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Hyper-Stable Social Welfare Functions*

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Abstract

We introduce a new consistency condition for neutral social welfare functions, called hyper-stability. A social welfare function α selects a complete weak order from a profile P_N of linear orders over any finite set of alternatives, given N individuals. Each linear order P in P_N generates a linear order over orders of alternatives,called hyper-preference, by means of a preference extension. Hence each profile P_N generates a hyper-profile $\dot{P}_N$. We assume that all preference extensions are separable: the hyper-preference of some order P ranks order Q above order Q' if the set of alternative pairs P and Q agree on contains the one P and Q' agree on. A special sub-class of separable extensions contains all Kemeny extensions, which build hyper-preferences by using the Kemeny distance criterion. A social welfare function α is hyper-stable (resp. Kemeny-stable) if at any profile P_N, at least one linearization of $\alpha(P_N)$ is ranked first by $\alpha(\dot{P}_N)$, where $\dot{P}_N$ is any separable (resp. Kemeny) hyper-profile induced from P_N. We show that no scoring rule is hyper-stable, and that no unanimous scoring rule is Kemeny-stable, while there exists a hyper-stable Condorcet social welfare function.

Keywords: *Hyper-preferences – Kemeny distance – Social Welfare Functions – Stability*

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1 Introduction

Many collective choice situations involve orderings of a finite set of m alternatives as resolute outcomes. Natural examples are choosing a social preference or a priority order over decisions, ranking candidates in sport or arts competitions (e.g. the Eurovision song contest) or assigning tasks to individuals. In the latter example, there are m positions to be filled by m individuals, each being assigned a specific position. Given the natural ranking $1 > ... > m$ of the positions, a social outcome is an order $f(1) \succ ... \succ f(m)$ over individuals obtained by means of a bijection f from the set of positions to the set of individuals.

The classical framework of social choice theory calls for individuals to report their preferences over social outcomes. When social outcomes are linear orders, preferences over outcomes are orders of orders, or *hyper-preferences*. However, reporting full preferences faces a problem of practical implementation: in the no-indifference case, individuals have to rank $m!$ outcomes. More generally, when outcomes are complex combinations of basic alternatives, likewise orderings or subsets, choosing from full preference profiles is hardly achievable in practice. This suggests to design procedures based on partial information about individual preferences.[1] An immediate option is asking each of the individuals to report only one order. Formally, this procedure reduces to using a Social Welfare Function (SWF) α, which maps every profile of linear orders to a weak order of alternatives, completed with a tie-breaking rule.

It follows that some normative properties of SWFs cannot be investigated without retaining assumptions on how individual orders over alternatives are extended to underlying hyper-preferences. A typical example is given by strategy-proofness, which can be defined only conditional to the way orders over alternatives are extended to hyper-preferences. Bossert and Storcken (1992) prove impossibility results for hyper-preferences generated by the criterion of *Kemeny distance*: given an order P over alternatives, the hyper-preference from P ranks an order Q above another order Q' if the Kemeny distance between P and Q is strictly lower than the one between P and Q'.[2] Bossert and Sprumont (2012) investigate strategy-proofness for hyper-preferences based on the following *betweenness criterion*: the hyper-preference from P ranks Q above Q'if the set of alternative pairs P and Q agree on contains the set of pairs P and Q' agree on.[3] Another property requiring extending orders to hyper-preferences is the Pareto property, which states that an SWF (with a tie-breaking rule) chooses at any profile over alternatives a linear order that is not unanimously less preferred than another order.

In this paper we introduce a new property for neutral SWFs, called *hyper-stability*, which also implies linking orders over alternatives to hyper-preferences. Hyper-stability is a consistency property relating two levels of choice, the one from profiles of orders over alternatives, called *basic profiles*, and the one from hyper-preference profiles, or *hyper-profiles*. Loosely speaking, an SWF is hyper-stable if its outcomes at any basic profile is top-ranked at the corresponding hyper-profile. More precisely, consider an SWF α defined for any finite number of alternatives. Hence, α provides a weak order at any basic profile over m alternatives as well as from any basic profile over $m!$ alternatives. Furthermore, suppose that α is neutral, meaning that its outcomes are non-sensitive to the labeling of alternatives. Thus, profiles over $m!$ alternatives can be also interpreted as hyper-preference profiles

[1] This is what prevails in the Eurovision song contest, where ballots are based on a partial scoring method.

[2] The Kemeny distance between two linear orders is the number of pairs of alternatives which they disagree on.

[3] See Duddy et al. (2010) for an analysis of strategy-proof SWFs based on ordinally fuzzy preferences.

over orders of m alternatives, or in short hyper-profiles. While a basic profile clearly entails a huge loss of information about preferences over outcomes, there may nonetheless exist, in the spirit of revealed-preference theory, a class of underlying hyper-profiles (over $m!$ orders) compatible with the basic profile at which α ranks at top at least one linearization of the weak order chosen from the basic profile. If this happens at every possible reduced profile, we say that α is hyper-stable.

As for strategy-proofness, a key-issue for hyper-stability is what is meant by a class of hyper-profiles compatible with a basic profile. We assume here that compatibility holds when hyper-preferences are generated from orders over alternatives in accordance with the betweenness criterion. Clearly, this criterion allows to compare only a small number of orders, therefore a basic profile generates a large class of compatible hyper-profiles. Nonetheless, we prove the existence of a unanimous and hyper-stable *Condorcet SWF*.[4] However, many well-known Condorcet SWFs are not hyper-stable.

We also pay attention to the sub-class of hyper-profiles built by means of the Kemeny distance criterion. Hyper-stability relative to this sub-class is called Kemeny-stability. We show that no *scoring rule* is Kemeny-stable, hence hyper-stable. Then, our main result is that ranking by scoring is incompatible with hyper-stability, while the Condorcet criterion is not.

To the best of our knowledge, hyper-stability is a new property for SWFs, although related properties appear in several studies of collective choice. The yeast of the present study can be found in Binmore (1975), who considers a stronger notion of hyper-stability, although in a different setting. Suppose that preferences are now weak orders over three alternatives, which are aggregated to a weak order by means of a neutral SWF α. Binmore does not comment on hyper-preferences beyond writing "if a rational entity holds a certain preference preordering over a set of alternatives, then that entity must also subscribe to a certain partial preordering of the set of all preorderings" (Binmore, 1975, page 379). Moreover, weak orders are compared according to their respective top-sets. All relevant top-sets in Binmore's analysis contain at most two elements and the criterion works as follows: Given a weak order R, sets $\{x\}$, $\{y\}$, and $\{x, y\}$ are ranked in the order $\{x\}, \{x, y\}, \{y\}$ if and only if xRy. Given the 13 possible weak orders over 3 alternatives, this criterion suffices to find a family $\mathcal{T}$ of triples of weak orders on which basic preferences generate a hyper-profile.[5] Since α is neutral, it can be applied to each of these hyper-profiles, leading to a weak order R_T over each triple T in $\mathcal{T}$. Furthermore, the weak order chosen from the basic profile also induces a weak order $\widetilde{R}_T$ over each triple T in $\mathcal{T}$. Binmore shows that R_T and $\widetilde{R}_T$ coincide for all T in $\mathcal{T}$ if and only if α is dictatorial, anti-dictatorial or constant. There are three main differences between Binmore's approach and the present one. First, basic preferences and hyper-preferences are weak orders in Binmore's study, while we assume both are linear orders. Second, Binmore's setting defines SWFs for three alternatives only. Using neutrality together with a way to generate hyper-preferences, this allows to choose from hyper-profiles over triples of orders. In contrast, our setting involves a variable

[4] An alternative is a Condorcet winner at some basic profile if it defeats all other alternatives according to the majority rule. An SWF is Condorcet if it uniquely ranks first a Condorcet winner whenever it exists.

[5] To see why, label alternatives as x, y and z, and consider the following weak orders R_1, R_2 and R_3 (with respective a-symmetric parts P_1, P_2 and P_3) defined by zP_1yP_1x, yP_2zP_2x and yR_3zP_3x. Denote by $\succsim_1$, $\succsim_2$ and $\succsim_3$the respective hyper-preferences induced on $\{R_1, R_2, R_3\}$ by R_1, R_2 and R_3. Then one gets $R_1 \succ_1 R_3 \succ_1 R_2$, $R_2 \succ_2 R_3 \succ_2 R_1$ and $R_1 \sim_3 R_2 \sim_3 R_3$. It is easily seen that for each of the 13 possible weak orders, R_1, R_2 and R_3 are ranked as in $\succsim_1$, or $\succsim_2$ or $\succsim_3$. Hence, any basic profile generates an hyper-profile over triple $\{R_1, R_2, R_3\}$.

number of alternatives, and defines hyper-preferences as linear orders over all orders. Again, using neutrality together with a way to generate hyper-preferences, this allows to have a well-deffined outcome at profiles over m alternatives and at hyper-profiles over $m!$ orders. Third, our definition of hyper-stability is clearly less demanding than Binmore's one, since it only requires that some social order chosen from basic profiles is top-ranked from hyper-profiles, imposing nothing about how this social order itself generates a social hyper-preference.

Another study related to hyper-stability can be found in Laffond and Lainé (2000), although the property is not explicitly stated there. Using the same framework as the present one, Laffond and Lainé characterize the class of (neutral and independent) hyper-preferences such that whenever the majority tournament at a basic profile is transitive, it is a Condorcet winner of any corresponding hyper-profile. This characterization result can be restated as follows in terms of hyper-stability. Call strongly Condorcet a SWF which gives as outcome the majority tournament whenever it is transitive. Then every strongly Condorcet SWF is hyper-stable relative to some class of hyper-preferences. .

Hyper-stability also appear, at least in watermark, in the literature of moral judgments.[6] Sen (1974) argues that morality requires to formulate judgments among preferences while rationality does not, and suggests using moral views, defined as hyper-preferences, as a way out of the Paretian liberal paradox *à la* Sen (1970).[7] If one accepts basic profiles as expressions of rationality (individuals reporting their first-best outcome) and hyper-profiles as expressions of moral judgments, hyper-stability can be interpreted as a property of moral consistency: choices made from rational preferences does not conflict with the one made from moral judgments.

Furthermore, hyper-stability can be related to a property of self-selectivity for SWFs. Self-selectivity is defined for a social choice function (SCF) by Koray (2000).[8] Roughly speaking, an SCF is self-selective if it chooses itself against any finite number of other social choice functions. Self-selectivity thus involves two levels of choice: choices from profiles over alternatives, and choices from profiles over choice functions. These two levels are connected by means of a consequentialist principle, which states that individuals preferring alternative x to alternative y will rank any function choosing x above any function choosing y. Koray (2000) shows that a neutral and unanimous SCF is self-selective if and only if it is dictatorial. While consequentialism allows for a canonical extension of preferences over alternatives to preferences over SCFs, this is no longer the case for SWFs. Nonetheless, self-selectivity for SWFs can be defined conditional to the definition of hyper-preferences. An individual with preference P in some basic profile P_N will prefer SWF α_1 to SWF α_2 if $\alpha_1(P_N)$ is "closer" to P than $\alpha_2(P_N)$, where closer can be in terms of the Kemeny or any other distance. More generally, once defined how P generates a hyper-preference $\dot{P}$, two SWFs being compared according to the way $\dot{P}$ ranks their respective outcomes. Hence the consequentialist principle applies, but conditional to the way basic preferences are extended to hyper-preferences. We say that a SWF is *SW self-selective* for some preference extension if, at any basic profile it ranks itself first when compared to any finite set of SWFs. We show below that hyper-stability is a necessary condition for SW self-selectivity.

[6]One can think of hyper-preferences also as preferences of individuals over others in the society.

[7]See Igersheim (2007). The reader may refer to Jeffrey (1974), McPherson (1982), and Sen (1977) for further discussion on the more general concept of a *meta-preference*.

[8]A social choice function picks one alternative at every profile of preferences over alternatives. For further studies of self-selectivity, see Koray and Unel (2003) and Koray and Slinko (2008).

The rest of the paper is organized as follows. Part 2 formally defines hyper-stability, and investigates its relation to self-selectivity. Hyper-stability of scoring rules is studied in Part 3. In particular, we provide examples showing that neither the Borda rule, nor the plurality and anti-plurality rules are Kemeny-stable, hence hyper-stable. Moreover, we show that no unanimous scoring rule is Kemeny-stable, and that no scoring rule is hyper-stable. Condorcet SWFs are considered in Part 4. We show that the Slater SWF, the Kemeny rule, and the Copeland SWF are not hyper-stable, whereas the transitive closure of the majority relation over alternatives is hyper-stable. The paper ends up with comments about alternative concepts of hyper-stability, together with open questions. Finally, all proofs are postponed to an appendix.

2 Hyper-stability

2.1 Notations and definitions

Let $\mathbb{N}$ be the set of non-zero natural numbers. We consider societies with variable numbers of individuals and of alternatives. Hence, $\mathbb{N}$ stands for the sets of potential alternatives and individuals, and each actual society involves finitely many individuals confronting finitely many alternatives. Given $m \in \mathbb{N}$, we define $A_m = \{1, ..., m\}$ as a set of m social alternatives. The set of linear (resp. weak) orders over A_m is denoted by $\mathcal{L}(A_m)$ (resp. $\mathcal{R}(A_m)$). An order $P \in \mathcal{L}(A_m)$ is *a linear extension* of $R \in \mathcal{R}(A_m)$ if for any $a, b \in A_m$, $aPb \Rightarrow aRb$. The set of all linear extensions of $R \in \mathcal{R}(A_m)$ is denoted by $\Delta(R)$. Given a set N of n individuals, a weak profile is an element R_N of $\mathcal{R}(A_m)^n$, and a profile is an element P_N of $\mathcal{L}(A_m)^n$. The set of all *linearizations of the weak profile* R_N is $\Delta(R_N) = \times_{i \in N}(\Delta(R_i))$.

A function α from $\cup_{m,n \in \mathbb{N}} \mathcal{L}(A_m)^n$ to $\cup_{m \in \mathbb{N}} \mathcal{R}(A_m)$ is a *social welfare function* (SWF) if, for all $n, m \in \mathbb{N}$ and all $P_N \in \mathcal{L}(A_m)^n$, $\alpha(P_N) \in \mathcal{R}(A_m)$. Moreover, a SWF α is neutral if for all $n, m \in \mathbb{N}$ and all $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, for all $a, b \in A_m$, $a\ \alpha(P_N)\ b$ if and only if $\gamma(a)\ \alpha(P_N^{\gamma})\ \gamma(b)$. Note that since α are defined for any number of alternatives, neutrality ensures that the precise labeling of alternatives does not matter. In particular, α is defined for profiles over $m!$ alternatives, which can be either basic alternatives or linear orders over m basic alternatives. Furthermore, a SWF α is *unanimous* if, for any $m, n \in \mathbb{N}$, for any profile $P_N \in \mathcal{L}(A_m)^n$, for any two alternatives $a, b \in A_m$, $[a\ P_i\ b$ for all $i = 1, ..., n]$ implies that $[a\ \alpha(P_N)\ b$ and $\neg(b\ \alpha(P_N)\ a)]$. Given a SWF α, the *α-induced correspondence* $f_\alpha : \cup_{n,m \in \mathbb{N}} \mathcal{L}(A_m)^n \to 2^{A_m} \setminus \varnothing$ is defined by: $\forall n, m \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$, $\forall a \in A_m$, $a \in f_\alpha(P_N) \iff a\ \alpha(P_N)\ b$ for all $b \in A_m$. Hence, the α-induced correspondence selects at each profile P_N the top-set for $\alpha(P_N)$.

2.2 Preference extensions

We turn now to the notion of hyper-preference. A *preference extension* is a function $e : \cup_{m \in \mathbb{N}} \mathcal{L}(A_m) \to \cup_{m \in \mathbb{N}} \mathcal{L}(\mathcal{L}(A_m))$ such that for all $m \in \mathbb{N}$ and all $P \in \mathcal{L}(A_m)$, $e(P) \in \mathcal{L}(\mathcal{L}(A_m))$. Hence, a preference extension maps each linear order over m alternatives to a linear order over all linear orders over alternatives. An element of $\mathcal{L}(\mathcal{L}(A_m))$ is called hyper-preference. An *extension domain* is a proper subset $\mathcal{E}$ of the set of all preference extensions. Given a profile $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$ together with a n-tuple $E = (e_1, ..., e_n) \in \mathcal{E}^n$, an *hyper-profile* of P_N is the element $P_N^E = (e_1(P_1), ..., e_n(P_n))$.

Given $P, Q \in \mathcal{L}(A_m)$, we define the set $A(P,Q) = \{(a,b) \in A_m \times A_m : aPb \text{ and } aQb\}$, which contains all alternative pairs P and Q agree on. We focus on the specific class of *separable* preference extensions.

Definition 1 *A preference extension e is separable if for all $m \in \mathbb{N}$ and all $P, Q, Q' \in \mathcal{L}(A_m)$, $A(P,Q) \supset A(P,Q')$ implies $Q\ e(P)\ Q'$.*

We denote by $\mathcal{S}$ the domain of separable preference extensions.

Given $P, Q \in \mathcal{L}(A_m)$, the *Kemeny distance* between P and Q is defined by $d_K(P,Q) = |\{(a,b) \in A_m \times A_m : aPb \text{ and } bQa\}|$, that is the number of pairs of alternatives P and Q disagree on.

Definition 2 *A preference extension e is Kemeny if for all $m \in \mathbb{N}$ and all $P, Q, Q' \in \mathcal{L}(A_m)$, $d_K(P,Q) < d_K(P,Q')$ only if $Q\ e(P)\ Q'$.*

We denote by $\mathcal{K}$ the domain of Kemeny preference extensions. Pick up any $P \in \mathcal{L}(A_m)$. Using Kemeny distance allows to induce from P the element $\succsim_P \in \mathcal{R}(\mathcal{L}(A_m))$ defined by: $\forall Q, Q' \in \mathcal{L}(A_m)$, $Q \succsim_P Q'$ iff $d_K(P,Q) \leq d_K(P,Q')$, and $Q \succ_P Q'$ iff $d_K(P,Q) < d_K(P,Q')$. In words, the weak order $\succsim_P$ induced by P ranks orders according to their respective distances to P. Given profile $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, the Kemeny weak profile for P_N is defined by $P_N^K = (\succsim_{P_1}, ..., \succsim_{P_n})$. Thus, a preference extension e is Kemeny if for all $m \in \mathbb{N}$ and all $P \in \mathcal{L}(A_m)$, e is a linear extension of $\succsim_P$. We call Kemeny hyper-profile any linearization of P_N^K. Clearly, every Kemeny extension is separable, and thus $\mathcal{K} \subset \mathcal{S}$.

The Kemeny distance criterion can be criticized by arguing that when comparing two orders, inversions in the lower tail of the ranking are less important that inversions in the upper tail. If three candidates a, b, c are to be ranked as gold, silver and bronze medal, and if your own ranking is $aPbPc$, then you should prefer order $aQcQb$ to order $bQ'aQ'c$, since reversing order for gold and silver seems appears as a more significant deviation than reversing order for silver and bronze. This calls for breaking symmetry by using weighted Kemeny distance (equivalently, this calls for some specific way to break ties in the Kemeny weak profiles). Note however that such a critic no longer holds if agendas are interpreted as task assignments. Indeed, suppose that $aQcQb$ stands for assigning task 1 to individual a task 2 to c and task 3 to b, a similar meaning being given to Q'. Provided that all task are given the same importance, Q and Q' involve only one mismatch from the viewpoint P, and nothing suggests why Q should be preferred to Q'.

The following example illustrates the construction of Kemeny hyper-profiles. Consider the following profile $P_N = (P_1, P_2, P_3)$ over 3 alternatives a, b, c:

$$P_N = \begin{pmatrix} P_1 & P_2 & P_3 \\ \hline a & c & c \\ b & b & a \\ c & a & b \end{pmatrix}$$

The Kemeny weak profile P_N^K of P_N is defined by

$$P_N^K = \left(\begin{array}{ccc} \succsim_{P_1} & \succsim_{P_2} & \succsim_{P_3} \\ \hline abc & cba & cab \\ acb, bac & bca, cab & cba, acb \\ bca, cab & acb, bac & abc, bca \\ cba & abc & bac \end{array} \right)$$

where xyz stands for the linear order $xPyPz$, and where two orders belonging to the same row and column are indifferent. A Kemeny hyper-profile for P_N is any element $\dot{P}_N$ of $\Delta(P_N^K)$. For instance,

$$\dot{P}_N = \left(\begin{array}{ccc} \dot{P}_1 & \dot{P}_2 & \dot{P}_3 \\ \hline abc & cba & cab \\ bac & cab & cba \\ acb & bca & acb \\ bca & bac & abc \\ cab & acb & bca \\ cba & abc & bac \end{array} \right)$$

Contrarily to the Kemeny distance criterion, separability does not automatically induces a weak order over orders. For instance, $e(P_1) \in \mathcal{S}$ only if the following conditions holds: (1) $e(P_1)$ uniquely ranks P_1 first and its inverse *cba* last, (2) *acb* is ranked above *cab*, and (3) *bac* is ranked above *bca*. The reader will easily check that hyper-profile $\widetilde{P}_N$ below is built from a vector of separable preference extensions which are not Kemeny.

$$\widetilde{P}_N = \left(\begin{array}{ccc} \widetilde{P}_1 & \widetilde{P}_2 & \widetilde{P}_3 \\ \hline abc & cba & cab \\ bac & cab & cba \\ bca & bac & acb \\ acb & bca & abc \\ cab & acb & bca \\ cba & abc & bac \end{array} \right)$$

2.3 Hyper-stability: definition

We are now ready to formally define hyper-stability:

Definition 3 *Given $n \in \mathbb{N}$, a neutral social welfare function α is hyper-stable for the domain $\mathcal{E}$ of preference extensions if for all $m, n \in \mathbb{N}$, for all $P_N \in \mathcal{L}(A_m)^n$, for all $E = (e_1, ..., e_n) \in \mathcal{E}^n$, we have $\Delta(\alpha(P_N)) \cap f_\alpha(P_N^E) \neq \varnothing$. Moreover, α is Kemeny-stable if it is hyper-stable for $\mathcal{K}$.*

A neutral SWF α is hyper-stable for domain E if at every finite profile P_N of linear orders over m alternatives, at least one linear extension of the weak order $\alpha(P_N)$ is ranked first by α when applied to any hyper-profile P_N^E induced from P_N by a vector of preference extensions in $\mathcal{E}$.

Figure 1 below illustrates hyper-stability.

A society with size n has to rank m alternatives, and has agreed on some SWF α as voting rule. Hence, individual ballots are linear orders of alternatives (profile P_N), and ballots are aggregated by

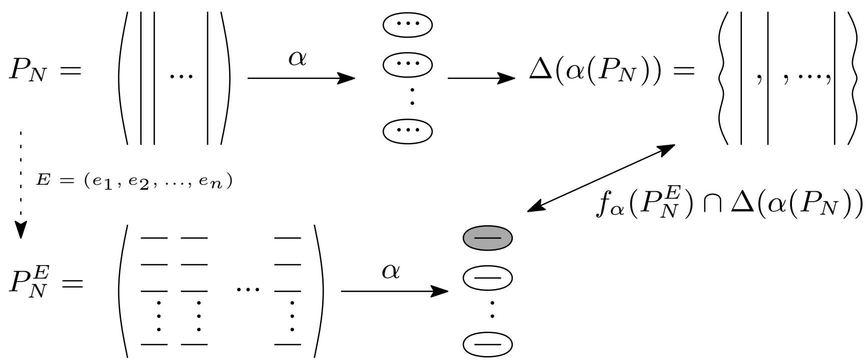

Figure 1: Hyper-stability.

means of α to a weak order $\alpha(P_N)$ of alternatives. Since $\alpha(P_N)$ may involve ties, and since resolute outcomes are linear orders, the final choice results from the use of some tie-breaking rule. The set $\Delta(\alpha(P_N))$ contains all possible outcomes obtained by a tie-breaking rule. Ballots provide little information about preferences over outcomes. We assume that "preferences behind ballots" are induced from ballots by some n-tuple $E = (e_1, ..., e_n)$ of preference extensions. Therefore, the set of ballots P_N together with E generates a profile P_N^E over orders, or hyper-profile. Since α is neutral and defined for any number of alternatives, it can be applied to P_N^E, leading to a weak order $\alpha(P_N^E)$ over outcomes. Hyper-stability prevails for $(e_1, ..., e_n)$ if at least one possible final outcome from ballots is ranked first by α (or, equivalently, chosen by f_α) at any full preference profile.

2.4 Hyper-stability and SW self-selectivity

While the main motivation for studying hyper-stability is that ballots can hardly indicate full preferences over outcomes, another one stems from its close relationship with self-selectivity. Self-selectivity is defined by Koray (2000) for SCFs.[9] Suppose that the society has to choose one alternative among finitely many, as well as the SCF itself. Moreover, suppose that given individual preferences over alternatives, individuals compare SCFs by considering only their respective outcomes. According to this consequentialist principle, initial preferences over alternatives naturally extend to preferences over SCFs: consider any finite subset $\mathcal{G}$ of neutral SCFs together with a profile $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$; define for all $i = 1, ..., n$ the weak order $R(P_i)$ over $\mathcal{G}$ by: $\forall F, G \in \mathcal{G}$, $F \; R^+(P_i) \; G \Leftrightarrow F(P_N) \; P_i \; G(P_N)$, and $F \; R^\sim(P_i) \; G \Leftrightarrow F(P_N) = G(P_N)$, where $R^+(P_i)$ (resp. $R^\sim(P_i)$) is the a-symmetric (resp. symmetric) part of $R(P_i)$. It follows that P_N induces a dual profile of weak orders $P_N^{\mathcal{G}} = (R(P_1), ..., R(P_n))$ over $\mathcal{G}$. *Self-selectivity* holds for a SCF F if, at any profile over alternatives, F selects itself some linearization of the dual profile over any finite set of SCFs. Formally, F is self-selective if for all $m, n \in \mathbb{N}$, for all $P_N \in \mathcal{L}(A_m)^n$, for all finite subsets $\mathcal{G}$ of neutral SCFs with $F \in \mathcal{G}$, there exists a linearization $\widetilde{P}_N^{\mathcal{G}}$ of $P_N^{\mathcal{G}}$ with $F(\widetilde{P}_N^{\mathcal{G}}) = F$. Koray (2000) proves that, given any fixed size

[9]Formally, a function $F : \cup_{m,n\in\mathbb{N}}\mathcal{L}(A_m)^n \to \cup_{m\in\mathbb{N}}A_m$ is a *social choice function* (SCF) if for all $n, m \in \mathbb{N}$ and all $P_N \in \mathcal{L}(A_m)^n$, $F(P_N) \in A_m$. Furthermore A SCF F is *neutral* if for all $n, m \in \mathbb{N}$ and all $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, for any permutation γ of A_m, $F(P_N^\gamma) = \gamma(F(P_N))$, where $P_N^\gamma = (P_1^\gamma, ..., P_n^\gamma) \in \mathcal{L}(A_m)^n$ is defined by: $\forall i \in \{1, ..., n\}$, $\forall a, b \in A_m$, aP_ib if and only if $\gamma(a)P_i^\gamma\gamma(b)$.

n of the society, a neutral and unanimous SCF is self-selective if and only if it is dictatorial.[10]

Self-selectivity for neutral SWFs is defined along the same lines: at any profile over alternatives, a self-selective SWF ranks itself first among finitely many other SWFs. However, since a SWF provides a weak order, there is no longer a natural duality between preferences over alternatives and preferences over SWFs. In order to make the consequentialist principle meaningful, we need to connect both preference levels by means of a preference extension. It follows that self-selectivity is defined conditional to some domain of preference extensions. This last point is the major difference between the SCF and the SWF settings: choosing preference extensions brings an extra degree of freedom in the analysis, which may allow to escape from Koray's impossibility result.

We formalize self-selectivity for SWFs as follows. A SWF α is called *strict* if for all $n, m \in \mathbb{N}$ and all $P_N \in \mathcal{L}(A_m)^n$, one has $\alpha(P_N) \in \mathcal{L}(A_m)$. A linearization of SWF α is a strict SWF α^* such that for all $n, m \in \mathbb{N}$, for all $a, b \in A_m$ and for all $P_N \in \mathcal{L}(A_m)$, one has , $a\ \alpha^*(P_N)\ b$ only if $a\ \alpha(P_N)\ b$. The set of all linearizations of α is denoted by $L(\alpha)$. Pick up a profile $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$ together with a domain $\mathcal{E}$, and consider any finite subset $\mathcal{A} = \{\alpha_1, ..., \alpha_K\}$ of neutral SWFs. A strict selection of $\mathcal{A}$ is a subset $\mathcal{A}^* = \{\alpha_1^*, ..., \alpha_K^*\}$ of linearizations of $\alpha_1, ..., \alpha_K$. For all $1 \leq i \leq n$, define the weak order $\succsim_{P_i}^{\mathcal{A}^*}$ over $\mathcal{A}^*$ by: $\forall 1 \leq k, k' \leq K$, $\alpha_k^* \succ_{P_i}^{\mathcal{A}^*} \alpha_{k'}^* \Leftrightarrow \alpha_k^*(P_N) e_i(P_i) \alpha_{k'}^*(P_N)$, and $\alpha_k^* \sim_{P_i}^{\mathcal{A}^*} \alpha_{k'}^* \Leftrightarrow \alpha_k^*(P_N) = \alpha_{k'}^*(P_N)$ for some $(e_1, ..., e_n) \in \mathcal{E}^n$. Thus, as for SCFs, P_N together with $E = (e_1, ..., e_n) \in \mathcal{E}^n$ induces a dual profile of weak orders $P_N^{E,\mathcal{A}^*} = (\succsim_{P_1}^{\mathcal{A}^*}, ..., \succsim_{P_n}^{\mathcal{A}^*})$ over $\mathcal{A}^*$.

Definition 4 *A neutral SWF α is SW self-selective for the domain of preference extensions E if and only if for all $m, n \in \mathbb{N}$, for all $P_N \in \mathcal{L}(A_m)^n$, for all finite subsets $\mathcal{A}$ of neutral SWFs that contain α, for all strict selection $\mathcal{A}^*$ of $\mathcal{A}$, for any $E = (e_1, ..., e_n) \in \mathcal{E}^n$, there exists a linearization $\widetilde{P}_N^{E,\mathcal{A}^*}$ of $P_N^{E,\mathcal{A}^*}$ for which $L(\alpha) \cap \mathcal{A}^* \cap f_\alpha(\widetilde{P}_N^{E,\mathcal{A}^*}) \neq \varnothing$.*

A neutral SWF α is SW self-selective for domain $\mathcal{E}$ if the following holds: pick up any strict selection $\mathcal{A}^*$ of any finite set $\mathcal{A}$ of neutral SWFs including α, together with any profile P_N over alternatives. Picking up n preference extensions in $\mathcal{E}$ generates from P_N a dual profile of weak orders over $\mathcal{A}^*$. Then, there exists a linearization of this dual profile at which α ranks first at least some of its linearizations in $\mathcal{A}^*$.

Note that, although it offers a natural adaptation of the original concept to SWFs, the formalization of SW self-selectivity sounds complex for two main reasons. First, two different SWFs may have the same outcome at some profile P_N. Therefore, choosing a domain $\mathcal{E}$ is not enough to provide a dual profile of linear orders over SWFs. Second, two SWFs may produce different weak orders at P_N that admit the same linearization. Moreover, note the crucial role played by neutrality, which allows for α to be well-defined for profiles over alternatives and for dual profiles over SWFs.

Proposition 1 below states that hyper-stability is a weaker property than SW self-selectivity.

Proposition 1 *A neutral SWF is SW self-selective for domain $\mathcal{E}$ only if it is hyper-stable for $\mathcal{E}$.*

[10] A SCF F is dictatorial if $\exists 1 \leq i \leq n$ such that, for all $P_N \in \mathcal{L}(A_m)^n$, $F(P_N) = a \Leftrightarrow aP_ib$ for all $b \in A_m/\{a\}$. Moreover, F is unanimous if for any m, for any $P_N \in \mathcal{L}(A_m)^n$, for all $a, b \in A_m$, $[aP_ib$ for all $1 \leq i \leq n] \Rightarrow b \notin F(P_N)$.

3 Scoring rules

We first study hyper-stability of scoring rules. Given a number m of alternatives, a score vector is an element $S^m = (s^{1,m}, s^{2,m}, ..., s^{m,m})$ of $\mathbb{R}^m_+$, where (1) $s^{m,m} = 0$, (2) $s^{1,m} \geq s^{2,m} \geq ... \geq s^{m,m}$, and (3) $s^{1,m} > s^{m,m}$. Given a profile $P_N \in \mathcal{L}(A_m)^n$ together with a score vector S^m, the score of the alternative $x \in A_m$ in P_N is $S^m(x, P_N) = \sum_{i \in N} s^{r_i(x,P_N),m}$, where $r_i(x, P_N)$ is the rank of x in P_i. A SWF α is a *scoring rule* if there exists a sequence $\{S^m_\alpha\}_{m \geq 3} = \{S^1_\alpha, S^2_\alpha, S^3_\alpha...\}$ of score vectors such that, for any $m, n \in \mathbb{N}$, for any $P_N \in \mathcal{L}(A_m)^N$, for any two alternatives $x, y \in A_m$, $x \; \alpha(P_N) \; y \iff S^m_\alpha(x, P_N) \geq S^m_\alpha(y, P_N)$. We begin with the analysis of well-known scoring rules, namely the Borda rule, the plurality rule and the anti-plurality rule.

The *Borda rule* $\mathcal{B}$ is defined by: for any $m \in \mathbb{N}$, for any $k \in \{1, ..., m-1\}$, $s^{k,m}_\mathcal{B} = s^{k+1,m}_\mathcal{B} + 1$. It is easily checked that $\mathcal{B}$ is not Kemeny-stable, hence not hyper-stable for $\mathcal{E}$. Indeed, consider the following profile P_N involving 3 alternatives a, b, c and 6 individuals, where the first row indicates the number of individuals sharing the same preference order

$$P_N = \begin{pmatrix} 3 & 1 & 2 \\ \hline a & c & c \\ b & b & a \\ c & a & b \end{pmatrix}$$

Next, consider the following linearization $\dot{P}_N$ of $P^\mathcal{K}_N$:

$$\dot{P}_N = \begin{pmatrix} 3 & 1 & 2 \\ \hline abc & cba & cab \\ bac & cab & cba \\ acb & bca & acb \\ bca & bac & abc \\ cab & acb & bca \\ cba & abc & bac \end{pmatrix}$$

Finally, $\mathcal{B}(P_N) = \{acb\} = \Delta(\mathcal{B}(P_N))$, whereas $S^6_\mathcal{B}(acb, \dot{P}_N) = 16 < S^6_\mathcal{B}(abc, \dot{P}_N) = 19$ implies that $acb \notin f_\mathcal{B}(\dot{P}_N)$. Since $\Delta(\mathcal{B}(P_N)) \cap f_\mathcal{B}(\dot{P}_N) = \emptyset$, then $\mathcal{B}$ is not Kemeny-stable.

The *plurality rule* is the scoring rule π, where, for any $m \in \mathbb{N}$, $s^{k,m}_\pi = 0$ for any $k = 2, ..., m$, and $s^{1,m}_\pi = 1$. Consider an alteration P'_N of the profile P_N above where the individual with preference cba changes to bca. Then $\pi(P'_N) = \{acb\}$, while, for any linearization $\dot{P}'_N$ of ${P'}^K_N$, $f_\pi(\dot{P}'_N) = \{abc\}$. Hence, π is not Kemeny-stable.

The *anti-plurality rule* is the scoring rule λ, where, for any $m \in \mathbb{N}$, $S^{k,m}_\lambda = 1$ for any $1 \leq k \leq m-1$. Consider the following profile $P_N \in \mathcal{L}(A_3)^{15}$, where $\lambda(P_N) = \{abc\}$, together with its associated Kemeny weak profile P^K_N:

$$P_N = \begin{pmatrix} 3 & 2 & 3 & 3 & 4 \\ \hline a & a & b & c & c \\ b & c & a & a & b \\ c & b & c & b & a \end{pmatrix} \quad P^K_N = \begin{pmatrix} 3 & 2 & 3 & 3 & 4 \\ \hline abc & acb & bac & cab & cba \\ acb, bac & abc, cab & abc, bca & cba, acb & cab, bca \\ bca, cab & cba, bac & acb, cba & abc, bca & bac, acb \\ cba & bca & cab & bac & abc \end{pmatrix}$$

We conclude that, for all $P \in \mathcal{L}(A_6)/\{abc\}$, $P\lambda(\dot{P}_N)abc$ for all $\dot{P}_N \in \Delta(P_N^K)$. Thus, $abc \notin f_\lambda(\dot{P}_N)$, which implies that λ is not Kemeny-stable.

We state below four negative results about Kemeny-stable scoring rules. The key-ingredient of the proofs is the following Theorem, which characterizes Kemeny-stable scoring rules for 3 alternatives.

Theorem 1 *A scoring rule α is Kemeny-stable only if $s_\alpha^{1,3} = 2.s_\alpha^{2,3} > 0$, and $s_\alpha^{1,6} = \frac{4}{3}s_\alpha^{2,6} = \frac{4}{3}s_\alpha^{3,6} = 4s_\alpha^{4,6} = 4s_\alpha^{5,6} > s_\alpha^{6,6} = 0$. Furthermore, this is also sufficient for a scoring rule to be Kemeny-stable for three alternative case.*

A scoring rule α is *non-truncated* if there exists no $m \in \mathbb{N}$ and no $k \in \{2, ..., m-1\}$ such that $s_\alpha^{k,m} = 0$: the score vector defined for some number m of alternatives gives a strictly positive score to any rank above the last one.

Theorem 2 *There is no Kemeny-stable and non-truncated scoring rule.*

A scoring rule α is *strict-at-top* if, for any $m \in \mathbb{N}$, $s_\alpha^{1,m} > s_\alpha^{2,m}$: all score vectors give a score to the top-ranked alternative strictly higher than any other score. Typical examples of strict-at-top scoring rules are the plurality and the Borda rules. Note that any convex scoring rule is also strict-at-top[11].

Theorem 3 *There is no Kemeny-stable and strict-at-top scoring rule.*

Since a unanimous scoring rule must be strict-at-top and non-truncated, we can state the following corollary of Theorems 2 and 3.

Theorem 4 *There is no Kemeny-stable and unanimous scoring rule.*

When enlarging the Kemeny domain $\mathcal{K}$ to the domain $\mathcal{S}$ of separable preference extensions, we get an even stronger negative result:

Theorem 5 *No scoring rule is hyper-stable for $\mathcal{S}$.*

4 Condorcet social welfare functions

We turn now to the analysis of Condorcet SWFs. We begin with some additional notations and definitions. Given a profile $P_N \in \mathcal{L}(A_m)^n$, where n is odd, the majority tournament for P_N is the complete and asymmetric binary relation $\mu(P_N)$ defined over $A_m \times A_m$ by: $\forall (x, y) \in A_m \times A_m$, $x \; \mu(P_N) \; y \Leftrightarrow |\{i \in N : xP_iy\}| > |\{i \in N : yP_ix\}|$. Given any P_N, the *Condorcet winner* of P_N is the element $CW(P_N) \in A_m$ such that $CW(P_N) \; \mu(P_N) \; a$ for all $a \in A_m/CW(P_N)$. A SWF α is *Condorcet* if, for any profile, α ranks the Condorcet winner at top whenever it exists.

We prove below the existence of a Condorcet SWF hyper-stable for $\mathcal{S}$. Beforehand, we show that three well-known Condorcet SWFs violate Kemeny stability. The *Copeland solution* is the SWF φ defined by: $\forall m \in \mathbb{N}$, $\forall n \in 2\mathbb{N}+1$, $\forall \; P_N \in \mathcal{L}(A_m)^n$, $\forall x, y \in A_m$, $x \; \varphi(P_N) \; y \Leftrightarrow c(x, P_N) \geq c(y, P_N)$, where $c(x, P_N) = |\{z \in A_m : x \; \mu(P_N) \; z\}|$. Consider the following profile P_N, together with the linearization $\dot{P}_N$ of P_N^K:

[11]A scoring rule α is convex if, for any $m \in \mathbb{N}$, the score vector $S_\alpha^m = (s_\alpha^{1,m}, ..., s_\alpha^{m,m})$ is such that $(s_\alpha^{1,m} - s_\alpha^{2,m}) \geq (s_\alpha^{2,m} - s_\alpha^{3,m}) \geq ... \geq (s_\alpha^{m-1,m} - s_\alpha^{m,m})$.

$$P_N = \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ \hline a & a & b & b & c \\ b & c & c & a & a \\ c & b & a & c & b \end{array}\right) \quad \dot{P}_N = \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ \hline abc & acb & bca & bac & cab \\ acb & cab & bac & bca & acb \\ bac & abc & cba & abc & cba \\ cab & cba & cab & acb & bca \\ bca & bac & abc & cba & abc \\ cba & bca & acb & cab & bac \end{array}\right)$$

Then, we have $\varphi(P_N) = abc$, while $c(abc, \dot{P}_N) = 3 < c(acb, \dot{P}_N) = 4$ implies that $\Delta(\varphi(P_N)) \cap f_\varphi(\dot{P}_N) = \varnothing$. Thus, φ is not Kemeny-stable.

The *Slater solution* is the social welfare correspondence[12] β defined by: $\forall m \in \mathbb{N}$, $\forall n \in 2\mathbb{N}+1$, $\forall\ P_N \in \mathcal{L}(A_m)^n$, $\forall P \in \mathcal{L}(A_m)$, $\beta(P_N) = ArgMin_{P \in \mathcal{L}(A_m)} d_K(P, \mu(P_N))$. A SWF α is *Slater-consistent* if, at any profile P_N, it always selects one linear order in $\beta(P_N)$. Consider the following profile $P_N \in \mathcal{L}(A_8)^5$:

$$P_N = \left(\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ \hline b & a & d & c & d \\ c & b & a & a & b \\ d & c & b & d & c \\ a & d & c & b & a \\ a' & b' & d' & d' & c' \\ b' & c' & a' & b' & a' \\ c' & d' & b' & c' & d' \\ d' & a' & c' & a' & b' \end{array}\right)$$

Define $X = \{a, b, c, d\}$ and $Y = \{a', b', c', d'\}$ and consider the restrictions $P_N|_X$ and $P_N|_Y$ of P_N to X and Y respectively. We have that $\mu(P_N|_X)$ and $\mu(P_N|_Y)$ are isomorphic. Moreover, we observe that (1) $a\mu(P_N)b\mu(P_N)c\mu(P_N)d\mu(P_N)a$, (2) $c\mu(P_N)a$, (3) $d\mu(P_N)b$, and (4) all alternatives in X defeat all alternatives in Y . This ensures that $\beta(P_N|_X) = \{cdab\}$ and $\beta(P_N|_Y) = \{c'd'a'b'\}$. Thus, $\beta(P_N) = \{cdabc'd'a'b'\}$. Now, consider $Q = dbcad'b'c'a'$. The next table gives the Kemeny distances between each of the 5 linear orders in $P = (P_1, ..., P_5)$ and respectively, $\beta(P_N)$ and Q:

P_i	$\beta(P_N)$	Q
P_1	$3+4$	$2+5$
P_2	$4+3$	$5+2$
P_3	$3+3$	$2+2$
P_4	$1+3$	$4+0$
P_5	$3+1$	$0+4$

.

It follows that in the Kemeny weak profile P_N^K, Q is strictly preferred to $\beta(P_N)$ by individual 3, while all other individuals are indifferent. Hence, there exists a linearization $\dot{P}_N$ of P_N^K where Q is

[12]A *social welfare correspondence* is a mapping δ from $\underset{n,m\in\mathbb{N}}{\cup}\mathcal{L}(A_m)^n$ to $\underset{m\in\mathbb{N}}{\cup}2^{\mathcal{R}(A_m)}$ such that, for any $n, m \in \mathbb{N}$, for any $P_N \in \mathcal{L}(A_m)^n$, $\delta(P_N) \in 2^{\mathcal{R}(A_m)}$, where $2^{\mathcal{R}(A_m)}$ is the set of all non-empty subsets of weak orders over A_m.

unanimously preferred to $\beta(P_N)$. Since the Slater solution is contained in the Pareto set, and since $\beta(P_N)$ is a singleton, we conclude that no Slater-consistent SWF is Kemeny-stable.

The *Kemeny rule* is the Condorcet social welfare correspondence ω defined by: $\forall\ P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n, \forall P \in \mathcal{L}(A_m), \omega(P_N) = ArgMin_{P\in\mathcal{L}(A_m)} \sum_{i\in N} d_K(P, P_i)$. A SWF α is *Kemeny-consistent* if, for any profile P_N, it always selects a linear order in $\omega(P_N)$. Consider the following profile $P_N \in \mathcal{L}(A_3)^9$ together with the linearization $\dot{P}_N$ of P_N^K:

$$P_N = \begin{pmatrix} 2 & 3 & 4 \\ \hline b & c & a \\ c & a & b \\ a & b & c \end{pmatrix} \quad \dot{P}_N = \begin{pmatrix} 2 & 3 & 4 \\ \hline bca & cab & abc \\ cba & cba & acb \\ bac & acb & bac \\ cab & bca & cab \\ abc & abc & bca \\ acb & bac & cba \end{pmatrix}$$

The reader will check that $\omega(P_N) = \{abc\}$, whereas $\omega(\dot{P}_N) = \{(cab)(abc)(acb)(bca)(cba)(bac)\}$ which leads to $f_\omega(\dot{P}_N) = \{cab\}$. Hence, there is no Kemeny-stable and Kemeny-consistent SWF.

We now establish the existence of a Condorcet and unanimous SWF which is hyper-stable for $\mathcal{S}$. The *transitive closure* $\theta(P_N)$ of $\mu(P_N)$ is defined by: $\forall x, y \in A_m$, $x\theta(P_N)y$ if and only if there exist $x_1, x_2, ..., x_H \in A_m$ such that $x\mu(P_N)x_1$, $x_1\mu(P_N)x_2$, ... , $x_H\mu(P_N)y$. Consider the SWF θ, which maps every profile $P_N \in \cup_{m,n}\mathcal{L}(A_m)^n$ (where n is odd) to the transitive closure $\theta(P_N)$ of $\mu(P_N)$. It is easily checked that θ is unanimous.

Theorem 6 *θ is hyper-stable for $\mathcal{S}$.*

5 Discussion

Our main result is that no unanimous scoring rule is Kemeny-stable, hence hyper-stable for the larger domain $\mathcal{S}$ of separable preference extensions. However, the transitive closure of the majority relation is a unanimous Condorcet SWF that is hyper-stable for $\mathcal{S}$.

Hyper-stability does not draw a clear border between scoring rules and Condorcet SWFs. Indeed, several Condorcet SWFs based on well-known tournament solutions, as well as the Kemeny SWF, are not Kemeny-stable. Characterizing the class of Condorcet SWFs hyper-stable for $\mathcal{S}$ is an open question worth being addressed. Another open problem is studying hyper-stability for non-unanimous scoring rules.

Further open questions relate to alternative concepts of hyper-stability.

5.1 Alternative hyper-stability concepts

Any *strongly Condorcet*[13] SWF α violates the following property of hyper Condorcet-stability: A SWF α is *hyper Condorcet-stable* if $\forall n, m \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$, $\forall E \in \mathcal{S}^n$, $\alpha(P_N) \in \mathcal{L}(A_m) \Rightarrow$

[13] A strongly Condorcet α is such that for any $m \in \mathbb{N}$, for any $n \in 2\mathbb{N}+1$ and for any $P_N \in \mathcal{L}(A_m)^n$ we have $\mu(P_N) \in \mathcal{L}(A_m)$ only if $\alpha(P_N) = \mu(P_N)$.

$[\alpha(P_N) = CW(P_N^E)]$. To see why, consider the following profile $P_N \in \mathcal{L}(A_m)^5$, together with the Kemeny hyper-profile $\dot{P}_N \in \Delta(P_N^K)$:

$$P_N = \begin{pmatrix} 1 & 1 & 1 \\ \hline a & a & b \\ b & c & c \\ c & b & a \end{pmatrix} \dot{P}_N = \begin{pmatrix} 1 & 1 & 1 \\ \hline abc & acb & bca \\ bac & cab & cba \\ acb & abc & bac \\ bca & cba & cab \\ cab & bac & abc \\ cba & bca & acb \end{pmatrix}$$

Since $\mu(P_N) = abc$, then $\alpha(P_N) = abc$ for any strongly Condorcet α. However, $\alpha(P_N)$ is defeated in $\mu(\dot{P}_N)$ by cab. An interesting question is whether any strongly Condorcet α satisfies the following weaker version of hyper Condorcet-stability: $\forall n, m \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$ such that $\alpha(P_N) \in \mathcal{L}(A_m)$, there exists $E \in \mathcal{S}^n$ for which $\alpha(P_N) = CW(P_N^E)$.

Note that there are strongly Condorcet SWFs that are hyper-stable for $\mathcal{S}$. Define the SWF ψ by: $\forall m, n \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$, $\psi(P_N) = \mu(P_N)$ if $\mu(P_N) \in \mathcal{L}(A_m)$, and otherwise, $a\ \psi(P_N)\ b$ and $b\ \psi(P_N)\ a$ for all $a, b \in A_m$. Then ψ is hyper stable for $\mathcal{S}$. This is an immediate corollary of the following proposition:

Proposition 2 *Let $P_N \in \mathcal{L}(A_m)^n$ be such that $\mu(P_N) \in \mathcal{L}(A_m)$. For any $E \in \mathcal{S}^n$, either $CW(P_N^E)$ does not exist, or $CW(P_N^E) = \mu(P_N)$.*

Remark that, in the Kemeny hyper-profile $\dot{P}_N$ above, all three individual preferences are extended through the same linearization of the Kemeny weak order. This common linearization can be defined as a linear order over the permutations of the set $\{1, 2, 3\}$ of ranks. Indeed, given two orders P and $Q = (a_1 a_2 ... a_m)$ in $\mathcal{L}(A_m)$, define $r_P(Q) = (r_P(a_1), ..., r_P(a_m))$ by $\forall h = 1, ..., m$, $r_P(a_h) = |\{b \in A_m : bPa_h\}| + 1$, that is, the rank given to a_h in P. Moreover, given $P_N = (P_1, ..., P_n) \in \mathcal{L}(A_m)^n$, we say that the hyper-profile $P_N^E = (e_1(P_1), ..., e_n(P_n))$ is *uniform* if there exists a linear order $\succ$ over the permutations of $\{1, ..., m\}$ such that, for any $i = 1, ..., n$, for any $Q, Q' \in \mathcal{L}(A_m)$, $[Q\ e^i(P_i)\ Q' \Leftrightarrow r_{P_i}(Q) \succ r_{P_i}(Q')]$. In the example above, $\succ$ is defined by: $(123) \succ (213) \succ (132) \succ (231) \succ (312) \succ (321)$. We say that a SWF α is *uniformly hyper-stable* for $\mathcal{S}$ if $\forall n, m \in \mathbb{N}$, $\forall P_N \in \mathcal{L}(A_m)^n$, $\Delta(\alpha(P_N)) \cap f_\alpha(P_N^E) \neq \emptyset$ for all uniform hyper-profiles P_N^E with $E \in \mathcal{S}^n$.

As a first step towards a complete study of uniform hyper-stability, we remark that the Borda rule $\mathcal{B}$ is not uniformly hyper-stable. To see why, consider the following profile P_N, together with the Kemeny hyper-profile $\dot{P}_N \in \Delta(P_N^K)$:

$$P_N = \begin{pmatrix} 1 & 1 & 2 \\ \hline a & b & c \\ b & a & b \\ c & c & a \end{pmatrix} \dot{P}_N = \begin{pmatrix} 1 & 1 & 2 \\ \hline abc & bac & cba \\ acb & bca & cab \\ bac & abc & bca \\ cab & cba & acb \\ bca & acb & bac \\ cba & cab & abc \end{pmatrix}$$

We get $\mathcal{B}(P_N) = bca$. Moreover, $\dot{P}_N$ is uniform (to see why, consider $(123) \succ (132) \succ (213) \succ (312) \succ (231) \succ (321)$). Finally, $S^6_{\mathcal{B}}(bca, \dot{P}_N) = 11 < S^6_{\mathcal{B}}(cba, \dot{P}_N) = 12$.

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Appendices

A Proofs

A.1 Proof of Proposition 1

Let α be a neutral SFW that is SW self-selective for some domain $\mathcal{E}$. Pick up any profile $P_N \in \mathcal{L}(A_m)^n$ where $\Delta(\alpha(P_N)) = \{Q_1, ..., Q_H\}$, together with any $E = (e_1, ..., e_n) \in \mathcal{E}^n$. Consider the set of SWFs $\mathcal{A} = \{\alpha_1, \alpha_2, ..., \alpha_H, \rho_1, ..., \rho_{m!-H}\}$ such that:

- $\alpha_1(P_N) = Q_1, ..., \alpha_H(P_N) = Q_H$
- $\forall k \neq k' \in \{1, ..., m! - H\}, \alpha_k(P_N) \neq \alpha_{k'}(P_N)$
- $\cup_{1 \leq k \leq m!-H} \rho_k(P_N) = \mathcal{L}(A_m) - \Delta(\alpha(P_N))$

Since all elements of $\mathcal{A}$ are strict SWFs, then $\mathcal{A}$ is a strict selection. Moreover, all elements of $\mathcal{A}$ having different outcomes from P_N, then $P_N^{E,\mathcal{A}}$ is a profile of linear orders over $\mathcal{A}$. Furthermore, $[\cup_{1\leq h\leq H}\alpha_h(P_N)]\cup[\cup_{1\leq k\leq m!-1}\rho_k(P_N)]=\mathcal{L}(A_m)$ implies that $P_N^{E,\mathcal{A}}$ is a profile over all $m!$ linear orders, so that $L(\alpha)\cap\mathcal{A}=L(\alpha)$. It follows from definition that $P_N^{E,\mathcal{A}}$ is isomorphic to P_N^E. Since α is SW self-selective for E, then $L(\alpha)\cap f_\alpha(\widetilde{P}_N^{E,\mathcal{A}})\neq\varnothing$. Since $P_N^{E,\mathcal{A}^*}$ is isomorphic to P_N^E, the neutrality of α ensures that $\Delta(\alpha(P_N))\cap f_\alpha(P_N^E)\neq\varnothing$ and the conclusion follows.

A.2 Proof of Theorem 1

We first prove the following three propositions, each providing a necessary condition for Kemeny-stability.

Proposition 3 *A scoring rule α is Kemeny-stable only if $s_\alpha^{2,3}>0$ and $s_\alpha^{1,6}>s_\alpha^{2,6}=s_\alpha^{3,6}>s_\alpha^{4,6}=s_\alpha^{5,6}>s_\alpha^{6,6}=0$.*

Proposition 4 *A scoring rule α is Kemeny-stable only if $s_\alpha^{1,6}=s_\alpha^{2,6}+s_\alpha^{4,6}$.*

Proposition 5 *A scoring rule α is Kemeny-stable only if $s_\alpha^{1,3}=2s_\alpha^{2,3}$.*

A.2.1 Proof of Proposition 3

The proof is organized in six 6 intermediate lemmas:

Lemma 1 *If α is a Kemeny-stable scoring rule, then $s_\alpha^{2,3}>0$.*

Proof: Suppose that $s_\alpha^{2,3}=0$, and consider the following profile $P_N\in\mathcal{L}(A_3)^{n_1+n_2+n_3+n_4}$, where $n_1>n_2>n_3+n_4$, together with the following linearization $\dot{P}_N$ of P_N^K:

$$P_N=\begin{pmatrix} n_1 & n_2 & n_3 & n_4\\ \hline a & b & c & c\\ c & c & a & b\\ b & a & b & a \end{pmatrix}\quad \dot{P}_N=\begin{pmatrix} n_1 & n_2 & n_3 & n_4\\ \hline acb & bca & cab & cba\\ cab & bac & acb & cab\\ abc & cba & cba & bca\\ bac & cab & bca & acb\\ cba & abc & abc & bac\\ bca & acb & bac & abc \end{pmatrix}$$

It follows that $\Delta(\alpha(P_N))=\{abc\}$. Kemeny-stability requires that $S_\alpha^6(abc,\dot{P}_N)=n_1s_\alpha^{3,6}+(n_2+n_3)s_\alpha^{5,6}\geq S_\alpha^6(cab,\dot{P}_N)=(n_1+n_4)s_\alpha^{2,6}+n_2s_\alpha^{4,6}+n_3s_\alpha^{1,6}$, hence that $n_1(s_\alpha^{3,6}-s_\alpha^{2,6})+n_2(s_\alpha^{5,6}-s_\alpha^{4,6})+n_3(s_\alpha^{5,6}-s_\alpha^{1,6})\geq n_4s_\alpha^{2,6}$, which is clearly impossible □

Lemma 2 *If α is a Kemeny-stable scoring rule, then $s_\alpha^{2,6}=s_\alpha^{3,6}$ and $s_\alpha^{4,6}=s_\alpha^{5,6}$.*

Proof: Suppose first that $s_\alpha^{1,3}>2s_\alpha^{2,3}$, and consider $P_N\in\mathcal{L}(A_3)^4$, and $\dot{P}_N\in\Delta(P_N^K)$:

$$P_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & b & a & a \end{pmatrix} \quad \dot{P}_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline abc & acb & bca & cba \\ bac & cab & cba & bca \\ acb & abc & bac & cab \\ bca & cba & cab & bac \\ cab & bac & abc & acb \\ cba & bca & acb & abc \end{pmatrix}$$

Since $S^3_\alpha(a, P_N) = 2s^{1,3}_\alpha$, and $S^3_\alpha(b, P_N) = S^3_\alpha(c, P_N) = s^{1,3}_\alpha + 2s^{2,3}_\alpha$, then $\Delta(\alpha(P_N)) = \{abc, acb\}$. Moreover, we have (1) $S^6_\alpha(abc, \dot{P}_N) = S^3_\alpha(acb, \dot{P}_N) = s^{1,6}_\alpha + s^{3,6}_\alpha + s^{5,6}_\alpha$, and (2) $S^6_\alpha(bca, \dot{P}_N) = s^{1,6}_\alpha + s^{2,6}_\alpha + s^{4,6}_\alpha$. Kemeny stability implies from (1) and (2) that $s^{3,6}_\alpha + s^{5,6}_\alpha \geq s^{2,6}_\alpha + s^{4,6}_\alpha$ (3), which in turn leads to $s^{2,6}_\alpha = s^{3,6}_\alpha$ and $s^{4,6}_\alpha = s^{5,6}_\alpha$.

Suppose now that $s^{1,3}_\alpha < 2s^{2,3}_\alpha$, and consider the same profile P_N above and the hyper-profile $\dot{P}'_N \in \Delta(P^K_N)$ obtained from $\dot{P}_N$ by switching in each order alternatives respectively ranked (1) second and third, and (2) fourth and fifth. We get $\Delta(\alpha(P_N)) = \{bca, cba\}$, and we reach the same conclusion as above by a symmetric argument. Finally, suppose that $s^{1,3}_\alpha = 2s^{2,3}_\alpha$, and consider the profile $P_N \in \mathcal{L}(A_3)^{4Z-1}$ below, where $Z > 1$, together with the Kemeny weak profile P^K_N:

$$P_N = \begin{pmatrix} Z & Z & Z-1 & Z \\ \hline a & b & c & c \\ b & a & b & a \\ c & c & a & b \end{pmatrix} \quad P^K_N = \begin{pmatrix} Z & Z & Z-1 & Z \\ \hline abc & bac & cba & cab \\ acb, bac & abc, bca & bca, cab & cba, acb \\ bca, cab & cba, acb & acb, bac & abc, bca \\ cba & cab & abc & bac \end{pmatrix}$$

Then $\alpha(P_N) = abc$. Moreover, there exists $\dot{P}_N \in \Delta(P^K_N)$ such that $S^6_\alpha(abc, \dot{P}_N) = Z(s^{1,6}_\alpha + s^{3,6}_\alpha + s^{5,6}_\alpha)$ and $S^6_\alpha(cab, \dot{P}_N) = Zs^{1,6}_\alpha + (Z-1)s^{2,6}_\alpha + Zs^{4,6}_\alpha$. Kemeny stability requires that $s^{3,6}_\alpha + s^{5,6}_\alpha \geq \frac{Z-1}{Z}s^{2,6}_\alpha + s^{4,6}_\alpha$ for all $Z > 1$. Thus, $s^{2,6}_\alpha + s^{4,6}_\alpha \leq s^{3,6}_\alpha + s^{5,6}_\alpha$, and hence $s^{2,6}_\alpha = s^{3,6}_\alpha$ and $s^{4,6}_\alpha = s^{5,6}_\alpha$ $\square$

We assume in the sequel that α is such that $s^{2,6}_\alpha = s^{3,6}_\alpha$ and $s^{4,6}_\alpha = s^{5,6}_\alpha$ (property $(*)$). Clearly, $(*)$ implies that given any profile P_N over 3 alternatives, given any Kemeny-stable SWF α, one has $\alpha(\dot{P}_N) = \alpha(\widetilde{P_N})$ for any two $\forall \dot{P}_N, \widetilde{P_N} \in \Delta(P^K_N)$.

Lemma 3 *If α is a Kemeny-stable scoring rule, then* $[s^{1,6}_\alpha = s^{2,6}_\alpha] \Rightarrow [s^{4,6}_\alpha = s^{5,6}_\alpha > 0]$.

Proof: Consider the following $P_N \in \mathcal{L}(A_3)^{3Z+W}$ below, where $Z, W \geq 1$ are chosen such that $W < \frac{s^{2,3}_\alpha}{s^{1,3}_\alpha}Z$:

$$P_N = \begin{pmatrix} Z & Z & Z & W \\ \hline a & b & c & a \\ b & a & b & c \\ c & c & a & b \end{pmatrix}$$

Then $\alpha(P_N) = bac$. Furthermore, using $(*)$ together with Kemeny stability and $s^{1,6}_\alpha = s^{2,6}_\alpha$, one must have $S^6_\alpha(bac, \dot{P}_N) = 2Zs^{1,6}_\alpha + (Z+W)s^{5,6}_\alpha \geq S^6_\alpha(abc, \dot{P}_N) = (2Z+W)s^{1,6}_\alpha$. Thus, $s^{1,6}_\alpha \leq \frac{Z+W}{W}s^{5,6}_\alpha$. Finally, since $s^{1,6}_\alpha > 0$, then $s^{5,6}_\alpha > 0$ $\square$

Lemma 4 *If α is a Kemeny-stable scoring rule, then* $[s_\alpha^{1,6} = s_\alpha^{2,6}] \Rightarrow [2s_\alpha^{1,3} = 3s_\alpha^{2,3}]$.

Proof: Define the two profiles $P_N \in \mathcal{L}(A_3)^5$ and $P'_N \in \mathcal{L}(A_3)^{3Z+1}$, where $Z > 1$, as follows:

$$P_N = \begin{pmatrix} 2 & 1 & 1 & 1 \\ \hline a & a & c & b \\ b & c & b & c \\ c & b & a & a \end{pmatrix} \quad P'_N = \begin{pmatrix} 2Z & 1 & Z \\ \hline a & c & c \\ b & a & b \\ c & b & a \end{pmatrix}$$

Suppose first that $2s_\alpha^{1,3} > 3s_\alpha^{2,3}$. It follows from $2s_\alpha^{1,3} > 3s_\alpha^{2,3}$ that $\alpha(P_N) = abc$. Using $(*)$, we have $s_\alpha^{1,6} = s_\alpha^{2,6} = s_\alpha^{3,6} \geq s_\alpha^{4,6} = s_\alpha^{5,6}$. Hence, $\forall \dot{P}_N \in \Delta(P_N^K)$, $S_\alpha^6(abc, \dot{P}_N) = 3s_\alpha^{1,6} + s_\alpha^{5,6}$, and $S_\alpha^6(bac, \dot{P}_N) = 3s_\alpha^{1,6} + 2s_\alpha^{5,6}$. Since Kemeny-stability requires $S_\alpha^6(abc, \dot{P}_N) \geq S_\alpha^6(bac, \dot{P}_N)$, then we get $s_\alpha^{5,6} = 0$, in contradiction with Lemma 3.

Similarly, suppose that $2s_\alpha^{1,3} < 3s_\alpha^{2,3}$. From $0 < 2s_\alpha^{1,3} < 3s_\alpha^{2,3}$, we get that $\alpha(P'_N) = bac$ for Z large enough. Moreover, $\forall \dot{P}_N \in \Delta(P_N'^K)$, $S_\alpha^6(bac, \dot{P}_N) = Z(2s_\alpha^{1,6} + s_\alpha^{5,6}) < S_\alpha^6(acb, \dot{P}_N) = Z(2s_\alpha^{1,6} + s_\alpha^{5,6}) + s_\alpha^{1,6}$, in contradiction with Kemeny stability □

Lemma 5 *If α is a Kemeny-stable scoring rule, then* $s_\alpha^{1,6} > s_\alpha^{2,6}$.

Proof: Suppose that $s_\alpha^{1,6} = s_\alpha^{2,6}$. From Lemma 3 and 4 together with $(*)$, we have $s_\alpha^{1,6} = s_\alpha^{2,6} = s_\alpha^{3,6}$, $2s_\alpha^{1,3} = 3s_\alpha^{2,3}$, and $s_\alpha^{4,6} = s_\alpha^{5,6} > 0$. Then, consider the following profile $P_N \in \mathcal{L}(A_3)^4$:

$$P_N = \begin{pmatrix} 2 & 1 & 1 & 1 \\ \hline a & b & c & c \\ b & a & b & a \\ c & c & a & b \end{pmatrix}$$

Since $S_\alpha^3(a, P_N) = 2s_\alpha^{1,3} + 2s_\alpha^{2,3}$, $S_\alpha^3(b, P_N) = s_\alpha^{1,3} + 3s_\alpha^{2,3}$, and $S_\alpha^3(c, P_N) = 2s_\alpha^{1,3}$, then, using Lemma 1 and Lemma 4, $\alpha(P_N) = abc$. From Kemeny-stability, we have that for any $\dot{P}_N \in \Delta(P_N^K)$, $S_\alpha^6(abc, \dot{P}_N) = 3s_\alpha^{1,6} + s_\alpha^{5,6} \geq S_\alpha^6(acb, \dot{P}_N) = 3s_\alpha^{1,6} + 2s_\alpha^{5,6}$. But this implies that $s_\alpha^{5,6} = 0$, in contradiction with Lemma 3 □

Lemma 6 *If α is a Kemeny-stable scoring rule, then* $s_\alpha^{3,6} > s_\alpha^{4,6}$.

Proof: Suppose that $s_\alpha^{3,6} = s_\alpha^{4,6}$. It follows from Lemma 2 together with Lemma 5 that $s_\alpha^{1,6} > s_\alpha^{2,6} = s_\alpha^{3,6} = s_\alpha^{4,6} = s_\alpha^{5,6} \geq s_\alpha^{6,6} = 0$. Using Lemma 1, we get the following possible cases:

Case 1: $s_\alpha^{1,3} = s_\alpha^{2,3} > 0$

Consider the 4 following profiles:

$$P_N = \begin{pmatrix} 3 & 2 & 3 & 3 & 4 \\ \hline a & a & b & c & c \\ b & c & a & a & b \\ c & b & c & b & a \end{pmatrix} \quad P'_N = \begin{pmatrix} 1 & 3 & 1 \\ \hline a & b & c \\ b & a & a \\ c & c & b \end{pmatrix} \quad P''_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & a & b \\ c & b & c & a \end{pmatrix} \quad P'''_N = \begin{pmatrix} 2 & 2 & 1 \\ \hline a & b & c \\ c & a & a \\ b & c & b \end{pmatrix}$$

If $s_\alpha^{2,6} > 0$, then $\alpha(P_N) = abc$. Since $S_\alpha^6(abc, \dot{P}_N)$ $= 3s_\alpha^{1,6} + 8s_\alpha^{2,6} < S_\alpha^6(cba, \dot{P}_N) = 4s_\alpha^{1,6} + 8s_\alpha^{2,6}$ for all $\dot{P}_N \in \Delta(P_N^K)$, then α is not Kemeny-stable. If $s_\alpha^{2,6} = 0$, then $\alpha(P'_N) = abc$. Since $f^\alpha(\dot{P}_N) = \{bac\}$ for all $\dot{P}_N \in \Delta(P_N'^K)$, then α is not Kemeny-stable.

Case 2: $s_\alpha^{1,3} > s_\alpha^{2,3} > 0$

If $s_\alpha^{2,6} > 0$, then $\alpha(P''_N) = abc$. Since $S_\alpha^6(abc, \dot{P}_N)$ $= s_\alpha^{1,6} + 2s_\alpha^{2,6} < S_\alpha^6(bac, \dot{P}_N) = s_\alpha^{1,6} + 3s_\alpha^{2,6}$ for all $\dot{P}_N \in \Delta(P_N''^K)$, then α is not Kemeny-stable. Finally, if $s_\alpha^{2,6} = 0$, then $\alpha(P'''_N) = abc$. Since $S_\alpha^6(abc, \dot{P}_N) = 0 < S_\alpha^6(acb, \dot{P}_N) = 2s_\alpha^{1,6}$ for all $\dot{P}_N \in \Delta(P_N'''^K)$, then α is not Kemeny-stable.

Thus, Kemeny-stability requires that $s_\alpha^{3,6} > s_\alpha^{4,6}$ □

By combining the six lemmas above, we get that any Kemeny-stable scoring rule α must satisfy (1) $s_\alpha^{1,6} > s_\alpha^{2,6} = s_\alpha^{3,6} > s_\alpha^{4,6} = s_\alpha^{5,6} \geq 0 = s_\alpha^{6,6}$, and (2) $s_\alpha^{1,3} \geq s_\alpha^{2,3} > 0 = s_\alpha^{3,3}$, hence Proposition 3.

A.2.2 Proof of Proposition 4

Suppose that $s_\alpha^{1,3} > 2s_\alpha^{2,3}$, and consider profiles $P_N, P'_N \in \mathcal{L}(A_3)^4$ below:

$$P_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & b & a & a \end{pmatrix} \quad P'_N = \begin{pmatrix} 1 & 2 & 1 \\ \hline a & a & b \\ b & c & c \\ c & b & a \end{pmatrix}$$

Since $S_\alpha^3(a, P_N) = 2s_\alpha^{1,3}$ and $S_\alpha^3(b, P_N) = S_\alpha^3(c, P_N) = s_\alpha^{1,3} + 2s_\alpha^{2,3}$, then $s_\alpha^{1,3} > 2s_\alpha^{2,3} \Rightarrow \Delta(\alpha(P_N) = \{abc, acb\}$. Using Proposition 3, we get that for any $\dot{P}_N \in \Delta(P_N^K)$, $S_\alpha^6(abc, \dot{P}_N) = s_\alpha^{1,6} + s_\alpha^{2,6} + s_\alpha^{5,6}$, while $S_\alpha^6(bac, \dot{P}_N) = 2s_\alpha^{2,6} + 2s_\alpha^{5,6}$. Therefore, Kemeny-stability requires $s_\alpha^{1,6} \geq s_\alpha^{2,6} + s_\alpha^{5,6}$. Similarly, since $S_\alpha^3(a, P'_N) = 3s_\alpha^{1,3}$, $S_\alpha^3(b, P'_N) = s_\alpha^{1,3} + s_\alpha^{2,3}$ and $S_\alpha^3(c, P'_N) = 3s_\alpha^{2,3}$, then $s_\alpha^{1,3} > 2s_\alpha^{2,3} \Rightarrow \alpha(P'_N) = abc$. For any $\dot{P}'_N \in \Delta(P_N'^K)$, $S_\alpha^6(abc, \dot{P}'_N) = s_\alpha^{1,6} + 2s_\alpha^{2,6} + s_\alpha^{5,6}$, while $S_\alpha^6(acb, \dot{P}'_N) = 2s_\alpha^{1,6} + s_\alpha^{2,6}$. Thus, Kemeny-stability requires $s_\alpha^{1,6} \leq s_\alpha^{2,6} + s_\alpha^{5,6}$. Therefore, if $s_\alpha^{1,3} > 2s_\alpha^{2,3}$, then $s_\alpha^{1,6} = s_\alpha^{2,6} + s_\alpha^{5,6}$.

Suppose that $s_\alpha^{1,3} < 2s_\alpha^{2,3}$, and consider profiles $\widetilde{P}_N \in \mathcal{L}(A_3)^{5Z+1}$, where $Z > 1$, and $\overline{P}_N \in \mathcal{L}(A_3)^4$ below:

$$\widetilde{P}_N = \begin{pmatrix} 2Z & Z+1 & Z & Z \\ \hline a & c & c & b \\ b & b & a & c \\ c & a & b & a \end{pmatrix} \quad \overline{P}_N = \begin{pmatrix} 1 & 1 & 1 & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & b & a & a \end{pmatrix}$$

Since $S_\alpha^3(a, \widetilde{P}_N) = 2Zs_\alpha^{1,3} + Zs_\alpha^{2,3}$, $S_\alpha^3(b, \widetilde{P}_N) = Zs_\alpha^{1,3} + (3Z+1)s_\alpha^{2,3}$ and $S_\alpha^3(c, \widetilde{P}_N) = (2Z+1)s_\alpha^{1,3} + Zs_\alpha^{2,3}$, then if Z is chosen large enough, $s_\alpha^{1,3} > 2s_\alpha^{2,3} \Rightarrow \alpha(P_N) = bca$. Moreover, using again Proposition 3, Kemeny-stability implies that for any $Z > 1$ and any $\dot{P}_N \in \Delta(\widetilde{P}_N^K)$, $S_\alpha^6(bca, \dot{P}_N) \geq S_\alpha^6(abc, \dot{P}_N)$. Thus, $Zs_\alpha^{1,6} + (Z+1)s_\alpha^{2,6} + 3Zs_\alpha^{5,6} \geq 2Z(s_\alpha^{1,6} + s_\alpha^{5,6})$, and therefore $s_\alpha^{1,6} \leq (1 + \frac{1}{Z})s_\alpha^{2,6} + s_\alpha^{5,6}$ for all $Z > 1$, leading to $s_\alpha^{1,6} \leq s_\alpha^{2,6} + s_\alpha^{5,6}$. Similarly, we get $\Delta(\alpha(\overline{P}_N)) = \{bca, cba\}$, while for any $\dot{P}_N \in \Delta(\overline{P}_N^K)$, $S_\alpha^6(bca, \dot{P}_N) = S_\alpha^6(cba, \dot{P}_N) = s_\alpha^{1,6} + s_\alpha^{2,6} + s_\alpha^{5,6}$, while $S_\alpha^6(bac, \dot{P}_N) = 2s_\alpha^{2,6} + 2s_\alpha^{5,6}$. Thus, Kemeny-stability implies $s_\alpha^{1,6} \geq s_\alpha^{2,6} + s_\alpha^{5,6}$. Therefore, if $s_\alpha^{1,3} < 2s_\alpha^{2,3}$, then $s_\alpha^{1,6} = s_\alpha^{2,6} + s_\alpha^{5,6}$.

Finally, suppose that $s_\alpha^{1,3} = 2s_\alpha^{2,3}$ and consider $Q_N \in \mathcal{L}(A_3)^{4Z+3}$ and $Q'_N \in \mathcal{L}(A_3)^{10Z+1}$, where $Z > 1$:

$$Q_N = \left(\begin{array}{cccc} Z & Z+1 & Z+1 & Z+2 \\ \hline a & b & b & c \\ c & c & a & a \\ b & a & c & b \end{array} \right) \quad Q'_N = \left(\begin{array}{cccc} 3Z & 3Z+1 & 2Z & 2Z \\ \hline a & b & b & c \\ c & c & a & a \\ b & a & c & b \end{array} \right)$$

Since $s_\alpha^{1,3} = 2s_\alpha^{2,3}$, then $\alpha(Q_N) = cba$. From Proposition 3, one has for any $\dot{P}_N \in \Delta(Q_N^K)$ that $S_\alpha^6(cba, \dot{P}_N) = (2Z+3)s_\alpha^{2,6} + (2Z+1)s_\alpha^{5,6}$ and $S_\alpha^6(acb, \dot{P}_N) = Zs_\alpha^{1,6} + (Z+2)s_\alpha^{2,6} + (Z+1)s_\alpha^{5,6}$. Kemeny-stability implies $s_\alpha^{1,6} \leq (1+\frac{1}{z})s_\alpha^{2,6} + s_\alpha^{5,6}$, and thus $s_\alpha^{1,6} \leq s_\alpha^{2,6} + s_\alpha^{5,6}$. Furthermore, we have $\alpha(Q'_N) = bca$, while Kemeny stability implies that for any $\dot{P}_N \in \Delta(Q'^K_N)$ that $S_\alpha^6(bca, \dot{P}_N) \geq S_\alpha^6(bac, \dot{P}_N)$. Hence, $(3Z+1)s_\alpha^{1,6} + 2Zs_\alpha^{2,6} + 2Zs_\alpha^{5,6} \geq 2Zs_\alpha^{1,6} + (3Z+1)s_\alpha^{2,6} + 3Zs_\alpha^{5,6}$, leading to $s_\alpha^{1,6} \geq s_\alpha^{2,6} + \frac{Z}{Z+1}s_\alpha^{5,6}$ for all $Z > 1$, Therefore, if $s_\alpha^{1,3} = 2s_\alpha^{2,3}$, then $s_\alpha^{1,6} = s_\alpha^{2,6} + s_\alpha^{5,6}$, and the proof is complete.

A.2.3 Proof of Proposition 5

Consider the following profiles $P_N \in \mathcal{L}(A_3)^{2Z+2}$ and $P'_N \in \mathcal{L}(A_3)^{56Z+1}$, where $Z > 1$:

$$P_N = \left(\begin{array}{ccc} Z+1 & 1 & Z \\ \hline a & a & c \\ b & c & b \\ c & b & a \end{array} \right) \quad P'_N = \left(\begin{array}{cccc} 11Z & 28Z & 17Z & 1 \\ \hline a & a & b & c \\ b & c & c & b \\ c & b & a & a \end{array} \right)$$

Suppose that $s_\alpha^{1,3} < 2s_\alpha^{2,3}$. Then $\alpha(P_N) = bac$ for Z large enough. Moreover, from Proposition 3, $S_\alpha^6(bac, \dot{P}_N) = (Z+1)(s_\alpha^{2,6} + s_\alpha^{5,6}) < S_\alpha^6(acb, \dot{P}_N) = s_\alpha^{1,6} + (Z+1)s_\alpha^{2,6} + Zs_\alpha^{5,6}$ for all $\dot{P}_N \in \Delta(P_N^K)$, in contradiction with Kemeny-stability.

Suppose that $s_\alpha^{1,3} > 2s_\alpha^{2,3}$. Then $\alpha(P'_N) = abc$ for Z large enough. Using again Proposition 3, $S_\alpha^6(abc, \dot{P}_N) = 11Zs_\alpha^{1,6} + 28Zs_\alpha^{2,6} + 17Zs_\alpha^{5,6}$ while $S_\alpha^6(acb, \dot{P}_N) = 28Zs_\alpha^{1,6} + 11Zs_\alpha^{2,6} + s_\alpha^{5,6}$ for all $\dot{P}_N \in \Delta(P'^K_N)$. Since $s_\alpha^{5,6} > 0$ from Proposition 3, we get by using Proposition 4, $S_\alpha^6(abc, \dot{P}_N) = 39Zs_\alpha^{2,6} + 28Zs_\alpha^{5,6} < S_\alpha^6(abc, \dot{P}_N) = 39Zs_\alpha^{2,6} + 28Zs_\alpha^{5,6} + s_\alpha^{5,6}$, in contradiction with Kemeny-stability.

A.2.4 End of proof of Theorem 1

(Necessary Part) Using Propositions 3,4 and 5, it suffices to prove that if α is Kemeny-stable, then $s_\alpha^{2,6} = 3s_\alpha^{5,6}$. Consider the following profiles $P_N \in \mathcal{L}(A_3)^{3Z+1}$ and $P'_N \in \mathcal{L}(A_3)^{3Z-4}$, where $Z > 2$:

$$P_N = \left(\begin{array}{cc} 2Z+1 & Z \\ \hline a & b \\ b & c \\ c & a \end{array} \right) \quad P'_N = \left(\begin{array}{ccc} Z-1 & Z-1 & Z-2 \\ \hline a & b & c \\ c & a & b \\ b & c & a \end{array} \right)$$

Suppose that $s_\alpha^{2,6} > 3s_\alpha^{5,6}$. Since $s_\alpha^{1,3} = 2s_\alpha^{2,3}$ from Proposition 5, then $\alpha(P_N) = abc$. For any $\dot{P}_N \in \Delta(P_N^K)$, we get from Proposition 3 together with Proposition 4 that $S_\alpha^6(abc, \dot{P}_N) = (2Z+1)s_\alpha^{1,6} + Zs_\alpha^{5,6} = (2Z+1)s_\alpha^{2,6} + (3Z+1)s_\alpha^{5,6}$, while $S_\alpha^6(bac, \dot{P}_N) = (3Z+1)s_\alpha^{2,6}$. But since $s_\alpha^{2,6} > 3s_\alpha^{5,6}$, we get $S_\alpha^6(bac, \dot{P}_N) > S_\alpha^6(abc, \dot{P}_N)$ for all $Z > 2$, in contradiction with Kemeny-stability.

Suppose that $s_\alpha^{2,6} < 3s_\alpha^{5,6}$. Using again $s_\alpha^{1,3} = 2s_\alpha^{2,3}$ from Proposition 5, we get $\alpha(P'_N) = abc$. For any $\dot{P}_N \in \Delta(P'^K_N)$, we get from Proposition 3 together with Proposition 4 that $S_\alpha^6(abc, \dot{P}_N) = (2Z-2)s_\alpha^{2,6}$, while $S_\alpha^6(cba, \dot{P}_N) = (Z-2)s_\alpha^{1,6} + (2Z-2)s_\alpha^{5,6} = (Z-2)s_\alpha^{2,6} + (3Z-4)s_\alpha^{5,6}$. Thus, $S_\alpha^6(bac, \dot{P}_N) > S_\alpha^6(abc, \dot{P}_N)$ for Z large enough, in contradiction with Kemeny-stability. Hence one must have $s_\alpha^{2,6} = 3s_\alpha^{5,6}$, which proves the Necessary Part.

(Sufficiency Part). Consider any $n \in \mathbb{N}$ together with any profile $P_N \in \mathcal{L}(A_3)^n$ having the form

$$P_N = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 \\ \hline a & a & b & b & c & c \\ b & c & a & c & a & b \\ c & b & c & a & b & a \end{pmatrix}$$

with $\sum_{h=1}^6 n_h = n$. Pick up any scoring rule α fulfilling the conditions $(*)$ $s_\alpha^{1,3} = 2s_\alpha^{2,3} > 0$, and $(**)$ $s_\alpha^{1,6} = \frac{4}{3}s_\alpha^{2,6} = \frac{4}{3}s_\alpha^{3,6} = 4s_\alpha^{4,6} = 4s_\alpha^{5,6} > s_\alpha^{6,6} = 0$. We get that:

- $S_\alpha^3(a, P_N) = (2n_1 + 2n_2 + n_3 + n_5)s_\alpha^{2,3}$
- $S_\alpha^3(b, P_N) = (2n_3 + 2n_4 + n_1 + n_6)s_\alpha^{2,3}$
- $S_\alpha^3(c, P_N) = (2n_5 + 2n_6 + n_2 + n_4)s_\alpha^{2,3}$

Moreover, suppose without loss of generality that $s_\alpha^{5,6} = 1$ and $abc \in \Delta(\alpha(P_N))$. It follows that:

- $n_1 + 2n_2 + n_5 \geq n_3 + 2n_4 + n_6$ *(1)*
- $2n_1 + n_2 + n_3 \geq n_4 + n_5 + 2n_6$ *(2)*
- $n_1 + 2n_3 + n_4 \geq n_2 + 2n_5 + n_6$ *(3)*

Now, pick up any $\dot{P}_N \in \Delta(P_N^K)$. Then we get from $(**)$ that:

- $S_\alpha^6(abc, \dot{P}_N) = 4n_1 + 3(n_2 + n_3) + (n_4 + n_5)$
- $S_\alpha^6(acb, \dot{P}_N) = 4n_2 + 3(n_1 + n_5) + (n_3 + n_6)$
- $S_\alpha^6(bac, \dot{P}_N) = 4n_3 + 3(n_1 + n_4) + (n_2 + n_6)$
- $S_\alpha^6(bca, \dot{P}_N) = 4n_4 + 3(n_3 + n_6) + (n_1 + n_5)$
- $S_\alpha^6(cab, \dot{P}_N) = 4n_5 + 3(n_2 + n_6) + (n_1 + n_4)$
- $S_\alpha^6(cba, \dot{P}_N) = 4n_6 + 3(n_4 + n_5) + (n_2 + n_3)$

Then one easily checks that *(3)* $\Rightarrow S_\alpha^6(abc, \dot{P}_N) \geq S_\alpha^6(acb, \dot{P}_N)$, *(1)* $\Rightarrow S_\alpha^6(abc, \dot{P}_N) \geq S_\alpha^6(bac, \dot{P}_N)$, *(1)+(2)* $\Rightarrow S_\alpha^6(abc, \dot{P}_N) \geq S_\alpha^6(bca, \dot{P}_N)$, *(2)+(3)* $\Rightarrow S_\alpha^6(abc, \dot{P}_N) \geq S_\alpha^6(cab, \dot{P}_N)$, and *(1)+(2)+(3)* $\Rightarrow S_\alpha^6(abc, \dot{P}_N) \geq S_\alpha^6(cba, \dot{P}_N)$. Hence $abc \in \Delta(\alpha(P_N)) \cap f_\alpha(\dot{P}_N)$, and the proof is complete.

A.3 Proof of Theorem 2

Let α be a non-truncated and Kemeny-stable scoring rule. Consider profile $P_N \in \mathcal{L}(A_6)^{A+B+C+1}$, where $A > B > C > 1$:

$$P_N = \begin{pmatrix} A & B & C & 1 \\ \hline a & b & c & f \\ c & c & a & e \\ b & a & b & d \\ d & d & d & c \\ e & e & f & b \\ f & f & e & a \end{pmatrix}$$

Using Theorem 1, and normalizing S^6_α by setting $s^{1,6}_\alpha = 1$, we get $S^6_\alpha(a, P_N) = A + \frac{3}{4}(B + C)$, $S^6_\alpha(b, P_N) = \frac{3}{4}(A + C) + B + \frac{1}{4}$, $S^6_\alpha(c, P_N) = \frac{3}{4}(A + B) + C + \frac{1}{4}$, $S^6_\alpha(d, P_N) = \frac{1}{4}(A + B + C) + \frac{3}{4}$, $S^6_\alpha(e, P_N) = \frac{1}{4}(A + B) + \frac{3}{4}$, and $S^6_\alpha(f, P_N) = \frac{1}{4}C + 1$. Obviously, A,B and C can be chosen to ensure that $\alpha(P_N) = abcdef$. Consider the following Kemeny hyper-profile $\dot{P}_N \in \Delta(P^K_N)$

$$\dot{P}_N = \left(\begin{array}{cccc} A & B & C & 1 \\ \hline acbdef & bcadef & cabdfe & fedcba \\ cabdef & cbadef & cabdef & ... \\ abcdef & bacdef & cbadfe & ... \\ ... & bcdaef & cadbfe & ... \\ ... & bcaedf & cabfde & ... \\ ... & bcadfe & acbdfe & ... \\ ... & cabdef & ... & ... \\ ... & abcdef & ... & ... \\ ... & ... & ... & ... \end{array}\right)$$

We get that $S^{6!}_\alpha(cabdef, \dot{P}_N) = (A + C)s^{2,6!}_\alpha + Bs^{7,6!}_\alpha + s^{z,6!}_\alpha$, where $z < 6!$, whereas $S^{6!}_\alpha(abcdef, \dot{P}_N) = As^{3,6!}_\alpha + Bs^{8,6!}_\alpha + Cs^{w,6!}_\alpha$, where $w > 6$. Finally, Kemeny-stability implies that $s^{1,6!}_\alpha = ... = s^{8,6!}_\alpha$, and $s^{z,6!}_\alpha = 0$, which contradicts that α is non-truncated.

A.4 Proof of Theorem 3

The proof is similar to the one above. Consider profile $P_N \in \mathcal{L}(A_6)^9$ below:

$$P_N = \left(\begin{array}{cccc} 4 & 3 & 1 & 1 \\ \hline a & b & c & c \\ c & c & a & b \\ b & a & b & a \\ d & d & d & d \\ e & e & e & e \\ f & f & f & f \end{array}\right)$$

We get $S^6_\alpha(a, P_N) = 4s^{1,6}_\alpha + s^{2,6}_\alpha + 4s^{3,6}_\alpha$, $S^6_\alpha(b, P_N) = 3s^{1,6}_\alpha + s^{2,6}_\alpha + 5s^{3,6}_\alpha$, $S^6_\alpha(c, P_N) = 2s^{1,6}_\alpha + 7s^{2,6}_\alpha$, $S^6_\alpha(d, P_N) = 9s^{4,6}_\alpha$, $S^6_\alpha(e, P_N) = 9s^{5,6}_\alpha$, and $S^6_\alpha(f, P_N) = 0$. If α is Kemeny-stable, it follows from Theorem 1 that $s^{2,6}_\alpha = s^{3,6}_\alpha$, which implies that $\Delta(\alpha(P_N)) \subseteq \{P \in \mathcal{L}(A_6) : P = (abc \to Q)$, where $Q \in \mathcal{L}(\{d, e, f\})\}$. Consider the following Kemeny hyper-profile $\dot{P}_N \in \Delta(P^K_N)$

$$
\dot{P}_N = \left(\begin{array}{cccc}
4 & 3 & 1 & 1 \\
\hline
acbdef & bcadef & cabdef & cbadef \\
cabdef & cbadef & acbdef & cabdef \\
abcdef & bacdef & cbadef & bcadef \\
... & bcdaef & cadbef & cbdaef \\
... & bcaedf & cabedf & cbaedf \\
... & bcadfe & cabdfe & cbadfe \\
... & cabdef & abcdef & abcdef \\
... & abcdef & ... & ... \\
... & ... & ... & ...
\end{array}\right)
$$

We get that $S^{6!}_\alpha(cabdef, \dot{P}_N) = 4s^{2,6!}_\alpha + 3s^{7,6!}_\alpha + s^{1,6!}_\alpha + s^{2,6!}_\alpha$, whereas $S^{6!}_\alpha(abcdef, \dot{P}_N) = 4s^{3,6!}_\alpha + 3s^{8,6!}_\alpha + s^{7,6!}_\alpha + s^{7,6!}_\alpha$. Using $s^{2,6!}_\alpha \geq s^{3,6!}_\alpha$ and $s^{7,6!}_\alpha \geq s^{8,6!}_\alpha$ together with the strict-at-top property, we have that $S^{6!}_\alpha(cabdef, \dot{P}_N) > S^{6!}_\alpha(abcdef, \dot{P}_N)$. The conclusion follows from the fact that $abcdef$ maximizes $S^{6!}_\alpha(P, \dot{P}_N)$ in $\Delta(\alpha(P_N))$.

A.5 Proof of Theorem 5

Consider profile $P_N \in \mathcal{L}(A_3)^5$ below:

$$
P_N = \left(\begin{array}{cccc}
2 & 1 & 1 & 1 \\
\hline
a & b & c & c \\
b & a & b & a \\
c & c & a & a
\end{array}\right)
$$

Pick up a scoring rule α hyper-stable for $\mathcal{S}$. Since $\mathcal{K} \subset \mathcal{S}$, then α is Kemeny-stable. It follows from Theorem 1 that score vectors must be such that $(*)$ $s^{1,3}_\alpha = 2s^{2,3}_\alpha > 0$, and $(**)$ $s^{1,6}_\alpha = \frac{4}{3}s^{2,6}_\alpha = \frac{4}{3}s^{3,6}_\alpha = 4s^{4,6}_\alpha = 4s^{5,6}_\alpha > s^{6,6}_\alpha = 0$. It follows that $\alpha(P_N) = abc$. It is straightforward to check that the following hyper-profile P^E_N is built from a 5-tuple $E = (e_1, e_1, e_3, e_4, e_5)$ of separable preference extensions:

$$
P^E_N = \left(\begin{array}{cccc}
2 & 1 & 1 & 1 \\
\hline
abc & bac & cba & cab \\
bac & bca & bca & acb \\
acb & abc & bac & cba \\
bca & acb & cab & bca \\
cab & cba & acb & abc \\
cba & cab & abc & bac
\end{array}\right)
$$

Note that all extensions in E but e_4 are Kemeny. We get $S^6_\alpha(abc, P^E_N) = 12s^{5,6}_\alpha < S^6_\alpha(bac, P^E_N) = 13s^{5,6}_\alpha$, which contradicts hyper-stability for $\mathcal{S}$.

A.6 Proof of Theorem 6

Given any $Q \in \mathcal{L}(A_m)$, we write $Q = (Q_1 \to Q_2 \to ... \to Q_H)$, where, for any $1 \le h \le H$, $Q_h \in \mathcal{L}(B_h)$ is a segment of Q, and where $\{B_1, B_2, ..., B_H\}$ is a partition of A_m into non-empty sets. Lemma 7 is a useful intermediate step towards the proof.

Lemma 7 *Let $Q, Q' \in \mathcal{L}(A_m)$ be respectively defined by $Q = (Q_1 \to x \to Q_2 \to y \to Q_3)$ and $Q' = (Q_1 \to y \to Q_2 \to x \to Q_3)$, where $Q_h \in \mathcal{L}(B_h)$, for $1 \le h \le 3$. Then, for any $P_N \in \mathcal{L}(A_m)^n$ with n is odd, and any $E = (e_1, ..., e_n) \in \mathcal{S}^n$, $[x\ \mu(P_N)\ y] \Rightarrow [Q\ \mu(P_N^E)\ Q']$,*

Proof: Define $B = \{x, y\} \cup B_2$, where $Q_2 \in \mathcal{L}(B_2)$. Pick up any $P_i \in \mathcal{L}(A_m)$ where xP_iy, and consider the restriction $P_i|_B$ of P_i to B. We can write $P_i|_B = (V_1 \to x \to V_2 \to y \to V_3)$, where V_1, V_2, and V_3 are segments of $P_i|_B$, with $V_h \in \mathcal{L}(B_{2h})$, $1 \le h \le 3$, and $\{B_{21}, B_{22}, B_{23}\}$ being a partition of B_2. Then $A(P_i|_B, Q|_B) = \{x, y\} \cup [\{x\} \times (B_{22} \cup B_{23})] \cup [(B_{21} \cup B_{22}) \times \{y\}] \cup A(P_i|_{B_2}, Q_2)$, while $A(P_i|_B, Q'|_B) = A(P_i|_B, Q|_B)/\{x, y\}$. Hence, $A(P_i|_B, Q'|_B) \subset A(P_i|_B, Q|_B)$. Since Q and Q' have the same segment Q_1 at top and the same segment Q_3 at bottom, then $A(P_i, Q') \subset A(P_i, Q)$. From separability of e_i, we get $Q\ e_i(P_i)\ Q'$. Finally, $x\ \mu(P_N)\ y$ implies that $|\{i : xP_iy\}| > \frac{n}{2}$, hence that $|\{i : Q\ e_i(P_i)\ Q'\}| > \frac{n}{2}$ and the conclusion follows. □

Given $P_N \in \mathcal{L}(A_m)^n$, the top-cycle for P_N is the subset $T(B, P_N)$ of A_m containing all maximal elements for $\theta(P_N)$. The transitive closure partition of A_m is the ordered set $S(\theta, P_N) = (S_1, S_2, ..., S_J)$ of indifference classes for $\theta(P_N)$, where $\forall j \le j' \in \{1, ..., J\}$, $\forall (x, x') \in S_j \times S_{j'}$, $x\theta(P_N)x'$ and $\neg(x'\theta(P_N)x$ if $j < j'$. By definition of θ, one has $\Delta(\theta(P_N)) = \{Q \in \mathcal{L}(A_m) : Q = (Q_1 \to Q_2 \to ... \to Q_J)$ where, for each $j = 1, ..., J$, $Q_j \in \mathcal{L}(S_j)\}$. The proof of Theorem 6 is complete if we show that for any $E = (e_1, ..., e_n) \in \mathcal{S}^n$, $\Delta(\theta(P_N)) \cap T(\mathcal{L}(A_m), P_N^E) \neq \varnothing$.

Pick up any $P \in \mathcal{L}(A_m)/\Delta(\theta(P_N))$ and any $E = (e_1, ..., e_n) \in \mathcal{S}^n$. Define $B(P) = \{x \in A_m : x \in S_j$ for some j and $\forall y \in S_{j'}/\{x\}, xPy \Rightarrow j' > j\}$, and $B = A_m/B(P)$. Consider order $Q(P) \in \Delta(\theta(P_N))$ such that:

- $Q(P) |_{B(P)} = P |_{B(P)}$
- $xPy \Rightarrow xQ(P)y$ for all $x, y \in B \cap S_j$ for some $j \in \{1, ..., J\}$.

Write $P |_B = b_1 b_2 ... b_T$, where $T = |B|$. There exists a permutation σ of $\{1, ..., T\}$ such that $Q(P) |_B = b_{\sigma(1)} b_{\sigma(2)} ... b_{\sigma(T)}$. Then, there is a finite sequence $\{\omega_h\}_{1 \le h \le H}$ of transpositions of A_m, where $H \le T$, such that ω_1 swaps b_1 and $b_{\sigma(1)}$ in $P |_B$, leading to $P^1 |_B = b_{\omega_1(1)} b_{\omega_1(2)} ... b_{\omega_1(T)}$, ω_2 swaps $b_{\omega_1(2)}$ and $b_{\sigma(2)}$ in $P^1 |_B$, leading to $P^2 |_B = b_{\omega_2 \circ \omega_1(1)} b_{\omega_2 \circ \omega_1(2)} ... b_{\omega_2 \circ \omega_1(T)}$, ..., ω_H swaps $b_{\sigma(T)}$ and $b_{\omega_{T-1} \circ ... \circ \omega_1(T)}$ in $P^{T-1} |_B$, leading to $P^T |_B = Q(P) |_B$. Since $b_{\sigma(1)} \mu(P_N) b_{\sigma(2)} \mu(P_N) ... \mu(P_N) b_{\sigma(T)}$, then Lemma 7 ensures that for all $1 \le h \le H$, either $P^{h+1} |_B = P^h |_B$ or $(P^{h+1} |_B) \mu(P_N^E |_B)(P^h |_B)$. Hence $(Q(P) |_B)\theta(P_N^E |_B)(P |_B)$, and thus $Q(P)\theta(P_N^E)P$. This proves that for any order P not in $\Delta(\theta(P_N))$, there exists $Q \in \Delta(\theta(P_N))$ such that $Q\theta(P_N^E)P$.

Finally, since $T(\mathcal{L}(A_m), P_N^E |_{\Delta(\theta(P_N))}) \neq \varnothing$, there exists $Q \in \Delta(\theta(P_N))$ such that $Q\theta(P_N^E)Q'$ for all $Q' \in \Delta(\theta(P_N))/\{Q\}$. Thus, there exists $Q \in \Delta(\theta(P_N))$ such that $Q\theta(P_N^E)Q'$ for all $Q' \in \mathcal{L}(A_m)/\{Q\}$. Thus $\Delta(\theta(P_N)) \cap T(\mathcal{L}(A_m), P_N^E)) \neq \varnothing$ and the proof is complete.

A.7 Proof of Proposition 2

Let $P_N = (P_1..., P_i, ..., P_n) \in \mathcal{L}(A_m)^n$. Choose any $E = (e_1, ..., e_n) \in \mathcal{S}^n$. Suppose without loss of generality that $\mu(P_N) = a_1 a_2 ... a_m$. Moreover, suppose that $CW(P_N^E) = Q = b_1 b_2 ... b_m$, with $b_1 \neq a_1 = b_h$ for some $2 \leq h \leq m$. Now define $Q' = a_1 b_2 ... b_{h-1} b_h b_{h+1} ... b_m \in \mathcal{L}(A_m)$. It follows from Lemma 7 above that $[1\ \mu(P_N)\ b_1] \Rightarrow [Q'\ \mu(P_N^E)\ Q']$, which contradicts $CW(P_N^E) = Q$. Thus, $b_1 = a_1$. We conclude by iterating the same argument for $b_2, ..., b_m$.