First hitting times for general non-homogeneous 1d diffusion processes: density estimates in small time

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FIRST HITTING TIMES FOR GENERAL NON-HOMOGENEOUS 1D DIFFUSION PROCESSES: DENSITY ESTIMATES IN SMALL TIME

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Abstract. Motivated by some applications in neurosciences, we here collect several estimates for the density of the first hitting time of a threshold by a non-homogeneous one-dimensional diffusion process and for the density of the associated process stopped at the threshold. We first remind the reader of the connection between both. We then provide some Gaussian type bounds for the density of the stopped process. We also discuss the stability of the density with respect to the drift. Proofs mainly rely on the parametrix expansion.

We believe that the collection of these results might be useful in the analysis of various models from biology, physics or finance.

Keywords: Hitting times; killed processes; density estimates; Fokker-Planck equation.

1. Background

In the course of a recent work [1] it became necessary to study the general problem of estimating the density of the first hitting time of a fixed level by a non-homogeneous diffusion process with general Lipschitz drift. To be precise, for a fixed $T > 0$ we consider the stochastic process $(X_t)_{t\geqslant 0}$ that satisfies

$$X_t = X_0 + \int_0^t b(X_s)ds + f(t) + W_t, \quad t \in [0,T], \tag{1.1}$$

where $(W_t)_{t\in[0,T]}$ is a standard one-dimensional Brownian motion (defined on some filtered probability space $(\Omega, ((\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}))$, $f \in \mathcal{C}^1([0,T])$ such that $f(0) = 0$, and $b : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous such that

$$|b(x)| \leqslant \Lambda(|x| + 1), \qquad |b(x) - b(y)| \leqslant K|x - y|, \qquad \forall x, y \in \mathbb{R},$$

for some constants Λ and K. Here $\mathcal{C}^1([0,T])$ denotes the space of real-valued continuously differentiable functions equipped with the usual norm $\|\cdot\|_{\mathcal{C}^1([0,T])}$. For a probability measure ν on $\mathbb{R}$, we make use of the (quite abusive) notation $\mathbb{P}^\nu$ in order to indicate that ν is precisely the distribution of X_0.

Considering 1 as a threshold, we moreover suppose that $X_0 < 1$ and define the stopping time

$$\tau = \inf\{t > 0 : X_{t\wedge T} \geqslant 1\} \tag{1.2}$$

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to be the first time that $(X_t)_{t\in[0,T]}$ reaches the level 1. Given a probability measure ν on $(-\infty, 1]$, the subject of these notes are the two related densities

$$\mathfrak{p}^{\nu}(t) = \frac{d}{dt}\mathbb{P}^{\nu}\left(\tau \leqslant t\right), \quad t \in [0,T], \tag{1.3}$$

i.e. the density of τ, and

$$p^{\nu}(t,y) = \frac{d}{dy}\mathbb{P}^{\nu}\left(X_t \leqslant y, t < \tau\right), \quad t \in [0,T], \quad y \in (-\infty, 1], \tag{1.4}$$

the density of the killed process, when they exist. When $\nu = \delta_{x_0}$ for some $x_0 < 1$, we will write $\mathfrak{p}^{x_0}(t)$ and $p^{x_0}(t,y)$ instead. In particular, the goal of these notes is to prove bounds on the behavior of $\mathfrak{p}^{x_0}(t)$ in terms of the function f.

In the case $b \equiv 0$, the question of the existence of $\mathfrak{p}^{x_0}$ for $x_0 < 1$ is a classical, though non-trivial, problem. Indeed, in this case $\mathfrak{p}^{x_0}$ is the density of the first hitting time of the curve $t \mapsto 1 - x_0 - f(t)$ by a Brownian path started from 0, which is known to exist and be continuous since f is assumed continuously differentiable and $f(0) = 0$ (see e.g. [7, Theorem 14.4]).

However, in the general case, we were unable to find a complete reference for this (certainly known) result when we impose a more general condition on the initial probability density ν. It is for this reason that in Section 2 below we thus provide a result guaranteeing the continuity of $\mathfrak{p}^{\nu}$ in the general case, under the condition that ν has a density close to the threshold that linearly decays to 0. The statement also gives detailed regularity properties of $p^{\nu}(t,y)$ in both t and y, and states that the classical link between the two densities holds in our general case.

The section 3 of these notes is devoted to estimates of $p^{x_0}(t,y)$ (and hence of $\mathfrak{p}^{x_0}(t)$) for $x_0 < 1$ in small time. We pay particular attention to the dependence of the bounds on t, the initial condition x_0 and the function f. Indeed to emphasize the dependence of the two densities on the function $f \in \mathcal{C}^1([0,T])$ we will write when necessary

$$\mathfrak{p}^{x_0}(t) = \mathfrak{p}^{x_0}_f(t), \quad p^{x_0}(t,y) = p^{x_0}_f(t,y), \quad t \in [0,T], y \in (-\infty, 1].$$

One of the main goals of these notes is to prove that $|\mathfrak{p}^{x_0}_{f_1}(t) - \mathfrak{p}^{x_0}_{f_1}(t)|$ for two functions f_1 and f_2 in $\mathcal{C}^1([0,T])$ is controlled by the difference $\|f_1 - f_2\|_{\mathcal{C}^1([0,T])}$, the shape of the ratio $|\mathfrak{p}^{x_0}_{f_1}(t) - \mathfrak{p}^{x_0}_{f_1}(t)|/\|f_1 - f_2\|_{\mathcal{C}^1([0,T])}$ being explicitly controlled when t is small and/or x_0 is large (in the negative). This is the result stated in Proposition 5.2.

Remark 1.1. *We have made two simplifications that are for notational reasons only, and result in no loss of generality. The first is that we have chosen to study the density of the first hitting time of* 1 *by the process from below, though we can just as easily study the first hitting of a general level* $a \in \mathbb{R}$ *by above or below by a simple shift* $X_t \to \pm(X_t - (a-1))$.

The second simplification is to take the intensity of the noise to be 1 *rather than a general* $\sigma > 0$. *Indeed, applying Brownian scaling and setting* $u = t/\sigma^2$, *we can write* (1.1) *as*

$$X_{\sigma^2 u} = X_0 + \int_0^u \sigma^2 b(X_{\sigma^2 s})ds + f\left(\sigma^2 u\right) + W_{\sigma^2 u}, \quad u \in [0, T/\sigma^2].$$

Thus setting $\widetilde{X}_u = X_{\sigma^2 u}$, $\widetilde{W}_u = (1/\sigma)W_{\sigma^2 u}$, $\widetilde{b} = \sigma^2 b$ *and* $\widetilde{f}(\cdot) = f(\sigma^2 \cdot)$ *yields*

$$\widetilde{X}_u = X_0 + \int_0^u \widetilde{b}(\widetilde{X}_s)ds + \widetilde{f}(u) + \sigma \widetilde{W}_u, \quad u \in [0, T/\sigma^2],$$

i.e. $(\widetilde{X}_t)$ *satisfies* (1.1) *with arbitrary noise intensity* $\sigma > 0$, *but where* b *and* f *have changed.*

The notation defined above will be used throughout the rest of the article.

2. Regularity of density of first hitting time and killed process

The following Lemma describes the regularity of the densities (1.3) and (1.4) under the condition that the initial probability measure ν has a density close to the threshold that linearly decays to 0.

Lemma 2.1. *Suppose* $(X_t)_{t\in[0,T]}$ *satisfies the SDE* (1.1), *where* b *is Lipschitz and* $f \in \mathcal{C}^1([0,T])$. *Let* ν *be a probability measure on* $(-\infty, 1]$ *such that*

$$\nu(dx) \leqslant \beta(1-x)dx, \quad x \in (1-\epsilon, 1],$$

for some $\beta, \epsilon > 0$. *Suppose further that the density of* ν *on the interval* $(1-\epsilon, 1]$ *is differentiable at point* 1. *Then:*

(i) *For any* $t \in (0,T]$, *the measure*

$$\mathcal{B}((-\infty,1]) \ni A \mapsto \mathbb{P}^\nu\big(X_t \in A, t < \tau\big)$$

has a density with respect to the Lebesgue measure. Here, $\mathcal{B}((-\infty,1])$ *denotes the Borel subsets of* $(-\infty,1]$. *The (unnormalized) density* $p^\nu(t,y)$, $t \in (0,T]$, $y \leqslant 1$ *given by* (1.4) *is thus well-defined.*

(ii) $p^\nu(t,y)$ *is continuous in* (t,y) *and continuously differentiable in* y *on* $(0,T] \times (-\infty,1]$ *and admits Sobolev derivatives of order 1 in* t *and of order 2 in* y *in any* L^ς, $\varsigma \geqslant 1$, *on any compact subset of* $(0,T]\times(-\infty,1)$. *When* $\mathrm{supp}(\nu) \subset (-\infty, 1-\epsilon]$, $p^\nu(t,y)$ *is actually continuous in* (t,y) *and continuously differentiable in* y *on any compact subset of* $([0,T]\times(-\infty,1]) \setminus (\{0\}\times(-\infty,1-\epsilon])$.

(iii) *The density* p^ν *satisfies (at least in the weak sense) the Fokker-Planck equation:*

$$\partial_t p^\nu(t,y) + \partial_y\big[\big(b(y)+f'(t)\big)p^\nu(t,y)\big] - \frac{1}{2}\partial^2_{yy}p^\nu(t,y) = 0, \quad t\in(0,T],\ y<1, \quad (2.1)$$

with the Dirichlet boundary condition $p^\nu(t,1) = 0$ *and the measure-valued initial condition* $p^\nu(0,y)dy = \nu(dy)$, *both* $p^\nu(t,y)$ *and* $\partial_y p^\nu(t,y)$ *decaying to* 0 *as* $y \to -\infty$.

(iv) *The first hitting time* $\tau = \inf\{t > 0 : X_t \geqslant 1\}$ *given that* $X_0 \sim \nu$ *has a density* $\mathfrak{p}^\nu$ *on* $[0,T]$, *given by*

$$\mathfrak{p}^\nu(t) = -\frac{1}{2}\partial_y p^\nu(t,1), \quad t \in [0,T], \quad (2.2)$$

the mapping $[0,T] \ni t \mapsto \partial_y p^\nu(t,1)$ *being continuous and its supremum norm being bounded in terms of* T, $\|f\|_{\mathcal{C}^1([0,T])}$, β *and the Lipschitz constant* K *of* b *only.*

Lemma 2.1 is quite standard when the coefficients are smooth but it is rather difficult to find a complete proof of it under our assumptions. For this reason, we provide a complete proof in the Appendix.

3. Relationship with density of homogeneous killed process

Throughout this section we assume that $f \in \mathcal{C}^1([0,T])$ and b is Lipschitz. Our aim is to seek information about $\mathfrak{p}^{x_0}(t) = \mathfrak{p}_f^{x_0}(t)$ given by (1.3) for $t \in [0,T]$ and $x_0 < 1$. Thanks to formula (2.2), it is equivalent to seek information about $\partial_y p_f^{x_0}(t,1)$ i.e. the derivative of the density of the process killed at the boundary.

The point of this section is to show that the density $p_f^{x_0}(t,y)$ of the killed inhomogeneous process $(X_t)_{t\in[0,T]}$ (given by (1.1)) started from x_0 can be represented in terms of the density of a *homogeneous* process killed at the boundary, for which Gaussian bounds independent of the function f may be proven (see Proposition 3.2). Indeed we have

Lemma 3.1 (Integral equation)**.** *Suppose $p_f^{x_0}(t,y)$, $t \in (0,T]$, $y \leqslant 1$ is the density of the process $(X_t)_{t\in[0,T]}$ killed at the barrier 1 started at $x_0 < 1$, given by (1.4) with $\nu = \delta_{x_0}$. Then*

$$p_f^{x_0}(t,y) = q(t,x_0,y) - \int_0^t \int_{-\infty}^1 \left(f'(s) + b(1)\right) \partial_z p_f^{x_0}(s,z) q(t-s,z,y) dz ds, \tag{3.1}$$

for $t \in (0,T]$ and $y < 1$, where $q(t,x,y)$ is the solution to the PDE

$$\begin{cases} \partial_t q(t,x,y) &= \frac{1}{2}\partial^2_{yy} q(t,x,y) - \partial_y\left[(b(y)-b(1))q(t,x,y)\right] \\ q(t,x,1) &= 0 \\ q(0,x,y) &= \delta_0(x-y) \end{cases} \tag{3.2}$$

on $[0,T]\times(-\infty,1]\times(-\infty,1]$.

Proof. By Lemma 2.1 the density $p_f^{x_0}(t,y)$, $t \in (0,T]$, $y \leqslant 1$ satisfies the Fokker-Planck equation (2.1) with boundary condition $p_f^{x_0}(t,1) = 0$ and measure-valued initial condition $p_f^{x_0}(0,y) = \delta_0(x_0 - y)$. The result follows from the paramatrix method in [4, Chapter 1]. It can also be verified by hand that the right-hand side of (3.1) must also satisfy the Fokker-Planck equation (2.1) with the same boundary and initial conditions, and so the result follows by uniqueness. $\square$

Since we want bounds on $\mathfrak{p}_f^{x_0}(t) = -[1/2]\partial_y p_f^{x_0}(t,1)$ by (2.2), we are also interested in $\partial_y p_f^{x_0}(t,y)$. Taking the derivative of (3.1) with respect to y, yields the formula

$$\partial_y p_f^{x_0}(t,y) = \partial_y q(t,x_0,y) - \int_0^t \int_{-\infty}^1 \left(f'(s) + b(1)\right) \partial_z p_f^{x_0}(s,z) \partial_y q(t-s,z,y) dz ds. \tag{3.3}$$

Evaluating the above at $y = 1$ then yields an expression for $\mathfrak{p}_f^{x_0}(t)$, which is the quantity we would like to estimate, and so formula (3.3) will provide the basis for our estimates.

3.1. **Bounds on the density of the killed homogeneous process.** We here prove some general Gaussian bounds on the behavior of the solution $q(t,x,y)$ to the homogeneous Fokker-Planck equation (3.2) on $[0,T]\times(-\infty,1]\times(-\infty,1]$. By Lemma 2.1, $q(t,x,y)$ is the density of the solution to (1.1) killed at the boundary when $f\equiv 0$ and b is replaced by $b-b(1)$.

Proposition 3.2. *Suppose $q(t,x,y)$ is the solution to the homogeneous Fokker-Planck equation* (3.2) *on $[0,T]\times(-\infty,1]\times(-\infty,1]$. Then there exists a constant $B_T>0$ such that*

$$|q(t,x,y)| \leqslant \frac{B_T}{\sqrt{t}} \exp\left(-\frac{|\xi_t^x-y|^2}{B_T t}\right), \tag{3.4}$$

and

$$|q(t,x,y)| \leqslant \frac{B_T}{\sqrt{t}} \exp\left(-\frac{|x-\xi_{-t}^y|^2}{B_T t}\right), \tag{3.5}$$

for all $t\in[0,T]$, $x,y\leqslant 1$, where

$$\xi_t^x = x+\int_0^t \widetilde{b}(\xi_r^x)dr, \quad t\in\mathbb{R}, \tag{3.6}$$

and $\widetilde{b}(v)=b(v)-b(1)$ if $v<1$ while $\widetilde{b}(v)=-\widetilde{b}(2-v)$ if $v\geqslant 1$. Moreover, we can also chose B_T large enough so that

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{t} \exp\left(-\frac{|\xi_t^x-y|^2}{B_T t}\right), \tag{3.7}$$

and

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{t} \exp\left(-\frac{|x-\xi_{-t}^y|^2}{B_T t}\right), \tag{3.8}$$

for all $t\in[0,T]$, $x,y\leqslant 1$.

Proof. Let $(\widetilde{X}_t)_{t\geqslant 0}$ denote the solution of the SDE:

$$d\widetilde{X}_t = \widetilde{b}(\widetilde{X}_t)dt + dW_t. \tag{3.9}$$

We can then see that $q(t,x,y)$ is the transition density of $\widetilde{X}_t$ killed at 1 (equation (3.2) is the Fokker-Planck equation for the killed process, as shown in Lemma 2.1).

Note that $\widetilde{b}(1+v)=-\widetilde{b}(1-v)$ for $v\in\mathbb{R}$, so that $\widetilde{b}$ is odd with respect to 1. Therefore, when initialised at $\widetilde{X}_0=1$ the process $(\widetilde{X}_t-1)_{0\leqslant t\leqslant T}$ satisfies

$$\widetilde{X}_t - 1 = \int_0^t \widetilde{b}(\widetilde{X}_s-1+1)ds + W_t, \qquad t\in[0,T]$$

and the process $(1-\widetilde{X}_t)_{0\leqslant t\leqslant T}$ satisfies

$$\begin{aligned}1-\widetilde{X}_t &= -\int_0^t \widetilde{b}(\widetilde{X}_s)ds - W_t = \int_0^t \widetilde{b}(2-\widetilde{X}_s)ds - W_t\\ &= \int_0^t \widetilde{b}(1-\widetilde{X}_s+1)ds - W_t,\end{aligned}$$

which is the same equation. Hence $(1-\widetilde{X}_t)_{0\leqslant t\leqslant T}$ and $(\widetilde{X}_t-1)_{0\leqslant t\leqslant T}$ have the same distribution when $\widetilde{X}_0=1$. Therefore the reflection principle (see [9, Section I.13]) applies and

$$q(t,x,y)=\widetilde{q}(t,x,y)-\widetilde{q}(t,x,2-y), \qquad t\in[0,T], \quad x,y\leqslant 1, \tag{3.10}$$

where $\widetilde{q}(t,x,y)$ is the transition density of the non-killed process.

We thus need to estimate the transition density $\widetilde{q}$ of the non-killed process. By Theorem 1.1 of [2] it follows that there exists a constant B_T such that

$$\widetilde{q}(t,x,y)\leqslant \frac{B_T}{\sqrt{t}}\exp\left(-\frac{|y-\xi_t^x|^2}{B_Tt}\right), \quad t>0,\ x,y\in\mathbb{R}, \tag{3.11}$$

where ξ is defined in (3.6). Thus

$$|q(t,x,y)|\leqslant \frac{B_T}{\sqrt{t}}\left[\exp\left(-\frac{|y-\xi_t^x|^2}{B_Tt}\right)+\exp\left(-\frac{|2-y-\xi_t^x|^2}{B_Tt}\right)\right], \quad t>0,\ x,y\leqslant 1.$$

We show this implies the bounds (3.4) and (3.5). Since $\widetilde{b}(1)=0$, we have $\xi_t^1=1$ for any $t\geqslant 0$. Therefore, by the comparison principle for ODEs, it must hold $\xi_t^x\leqslant\xi_t^1=1$ for any $t\geqslant 0$ and $x\leqslant 1$. As a consequence, for $t>0$ and $x,y\leqslant 1$,

$$\begin{aligned}|y-\xi_t^x|=|(y-1)-(\xi_t^x-1)|&\leqslant|y-1|+|\xi_t^x-1|\\&=1-y+1-\xi_t^x=2-y-\xi_t^x=|2-y-\xi_t^x|.\end{aligned}$$

Therefore, the largest exponential term in the above bound for $|q(t,x,y)|$ is the first one. The bound (3.4) then follows directly (adjusting B_T as necessary). Moreover, by Gronwall's Lemma and thanks to the Lipschitz property of b, we have:

$$\exp(-Ks)|\xi_t^x-y|\leqslant|\xi_{t-s}^x-\xi_{-s}^y|\leqslant\exp(Ks)|\xi_t^x-y| \tag{3.12}$$

for all $s\in[0,t]$, so that, by choosing $s=t$, we see that (3.5) also holds (again by increasing the constant B_T).

By the same argument, (3.7) and (3.8) will follow if we can show that a bound similar to (3.11) holds for $|\partial_y\widetilde{q}(t,x,y)|$, i.e.

$$|\partial_y\widetilde{q}(t,x,y)|\leqslant\frac{B_T}{t}\exp\left(-\frac{|y-\xi_t^x|^2}{B_Tt}\right), \quad t>0,\ x,y\in\mathbb{R}. \tag{3.13}$$

The rest of the proof is thus concerned with showing (3.13). Without any loss of generality, we can assume $\widetilde{b}$ to be a twice continuously differentiable function with a bounded second-order derivative. Indeed, if we can prove that, in such a case, (3.13) holds with respect to a constant B_T that does not refer to the second-order differentiability of $\widetilde{b}$, then (3.13) holds in the original setting as well by a standard mollification argument. Since $\widetilde{q}$ is the transition probability of the solution to (3.9), it satisfies the Fokker-Planck equation

$$\partial_t\widetilde{q}(t,x,y)=\frac{1}{2}\partial^2_{yy}\widetilde{q}(t,x,y)-\widetilde{b}(y)\partial_y\widetilde{q}(t,x,y)-\widetilde{b}'(y)\widetilde{q}(t,x,y),$$

for $t \in [0,T]$ and $x, y \in \mathbb{R}$. Define $(\widetilde{Y}_t^y)_{t \geqslant 0}$ to be the solution of the SDE

$$\begin{cases} d\widetilde{Y}_t^y &= -\widetilde{b}(\widetilde{Y}_t^y)dt + dW_t, \\ \widetilde{Y}_0^y &= y. \end{cases} \tag{3.14}$$

By Theorem 39 [8, p. 312], we know that, a.s., the mapping $[0,+\infty) \times \mathbb{R} \ni (t,y) \mapsto Y_t^y$ is continuously differentiable with respect to y and that

$$\partial_y \widetilde{Y}_s^y = \exp\biggl(-\int_0^s \widetilde{b}'(\widetilde{Y}_u^y)du\biggr). \tag{3.15}$$

Moreover, by Lemma 2.1, we know that $\widetilde{q}$ is regular enough in order to apply the Itô-Krylov formula (see [6, Section II.10]) to $[0,t/2] \ni s \mapsto \widetilde{q}\left(t-s,x,\widetilde{Y}_s^y\right)$, which yields

$$\widetilde{q}(t,x,y) = \mathbb{E}\left[\widetilde{q}\left(t/2,x,\widetilde{Y}_{t/2}^y\right)\right] - \int_0^{t/2} \mathbb{E}\left[\widetilde{b}'(\widetilde{Y}_s^y)\widetilde{q}\left(t-s,x,\widetilde{Y}_s^y\right)\right] ds.$$

By the Malliavin-Bismut-Elworthy formula (see, for example, Theorem 2.1 of [3]),

$$\begin{aligned} \partial_y \widetilde{q}(t,x,y) = {}& \frac{2}{t}\mathbb{E}\left[\widetilde{q}(t/2,x,\widetilde{Y}_{t/2}^y)\int_0^{t/2} \partial_y \widetilde{Y}_s^y dW_s\right] \\ & - \int_0^{t/2} \frac{1}{s}\mathbb{E}\left[\widetilde{b}'(\widetilde{Y}_s^y)\widetilde{q}\left(t-s,x,\widetilde{Y}_s^y\right)\int_0^s \partial_y \widetilde{Y}_r^y dW_r\right] ds. \end{aligned} \tag{3.16}$$

By (3.15), $|\partial_y \widetilde{Y}_s^y| \leqslant \exp(Ks)$. Thus using (3.11) we can compute for any $0 < s \leqslant t/2$ (where the constant B_T changes from line to line below)

$$\begin{aligned} I(s,t) &:= \left|\frac{1}{s}\mathbb{E}\left[\widetilde{q}(t-s,x,\widetilde{Y}_s^y)\int_0^s \partial_y \widetilde{Y}_r^y dW_r\right]\right| \\ &\leqslant \frac{1}{s}\left[\mathbb{E}\left(\widetilde{q}^2(t-s,x,\widetilde{Y}_s^y)\right)\right]^{1/2}\left[\mathbb{E}\left(\int_0^s \partial_y \widetilde{Y}_r^y dW_r\right)^2\right]^{1/2} \\ &\leqslant \frac{B_T}{\sqrt{s}}\left[\mathbb{E}\left(\widetilde{q}^2(t-s,x,\widetilde{Y}_s^y)\right)\right]^{1/2} \\ &\leqslant \frac{B_T}{\sqrt{s}}\left[\frac{1}{t-s}\mathbb{E}\left(\exp\left(-\frac{|\widetilde{Y}_s^y - \xi_{t-s}^x|^2}{B_T(t-s)}\right)\right)\right]^{1/2} \\ &= \frac{B_T}{\sqrt{s}}\left[\frac{1}{t-s}\int_{\mathbb{R}} \exp\left(-\frac{|z-\xi_{t-s}^x|^2}{B_T(t-s)}\right)\mathbb{P}(\widetilde{Y}_s^y \in dz)\right]^{1/2}. \end{aligned}$$

By an estimate similar to (3.11) but for the density of $\widetilde{Y}$, we deduce

$$\begin{aligned} I(s,t) &\leqslant \frac{B_T}{\sqrt{s}}\left[\frac{1}{(t-s)\sqrt{s}}\int_{\mathbb{R}} \exp\left(-\frac{|z-\xi_{t-s}^x|^2}{B_T(t-s)}\right)\exp\left(-\frac{|z-\xi_{-s}^y|^2}{B_T s}\right)dz\right]^{1/2} \\ &\leqslant \frac{B_T}{\sqrt{s}}\left[\frac{1}{\sqrt{t-s}\sqrt{t}}\exp\left(-\frac{|\xi_{t-s}^x - \xi_{-s}^y|^2}{B_T t}\right)\right]^{1/2}, \end{aligned}$$

where, for the first line, we have used the fact the flow associated with the ODE driven by $-\widetilde{b}$ is nothing but the backward flow associated with the ODE driven by $\widetilde{b}$, i.e. $(\xi^x_{-t})_{t\geqslant 0}$ satisfies

$$\xi^x_{-t} = x - \int_0^t \widetilde{b}(\xi^x_{-s})ds, \quad t \geqslant 0,$$

and, to pass from the first to the second line, we have recognised a Gaussian convolution. By (3.12), we deduce that

$$I(s,t) \leqslant \frac{B_T}{\sqrt{s}}\left[\frac{1}{\sqrt{t-s}\sqrt{t}}\exp\left(-\frac{|\xi^x_t - y|^2}{B_T t}\right)\right]^{1/2},$$

from which (3.16) yields

$$\begin{aligned}
|\partial_y \widetilde{q}(t,x,y)| &\leqslant \frac{B_T}{t}\exp\left(-\frac{|\xi^x_t - y|^2}{B_T t}\right) \\
&\quad + B_T \exp\left(-\frac{|\xi^x_t - y|^2}{B_T t}\right)\int_0^{t/2}\frac{1}{\sqrt{s}}\left[\frac{1}{\sqrt{t-s}\sqrt{t}}\right]^{1/2} ds \\
&\leqslant \frac{B_T}{t}\exp\left(-\frac{|\xi^x_t - y|^2}{B_T t}\right) + \frac{B_T}{\sqrt{t}}\exp\left(-\frac{|\xi^x_t - y|^2}{B_T t}\right)\int_0^{t/2}\frac{1}{\sqrt{s}}ds.
\end{aligned}$$

This proves (3.13). □

4. Bounds on the density of the first hitting time in small time

As above, for $x_0 < 1$, $f \in \mathcal{C}^1([0,T])$, $\mathfrak{p}^{x_0}_f(t)$ represents the density of the first hitting time of the threshold 1 by the process $(X_t)_{t\in[0,T]}$ given by (1.1) and started at x_0 (see (1.3)).

In this section we pursue our aim of bounding on $\mathfrak{p}^{x_0}_f(t)$ in terms of f and x_0 in small time by exploiting the relationship $\mathfrak{p}^{x_0}_f(t) = -[1/2]\partial_y p^{x_0}_f(t,1)$ given by (2.2), where $p^{x_0}_f(t,y)$, $t \in (0,T]$, $y \leqslant 1$, is the density of the killed process started at x_0 and killed at the boundary 1. Our starting point is the representation (3.3), and throughout the section $q(t,x,y), t \in [0,T], x,y \leqslant 1$ is the solution to (3.2) that appears within this representation.

We also introduce the notation

$$\|g\|_{\infty,t} := \sup_{s\in[0,t]}|g(s)|, \qquad \|G(\cdot)\|_\infty = \sup_{x\in\mathcal{B}}|G(x)|$$

for any continuous functions $g : [0,t] \to \mathbb{R}$, and $G : \mathcal{B} \to \mathbb{R}$, with $\mathcal{B} \subset \mathbb{R}$.

The first result is an L^∞-bound.

Lemma 4.1. *Let $f \in \mathcal{C}^1([0,T])$. Then there exists a constant $\kappa_1(T)$ (independent of f) which increases with T such that for all $x_0 < 1$,*

$$\sup_{0\leqslant s\leqslant t}\left[\sqrt{s}\left\|\partial_y p^{x_0}_f(s,.) - \partial_y q(s,x_0,.)\right\|_\infty\right] \leqslant (\|f'\|_{\infty,T} + |b(1)|)\kappa_1(T),$$

for all $t \leqslant \min\{[(\|f'\|_{\infty,T} + |b(1)|)\kappa_1(T)]^{-2}, T\}$.

Proof. For notational sake, define

$$F_f^{x_0}(t,y) := \partial_y p_f^{x_0}(t,y) - \partial_y q(t,x_0,y).$$

By (3.3), we see that

$$\begin{aligned}\left|F_f^{x_0}(t,y)\right| &\leqslant (\|f'\|_{\infty,T} + |b(1)|) \int_0^t \int_{-\infty}^1 |\partial_z q(s,x_0,z)\partial_y q(t-s,z,y)| dz ds \\ &+ (\|f'\|_{\infty,T} + |b(1)|) \int_0^t \int_{-\infty}^1 \left|\partial_z p_f^{x_0}(s,z) - \partial_z q(s,x_0,z)\right| |\partial_y q(t-s,z,y)| \, dz ds.\end{aligned} \tag{4.1}$$

Thus defining $\widetilde{A} := \|f'\|_{\infty,T} + |b(1)|$ and allowing B_T to increase as necessary from line to line below, by Proposition 3.2 we see that for $t \leqslant T$

$$\begin{aligned}&\left|F_f^{x_0}(t,y)\right| \\ &\leqslant \widetilde{A}B_T \int_0^t \int_{-\infty}^1 \frac{1}{s(t-s)} \exp\left(-\frac{|z-\xi_s^{x_0}|^2}{B_T s}\right) \exp\left(-\frac{|z-\xi_{-(t-s)}^y|^2}{B_T(t-s)}\right) dz ds \\ &\quad + \widetilde{A}B_T \int_0^t \frac{1}{t-s} \left\|F_f^{x_0}(s,\cdot)\right\|_\infty \int_{-\infty}^1 \exp\left(-\frac{|z-\xi_{-(t-s)}^y|^2}{B_T(t-s)}\right) dz ds.\end{aligned}$$

We can recognise the first term in the above as a Gaussian convolution. Thus

$$\begin{aligned}\left|F_f^{x_0}(t,y)\right| &\leqslant \frac{\widetilde{A}B_T}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} \exp\left(-\frac{|\xi_{-(t-s)}^y - \xi_s^{x_0}|^2}{B_T t}\right) ds \\ &\quad + \widetilde{A}B_T \int_0^t \frac{\sqrt{s}}{\sqrt{t-s}\sqrt{s}} \left\|F_f^{x_0}(s,\cdot)\right\|_\infty ds \\ &\leqslant \widetilde{A}B_T \left(\frac{1}{\sqrt{t}} \exp\left(-\frac{|\xi_{-t}^y - x_0|^2}{B_T t}\right) + \sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\left\|F_f^{x_0}(s,\cdot)\right\|_\infty\right]\right),\end{aligned} \tag{4.2}$$

where again we have used (3.12). Therefore, by multiplying both sides of the inequality by $\sqrt{t}$, we deduce

$$\sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\left\|F_f^{x_0}(s,\cdot)\right\|_\infty\right] \leqslant \widetilde{A}B_T + \sqrt{t}\widetilde{A}B_T \sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\left\|F_f^{x_0}(s,\cdot)\right\|_\infty\right],$$

We can conclude that, for $t \in [0,T]$ such that $t \leqslant [2\widetilde{A}B_T]^{-2} = [2(\|f'\|_{\infty,T} + |b(1)|)B_T]^{-2}$, we have that

$$\sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\left\|\partial_y p_f^{x_0}(s,.) - \partial_y q(s,x_0,.)\right\|_\infty\right] \leqslant 2\widetilde{A}B_T. \tag{4.3}$$

□

The problem with this bound is that it is crude with respect to the initial point x_0. However, it does allow us to prove the following refinement.

Proposition 4.2. *Let $f \in \mathcal{C}^1([0,T])$. There exists a constant $\kappa_2(T)$ (independent of f) which increases with T such that for all $x_0 < 1$,*

$$\left|\partial_y p_f^{x_0}(t,y) - \partial_y q(t,x_0,y)\right| \leqslant \kappa_2(T)(\|f'\|_{\infty,T} + |b(1)|)\frac{1}{\sqrt{t}}\exp\left(-\frac{\left|\xi_t^{x_0} - y\right|^2}{\kappa_2(T)t}\right),$$

for all $t \leqslant \min\{[(\|f'\|_{\infty,T} + |b(1)|)\kappa_2(T)]^{-2}, T\}$ and $y \leqslant 1$, where $(\xi_t^{x_0})_{t\geqslant 0}$ is as in Proposition 3.2.

Proof. Throughout the proof, we will use the fact the mapping

$$\varphi : \mathbb{R} \ni x \mapsto \frac{\exp(x) - 1}{x} \quad (\varphi(0) = 1), \tag{4.4}$$

is non-decreasing. We will also use the same notation as in the proof of Lemma 4.1:

$$F_f^{x_0}(t,y) := \partial_y p_f^{x_0}(t,y) - \partial_y q(t,x_0,y).$$

First Step. As noted in (4.1), we have for $t \in [0,T]$

$$\begin{aligned}\left|F_f^{x_0}(t,y)\right| \leqslant \widetilde{A}\int_0^t\int_{-\infty}^1 |\partial_z q(s,x_0,z)\partial_y q(t-s,z,y)|dzds \\ + \widetilde{A}\int_0^t\int_{-\infty}^1 \left|F_f^{x_0}(t_1,y_1)\right| |\partial_y q(t-t_1,y_1,y)|\, dy_1 dt_1\end{aligned} \tag{4.5}$$

where we have denoted $\widetilde{A} = \|f'\|_{\infty,T} + |b(1)|$. The fact that φ given by (4.4) is non-decreasing says that, for $t \in [0,T]$,

$$\frac{1-\exp(-2KT)}{2KT} \leqslant \frac{1-\exp(-2Kt)}{2Kt} \leqslant 1 \leqslant \frac{\exp(2Kt)-1}{2Kt} \leqslant \frac{\exp(2KT)-1}{2KT},$$

from which, together with Proposition 3.2, we deduce there exists $B_T > 0$ such that

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{1-e^{-2Kt}}\exp\left(-\frac{|x-\xi_{-t}^y|^2}{2B_T(1-e^{-2Kt})}\right) \tag{4.6}$$

$$|\partial_y q(t,x,y)| \leqslant \frac{B_T}{e^{2Kt}-1}\exp\left(-\frac{|\xi_t^x - y|^2}{2B_T(e^{2Kt}-1)}\right). \tag{4.7}$$

Unlike in the previous lemma, B_T is now fixed. We then introduce the kernel

$$\mathcal{K}(t,x,y) = \begin{cases} \dfrac{1}{\sqrt{B_T(e^{2Kt}-1)}}\exp\left(-\dfrac{|\xi_t^x - y|^2}{2B_T(e^{2Kt}-1)}\right) & \text{if } t > 0, \\ \dfrac{1}{\sqrt{B_T(1-e^{2Kt})}}\exp\left(-\dfrac{|x-\xi_t^y|^2}{2B_T(1-e^{2Kt})}\right) & \text{if } t < 0. \end{cases} \tag{4.8}$$

For $0 < s < t \leqslant T$, $\mathcal{K}$ satisfies the Gaussian convolution property:

$$\begin{aligned}\mathcal{K}(s,x,\cdot)\otimes\mathcal{K}\big(-(t-s),\cdot,y\big) &:= \int_{\mathbb{R}}\mathcal{K}(s,x,z)\mathcal{K}\big(-(t-s),z,y\big)dz \\ &= \frac{\sqrt{2\pi}}{\sqrt{B_T(e^{2Ks}-e^{-2K(t-s)})}}\exp\left(-\frac{|\xi_s^x - \xi_{-(t-s)}^y|^2}{2B_T(e^{2Ks}-e^{-2K(t-s)})}\right).\end{aligned}$$

By (3.12), we know that $|\xi_t^x - y| \leqslant e^{K(t-s)}|\xi_s^x - \xi_{-(t-s)}^y|$ for all $0 < s < t$. Therefore,

$$\begin{aligned}\mathcal{K}(s,x,\cdot)\otimes\mathcal{K}\big(-(t-s),\cdot,y\big) &\leqslant \frac{\sqrt{2\pi}}{\sqrt{B_T(1-e^{-2Kt})}}\exp\left(-\frac{|\xi_t^x - y|^2}{2B_T(e^{2Kt}-1)}\right)\\ &\leqslant \sqrt{2\pi}e^{KT}\mathcal{K}(t,x,y).\end{aligned} \tag{4.9}$$

By (4.5), (4.6) and (4.7), we deduce that, for $t \in [0,T]$,

$$\begin{aligned}\big|F_f^{x_0}(t,y)\big| &\leqslant \widetilde{A}B_T^3 e^{KT}\mathcal{K}(t,x_0,y)\int_0^t \frac{\sqrt{2\pi}}{\sqrt{e^{2Ks}-1}\sqrt{1-e^{-2K(t-s)}}}ds\\ &\quad + \widetilde{A}B_T^{3/2}\int_0^t\int_{-\infty}^1 \frac{\big|F_f^{x_0}(t_1,y_1)\big|}{\sqrt{1-e^{-2K(t-t_1)}}}\mathcal{K}(-(t-t_1),y_1,y)dy_1dt_1.\end{aligned} \tag{4.10}$$

Using (4.4), we notice that

$$\begin{aligned}\int_0^t \frac{1}{\sqrt{e^{2Ks}-1}\sqrt{1-e^{-2K(t-s)}}}ds &\leqslant \frac{\sqrt{2KT}}{\sqrt{1-\exp(-2KT)}}\int_0^t \frac{1}{\sqrt{2Ks}\sqrt{2K(t-s)}}ds\\ &= \frac{\sqrt{T}}{\sqrt{2K(1-\exp(-2KT))}}\int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}}ds\\ &\leqslant \frac{4\sqrt{T}}{\sqrt{2K(1-\exp(-2KT))}} =: C_T,\end{aligned} \tag{4.11}$$

using $\int_0^t(\sqrt{s}\sqrt{t-s})^{-1}ds = \int_0^1(\sqrt{u}\sqrt{1-u})^{-1}du = \pi \leqslant 4$. Finally, (4.10) yields

$$\begin{aligned}\big|F_f^{x_0}(t,y)\big| &\leqslant \widetilde{A}\widetilde{C}_T B_T^3\mathcal{K}(t,x_0,y)\\ &\quad + \widetilde{A}B_T^{3/2}\int_0^t\int_{-\infty}^1 \frac{\big|F_f^{x_0}(t_1,y_1)\big|}{\sqrt{1-e^{-2K(t-t_1)}}}\mathcal{K}(-(t-t_1),y_1,y)dy_1dt_1,\end{aligned} \tag{4.12}$$

with $\widetilde{C}_T := \sqrt{2\pi}C_T\exp(KT)$.

Second Step. We now prove by induction, that for any $N \geqslant 0$,

$$\big|F_f^{x_0}(t,y)\big| \leqslant \widetilde{A}\widetilde{C}_T B_T^3\mathcal{K}(t,x_0,y)\sum_{i=0}^N\left(\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}\sqrt{t}\right)^i + R_{N+1}(t,y), \tag{4.13}$$

where

$$\begin{aligned}R_N(t,y) = (\widetilde{A}B_T^{3/2})^N &\int_{0\,\leqslant\,t_N\,\leqslant\,\ldots\,\leqslant\,t_0} dt_N\ldots dt_1\Bigg\{\prod_{i=0}^{N-1}\frac{1}{\sqrt{1-e^{-2K(t_i-t_{i+1})}}}\\ &\int_{(-\infty,1]^N} dy_N\ldots dy_1\,\big|F_f^{x_0}(t_N,y_N)\big|\prod_{i=0}^{N-1}\mathcal{K}\big(-(t_i-t_{i+1}),y_{i+1},y_i\big)\Bigg\},\end{aligned}$$

with the convention $t_0 = t$ and $y_0 = y$. In the first step, we established (4.13) when $N = 0$. Assume now that, for some $N \geqslant 0$, (4.13) holds for any $t \in [0,T]$ and $y \leqslant 1$.

Then, plugging the induction assumption at rank N into (4.12), we get:

$$
\begin{aligned}
&\left|F_f^{x_0}(t,y)\right| \\
&\leqslant \widetilde{A}\widetilde{C}_T B_T^3 \mathcal{K}(t,x_0,y) \\
&\quad + \widetilde{A}\widetilde{C}_T B_T^3 \left(\sum_{i=0}^{N} \left(\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}\sqrt{t} \right)^i \right) \widetilde{A} B_T^{3/2} \int_0^t \frac{\mathcal{K}(t_1,x_0,\cdot)\otimes \mathcal{K}(-(t-t_1),\cdot,y)}{\sqrt{1-e^{-2K(t-t_1)}}} dt_1 \\
&\quad + R_{N+2}(t,y).
\end{aligned}
$$

By (4.9), we deduce that

$$
\begin{aligned}
&\left|F_f^{x_0}(t,y)\right| \\
&\leqslant \widetilde{A}\widetilde{C}_T B_T^3 \mathcal{K}(t,x_0,y) \\
&\quad + \widetilde{A}\widetilde{C}_T B_T^3 \left(\sum_{i=0}^{N} \left(\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}\sqrt{t} \right)^i \right) \widetilde{A} B_T^{3/2} \sqrt{2\pi} e^{KT} \mathcal{K}(t,x_0,y) \int_0^t \frac{1}{\sqrt{1-e^{-2K(t-t_1)}}} dt_1 \\
&\quad + R_{N+2}(t,y).
\end{aligned}
$$

Following (4.11), we have:

$$
\begin{aligned}
\int_0^t \frac{ds}{\sqrt{1-e^{-2K(t-s)}}} &\leqslant \frac{\sqrt{2KT}}{\sqrt{1-\exp(-2KT)}} \int_0^t \frac{1}{\sqrt{2K(t-s)}} ds \\
&= \frac{\sqrt{T}}{\sqrt{1-\exp(-2KT)}} \int_0^t \frac{1}{\sqrt{t-s}} ds \\
&\leqslant \frac{2\sqrt{t}\sqrt{T}}{\sqrt{1-\exp(-2KT)}} \leqslant \sqrt{t}\sqrt{K} C_T.
\end{aligned}
$$

Since $\widetilde{C}_T = \sqrt{2\pi}\exp(KT)C_T$, we deduce that (4.13) holds at rank $N+1$.

Third Step. Suppose that $\sqrt{t} \leqslant [\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}]^{-1}/2$. Then, the series in (4.13) is convergent (and bounded above by 2). The point is thus to prove that $R_N(t,y) \to 0$ as $N \to \infty$ for all $y \leqslant 1$. By Lemma 4.1, we deduce that, for $\sqrt{t} \leqslant [\widetilde{A}\kappa_1(T)]^{-1}$,

$$
\begin{aligned}
&R_N(t,y) \\
&\leqslant (\sqrt{2\pi}\widetilde{A}B_T^{3/2})^N \widetilde{A}\kappa_1(T) \int_{0 \leqslant t_N \leqslant t_{N-1} \leqslant \ldots \leqslant t_0} \frac{1}{\sqrt{t_N}} \prod_{i=0}^{N-1} \frac{1}{\sqrt{1-e^{-2K(t_i-t_{i+1})}}} dt_1 \ldots dt_N.
\end{aligned}
$$

Using (4.4), we obtain

$$
\begin{aligned}
R_N(t,y) \leqslant & \left(\sqrt{2\pi}\widetilde{A}B_T^{3/2} \frac{\sqrt{T}}{\sqrt{1-e^{-2KT}}} \right)^N \widetilde{A}\kappa_1(T) \\
& \times \int_{0 \leqslant t_N \leqslant t_{N-1} \leqslant \ldots \leqslant t_0} \frac{1}{\sqrt{t_N}} \prod_{i=0}^{N-1} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \ldots dt_N.
\end{aligned}
\tag{4.14}
$$

To compute the integral in the right-hand side of the above, define

$$J_n(t) = \int_0^t u^{(n-1)/2}(t-u)^{-1/2}du, \quad t \geqslant 0,\ n \in \mathbb{N}.$$

By the change of variable $u = ts$, we have $J_n(t) = t^{n/2}J_n(1)$. Note that $J_0(1) = \pi \leqslant 4$. Moreover, for $n \geqslant 1$,

$$J_n(1) \leqslant n^{1/2}\int_0^{1-1/n} u^{(n-1)/2}du + \int_{1-1/n}^1 (1-u)^{-1/2}du \leqslant \frac{2n^{1/2}}{n+1} + 2n^{-1/2} \leqslant 4n^{-1/2}. \tag{4.15}$$

With this notation

$$\begin{aligned} I_{N,0}(t) &:= \int_{0 \,\leqslant\, t_N \,\leqslant\, t_{N-1} \,\leqslant\, \dots \,\leqslant\, t_0} \frac{1}{\sqrt{t_N}} \prod_{i=0}^{N-1} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \dots dt_N \\ &= \int_{0 \,\leqslant\, t_{N-1} \,\leqslant\, \dots \,\leqslant\, t_0} J_0(t_{N-1}) \prod_{i=0}^{N-2} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \dots dt_{N-1}, \end{aligned}$$

where the second equality holds for $N \geqslant 2$. More generally, setting for any $n \leqslant N-2$

$$I_{N,n}(t) := \int_{0 \,\leqslant\, t_{N-(n+1)} \,\leqslant\, \dots \,\leqslant\, t_0} J_n(t_{N-(n+1)}) \prod_{i=0}^{N-(n+2)} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \dots dt_{N-(n+1)},$$

we have, for any $0 \leqslant n \leqslant N-3$,

$$\begin{aligned} I_{N,n}(t) &= J_n(1) \int_{0 \,\leqslant\, t_{N-(n+1)} \,\leqslant\, \dots \,\leqslant\, t_0} t_{N-(n+1)}^{n/2} \prod_{i=0}^{N-(n+2)} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \dots dt_{N-(n+1)} \\ &= J_n(1) \int_{0 \,\leqslant\, t_{N-(n+2)} \,\leqslant\, \dots \,\leqslant\, t_0} \left[\int_0^{t_{N-(n+2)}} \frac{t_{N-(n+1)}^{n/2}}{\sqrt{t_{N-(n+2)} - t_{N-(n+1)}}} dt_{N-(n+1)} \right] \\ &\qquad \times \prod_{i=0}^{N-(n+3)} \frac{1}{\sqrt{t_i - t_{i+1}}} dt_1 \dots dt_{N-(n+2)} \\ &= J_n(1) I_{N,n+1}(t), \end{aligned}$$

so that, by (4.15),

$$\begin{aligned} I_{N,0}(t) &= \left[\prod_{n=0}^{N-3} J_n(1)\right] I_{N,N-2}(t) = \left[\prod_{n=0}^{N-2} J_n(1)\right] J_{N-1}(t) \\ &= \left[\prod_{n=0}^{N-1} J_n(1)\right] t^{(N-1)/2} \leqslant 4[4t^{1/2}]^{N-1}[(N-1)!]^{-1/2}, \end{aligned}$$

using the fact that $I_{N,N-2}(t) = J_{N-2}(1)J_{N-1}(t)$. Going back to (4.14), we deduce that $R_N(t,y) \to 0$ as $N \to \infty$. Hence taking the limit as $N \to \infty$ in (4.13) yields (with $\sqrt{t} \leqslant [\sqrt{K}\widetilde{A}\widetilde{C}_T B_T^{3/2}]^{-1}/2$)

$$\left|F_f^{x_0}(t,y)\right| \leqslant 2\widetilde{A}\widetilde{C}_T B_T^3 \mathcal{K}(t,x_0,y),$$

from which the result follows. □

As a corollary, we get the following bound on the behavior of $\mathfrak{p}_f^{x_0}(t)$ in small time:

Corollary 4.3. *Let $f \in \mathcal{C}^1[0,T]$. There exists a constant $\kappa(T)$, depending only on T and the drift function b, which increases with T such that for all $x_0 < 1$,*

$$\frac{1}{2}|\partial_y p_f^{x_0}(t,1)| = |\mathfrak{p}_f^{x_0}(t)| \leqslant \kappa(T)(\|f'\|_{\infty,T}+1)\frac{1}{t}\exp\left(-\frac{(1-x_0)^2}{\kappa(T)t}\right)$$

for all $t \leqslant \min\{[(\|f'\|_{\infty,T}+1)\kappa(T)]^{-2}, T\}$.

Proof. The result follows from Proposition 4.2, Proposition 3.2 and (3.12) by taking $y = 1$, and using the fact that $\xi_{-t}^1 = 1$ for all $t > 0$. We have also now incorporated the term involving b into the constant κ, using the fact that $\|f'\|_{\infty,T} + |b(1)| \leqslant \max\{1, |b(1)|\}(\|f'\|_{\infty,T}+1)$. □

5. Bounds on the difference of densities

Once again, throughout the section we will adopt the notation of the previous sections, so that, for $x_0 < 1$ and $f \in \mathcal{C}^1([0,T])$, $\mathfrak{p}_f^{x_0}(t)$ will represent the density of the first hitting time of the threshold 1 by the process $(X_t)_{t\in[0,T]}$ given by (1.1) and started at x_0 (see (1.3)), and $p_f^{x_0}(t,y)$, $t \in (0,T]$, $y \leqslant 1$, will be the density of the killed process started at x_0 and killed at the boundary 1 (see (1.3) and (1.4)). These densities are guaranteed to exist by Lemma 2.1.

We now look for bounds on $|\mathfrak{p}_{f_1}^{x_0}(t) - \mathfrak{p}_{f_2}^{x_0}(t)|$ for $f_1, f_2 \in \mathcal{C}^1([0,T])$ in terms of $\|f_1 - f_2\|_{\mathcal{C}^1([0,T])}$ and x_0 in small time. As in the previous section we again exploit the relationship $\mathfrak{p}_f^{x_0}(t) = -[1/2]\partial_y p_f^{x_0}(t,1)$ given by (2.2) and the representation (3.3) of $\partial_y p_f^{x_0}(t,y)$, $t \in (0,T]$, $y \leqslant 1$.

Lemma 5.1. *Let $f_1, f_2 \in \mathcal{C}^1[0,T]$ and let $A = \max\{\|f_1'\|_{\infty,T}, \|f_2'\|_{\infty,T}\}$. Then there exists a constant $\kappa_3(T)$ (independent of f_1 and f_2) which increases with T such that for all $x_0 < 1$,*

$$\sup_{0\leqslant s\leqslant t}\left[\sqrt{s}\left\|\partial_y p_{f_1}^{x_0}(s,.) - \partial_y p_{f_2}^{x_0}(s,.)\right\|_\infty\right] \leqslant \widetilde{A}\kappa_3(T)\|f_1' - f_2'\|_{\infty,t},$$

for all $t \leqslant \min\{[\widetilde{A}\kappa_3(T)]^{-2}, T\}$, where $\widetilde{A} := \max(A + |b(1)|, 1)$.

Proof. Let $y \leqslant 1$. Again by (3.3), we see that

$$\begin{aligned}
&\left|\partial_y p_{f_1}^{x_0}(t,y) - \partial_y p_{f_2}^{x_0}(t,y)\right| \\
&\leqslant (A + |b(1)|) \int_0^t \int_{-\infty}^1 \left|\partial_z p_{f_1}^{x_0}(s,z) - \partial_z p_{f_2}^{x_0}(s,z)\right| |\partial_y q(t-s,z,y)|\, dz ds \\
&\quad + \|f_1' - f_2'\|_{\infty,t} \int_0^t \int_{-\infty}^1 \left|\partial_z p_{f_2}^{x_0}(s,z) - \partial_z q(s,x_0,z)\right| |\partial_y q(t-s,z,y)|\, dz ds \\
&\quad + \|f_1' - f_2'\|_{\infty,t} \int_0^t \int_{-\infty}^1 |\partial_z q(s,x_0,z)|\, |\partial_y q(t-s,z,y)|\, dz ds, \qquad (5.1)
\end{aligned}$$

where $q(t,x,y), t \in [0,T], x, y \leqslant 1$ is the solution to (3.2), as before. Using Proposition 3.2 once again and the notation (4.8) (see also (4.6) and (4.7)) together with (4.4), we see that

$$\begin{aligned}
&\left|\partial_y p_{f_1}^{x_0}(t,y) - \partial_y p_{f_2}^{x_0}(t,y)\right| \\
&\leqslant \widetilde{A}\kappa_3(T) \sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\left\|\partial_z p_{f_1}^{x_0}(s,.) - \partial_z p_{f_2}^{x_0}(s,.)\right\|_\infty\right] \\
&\quad + \kappa_3(T)\, \|f_1' - f_2'\|_{\infty,t} \int_0^t \int_{-\infty}^1 \frac{\left|\partial_z p_{f_2}^{x_0}(s,z) - \partial_z q(s,x_0,z)\right|}{\sqrt{t-s}} \mathcal{K}(-(t-s),z,y) dz ds \\
&\quad + \kappa_3(T)\, \|f_1' - f_2'\|_{\infty,t} \int_0^t \int_{-\infty}^1 \frac{1}{\sqrt{s}} \frac{1}{\sqrt{t-s}} \mathcal{K}(s,x_0,z)\mathcal{K}(-(t-s),z,y) dz ds,
\end{aligned}$$

for some constant $\kappa_3(T) > 0$, where $\widetilde{A} = \max(A + |b(1)|, 1)$. Using Proposition 4.2 and (4.9), and allowing the constant $\kappa_3(T)$ to increase as necessary from line to line below, it follows that

$$\begin{aligned}
&\left|\partial_y p_{f_1}^{x_0}(t,y) - \partial_y p_{f_2}^{x_0}(t,y)\right| \\
&\leqslant \widetilde{A}\kappa_3(T) \sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\left\|\partial_y p_{f_1}^{x_0}(s,.) - \partial_y p_{f_2}^{x_0}(s,.)\right\|_\infty\right] \\
&\quad + \kappa_3(T)\widetilde{A}\, \|f_1' - f_2'\|_{\infty,t} \int_0^t \int_{-\infty}^1 \frac{1}{\sqrt{t-s}} \mathcal{K}(s,x_0,z)\mathcal{K}(-(t-s),z,y) dz ds \\
&\quad + \kappa_3(T)\, \|f_1' - f_2'\|_{\infty,t}\, \mathcal{K}(t,x_0,y),
\end{aligned}$$

for all $t \leqslant \min\{[\widetilde{A}\kappa_2(T)]^{-2}, T\}$. Using (4.9) again, together with the bound $\widetilde{A} \geqslant 1$, we deduce:

$$\begin{aligned}
\left|\partial_y p_{f_1}^{x_0}(t,y) - \partial_y p_{f_2}^{x_0}(t,y)\right| &\leqslant \widetilde{A}\kappa_3(T) \sup_{0 \leqslant s \leqslant t} \left[\sqrt{s}\left\|\partial_y p_{f_1}^{x_0}(s,.) - \partial_y p_{f_2}^{x_0}(s,.)\right\|_\infty\right] \\
&\quad + \kappa_3(T)\widetilde{A}\, \|f_1' - f_2'\|_{\infty,t}\, \mathcal{K}(t,x_0,y). \qquad (5.2)
\end{aligned}$$

Assuming that $\kappa_3(T) \geqslant \kappa_2(T)$ and multiplying the above inequality by $\sqrt{t}$, we deduce that, for $t \leqslant \min\{[2\widetilde{A}\kappa_3(T)]^{-2}, T\}$ we have

$$\begin{aligned}\sup_{0 \leqslant s \leqslant t} &\left[\sqrt{s}\left\|\partial_y p^{x_0}_{f_1}(s,.) - \partial_y p^{x_0}_{f_2}(s,.)\right\|_\infty\right] \\ &\leqslant 2\kappa_3(T) \sup_{0<t \leqslant T}\left[\frac{\sqrt{t}}{\sqrt{B_T(\exp(2Kt)-1)}}\right]\widetilde{A}\left\|f_1' - f_2'\right\|_{\infty,t}.\end{aligned}$$

□

Proposition 5.2. *Let $f_1, f_2 \in \mathcal{C}^1[0,T]$ and let $A = \max\{\|f_1'\|_{\infty,T}, \|f_2'\|_{\infty,T}\}$. Then there exists a constant $\kappa_4(T)$ (independent of f_1, f_2) which increases with T such that for all $x_0 < 1$,*

$$\left|\partial_y p^{x_0}_{f_1}(t,y) - \partial_y p^{x_0}_{f_2}(t,y)\right| \leqslant \widetilde{A}\kappa_4(T)\frac{1}{\sqrt{t}}\exp\left(-\frac{\left|\xi^{x_0}_t - y\right|^2}{\kappa_4(T)t}\right)\|f_1' - f_2'\|_{\infty,t},$$

for all $t \leqslant \min\{[\widetilde{A}\kappa_4(T)]^{-2}, T\}$ and $y \leqslant 1$, where $(\xi^{x_0}_t)_{t \geqslant 0}$ is as in Proposition 3.2 and $\widetilde{A} := \max(A + |b(1)|, 1)$.

Proof. We follow the strategy of the proof of Proposition 4.2. We thus define

$$G^{x_0}_{f_1,f_2}(t,y) = \partial_y p^{x_0}_{f_1}(t,y) - \partial_y p^{x_0}_{f_2}(t,y).$$

Going back to (5.1) we can proceed in the same way as in Lemma 5.1 in order to bound the second two terms of this expression. We then get that, for some constant $\kappa_4(T) > 0$,

$$\begin{aligned}\left|G^{x_0}_{f_1,f_2}(t,y)\right| &\leqslant \widetilde{A}\kappa_4(T)\,\|f_1' - f_2'\|_{\infty,t}\,\mathcal{K}(t,x_0,y) \\ &\quad + \widetilde{A}\kappa_4(T)\int_0^t\int_{-\infty}^1 \frac{\left|G^{x_0}_{f_1,f_2}(t_1,y_1)\right|}{\sqrt{1-e^{-2K(t-t_1)}}}\mathcal{K}(-(t-t_1),y_1,y)dy_1dt_1.\end{aligned}$$

We can iterate this inequality in exactly the same way as in Proposition 4.2 (precisely, we can divide both sides by $\|f_1' - f_2'\|_{\infty,t}$ in order to recover (4.12) with $|F^{x_0}_e(\cdot,y)|$ replaced by $|G^{x_0}_{f_1,f_2}(\cdot,y)|/\|f_1' - f_2'\|_{\infty,t}$ therein and then follow (4.13)), and the convergence follows from Lemma 5.1. This yields the result. □

This yields the following corollary.

Corollary 5.3. *Let $f_1, f_2 \in \mathcal{C}^1[0,T]$ and let $A = \max\{\|f_1'\|_{\infty,T}, \|f_2'\|_{\infty,T}\}$. Then there exists a constant $\kappa(T)$ (independent of f_1 and f_2) such that for all $x_0 < 1$,*

$$\left|\mathfrak{p}^{x_0}_{f_1}(t) - \mathfrak{p}^{x_0}_{f_2}(t)\right| \leqslant \kappa(T)(A+1)\frac{1}{\sqrt{t}}\exp\left(-\frac{(1-x_0)^2}{\kappa(T)t}\right)\|f_1' - f_2'\|_{\infty,t},$$

for all $t \leqslant \min\{[(1+A)\kappa(T)]^{-2}, T\}$ and $y \leqslant 1$. Note that the left-hand side is also equal to $\left|\partial_y p^{x_0}_{f_1}(t,1) - \partial_y p^{x_0}_{f_2}(t,1)\right|$.

Proof. This follows from taking $y = 1$ in Proposition 5.2, before using (3.12) and the fact that $\xi^1_{-t} = 1$ for all $t \geqslant 0$. □

6. Appendix: Proof of Lemma 2.1

First Step. We first discuss the solvability of the Fokker-Planck equation (2.1). We start with the following case: we assume that $\nu = \delta_{x_0}$ for $x_0 < 1$, and that b and f are smooth and bounded, with bounded derivatives of any order. Then, by [5, Th 1.10, Chap. VI], we know that the generator of the process $(X_t)_{t\in[0,T]}$, namely the family of second-order differential operators

$$\left(\mathcal{L}_{s,x} := \big(b(\cdot) + f'(s)\big)\partial_x + \frac{1}{2}\partial^2_{xx}\right)_{s\in[0,T]},$$

admits a Green function $G : [0,T]^2 \times (-\infty,1]^2 \ni (s,t,x,y) \mapsto G(s,x,t,y)$. For a given $(t,y) \in [0,T] \times (-\infty,1]$, the function $[0,t) \times (-\infty,1] \ni (s,x) \mapsto G(s,x,t,y)$ is a classical solution of the PDE

$$\partial_s G(s,x,t,y) + \mathcal{L}_{s,x} G(s,x,t,y) = 0,$$

with $G(s,1,t,y) = 0$, for $s \in [0,t)$ and $G(s,x,t,y) \to \delta_0(x-y)$ as $s \nearrow t$, where δ_0 is the Dirac mass at point 0 (pay attention that our definition of the Green function obeys the convention used in probability theory: it is thus reversed in time in comparison with the standard notation used in the PDE literature). Following [4, Th. 5, Sec. 5, Chap. 9], for a given $(s,x) \in [0,T) \times (-\infty,1)$, the function $(s,T] \times (-\infty,1] \ni (t,y) \mapsto G(s,x,t,y)$ is also known to be the Green function of the adjoint operator

$$\partial_t \cdot + \partial_y\big[\big(b(y) + f'(t)\big)\cdot\big] - \frac{1}{2}\partial^2_{yy}\cdot,$$

with a Dirichlet boundary condition on $[0,T] \times \{1\}$. In particular, $G(s,x,t,1) = 0$ and $G(s,x,t,y) \to \delta_0(y-x)$ as $t \searrow s$. We then set

$$p^{x_0}(t,y) = G(0,x_0,t,y), \quad t \in (0,T],\ y \in (-\infty,1]. \tag{6.1}$$

By [5, Th. 1.10, Chap. VI] (applied to the adjoint operator), we know that $p^{x_0}(t,y)$ (as the solution to the Fokker-Planck equation (2.1) under the current smoothness assumptions with $p^{x_0}(0,y) = \delta_0(x_0 - y)$) decays exponentially fast as t tends to 0 and y stays away from x_0. This proves that p^{x_0} is continuous on any compact subset of $([0,T] \times (-\infty,1]) \setminus \{(0,x_0)\}$.

Second Step. Still in the smooth framework, we now make the connection with the diffusion process $(X_t)_{t\in[0,T]}$ when $X_0 = x_0 < 1$. Precisely, the point is to prove that the definition of p^{x_0} in (1.4) is coherent with that in (6.1). To put it differently, we must check that the right-hand side of (1.4) coincides with the definition of p^{x_0} in (6.1). Given a smooth function $\phi : [0,T] \times (-\infty,1] \to \mathbb{R}$, with a compact support, the analysis of the Green function in [5, Th. 1.10, Chap. VI] says that the PDE

$$\partial_s u(s,x) + \big(b(x) + f'(s)\big)\partial_x u(s,x) + \frac{1}{2}\partial^2_{xx} u(s,x) + \phi(s,x) = 0,$$

for $(s,x) \in (0,T] \times (-\infty,1)$, with $u(T,x) = 0$, $x \in (-\infty,1)$, as initial condition and $u(s,1) = 0$, $s \in [0,T]$, as Dirichlet boundary condition, admits a (unique) classical solution

$$u(s,x) = \int_s^T \int_{-\infty}^1 G(s,x,t,y)\phi(t,y)dydt, \quad s \in [0,T), \ x \leqslant 1. \tag{6.2}$$

Moreover, u is bounded and continuous on $[0,T] \times (-\infty, 1]$ and is once continuously differentiable in time and twice differentiable in space on $[0,T) \times (-\infty,1)$. Therefore, we can expand $(u(t \wedge \tau, X_{t\wedge\tau}))_{t\in[0,T]}$ by Itô's formula, where $\tau = \inf\{t \geqslant 0 : X_{t\wedge T} \geqslant 1\}$. We then have the well-known representation formula:

$$u(0,x_0) = \mathbb{E}^{x_0}\left(\int_0^{T\wedge\tau} \phi(t,X_t)dt\right). \tag{6.3}$$

By equalizing (6.2) and (6.3), we deduce that

$$\mathbb{E}^{x_0}\left(\int_0^{T\wedge\tau} \phi(t,X_t)dt\right) = \int_0^T \int_{-\infty}^1 p^{x_0}(t,y)\phi(t,y)dydt. \tag{6.4}$$

Writing

$$\begin{aligned}\mathbb{E}x_0\left(\int_0^{T\wedge\tau} \phi(t,X_t)dt\right) &= \int_0^T \mathbb{E}^{x_0}\left[\phi(t,X_t)\mathbf{1}_{\{t<\tau\}}\right]dt \\ &= \int_0^T \int_{-\infty}^1 \phi(t,y)\mathbb{P}^{x_0}\big(X_t \in dy, t < \tau\big)dt,\end{aligned}$$

we deduce that

$$\int_0^T \int_{-\infty}^1 \phi(t,y)\mathbb{P}^{x_0}\big(X_t \in dy, t<\tau\big)dt = \int_0^T \int_{-\infty}^1 p^{x_0}(t,y)\phi(t,y)dydt, \tag{6.5}$$

so that (1.4) holds in the smooth setting.

In the same framework, we then prove (2.2). The cumulative distribution function of τ is given by

$$\mathbb{P}^{x_0}\big(\tau \leqslant t\big) = 1 - \mathbb{P}^{x_0}\big(\tau > t\big) = 1 - \int_{-\infty}^1 p^{x_0}(t,y)dy.$$

By [5, Th. 1.10, Chap. VI], we can differentiate the above expression with respect to t and exchange the derivative and the integral. From (2.1), we deduce:

$$\begin{aligned}\frac{d}{dt}\mathbb{P}^{x_0}\big(\tau \leqslant t\big) &= -\int_{-\infty}^1 \partial_t p^{x_0}(t,y)dy \\ &= \int_{-\infty}^1 \partial_y\big([f'(t)+b(y)]p^{x_0}(t,y)\big)dy - \frac{1}{2}\int_{-\infty}^1 \partial^2_{yy}p^{x_0}(t,y)dy,\end{aligned}$$

for $t \in [0,T]$. Again by [5, Th. 1.10, Chap. VI], we know that both $p^{x_0}(t,y)$ and $\partial_y p^{x_0}(t,y)$ tend to 0 exponentially fast as $y \to -\infty$. So, using the boundary condition $p^{x_0}(t,1) = 0$, we obtain (2.2).

Third Step. We now aim at proving the same results, still with $X_0 = x_0 < 1$, but under the original assumptions on b and f. The strategy is to use a mollification argument. In order to do so, we must prove that, in the smooth setting, p^{x_0} and

$\partial_y p^{x_0}$ can be bounded by constants that only depend upon T, x_0, Λ, K (the Lipschitz constants associated with b) and A, where $A = \sup_{t \leqslant T} |f'(t)|$.

We thus go back to the case when b and f are smooth and bounded with bounded derivatives of any order. By Proposition 4.2, for $t \in [0,T]$ small enough, we already have a Gaussian bound for $\partial_y p^{x_0}(t,y)$ in terms of T, A, K and Λ only (notice that the argument applies since we know that p^{x_0} indeed satisfies PDE (2.1) in the smooth setting). With the same notation as in the statement of Proposition 4.2 (using in addition Proposition 3.2), the bound is of the form

$$|\partial_y p^{x_0}(t,y)| \leqslant \frac{C}{t} \exp\left(-\frac{|\xi_t^{x_0} - y|^2}{Ct}\right), \tag{6.6}$$

for $t \in (0,\delta]$, where $(\xi_t^{x_0})_{t \geqslant 0}$ is given by (3.6) and C and δ are constants that depend on T, A, K and Λ only. Plugging this bound into (3.3) and repeating the Gaussian convolution argument used in (4.9), we deduce that, for $t \in (0,\delta]$,

$$p^{x_0}(t,y) \leqslant \frac{C}{\sqrt{t}} \exp\left(-\frac{|\xi_t^{x_0} - y|^2}{Ct}\right), \tag{6.7}$$

up to a new value of C.

Actually, (6.6) and (6.7) can be seen as bounds for the Green function G and its derivative in small time since $p^{x_0}(t,y) = G(0,x_0,t,y)$. In the same way, we could prove similar bounds for $G(s,x,t,y)$ and $\partial_y G(s,x,t,y)$ when $t - s \in (0,\delta]$. By a standard chaining argument, we then deduce that (6.6) and (6.7) are valid on the whole $[0,T]$, for a possibly new value of C. Indeed, for a given $t \in (0,T]$, we can consider a sequence $0 = t_0 < t_1 < \cdots < t_N = t$ such that $t_{i+1} - t_i \leqslant \delta$. Then, by using the Markov structure, we have:

$$p^{x_0}(t,y) = \int_0^t \int_{(-\infty,1]^{N-1}} \prod_{i=1}^N G(t_{i-1},t_i,z_{i-1},z_i) dz_1 \dots dz_{N-1},$$

with the convention $z_0 = x_0$ and $z_N = y$. Noticing that N can be assumed to be bounded from above and using again a Gaussian convolution argument, we deduce that (6.6) and (6.7) can be extended to the whole $(0,T]$.

Fourth Step. We still assume that the coefficients are smooth and bounded, with bounded derivatives of any order and that $X_0 = x_0 < 1$. By the third step, we are then able to reduce the PDE (2.1) to an heat PDE with a non-trivial source term:

$$\partial_t p^{x_0}(t,y) - \frac{1}{2}\partial^2_{yy} p^{x_0}(t,y) = -\partial_y\big[\big(b(y) + f'(t)\big)p^{x_0}(t,y)\big], \quad t \in (0,T],\ y < 1.$$

For any compact subset $\mathcal{K} \subset (0,T] \times (-\infty,1]$, we can consider a smooth cut-off function $\eta : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ matching 1 on $\mathcal{K}$ and vanishing outside another compact subset $\mathcal{K}' \subset (0,T] \times \mathbb{R}$, $\mathcal{K} \subset \mathcal{K}'$. Then, the function $(0,T] \times (-\infty,1] \ni (t,y) \mapsto [\eta p^{x_0}](t,y)$ satisfies the heat equation:

$$\big(\partial_t - \frac{1}{2}\partial^2_{yy}\big)\big[\eta p^{x_0}\big](t,y) = h(t,y), \tag{6.8}$$

with $[\eta p^{x_0}](0,\cdot) = 0$ as initial condition and with $[\eta p^{x_0}](t,1) = 0$ and $[\eta p^{x_0}](t,y) = 0$ for $t \in [0,T]$ and $|y|$ large enough as boundary conditions, where h is a smooth function which can be bounded in terms of T, A, η, K and Λ only. By [4, Th. 4, Sec. 2, Chap. 7], we deduce that ηp^{x_0} and $\partial_y[\eta p^{x_0}]$ are $(1/4,1/2)$-Hölder continuous in (t,y), the Hölder constant depending on T, A, η, K and Λ only. Therefore, on any compact subset $\mathcal{K}$ of $(0,T]\times(-\infty,1]$, p^{x_0} and $\partial_y p^{x_0}$ are $(1/4,1/2)$-Hölder continuous in (t,y), the Hölder constant depending on T, A, $\mathcal{K}$, K and Λ only. By the same argument, on any compact subset $\mathcal{K}$ of $[0,T]\times\mathcal{I}$, $\mathcal{I}$ being a finite interval included in $(-\infty,1]$ not containing x_0, p^{x_0} and $\partial_y p^{x_0}$ are also $(1/4,1/2)$-Hölder continuous in (t,y), the Hölder constant depending on T, A, $\mathcal{K}$, K and Λ only. Indeed, in such a case, $p^{x_0}(t,y)$ and $\partial_y p^{x_0}(t,y)$ tend to 0 as t tends to 0 and y stays in $\mathcal{I}$, so that the compact support $\mathcal{K}'$ of η can be assumed to be included in $[0,T]\times\mathbb{R}$ (and not necessarily in $(0,T]\times\mathbb{R}$). In the end, we deduce that, on any compact subset $\mathcal{K}$ of $([0,T]\times(-\infty,1])\setminus\{(0,x_0)\}$, p^{x_0} and $\partial_y p^{x_0}$ are $(1/4,1/2)$-Hölder continuous in (t,y), the Hölder constant depending on T, A, $\mathcal{K}$, K and Λ only.

In order to tackle the second-order derivatives in space, we assume that $\mathcal{K}'\subset(0,T]\times(-\infty,1)$, $\mathcal{K}'$ being the support of η. Then, the function $(0,T]\times\mathbb{R}\ni(t,y)\mapsto[\eta p^{x_0}](t,y)$ (with $[\eta p^{x_0}](t,y)=0$ for $y>1$) satisfies (6.8) on the whole $(0,T]\times\mathbb{R}$, so that it can be represented as a standard Gaussian convolution. Then, by Calderon and Zygmund estimates, see [10, Eq. (0.4), App. A], for any $\varsigma\geqslant 1$, the $L^{\varsigma}([0,T]\times\mathbb{R},dt\otimes dy)$-norms of $\partial_t[\eta p^{x_0}]$ and $\partial_{yy}[\eta p^{x_0}]$ are bounded in terms of A, η, K, Λ and T only. Therefore, on any compact subset $\mathcal{K}$ of $(0,T]\times(-\infty,1)$, for any $\varsigma\geqslant 1$, the $L^{\varsigma}(\mathcal{K},dt\otimes dy)$-norms of $\partial_t p^{x_0}$ and $\partial_{yy}p^{x_0}$ are bounded in terms of T, A, $\mathcal{K}$, K and Λ only. For example, when $\mathcal{K}$ is a cylinder of the form $[\delta,T]\times[y-1,y+1]$, (6.6) and (6.7) say that

$$\int_{\mathcal{K}}\big(|\partial_t p^{x_0}(t,z)|^{\varsigma}+|\partial^2_{yy}p^{x_0}(t,z)|^{\varsigma}\big)dtdz\leqslant C_{\varsigma,\delta}\int_{\delta}^{T}\exp\left(-\frac{|\xi_s^{x_0}-y|^2}{C_{\varsigma,\delta}}\right)ds,\qquad(6.9)$$

for a constant $C_{\varsigma,\delta}$ that is independent of x_0 and y.

Fifth Step. We now have all the required ingredients to go back to the original framework. The point is to approximate b and f by two sequences $(b^n)_{n\geqslant 1}$ and $(f^n)_{n\geqslant 1}$ (for the topology of uniform convergence on compact sets) that satisfy the previous smoothness conditions. The associated solutions to the PDE (2.1) are denoted by $(p^{x_0,n})_{n\geqslant 1}$. On any compact subset $\mathcal{K}\subset([0,T]\times(-\infty,1])\setminus\{(0,x_0)\}$, the sequences $(p^{x_0,n})_{n\geqslant 1}$ and $(\partial_y p^{x_0,n})_{n\geqslant 1}$ are uniformly bounded and $(1/4,1/2)$-Hölder continuous in (t,y). Therefore, there exists a subsequence $(\varphi(n))_{n\geqslant 1}$ such that $(p^{x_0,\varphi(n)})_{n\geqslant 1}$ and $(\partial_y[p^{x_0,\varphi(n)}])_{n\geqslant 1}$ are uniformly convergent on compact subsets of $([0,T]\times(-\infty,1])\setminus\{(0,x_0)\}$. Similarly, we can assume that the sequences $(\partial_t[p^{x_0,\varphi(n)}])_{n\geqslant 1}$ and $(\partial^2_{yy}[p^{x_0,\varphi(n)}])_{n\geqslant 1}$ are weakly convergent for the $L^{\varsigma}(\mathcal{K},dt\otimes dy)$ topology, for any $\varsigma\geqslant 1$ and any compact subset $\mathcal{K}\subset(0,T]\times(-\infty,1)$. The limit function of the sequence $(p^{x_0,\varphi(n)})_{n\geqslant 1}$ is denoted by p^{x_0}: clearly, it is a solution of (2.1) (in the Sobolev sense), with $p^{x_0}(t,1)=0$ for $t>0$, as Dirichlet boundary condition. By (6.7), it tends to 0 as t tends to 0 and y stays away from x_0. Moreover, $\partial_y p^{x_0}$ exists and is continuous on $([0,T]\times(-\infty,1])\setminus\{(0,x_0)\}$.

Then, we can prove (1.4). Indeed, we know that $(p^{x_0,\varphi(n)})_{n\geqslant 1}$ converges toward p^{x_0} uniformly on compact subsets of $([0,T]\times(-\infty,1])\setminus\{(0,x_0)\}$. Moreover, it is standard to prove that the solution X^n to (1.1) started at x_0, but driven by b^n and f^n instead of b and f, converges in law (as $n \to \infty$) toward X on the space $\mathcal{C}([0,T],\mathbb{R})$ of continuous functions on $[0,T]$ endowed with the uniform topology. Denoting by $\tau^n := \inf\{t \geqslant 0 : X_t^n \geqslant 1\}$, we claim that the pair $(\tau^n \wedge T, X^n)$ converges in law toward $(\tau \wedge T, X)$ on the product space $\mathbb{R} \times \mathcal{C}([0,T],\mathbb{R})$ endowed with the product topology. Indeed, the mapping

$$\mathcal{C}([0,T],\mathbb{R}) \ni x \mapsto (\inf\{t \geqslant 0 : x(t) \geqslant 1\}) \wedge T$$

is continuous at any path x satisfying the "crossing property":

$$\big(\exists t \in [0,T) : x(t) \geqslant 1\big) \Rightarrow \big(\forall p \geqslant 1,\ \exists t_p \in (t, t+1/p) : x(t_p) > 1\big), \tag{6.10}$$

and (6.10) holds for a.e. trajectory of X because of the Brownian part in the dynamics of X. Recalling that (6.4) holds true in the smooth setting, that is with p^{x_0} and τ therein replaced by $p^{x_0,n}$ and τ^n, $n \geqslant 1$, we can pass to the limit along the subsequence $(\varphi(n))_{n \geqslant 1}$ (to pass to the limit in the right-hand side of (6.4), use the boundedness of the support of ϕ and the uniform convergence of $(p^{x_0,(\varphi(n))})_{n \geqslant 1}$ toward p^{x_0} on compact subsets of $(0,T] \times (-\infty,1]$). We deduce that (6.4) holds in the general setting as well, from which we deduce that (6.5) also holds in the general setting.

By the same approximation argument, we can prove that when $\nu = \delta_{x_0}$, (2.2) holds in the general setting as well.

By (1.4), p^{x_0} depends upon x_0 in a measurable way.

Sixth Step. To complete the proof, it remains to discuss what happens when ν does not reduce to a Dirac mass. The point is then prove that

$$p^{\nu}(t,y) = \int_{-\infty}^{1} p^{x}(t,y)\nu(dx), \quad t \geqslant 0,\ y \leqslant 1, \tag{6.11}$$

is the right candidate for solving the Fokker-Planck equation and for making the connection with X.

We first check that the definition (6.11) makes sense when $t > 0$. By applying the Markov property for X, this will directly prove (1.4). From (6.7), we know that $p^x(t,y) \leqslant (C/t^{1/2}) \exp[-|\xi_t^x - y|^2/(Ct)]$: this is enough to check that (6.11) indeed makes sense. By Lebesgue dominated convergence Theorem, this also proves that $p^{\nu}(t,y) \to 0$ when $y \to -\infty$. Similarly, p^{ν} is continuous on $(0,T] \times (-\infty,1]$. When $\mathrm{supp}(\nu) \subset (-\infty, 1-\epsilon]$, the same domination argument shows that continuity holds on any compact subset of $([0,T] \times (-\infty,1]) \setminus (\{0\} \times (-\infty,1-\epsilon])$.

Using (6.6), we can prove in a similar way that p^{ν} is continuously differentiable in y on $(0,T] \times (-\infty,1]$, with

$$\partial_y p^{\nu}(t,y) = \int_{-\infty}^{1} \partial_y p^{x}(t,y)\nu(dx), \quad t > 0,\ y \leqslant 1, \tag{6.12}$$

and that $\partial_y p^{\nu}(t,y) \to 0$ when $y \to -\infty$ When $\mathrm{supp}(\nu) \subset (-\infty,1-\epsilon]$, continuous differentiability holds on any compact subset of $([0,T] \times (-\infty,1]) \setminus (\{0\} \times (-\infty,1-\epsilon])$.

By combining the same domination argument with (6.9), we can also prove that p^{ν} admits Sobolev derivatives of order 1 in t and of order 2 in y in any L^{ς}, $\varsigma \geqslant 1$, on any compact subsets of $(0,T] \times (-\infty, 1)$, the derivatives being given as the integrals of the derivatives of p^x with respect to the law of ν. It is then plain to check that p^ν satisfies (2.1).

The last point is thus to check that p^ν satisfies (2.2). By Markov's property,

$$\begin{aligned}\mathbb{P}^\nu(\tau \leqslant t) &= \int_{-\infty}^1 \mathbb{P}^x(\tau \leqslant t)\,\nu(dx) \\ &= -\frac{1}{2}\int_{-\infty}^1 \left[\int_0^t \partial_y p^x(s,1)ds\right]\nu(dx).\end{aligned} \tag{6.13}$$

In order to prove (2.2), we must exchange the two integrals. To exchange the two integrals, we specify the upper bound for $|\partial_y p^x(t,1)|$. Using (6.6) and (3.12) and recalling that $\xi^1_\cdot \equiv 1$, we deduce that $|\partial_y p^x(t,1)| \leqslant (C/t)\exp[-(1-x)^2/(Ct)]$, from which it holds that

$$\begin{aligned}&\int_{-\infty}^1 \int_0^t |\partial_y p^x(s,1)| ds\,\nu(dx) \\ &\leqslant \int_{-\infty}^{1-\epsilon}\int_0^t \frac{C}{s}\exp\left(-\frac{(1-x)^2}{Cs}\right)ds\,\nu(dx) \\ &\quad + \int_{1-\epsilon}^1 \int_0^t \frac{C}{s}\exp\left(-\frac{(1-x)^2}{Cs}\right)ds\,\nu(dx).\end{aligned} \tag{6.14}$$

The first integral in the right-hand side is clearly bounded. To tackle the second one, we make use of the assumption $\nu(dx) \leqslant \beta(1-x)dx$ for $x \in (1-\epsilon, 1]$. We obtain

$$\int_{1-\epsilon}^1 \int_0^t \frac{C}{s}\exp\left(-\frac{(1-x)^2}{Cs}\right)ds\,\nu(dx) \leqslant \beta\int_0^\epsilon \int_0^t \frac{Cx}{s}\exp\left(-\frac{x^2}{Cs}\right)ds\,dx < \infty. \tag{6.15}$$

Going back to (6.13), we obtain

$$\mathbb{P}^\nu(\tau \leqslant t) = -\frac{1}{2}\int_0^t \left(\int_{-\infty}^1 \partial_y p^x(s,1)\nu(dx)\right)ds = -\frac{1}{2}\int_0^t \partial_y p^\nu(s,1)ds,$$

the last part following from (6.12). Since the mapping $(0,T] \ni t \mapsto \partial_y p^\nu(t,1)$ is continuous, we deduce that the mapping $(0,T] \ni t \mapsto \mathbb{P}^\nu(\tau \leqslant t)$ is continuously differentiable. It then remains to check the continuous differentiability at time $t = 0$. To this end, it is sufficient to prove that $\partial_y p^\nu(\cdot, 1)$ is continuous at $t = 0$. Following (6.14) and (6.15), it is clear that, for any $\epsilon' \in (0, \epsilon)$,

$$\lim_{t\searrow 0}\partial_y p^\nu(t,1) = \lim_{t\searrow 0}\int_{1-\epsilon'}^1 \partial_y p^x(t,1)\nu(dx) = \lim_{t\searrow 0}\int_{1-\epsilon'}^1 \partial_y p^x(t,1)p_0^\nu(x)dx, \tag{6.16}$$

where $p_0^\nu(x)dx = \nu(dx)$, which is assumed to make sense on $(1-\epsilon', 1)$. By Proposition 4.2, we know that, for some constant $C > 0$,

$$|\partial_y p^x(t,y) - \partial_y q(t,x,y)| \leqslant \frac{C}{\sqrt{t}}\exp\left(-\frac{|x-1|^2}{Ct}\right),$$

which proves, by using the bound $p_0^\nu(x) \leqslant \beta(1-x)$ and by letting ϵ' tend to 0 in (6.16), that

$$\lim_{t\searrow 0} \partial_y p^\nu(t,1) = \lim_{\epsilon'\searrow 0}\lim_{t\searrow 0} \int_{1-\epsilon'}^1 \partial_y q(t,x,1) p_0^\nu(x)dx. \tag{6.17}$$

We then introduce the killed Gaussian kernel with reflection at point 1:

$$\hat{q}(t,x,y) = \frac{1}{\sqrt{2\pi t}}\left[\exp\left(-\frac{(x-y)^2}{2t}\right) - \exp\left(-\frac{(x-(2-y))^2}{2t}\right)\right].$$

Now a formula like (3.3) for expressing $\partial_y p^x(t,y)$ in terms of $\partial_y q(t,x,y)$ can be proven for expressing $\partial_y q(t,x,y)$ in terms of $\partial_y \hat{q}(t,x,y)$ (replacing $f'(s)+b(1)$ by $b(z)-b(1)$ in (3.3)). Recalling $|b(z)-b(1)| \leqslant K|z-1|$, we obtain (for a possibly new value of C which is allowed to increase from line to line)

$$\begin{aligned}
&|\partial_y q(t,x,1) - \partial_y \hat{q}(t,x,1)| \\
&\leqslant C\int_0^t \int_{-\infty}^1 \frac{C(1-z)}{s(t-s)} \exp\left(-\frac{|\xi_s^x - z|^2}{Cs}\right)\exp\left(-\frac{|1-z|^2}{C(t-s)}\right) dzds \\
&\leqslant C\int_0^t \int_{-\infty}^1 \frac{C}{s(t-s)^{1/2}} \exp\left(-\frac{|\xi_s^x - z|^2}{Cs}\right) dzds \\
&\leqslant C.
\end{aligned}$$

Recalling (6.17), we deduce that:

$$\begin{aligned}
\lim_{t\searrow 0} \partial_y p^\nu(t,1) &= \lim_{\epsilon'\searrow 0}\lim_{t\searrow 0} \int_{1-\epsilon'}^1 \partial_y \hat{q}(t,x,1) p_0^\nu(x)dx, \\
&= -\lim_{\epsilon'\searrow 0}\lim_{t\searrow 0} 2t^{-1/2} \int_{1-\epsilon'}^1 g'\left(\frac{x-1}{t^{1/2}}\right) p_0^\nu(x)dx,
\end{aligned}$$

where g stands for the standard Gaussian kernel. Since p_0^ν is assumed to be differentiable at 1, with $p_0^\nu(1)=0$, we can write $p_0^\nu(x) = (x-1)[d/dx]p_0^\nu(1) + o(x-1)$, where $o(\cdot)$ stands for the Landau notation. We deduce that the limit of $\partial_y p^\nu(t,1)$ as $t\searrow 0$ must be $[d/dx]p_0^\nu(1)$. Following (6.14) and (6.15), we have an explicit bound for the supremum norm of $\partial_y p^\nu(\cdot,1)$ in terms of β, K, Λ, T and the supremum norms of f and f' only.

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