^n + \overline{z}^e)^l = \overline{t}^m$. So $\overline{x}^n + \overline{z}^e \in \ker(\overline{D})$ which implies that $\overline{x}, \overline{z} \in \ker(\overline{D})$ ([1] Lemma 9.3). This means $\overline{D} = 0$, a contradiction.

Finally, if $\overline{y}$ divides $\overline{f}$, then $\overline{D}(\overline{y}) = 0$. Choose $H \in \ker(\overline{D})$ which is homogeneous and algebraically independent of $\overline{y}$, which is possible, since $\mathrm{tr.deg}_{\mathrm{k}} \ker(\overline{D}) = 2$ and $\ker(\overline{D})$ is generated by homogeneous elements. Then by §4.1.2, H has the form $H = h(\overline{x},\overline{z}).\overline{y}^l$. By algebraic dependence, we may assume $H = h(\overline{x},\overline{z})$, which is non-constant, and that $h(0,0) = 0$. So $\overline{D}$ extends to a locally nilpotent derivation $\widetilde{D}$ of the ring $\widetilde{B} = k(\overline{y},H)[\overline{x},\overline{z},\overline{t}]/\langle h(\overline{x},\overline{z}) - H, \overline{y}(\overline{x}^n + \overline{z}^e)^l - \overline{t}^m\rangle$. But $\widetilde{B}$ is of transcendence degree one over the field $k(\overline{y},H)$ whose spectrum has a singular point at 0. This means that $\widetilde{B}$ is rigid (Lemma 3.3). Thus $\widetilde{D} = 0$, which implies $\overline{D} = 0$, a contradiction.

So the only possibility is that $\overline{f} \in k[\overline{x},\overline{z}]$, and this means $\deg_\partial(f) = 0$, thus $f \in k[x,z]$ and $\ker(D) \subset k[x,z]$. Finally, $k[x,z] = \ker(D)$ because $\mathrm{tr.deg}_{\mathrm{k}}(\ker(D)) = 2$. So we get $\mathrm{ML}(B) = k[x,z]$

(2) follows again from Proposition 3.2. □

5. A new class of semi-rigid rings

In this section, we use the LND-filtration method to establish the semi-rigidity of new families of two dimensional domains of the form

$$R = k[X,Y,Z]/\left\langle X^nY - P\left(X, Q(X,Y) - X^eZ\right)\right\rangle$$

for suitable integers $n, e \geq 2$ and polynomials $P(X,T), Q(X,T) \in k[X,T]$. They share with Danielewski hypersurfaces discussed in 4.1.1 above, the property to come naturally equipped with an irreducible locally nilpotent derivation induced by a locally nilpotent derivation of $k[X,Y,Z]$. But in contrast with the Danielewski hypersurfaces case, the corresponding derivation on $k[X,Y,Z]$ are no longer triangular, in fact not even triangulable by virtue of characterization due to Daigle [9].

We will begin with a very elementary example illustrating the steps needed to determine the LND-filtration and its associated graded algebra, and then we proceed to the general case.

5.1. A toy example.

We let

$$R = k[X,Y,Z]/\langle X^2Y - (Y^2 - XZ)^2\rangle$$

and we let x, y, z be the images of X, Y, Z in R. A direct computation reveals that the derivation

$$2XS\partial_Y + (4YS - X^2)\partial_Z$$

of $k[X,Y,Z]$ where $S := Y^2 - XZ$ is locally nilpotent and annihilates the polynomial $X^2Y - (Y^2 - XZ)^2$. It induces a locally nilpotent derivation ∂ of R for which we have $\partial(x) = 0$, $\partial^3(y) = 0, \partial^5(z) = 0$. Furthermore, the element $s = y^2 - xz$ is a local slice for ∂ with $\partial(s) = x^3$. The kernel of ∂ is $k[x]$ and the plinth ideal is the principal ideal generated by x^3. We have $\deg_\partial(x) = 0$, $\deg_\partial(y) = 2$, $\deg_\partial(z) = 4$, $\deg_\partial(s) = 1$.

Proposition 5.1. *With the notation above, we have:*

(1) *The ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is given by :*

$$\mathcal{F}_{4i+2j+l} = k[x]s^l y^j z^i + \mathcal{F}_{4i+2j+l-1}$$

where $i \in \mathbb{N}$, $j \in \{0,1\}$, $l \in \{0,1\}$.

(2) *The associated graded algebra $\mathrm{Gr}_\partial(R) = \oplus_{i\in\mathbb{N}}\overline{R}_i$, where $\overline{R}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$, is generated by $\overline{x} = gr_\partial(x)$, $\overline{y} = gr_\partial(y)$, $\overline{z} = gr_\partial(z)$, $\overline{s} = gr_\partial(s)$ as an algebra over k with relations $\overline{x}^2\overline{z} = \overline{s}^2$ and $\overline{x}\,\overline{z} = \overline{y}^2$, i.e. $\mathrm{Gr}_\partial(R) = k[\overline{X},\overline{Y},\overline{Z},\overline{S}]/\langle \overline{X}^2\overline{Z} - \overline{S}^2, \overline{XZ} - \overline{Y}^2\rangle$. Furthermore:*

$$\overline{R}_{4i+2j+l} = k[\overline{x}]\overline{s}^l\overline{y}^j\overline{z}^i$$

where $i \in \mathbb{N}$, $j \in \{0,1\}$, $l \in \{0,1,2,3\}$.

Proof. 1) First, the ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is given by $\mathcal{F}_r = \sum_{h\le r} H_h$ where $H_h := \sum_{u+2v+4w=h} k[x]\,(s^u y^v z^w)$ and $u,v,w,h \in \mathbb{N}$. To show this, let J be the ideal in $k^{[4]} = k[X,Y,Z,S]$ defined by $J = \big(X^2Y - (Y^2 - XZ)^2, Y^2 - XZ - S\big)$. Define an $\mathbb{N}$-degree function ω on $k^{[4]}$ by declaring that $\omega(X) = 0$, $\omega(S) = 1$, $\omega(Y) = 2$, and $\omega(Z) = 4$. By Lemma 2.4, the $\mathbb{N}$-filtration $\{\mathcal{G}_r\}_{i\in\mathbb{N}}$ where $\mathcal{G}_r = \sum_{h\le r} H_h$ is proper if and only if $\hat{J}$ is prime. Which is the case since $\hat{J} = \big\langle X^2Y - S^2, Y^2 - XZ\big\rangle$ is prime. Thus by Lemma 2.3 we get the desired description.

Second, let $l \in \{0,1\}$ and $j \in \{0,1,2,3\}$ be such that $l := r \bmod 2$, $j := r - l \bmod 4$, and $i := \frac{r-2j-l}{4}$. Then we get the following unique expression $r = 4i + 2j + l$. Since $\mathcal{F}_r = \sum_{u+2v+4w=r} k[x]\,(s^u y^v z^w) + \mathcal{F}_{r-1}$, we conclude in particular that $\mathcal{F}_r \supseteq k[x]s^l y^j z^i + \mathcal{F}_{r-1}$. For the other inclusion, the relation $x^2y = s^2$ allows to write $s^u y^v z^w = x^e s^l y^{v_0} z^w$ and from the relation $y^2 = s + zx$ we get $x^e s^l y^{v_0} z^w = x^e s^l y^j (s+xz)^n z^w$. Since the monomial with the highest degree relative to $\deg_\partial$ in $(s+xz)^n$ is $x^n.z^n$, we deduce that $x^e s^l y^j (s+xz)^n z^w = x^{e+n} s^l y^j z^{w+n} + \sum M_\beta$ where M_β is monomial in x, y, s, z of degree less than r. Since the expression $r = 4i+2j+l$ is unique, we get $w + n = i$. So $s^u y^v z^w = x^{e+n} s^l y^j z^i + f$ where $f \in \mathcal{F}_{r-1}$. Thus $k[x]\,(s^u y^v z^w) \subseteq k[x]s^l y^j z^i + \mathcal{F}_{r-1}$ and finally $\mathcal{F}_r = k[x]s^l y^j z^i + \mathcal{F}_{r-1}$.

2) By part (1), an element f of degree r can be written as $f = g(x)s^l y^j z^i + f_0$ where $f_0 \in \mathcal{F}_{r-1}$, $l = r \bmod 2$, $j = r - l \bmod 4$, $i = \frac{r-2j-l}{4}$, and $i \in \mathbb{N}$, $j \in \{0,1\}$, $l \in \{0,1\}$. So by Lemma 1.4, P2, P1, P3 respectively we get

$$\overline{f} = \overline{g(x)s^l y^j z^i + h} = \overline{g(x)s^l y^j z^i} = \overline{g(x)}\,\overline{s}^l\overline{y}^j\overline{z}^i = g(\overline{x})\overline{s}^l\overline{y}^j\overline{z}^i$$

and therefore $\overline{B}_{4i+2j+l} = k[\overline{x}]\overline{s}^l\overline{y}^j\overline{z}^i$.

Finally, by Proposition 2.5, $\mathrm{Gr}_\partial(B) = k[\overline{X},\overline{Y},\overline{Z},\overline{S}]/\langle \overline{X}^2\overline{Z} - \overline{S}^2, \overline{X}\,\overline{Z} - \overline{Y}^2\rangle$. $\square$

5.2. **A more general family.**

We now consider more generally rings R of the form

$$k[X,Y,Z]/\left\langle X^nY - P\left(X, Q(X,Y) - X^eZ\right)\right\rangle$$

where

$$P(X,S) = S^d + f_{d-1}(X)S^{d-1} + \cdots + f_1(X)S + f_0(X)$$

$$Q(X,Y) = Y^m + g_{m-1}(X)Y^{m-1} + \cdots + g_1(X)Y + g_0(X)$$

$n \geq 2$, $d \geq 2$, $m \geq 1$, and $e \geq 1$. Up to a change of variable of the form $Y \mapsto Y - c$ where $c \in k$, we may assume that $0 \in \mathrm{Spec}(R)$.

Let x, y, z be the images of X, Y, Z in R. Define ∂ by: $\partial(x) = 0$, $\partial(s) = x^{n+e}$ where $s := Q(x,y) - x^e z$. Considering the relation $x^n y = P(x, Q(x,y) - x^e z)$, a simple computation leads to $\partial(y) = x^e \frac{\partial P}{\partial s}$,$\partial(z) = \frac{\partial Q}{\partial y}\frac{\partial P}{\partial s} - x^n$, i.e.

$$\partial := x^e \frac{\partial P}{\partial s}\partial_y + (\frac{\partial Q}{\partial y}\frac{\partial P}{\partial s} - x^n)\partial_z$$

where $\frac{\partial P}{\partial s} = ds^{d-1} + (d-1)f_{d-1}(x)s^{d-2} + \cdots + f_1(x)$, and $\frac{\partial Q}{\partial y} = my^{m-1} + (m-1)g_{m-1}(x)y^{m-2} + \cdots + g_1(x)$. Since $\partial(x^n y - P(x, Q(x,y) - x^e z)) = 0$ and $\partial^{d+1}(y) = 0, \partial^{md+1}(z) = 0$, ∂ is a well-defined locally nilpotent derivation of R. The kernel of ∂ is equal to $k[x]$ and the element s is a local slice for ∂ by construction. One checks further that the plinth ideal is equal to $\mathrm{pl}(\partial) = \langle x^{n+e} \rangle$. A direct computation shows that $\deg_\partial(x) = 0$, $\deg_\partial(y) = d$, $\deg_\partial(z) = md$ and $\deg_\partial(s) = 1$. Furthermore:

1- The ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is given by :

$$\mathcal{F}_{mdi+dj+l} = k[x]s^l y^j z^i + \mathcal{F}_{mdi+dj+l-1}$$

where $i \in \mathbb{N}$, $j \in \{0, \ldots, m-1\}$, $l \in \{0, \ldots, d-1\}$.

2- The associated graded algebra $\mathrm{Gr}(R) = \oplus_{i\in\mathbb{N}} \overline{R}_i$, where $\overline{R}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$, is generated by $\overline{x} = gr_\partial(x)$, $\overline{y} = gr_\partial(y)$, $\overline{z} = gr_\partial(z)$, $\overline{s} = gr_\partial(s)$ as an algebra over k with relations $\overline{x}^n\overline{z} = \overline{s}^d$ and $\overline{x}^e\,\overline{z} = \overline{y}^m$, i.e. $\mathrm{Gr}(R) = k[\overline{X}, \overline{Y}, \overline{Z}, \overline{S}]/\left\langle \overline{X}^n\overline{Z} - \overline{S}^d, \overline{X}^e\,\overline{Z} - \overline{Y}^m \right\rangle$. And we have :

$$\overline{R}_{mdi+dj+l} = k[\overline{x}]\overline{s}^l\overline{y}^j\overline{z}^i$$

where $i \in \mathbb{N}$, $j \in \{0, \ldots, m-1\}$, $l \in \{0, \ldots, d-1\}$.

Theorem 5.2. *With the above notation the following hold:*

(1) $\mathrm{ML}(R) = k[x]$. *Consequently R is semi-rigid.*

(2) *Every $D \in \mathrm{LND}(R)$ has form $D = f(x)\partial$, i.e. R is almost rigid.*

Proof. (1) Given a non-zero $D \in \mathrm{LND}(R)$. By Proposition 2.2, D respects the ∂-filtration and induces a non-zero locally nilpotent derivation $\overline{D}$ of $\mathrm{Gr}(R)$. Suppose that $f \in \ker(D) \setminus k$, then $\overline{f} \in \ker(\overline{D}) \setminus k$ is an homogenous element of $\mathrm{Gr}(R)$. So there exists $i \in \mathbb{N}$ such that $\overline{f} \in \overline{R}_i$.

Assume that $\overline{f} \notin k[\overline{x}] = \overline{R}_0$, then one of the elements $\overline{s}$, $\overline{y}$, $\overline{z}$ must divides $\overline{f}$ by 5.2,2. Which leads to a contradiction as follows :

If $\overline{s}$ divides $\overline{f}$, then $\overline{s} \in \ker(\overline{D})$ as $\ker(\overline{D})$ is factorially closed, and for the same reason $\overline{x}, \overline{y} \in \ker(\overline{D})$ due to the relation $\overline{x}^n\overline{y} = \overline{s}^d$. Then by the relation $\overline{x}^e\,\overline{z} = \overline{y}^m$, we must have $\overline{z} \in \ker(\overline{D})$, which means $\overline{D} = 0$, a contradiction. In the same way, we get a contradiction if $\overline{y}$ divides $\overline{f}$.

Finally, if $\overline{z}$ divides $\overline{f}$, then $\overline{D}(\overline{z}) = 0$. So $\overline{D}$ induces in a natural way a locally nilpotent derivation $\widetilde{D}$ of the ring $\widetilde{R} = k(\overline{z})[\overline{x}, \overline{y}, \overline{s}]/\langle \overline{x}^n\,\overline{z} - \overline{s}^d, \overline{x}^e\,\overline{z} - \overline{y}^m \rangle$. But since $0 \in \mathrm{Spec}(\widetilde{\mathrm{R}})$ is a singular point, $\widetilde{R}$ is rigid (Lemma 3.3). So $\widetilde{D} = 0$, which implies $\overline{D} = 0$, a contradiction.

So the only possibility is that $\overline{f} \in k[\overline{x}]$, and this means $\deg_\partial(f) = 0$, thus $f \in k[x]$ and $\ker(D) \subset k[x]$. Finally, $k[x] = \ker(D)$ because $\mathrm{tr.deg}_{\mathrm{k}}(\ker(D)) = 1$ and $k[x]$ is algebraically closed in B. So $\mathrm{ML}(R) = k[x]$.

(2) follows again from Proposition 3.2. □

6. Further applications of the LND-filtratoin

Given a commutative domain B over an algebraically closed field k of characteristic zero, we denote $\mathrm{Aut}_{\mathrm{k}}(B)$ the group of algebraic k-automorphisms of B. This group acts by conjugation on $\mathrm{LND}(B)$. An immediate consequence is that $\alpha(\mathrm{ML}(B)) = \mathrm{ML}(B)$ for every $\alpha \in \mathrm{Aut}_{\mathrm{k}}(B)$ which yield in particular an induced action of $\mathrm{Aut}_{\mathrm{k}}(B)$ on $\mathrm{ML}(B)$. Let $\partial_\alpha = \alpha^{-1}\partial\alpha$ be the conjugate of ∂ by a given automorphism α of B, it is straightforward to check that $\alpha\{\ker(\partial_\alpha)\} = \ker(\partial)$ and more generally that $\deg_{\partial_\alpha}(b) = \deg_\partial(\alpha(b))$ for any $b \in B$. In other words, α respects $\deg_\partial$ and $\deg_{\partial_\alpha}$(i.e. α sends an element of degree n relative to $\deg_{\partial_\alpha}$, to an element of the same degree n relative to $\deg_\partial$.

Definition 6.1. We say that an algebraic k-automorphism α preserves the *∂-filtration* for some $\partial \in \mathrm{LND}(B)$ if $\deg_\partial(\alpha(b)) = \deg_\partial(b)$ for any $b \in B$.

Lemma 6.2. *Let $\partial \in \mathrm{LND}(B)$ and $\alpha \in \mathrm{Aut}_{\mathrm{k}}(B)$. Then ∂ and ∂_α are equivalent, i.e. have the same kernel, if and only if α preserve the ∂-filtration.*

Proof. Suppose that ∂ and ∂_α are equivalents, then $\deg_\partial(\alpha(b)) = \deg_{\partial_\alpha}(b)$ for every $b \in B$, and by hypothesis $\ker(\partial) = \ker(\partial_\alpha)$, so $\deg_\partial = \deg_{\partial_\alpha}$. Then we obtain $\deg_\partial(\alpha(b)) = \deg_\partial(b)$. Thus α preserves the ∂-filtration. Since the other direction is obvious we are done. □

The following Corollary shows a nice property of a semi-rigid ring. That is, every algebraic automorphism α has to preserve the unique filtration induced by any locally nilpotent derivation ∂, i.e. α sends an element of degree i relative to ∂ to an element of the same degree relative to ∂. Which makes the computation of the group of automorphisms easier up to the automorphism group of $\ker(\partial)$.

Corollary 6.3. *Let B be a semi-rigid k-domain, then every k-automorphism of B preserve the ∂-filtration for every $\partial \in \mathrm{LND}(B)$.*

Proof. A direct consequence of Definition 3.1 and Lemma 6.2. □

6.1. The group of algebraic k-automorphisms of a semi-rigid k-domain.

Suppose that B is a semi-rigid k-domain. Then, it has a unique proper filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ which is the ∂-filtration corresponding to any non-zero locally nilpotent derivation ∂ of B. Since every algebraic k-automorphism of B preserves this filtration (Corollary 6.3), we obtain an exact sequence

$$0 \to \mathrm{Aut}_{\mathrm{k}}(B, \mathrm{ML}(B)) \to \mathrm{Aut}_{\mathrm{k}}(B) \to \mathrm{Aut}_{\mathrm{k}}(\mathrm{ML}(B))$$

where $\mathrm{Aut}_{\mathrm{k}}(B, \mathrm{ML}(B))$ is by definition the sub-group of $\mathrm{Aut}_{\mathrm{k}}(B)$ consisting of elements whose induced action on $\mathrm{ML}(B)$ is trivial. Furthermore, every element of $\mathrm{Aut}_{\mathrm{k}}(B, \mathrm{ML}(B))$ induces for every $i \geq 1$ an automorphism of $\mathcal{F}_0$-module of each $\mathcal{F}_i$. In this section we illustrate how to exploit these information to compute $\mathrm{Aut}_{\mathrm{k}}(B)$ for certain semi-rigid k-domains B.

6.1.1. Aut_k **for example** *4.1.1.*
In [7] Makar-Limanov computed the k-automorphism group for surfaces in $k^{[3]}$ defined by equation $X^nZ - P(Y) = 0$ where $n > 1$ and $\deg_Y P(Y) > 1$. Then Poloni, see [10], generalized Makar-Limanov's method to obtained similar results for the rings considered in 4.1.1 above. Here we briefly indicate how to recover these results using LND-filtrations. So let

$$B_{n,P} = k[X, Y, Z]/\left\langle X^nZ - P(X, Y)\right\rangle$$

where $P(X, Y) = Y^m + f_{m-1}(X)Y^{m-1} + \cdots + f_0(X)$, $f_i(X) \in k[X]$, $n \geq 2$, and $m \geq 2$. Up to change of variable of the form Y by $Y - \frac{f_{m-1}(X)}{m}$ we may assume without loss of generality that $f_{m-1}(X) = 0$.

Proposition 6.4. *Let $B_{n,P}$ be as above. Then every algebraic k-automorphism α of B has the form*

$$\alpha(x, y, z) = (\lambda x,\ \mu y + x^n a(x),\ \frac{\mu^m}{\lambda^n}z + \frac{P(\lambda x, \mu y + x^n a(x)) - \mu^m P(x, y)}{\lambda^n x^n})$$

where $\lambda, \mu \in k^*$ satisfy $f_{m-i}(\lambda x) \equiv \mu^i.f_{m-i}(x)$ mod x^n for all i, and $a(x) \in k[x]$.

Proof. By Proposition 4.1, (1) and §4.1.1, $\mathrm{ML}(B) = k[x]$ and the ∂-filtration $\{\mathcal{F}_i\}_{i\in\mathbb{N}}$ is given by $\mathcal{F}_{im+j} = k[x]y^{im+j} + k[x]y^{im+j-m}z + \ldots + k[x]y^jz^i + \mathcal{F}_{im+j-1}$, where $\partial = x^n.\partial_y + \frac{\partial P}{\partial y}.\partial_z$, $\deg_\partial(x) = 0$, $\deg_\partial(y) = 1$, and $\deg_\partial(z) = m$. In particular $\mathcal{F}_0 = k[x]$, $\mathcal{F}_1 = k[x]y + \mathcal{F}_0$, and $\mathcal{F}_m = k[x]y^m + k[x]z + \mathcal{F}_{m-1}$.

Now by Corollary 6.3 α preserve $\deg_\partial$, so we must have $\alpha(x) \in \mathcal{F}_0 = k[x]$, $\alpha(y) \in \mathcal{F}_1 = k[x]y + k[x]$ and $\alpha(z) \in \mathcal{F}_m = k[x]y^m + k[x].z + \mathcal{F}_{m-1}$. Since α is invertible we get $\alpha(x) = \lambda x + c$, $\alpha(y) = \mu y + b(x)$, and $\alpha(z) = \xi z + h(x, y)$ where $\lambda, \mu, \xi \in k^*$, $c \in k$, $b \in k[x]$, $h(x, y) \in k[x, y]$, and $\deg_y h(x, y) \leq m$.

By Proposition 4.1 (2) every $D \in \mathrm{LND}(B)$ has the form $D = f(x)\partial$. In particular, $\partial_\alpha = f(x)\partial$ for some $f(x) \in k[x]$. Since $\alpha\partial_\alpha = \partial\alpha$ we have $\partial(\alpha(y)) = \alpha(f(x)\partial(y)) = f(\alpha(x))\alpha(x^n)$ where $(\partial(y) = x^n)$. So we get $\partial(\mu y + b(x)) = f(\alpha(x))\,(\lambda x + c)^n$. Since $\partial(\mu y + b(x)) = \mu x^n$, x divides $(\lambda x + c)^n$ in $k[x]$, and this is possible only if $c = 0$, so we get $\alpha(x) = \lambda x$.

Applying α to the relation $x^n z = P(x, y)$ in $B_{P,n}$, we get $\lambda^n x^n \alpha(z) = P(\lambda x, \mu y + b(x)) = \mu^m P(x, y) + m\mu^{m-1}y^{m-1}b(x) + H(x, y)$ where $\deg_y H \leq m - 2$. Since x^n divides $P = x^n z$ and $\deg_y H \leq m - 2$, x^n divides $m\mu^{m-1}y^{m-1}b(x) + H(x, y)$ in $k[x, y]$. So x^n divides $b(x)$, i.e. $\alpha(y) = \mu y + x^n a(x)$.

In addition, x^n divides every coefficient of H as a polynomial in y, so x^n divides $-\mu^m f_{m-i}(x) + \mu^{m-i}f_{m-i}(\lambda x)$ because coefficients of $H(y)$ are of the form $q(x, y)b(x) - \mu^m f_{m-i}(x) + \mu^{m-i}f_{m-i}(\lambda x)$ and $b(x)$ is divisible by x^n. So x^n divides $-\mu^i f_{m-i}(x) + f_{m-i}(\lambda x)$ for every i. And we are done. □

6.1.2. Aut_k **for example** *5.2.*
The same method as in 6.1.1 can be applied to compute automorphism groups of rings R defined as in Theorem 5.2. For simplicity we only deal with the case where $Q(X, Y) = Y^m$, the general case can be deduced in the same way at the cost of longer and more complicated computation. Again we make a substitution in S as in 6.1.1 to get relation of the form presented in the following result:

Theorem 6.5. *Let R denote the ring $R = k[X, Y, Z]/\left\langle X^nY - P(X, Y^m - X^eZ)\right\rangle = k[x, y, z]$ where $P(X, S) = S^d + f_{d-2}(X)S^{d-2} + \cdots + f_1(X)S + f_0(S)$, $n \geq 2$, $d \geq 2$,*

$m \geq 1$, *and* $e \in \mathbb{N}$. *Then, every algebraic k-automorphism of B has the form*

$$\alpha(x,y,z) = (\lambda x,\ \frac{\mu^d}{\lambda^n} y + F,\ \frac{\mu^{dm}}{\lambda^{nm+e}}.z + \frac{(\frac{\mu^d}{\lambda^n} y + F)^m - \frac{\mu^{dm}}{\lambda^{nm}} y^m + x^{n+e} a(x)}{\lambda x^e})$$

where:

$\lambda, \mu \in k^*$ *verify both* $\frac{\mu^{dm}}{\lambda^{nm}} = \mu$ *and* $f_{d-i}(\lambda x) \equiv \mu^i f_{d-i}(x)$ *mod* x^{n+e} *for every* $i \in \{2, \dots, d\}$. $s = y^m - x^e z$ *, and* $F = \frac{P(\lambda x, \mu s + x^{n+e} a(x)) - \mu^d P(x,s)}{\lambda^n x^n}$, $a(x) \in k[x]$.

Proof. A similar argument as in the proof of Proposition 6.4 leads to $\alpha(x) = \lambda x$ and $\alpha(s) = \mu s + x^n a(x)$ where $\lambda, \mu \in k^*$ verify $f_{d-i}(\lambda x) \equiv \mu^i f_{d-i}(x)$ modx^n for all i. Now $\alpha(x)$ and $\alpha(s)$ determine

$$\alpha(y) = \frac{\mu^d}{\lambda^n} y + \frac{P(\lambda x, \mu s + x^n a(x)) - \mu^d P(x,s)}{\lambda^n x^n}.$$

Apply α to $x^e z = y^m - s$ to get $\lambda^e x^e \alpha(z) = (\frac{\mu^h}{\lambda^n} y + F)^m - \mu s - x^n a(x)$ where $F = \frac{P(\lambda x, \mu s + x^n a(x)) - \mu^h P(x,s)}{\lambda^n x^n} \in k[x,s]$. So we have $\lambda^e x^e \alpha(z) = [\frac{\mu^{hm}}{\lambda^{nm}} y^m - \frac{\mu^{hm}}{\lambda^{nm}} s] + (\frac{\mu^{hm}}{\lambda^{nm}} - \mu)s + m(\frac{\mu^h}{\lambda^n} y)^{m-1} F + \ldots + F^m - x^n a(x)$. Since $\frac{\mu^{hm}}{\lambda^{nm}} y^m - \frac{\mu^{hm}}{\lambda^{nm}} s = \frac{\mu^{hm}}{\lambda^{nm}} x^e z$, we see that x^e divides $G := (\frac{\mu^{hm}}{\lambda^{nm}} - \mu)s + m(\frac{\mu^h}{\lambda^n} y)^{m-1} F + \ldots + F^m - x^n a(x)$ in $k[x,s,y] \subset B$ because $F \in k[x,s]$. Thus x^e divides every coefficients of G as a polynomial in y because $\deg_s G \leq d-1$. So x^e divides F and $\frac{\mu^{hm}}{\lambda^{nm}} - \mu = 0$. This means that x^{n+e} divides $f_{d-i}(\lambda x) - \mu^i f_{d-i}(x)$ for all $i \in \{2, \dots, d\}$ (see proof of Proposition 6.4), and $\frac{\mu^{hm}}{\lambda^{nm}} = \mu$. Finally, by the relation $s = y^m - x^e z$, we get $\alpha(z) = \frac{\mu^{hm}}{\lambda^{nm+e}} z + \frac{m(\frac{\mu^h}{\lambda^n} y)^{m-1} F + \ldots + F^m + x^n a(x)}{\lambda x^e}$, and we are done. □

6.2. Isomorphisms.

We are going to use the previous facts about semi-rigid k-domains to give a necessary and sufficient condition for two hypersurfaces of the family 4.1.1 to be isomorphic.

Let $\Psi : A \longrightarrow B$ be an algebraic isomorphism, we refer to this by $\Psi \in \mathrm{Isom}_k(A,B)$, between two finitely generated k-domains $A = k[y_1, \dots, y_r]$, $B = k[x_1, \dots, x_r]$ where $y_1, \dots, y_r$ and $x_1, \dots, x_r$ are minimal sets of generators. Since $\Psi(A) = k[\Psi(y_1), \dots, \Psi(y_r)] = k[x_1, \dots, x_r]$, there exists an automorphism $\psi : B \longrightarrow B$ such that for every $i \in \{1, \dots, r\}$ there exists $j \in \{1, \dots, r\}$ such that $\psi(x_i) = \Psi(y_j)$.

Given $\partial \in \mathrm{LND}(B)$ and $\Psi \in \mathrm{Isom}_k(A,B)$. For any $n \in \mathbb{N}$ we have $(\Psi^{-1}\partial\Psi)^n = \Psi^{-1}\partial^n\Psi$. So we see that $\Psi^{-1}\partial\Psi \in \mathrm{LND}(A)$. An immediate result is $\Psi(\mathrm{ML}(A)) = \mathrm{ML}(B)$ for any $\Psi \in \mathrm{Isom}_k(A,B)$. Denote $\partial_\Psi := \Psi^{-1}\partial\Psi$, we have the following properties

1. $\Psi\{\ker(\partial_\Psi)\} = \ker(\partial)$.
2. $\deg_{\partial_\Psi}(a) = \deg_\partial(\Psi(a))$ for all $a \in A$.

In [10], Poloni obtained similar results to the next Proposition.

Proposition 6.6. *Let* $B_1 = k[x_1, y_1, z_1]$ *and* $B_2 = k[x_2, y_2, z_2]$ *be as in 4.1.1 where*

$$P_1 = y_1^{m_1} + f_{m_1-2}(x_1) y_1^{m_1-2} + \cdots + f_0(x_1)$$

$$P_2 = y_2^{m_2} + g_{m_2-2}(x_2) y_2^{m_2-2} + \cdots + g_0(x_2)$$

$f_i(x_1) \in k[x_1]$, $g_i(x_2) \in k[x_2]$ *and* $n_i > 1, m_i > 1$. *Then*

B_1 and B_2 are isomorphic if and only if $n = n_1 = n_2$, $m = m_1 = m_2$, and $f_{m-i}(\lambda x) \equiv \mu^i g_{m-i}(x)$ mod x^n for every $i \in \{2, \dots, m\}$.

In addition, every isomorphism between B_1 and B_2 takes the form

$$\Psi(x_1, y_1, z_1) = (\lambda x_2,\ \mu y_2 + x_2^n a(x_2),\ \frac{\mu^m}{\lambda^n} z_2 + \frac{P_1(\lambda x_2, \mu y_2 + x_2^n a(x_2)) - \mu^m P_2(x_2, y_2)}{\lambda^n x_2^n}),$$

where $a(x) \in k[x]$ and $\lambda \in k^*$, $\mu \in k^*$ satisfy $f_{m-i}(\lambda x) \equiv \mu^i g_{m-i}(x)$ mod x^n for all i.

Proof. Let $D \in \mathrm{LND}(B_2)$, and let $\Psi \in \mathrm{Isom}_k(B_1, B_2)$. By property 2, $\deg_{D_\Psi}(x_1) = \deg_D(\Psi(x_1))$, but we saw before that $\deg_E(x_1) = 0$ for all $E \in \mathrm{LND}(B_1)$, so we get $\deg_{D_\Psi}(x_1) = 0 = \deg_D(\Psi(x_1))$ and $\Psi(x_1) \in \mathcal{F}_0 = k[x_2]$. The same argument shows that $\deg_{D_\Psi}(y_1) = 1 = \deg_D(\Psi(y_1))$, $\Psi(y_1) \in \mathcal{F}_1 - \mathcal{F}_0$, and $\deg_{D_\Psi}(z_1) = m_1 = \deg_D(\Psi(z_1))$. This implies that the only possibility for ψ defined as in 6.2 is $\psi(x_2) = \Psi(x_1)$, $\psi(y_2) = \Psi(y_1)$ and $\psi(z_2) = \Psi(z_1)$.

Now by Proposition 6.4

$$\psi(x_2, y_2, z_2) = (\lambda x_2,\ \mu y_2 + x_2^{n_2} a(x_2),\ \frac{\mu^{m_2}}{\lambda^{n_2}} z_2 + \frac{Q_2(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2)) - \mu^{m_2} Q_2(x_2, y_2)}{\lambda^{n_2} x_2^{n_2}}),$$

where $a(x_2) \in k[x_2]$ and $\lambda \in k^*$, $\mu \in k^*$ such that $g_{m-i}(\lambda x) \equiv \mu^i g_{m-i}(x)$ mod x^{n_2} for all i. So we get

$$\Psi(x_1, y_1, z_1) = (\lambda x_2,\ \mu y_2 + x_2^{n_2} a(x_2),\ \frac{\mu^{m_2}}{\lambda^{n_2}} z_2 + \frac{Q_2(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2)) - \mu^{m_2} Q_2(x_2, y_2)}{\lambda^{n_2} x_2^{n_2}}).$$

Since $\deg_D(z_2) = m_2$ for any non-zero $D \in \mathrm{LND}(B_2)$, and since ψ preserves $\deg_D$, we get $m_2 = \deg_D(z_2) = \deg_D(\psi(z_2)) = \deg_D(\Psi(z_1)) = \deg_{D_\Psi}(z_1) = m_1$, i.e. $m_1 = m_2$.

By applying Ψ to relation $x_1^{n_1} z_1 = Q_1(x_1, y_1)$ in B_1, we obtain

$$\lambda^{n_1} x_2^{n_1} (\frac{\mu^{m_2}}{\lambda^{n_2}} z_2 + \frac{Q_2(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2)) - \mu^{m_2} Q_2(x_2, y_2)}{\lambda^{n_2} x_2^{n_2}}) = Q_1(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2)).$$

Applying the map gr_D to the last equation, we get

$$\lambda^{n_1} \frac{\mu^{m_2}}{\lambda^{n_2}} \overline{x}_2^{n_1} \overline{z}_2 = \mu^{m_2} \overline{y_2}^{m_2}.$$

On the other hand $\overline{x}_2^{n_2} \overline{z}_2 = \overline{y}_2^{m_2}$ (apply gr_D to $x_2^{n_2} z_2 = Q_2(x_2, y_2)$), the last two equations give $\lambda^{n_1} \frac{\mu^{m_2}}{\lambda^{n_2}} \overline{x}_2^{n_1} = \mu^{m_2} \overline{x}_2^{n_2}$ which means that $n_1 = n_2$. We could have obtained that $n_1 = n_2$ in this way: from $\lambda^{n_1} x_2^{n_1} \Psi(z_1) = Q_1(\lambda x_2, \mu y_2 + x_2^{n_2} a(x_2))$ we conclude that $\lambda^{n_1} x_2^{n_1} \partial(\Psi(z_1)) = \partial(y_2) H(x_2, y_2)$ where $\deg_{y_2} H < m_2$ and x_2 does not divide H. So $x_2^{n_1}$ divides $\partial(y_2) = x_2^{n_2}$ where ∂ is defined as in 4.1.1, which mean that $n_1 \leq n_2$. Since B_1 and B_2 play symmetric roles the equality follows. $\square$

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