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# VARIATIONAL PRINCIPLE FOR WEIGHTED POROUS MEDIA EQUATION

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Abstract. In this paper we state the variational principle for the weighted porous media equation. It extends V.I. Arnold's approach to the description of Euler flows as a geodesics on some manifold, i.e. as a critical points of some energy functional.

## 1. Introduction

In the beginning 18th century Leibniz, Maupertuis, Euler claimed that all physical phenomenons may be obtained from the Least Action Principle and since Lagrange and Hamilton it was well understood for the classical mechanics. However only in 1966 V.I. Arnold in [2] achieved it for the fluid dynamics. To do this he remarked that the group of volume preserving diffeomorphisms $\mathcal{D}_\mu(M)$ of a manifold $M$ ($\mu$ being a given volume element on $M$) is the appropriate configuration space for the hydrodynamics of an incompressible fluid. In this framework the solutions to the Euler equation become geodesic curves with respect to the right invariant metric on $\mathcal{D}_\mu$, which at $g \in \mathcal{D}_\mu$ is given by $\big(X, Y\big) = \int_M < X(x), Y(x) >_x d\mu(x)$, for $X, Y \in T_g\mathcal{D}_\mu$, $< \cdot, \cdot >_x$ is a metric on $T_xM$, and $\mu$ is the volume element on $M$ induced by the metric. The relation between geodesics on $\mathcal{D}_\mu$ and the Euler equation was further studied in [7] and shortly may be expressed in the following way. Let $t \mapsto g_t \in \mathcal{D}_\mu$ be a geodesic with respect to the right invariant metric $(\cdot, \cdot)$, $v_t = \frac{d}{dt} g_t$ be the corresponding velocity, and $u_t = v_t \circ g_t^{-1}$ be a time dependent vector field on M. Then $u_t$ is a solution to the Euler equation for perfect fluid. In particular the map $t \mapsto g_t$ defined on some time interval $[0, T]$ minimizes the energy functional

$$S(g) = \frac{1}{2} \int_0^T \Big( \int_M \| \frac{dg_t}{dt} \|^2 d\mu(x) \Big) \, dt$$

and the Euler-Lagrange equations for this functional are precisely the Euler equation for perfect fluid.

Developing this approach in [1], [3], [9], by means of stochastic methods it was shown that an incompressible stochastic flow $g(u)$ with generator $\frac{1}{2}\Delta + u_t$ is critical for some energy functional if and only if $u$ solves Navier-Stokes equation for viscous incompressible fluid. See also [4] and [8] for other stochastic characterizations of solutions to Navier-Stokes equation. The purpose of this article is to show that the weighted porous media equation ([5], [6]), which generalizes the standard porous media equation,

$$\frac{\partial u}{\partial t} = \left( -u \cdot \nabla + \frac{1}{2} \Delta \right) \left( \|u\|^{q-2} u \right) + \nabla P. \tag{1}$$

may be also obtained in the framework of Least Action Principle for specially chosen energy functional. In the particular case of $q = 2$ this recovers the Navier-Stokes equation.

## 2. Operator formulation of variational principle.

For simplicity we work on the torus $\mathbb{T}$ of dimension $N$. From now on, when integrating in the torus, $dx$ will stand for the normalized Lebesgue measure.

**Definition 2.1.** For some smooth divergence free time dependent vector field $(t, x) \mapsto v_t(x) \in T_x\mathbb{T}$ we define the flow of $\dot{v}_t$: $e_t(v) \in \mathcal{D}_\mu(\mathbb{T})$ as a solution of the ordinary differential equation

$$\frac{de_t(v)}{dt} = \dot{v}_t(e_t(v)), \quad e_0(v) = \boldsymbol{I}_{\mathbb{T}}. \tag{2}$$

Let us remark that in some sense $e_t(v)$ is a perturbation of identity map in space $\mathcal{D}_\mu(\mathbb{T})$. The solvability of this equation easily follows from the compactness of $\mathbb{T}$ and smoothness of $v$.

Consider a time-dependent divergence-free vector field $u$ on $[0, T] \times \mathbb{T}$. So $u$ takes its values in the tangent bundle of $\mathbb{T}$ which can at every point be identified with $\mathbb{R}^N$. Divergence-free means that $\sum_{j=1}^N \partial_j u^j \equiv 0$. Define the operator $L(u_t) : C^\infty(\mathbb{T}, \mathbb{R}^N) \to C^\infty(\mathbb{T}, \mathbb{R}^N)$ by $L(u_t)f = \frac{1}{2}\Delta f + u_t \cdot \nabla f$.

**Definition 2.2.** *The energy functional* is defined for $q > 1$ as

$$\mathcal{E}_q(u, v) = \frac{1}{q}\int_0^T \int_{\mathbb{T}} \Big\| \Big[\big(\partial_t + L(u_t)\big)e_t(v)\Big](e_t^{-1}(v)(x))\Big\|^q dx\, dt, \tag{3}$$

where $e_t^{-1}(v)$ is the inverse map of the diffeomorphism $e_t(v) : \mathbb{T} \to \mathbb{T}$.

**Definition 2.3.** We say that $u$ is a critical point of $\mathcal{E}_q$ if for all divergence-free time dependent vector field $v$ such that $v_0 = 0$ and $v_T = 0$, $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_q(u, \varepsilon v) = 0$.

**Theorem 2.1.** A divergence-free time dependent vector field $u$ is a critical point of $\mathcal{E}_q$, $q \geq 2$ if and only if there exists a function $P(x)$ such that (1) is satisfied.

Proof. For $e_t(\varepsilon v)_*\big(u_t\big)(x) = T_{e_t^{-1}(\varepsilon v)(x)} e_t(\varepsilon v)\left(u_t\left(e_t^{-1}(\varepsilon v)(x)\right)\right)$, we compute

$$\begin{aligned}\Big[\big(\partial_t + L(u_t)\big)e_t(\varepsilon v)\Big]\big(e_t^{-1}(\varepsilon v)(x)\big) &= \varepsilon \dot{v}(t, e_t^{-1}(\varepsilon v)(x)) + e_t(\varepsilon v)_*\big(u_t\big)(x) + \\ &+ \frac{1}{2}\big(\Delta e_t(\varepsilon v)\big)\big(e_t^{-1}(\varepsilon v)(x)\big),\end{aligned}$$

where $T_y e_t(\varepsilon v)(\cdot)$ being the tangent map of $e_t(\varepsilon v)$ at point $y$. Therefore we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big[\big(\partial_t + L(u_t)\big)e_t(\varepsilon v)\Big]\big(e_t^{-1}(\varepsilon v)(x)\big) = \dot{v}_t(x) + [u_t, v_t](x) + \frac{1}{2}\Delta v_t(x).$$

Since $u_t = \big(\partial_t + L(u_t)\big)(I)$, for $\boldsymbol{I} = e_t(0) : \mathbb{T} \to \mathbb{T}$ the identity map, $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_q(u, \varepsilon v)$ equals

$$\int_0^T \int_{\mathbb{T}} \|\big(\partial_t + L(u_t)\big)(id)\|^{q-2}\big\langle \dot{v}_t + [u_t, v_t] + \frac{1}{2}\Delta v_t, u_t\big\rangle dx\, dt =$$

$$= \int_0^T \int_{\mathbb{T}} \|u_t\|^{q-2} \big\langle \dot{v}_t + [u_t, v_t] + \frac{1}{2}\Delta v_t, u_t \big\rangle \, dx \, dt.$$

On the other hand

$$0 = \int_{\mathbb{T}} \|u_T\|^{q-2} \big\langle u_T, v_T \big\rangle dx$$

$$= \int_0^T \int_{\mathbb{T}} \Big\{ \|u_t\|^{q-2} \big\langle u_t, \dot{v}_t \big\rangle + \big\langle \|u_t\|^{q-4}(q-2) < \dot{u}_t, u_t > u_t + \|u_t\|^{q-2} \dot{u}_t, v_t \big\rangle \Big\} dx \, dt.$$

Therefore, writing $u = u_t$ and $v = v_t$,

$$0 = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_q(u, \varepsilon v) +$$

$$+ \int_0^T \int_{\mathbb{T}} \Big\{ \|u\|^{q-2} \Big( \big\langle \dot{u}, v \big\rangle - \big\langle [u, v], u \big\rangle - \frac{\big\langle \Delta v, u \big\rangle}{2} \Big) + (q-2)\|u\|^{q-4} \big\langle \dot{u}, u \big\rangle \big\langle u, v \big\rangle \Big\} dx \, dt.$$

Due to

$$\int_{\mathbb{T}} \|u\|^{q-2} < \nabla_v u, u > dx = \frac{1}{q} \int_{\mathbb{T}} < \nabla \|u\|^q, v > dx = -\frac{1}{q} \int_{\mathbb{T}} \|u\|^q \operatorname{div} v \, dx = 0$$

for $\operatorname{div} v = 0$, we have, using $[u, v] = \nabla_u v - \nabla_v u$,

$$\begin{aligned}
-\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_q\Big(u, (\varepsilon v)\Big) &= \int_0^T \int_{\mathbb{T}} \Big\{ - \|u\|^{q-2} \big\langle \nabla_u v, u \big\rangle - \frac{1}{2} \big\langle v, \Delta\Big( \|u\|^{q-2} u \Big) \big\rangle + \\
&+ (q-2)\|u\|^{q-4} \big\langle \dot{u}, u \big\rangle \big\langle u, v \big\rangle + \|u\|^{q-2} \big\langle \dot{u}, v \big\rangle \Big\} \, dx \, dt \\
&= \int_0^T \int_{\mathbb{T}} \Big\langle \nabla_u \Big( \|u\|^{q-2} u \Big) - \frac{1}{2} \Delta \Big( \|u\|^{q-2} u \Big) + \\
&+ (q-2)\|u\|^{q-4} \big\langle \dot{u}, u \big\rangle u + \|u\|^{q-2} \dot{u}, v \Big\rangle \, dx \, dt = \\
&= \int_0^T \int_{\mathbb{T}} \big\langle \big( \partial_t + u \cdot \nabla - \frac{1}{2}\Delta \big) \|u\|^{q-2} u, v \big\rangle \, dx \, dt
\end{aligned}$$

(notice that in the second equality we used the fact that $\int_{\mathbb{T}} u \left( \langle v, \|u\|^{q-2} \rangle \right) dx = \int_{\mathbb{T}} \operatorname{div} u \langle v, \|u\|^{q-2} \rangle dx = 0$). This equality is true for all time dependent divergence free vector field $v$, so it gives the equivalence between $u$ critical point of $\mathcal{E}_q$ and solution to equation (1). ■

## 3. Stochastic variational principle for incompressible diffusion flows

We define a *diffusion flow* $g_t(x)$ on $\mathbb{T}$, $x \in \mathbb{T}$, $t \in [0, T]$, $T > 0$ as a stochastic process, which satisfies the Itô stochastic equation:

$$dg_t(x) = \sigma(g_t(x)) \, dW_t + u_t(g_t(x)) \, dt, \quad g_0(x) = x \tag{4}$$

where $u_t$ is a time dependent vector field on $\mathbb{T}$, $\sigma \in \Gamma(Hom(\mathcal{H}, T\mathbb{T}))$ is a $C^2$-map satisfying for all $x \in \mathbb{T}$ $(\sigma\sigma^*)(x) = I_{T_x\mathbb{T}}$, $W_t$ is a cylindric Brownian motion in Hilbert space $\mathcal{H}$.

Let us remark that a diffusion flow is a diffusion process $\{g_t(u)(x)\}_{t\geq 0}$ with generator $L(u_t) = \frac{1}{2}\Delta + u_t$. We define an *incompressible diffusion flow* $g_t(u)(x)(\omega)$ as a diffusion flow such that a.s. $\omega$ for all $t \geq 0$, the map $x \mapsto g_t(u)(x)(\omega)$ is a volume preserving diffeomorphism of $\mathbb{T}$. Examples of incompressible diffusion flows can be found in [3]. Notice that a necessary condition is $\operatorname{div} u_t = 0$.

For the diffusion flow $g_t$ (4) we define the *drift* as the time derivative of the finite variation part by $Dg_t(\omega) := u_t(g_t, \omega)$, and the *energy functional* by

$$\mathcal{E}_q(g) := \frac{1}{q}\mathbb{E}\Big[\int_0^T \int_{\mathbb{T}} \Big\|Dg_t(x)(\omega)\Big\|^q dx\, dt\Big], \quad q > 1. \tag{5}$$

We make a perturbation by letting $g_t^v(u) = e_t(v) \circ g_t(u)$, where $v$ is a smooth divergence free time dependent vector field and $e_t(v)$ is defined in (2).

**Definition 3.1.** We say that $g_t(u)$ is a critical point for the energy functional $\mathcal{E}_q$ if for all smooth time dependent divergence free vector field $v$ on $T\mathbb{T}$ such that $v_0 = v_T = 0$,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{E}_q(g^{\varepsilon v}(u)) = 0.$$

**Theorem 3.2.** *Let $q \geq 2$. An incompressible diffusion flow $g_t(u)$ with generator $L(u_t)$ is a critical point for the energy functional $\mathcal{E}_q$ if and only if there exists a function $P(x)$ such that $u_t$ satisfies equation (1).*

**Proof** of this theorem is a consequence of Theorem 2.1 and the Itô's formula. ■

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