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# Deformations of the seventh order Peregrine breather solutions of the NLS equation with twelve parameters.

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**Abstract**

We study the solutions of the one dimensional focusing NLS equation. Here we construct new deformations of the Peregrine breather of order 7 with 12 real parameters. We obtain new families of quasi-rational solutions of the NLS equation. With this method, we construct new patterns of different types of rogue waves. We recover triangular configurations as well as rings isolated. As already seen in the previous studies, one sees appearing for certain values of the parameters, new configurations of concentric rings.

## 1 Introduction

The nonlinear Schrödinger equation was first solved by Zakharov and Shabat [1] in 1972 by the inverse scattering method. The first expressions of the quasi-rational solutions were given by Peregrine [2] in 1983. From this time, a considerable number of studies were carried out. Eleonski, Akhmediev and Kulagin obtained the first higher order analogue of the Peregrine breather [3] in 1986. Akhmediev et al. [4, 5], constructed other analogues of order 3 and 4, using Darboux transformations.
Rational solutions of the NLS equation have been written in 2010, as a quotient of two Wronskians in [6]. An other representation of the solutions of

the NLS equation has been constructed in [7] in 2011, also in terms of a ratio of two Wronskians determinants of order $2N$.
In 2012, Guo, Ling and Liu constructed an other representation of the solutions of the focusing NLS equation, as a ratio of two determinants has been given in [8] using generalized Darboux transform.
In the same year, Ohta and Yang [9] have given a new approach where solutions of the focusing NLS equation by means of a determinant representation, obtained from Hirota bilinear method.
A the beginning of the year 2012, one obtained a representation in terms of determinants which does not involve limits [10].
The two formulations given in [7, 10] did depend in fact only on two parameters; this remark was first made by V.B. Matveev. Then we found for the order $N$ (for determinants of order 2N), solutions depending on $2N-2$ real parameters.
In this article, we restrict ourself the study to the case of the solutions of NLS of order 7; because of the constraints of the publication, we do not have the space to publish all the deformations. With this new method, we construct news deformations at order 7 with 12 real parameters. The explicit representation in terms of polynomials is found, but is too monstrous to be published.
One constructs various drawings to illustrate the evolution of the solutions according to the parameters. One obtains at the same time triangular configurations and ring structures with a maximum of 28 peaks. These deformations are completely new and gives by new patterns a better understanding of the NLS equation.

# 2 Determinant representation of solutions of NLS equation

We recall the results obtained in [7] and [10]. We consider the focusing NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0. \tag{1}$$

In the following, we consider $2N$ parameters $\lambda_\nu$, $\nu = 1, \ldots, 2N$ satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \leq j \leq N. \tag{2}$$

We define the terms $\kappa_\nu$, $\delta_\nu$, $\gamma_\nu$ by the following equations,

$$\kappa_\nu = 2\sqrt{1-\lambda_\nu^2}, \quad \delta_\nu = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \tag{3}$$

and

$$\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = 1/\gamma_j, \quad j = 1 \ldots N. \tag{4}$$

The terms $x_{r,\nu}$ ($r = 3,\ 1$) are defined by

$$x_{r,\nu} = (r-1) \ln \frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad 1 \le j \le 2N. \tag{5}$$

The parameters $e_\nu$ are defined by

$$e_j = i a_j - b_j, \quad e_{N+j} = i a_j + b_j, \quad 1 \le j \le N, \tag{6}$$

where $a_j$ and $b_j$, for $1 \le j \le N$ are arbitrary real numbers.
We use the following notations :

$$\begin{array}{l} A_\nu = \kappa_\nu x/2 + i\delta_\nu t - i x_{3,\nu}/2 - i e_\nu/2, \\ B_\nu = \kappa_\nu x/2 + i\delta_\nu t - i x_{1,\nu}/2 - i e_\nu/2, \end{array} \tag{7}$$

for $1 \le \nu \le 2N$, with $\kappa_\nu$, $\delta_\nu$, $x_{r,\nu}$ defined in (3), (4) and (5).
The parameters $e_\nu$ are defined by (6).
Here, the parameters $a_j$ and $b_j$, for $1 \le N$ are chosen in the form

$$a_j = \sum_{k=1}^{N-1} \tilde{a_k} \epsilon^{2k+1} j^{2k+1}, \quad b_j = \sum_{k=1}^{N-1} \tilde{b_k} \epsilon^{2k+1} j^{2k+1}, \quad 1 \le j \le N. \tag{8}$$

We consider the following functions :

$$\begin{array}{ll} f_{4j+1,k} = \gamma_k^{4j-1} \sin A_k, & f_{4j+2,k} = \gamma_k^{4j} \cos A_k, \\ f_{4j+3,k} = -\gamma_k^{4j+1} \sin A_k, & f_{4j+4,k} = -\gamma_k^{4j+2} \cos A_k, \end{array} \tag{9}$$

for $1 \le k \le N$, and

$$\begin{array}{ll} f_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos A_{N+k}, & f_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin A_{N+k}, \\ f_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos A_{N+k}, & f_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin A_{N+k}, \end{array} \tag{10}$$

for $1 \leq k \leq N$.
We define the functions $g_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way, we replace only the term $A_k$ by $B_k$.

$$\begin{array}{ll} g_{4j+1,k} = \gamma_k^{4j-1} \sin B_k, & g_{4j+2,k} = \gamma_k^{4j} \cos B_k, \\ g_{4j+3,k} = -\gamma_k^{4j+1} \sin B_k, & g_{4j+4,k} = -\gamma_k^{4j+2} \cos B_k, \end{array} \tag{11}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll} g_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos B_{N+k}, & g_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin B_{N+k}, \\ g_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos B_{N+k}, & g_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin B_{N+k}, \end{array} \tag{12}$$

for $1 \leq k \leq N$.
Then we get the following result :

**Theorem 2.1** *The function $v$ defined by*

$$v(x,t) = \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} e^{2it-i\varphi} \tag{13}$$

*is a quasi-rational solution of the NLS equation (1)*

$$iv_t + v_{xx} + 2|v|^2 v = 0,$$

*depending on $2N-2$ parameters $\tilde{a}_j$, $\tilde{b}_j$, $1 \leq j \leq N-1$, where*

$$\begin{array}{ll} n_{j1} = f_{j,1}(x,t,0), & n_{jk} = \frac{\partial^{2k-2} f_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0), \\ n_{jN+1} = f_{j,N+1}(x,t,0), & n_{jN+k} = \frac{\partial^{2k-2} f_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0), \\ d_{j1} = g_{j,1}(x,t,0), & d_{jk} = \frac{\partial^{2k-2} g_{j,1}}{\partial \epsilon^{2k-2}}(x,t,0), \\ d_{jN+1} = g_{j,N+1}(x,t,0), & d_{jN+k} = \frac{\partial^{2k-2} g_{j,N+1}}{\partial \epsilon^{2k-2}}(x,t,0), \\ 2 \leq k \leq N,\ 1 \leq j \leq 2N & \end{array} \tag{14}$$

*The functions $f$ and $g$ are defined in (9),(10), (11), (12).*

We don't have the space to give the proof in this publication. We will give it in an other forthcoming paper.
The solutions of the NLS equation can also be written in the form :

$$v(x,t) = \exp(2it - i\varphi) \times Q(x,t)$$

where $Q(x,t)$ is defined by :

$$Q(x,t) := \frac{\begin{vmatrix} f_{1,1}[0] & \dots & f_{1,1}[N-1] & f_{1,N+1}[0] & \dots & f_{1,N+1}[N-1] \\ f_{2,1}[0] & \dots & f_{2,1}[N-1] & f_{2,N+1}[0] & \dots & f_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{2N,1}[0] & \dots & f_{2N,1}[N-1] & f_{2N,N+1}[0] & \dots & f_{2N,N+1}[N-1] \end{vmatrix}}{\begin{vmatrix} g_{1,1}[0] & \dots & g_{1,1}[N-1] & g_{1,N+1}[0] & \dots & g_{1,N+1}[N-1] \\ g_{2,1}[0] & \dots & g_{2,1}[N-1] & g_{2,N+1}[0] & \dots & g_{2,N+1}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{2N,1}[0] & \dots & g_{2N,1}[N-1] & g_{2N,N+1}[0] & \dots & g_{2N,N+1}[N-1] \end{vmatrix}} \quad (15)$$

# 3 Quasi-rational solutions of order 7 with twelve parameters

Wa have already constructed in [7] solutions for the cases from $N = 1$ until $N = 6$, and in [10] with two parameters.
Because of the length of the expression $v$ of the solution of NLS equation with 12 parameters, we can't give here. We only construct figures to show deformations of the analogue of the seventh Peregrine breather; in the following we will call it for simplicity, the seventh Peregrine breather.
Conversely to the study with two parameters given in preceding works [7, 10], we get other type of symmetries in the plots in the $(x,t)$ plane. We give some examples of this fact in the following.
It is important to note the similar role played by the parameters $\tilde{a}_j$ and $\tilde{b}_j$ for a same $j$; the same configuration of the peaks is obtained. For this reason one will give the figures only for a parameter $\tilde{a}_j$ or $\tilde{b}_j$. On the other hand, to understand the configuration for a value of the parameter, one will give two sights to see the distribution of the peaks.
With different choices of parameters, we obtain all types of configurations : triangles, rings and concentric rings with a maximum of 28 peaks.

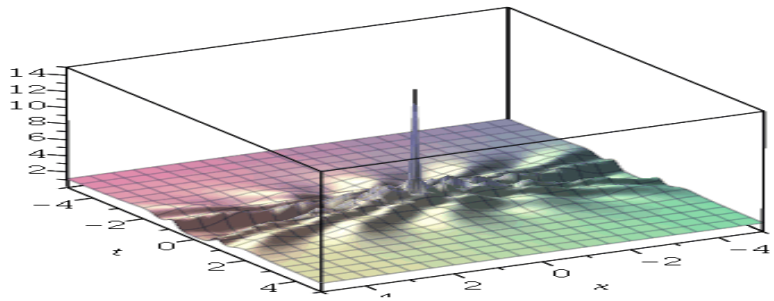


Figure 1: Solution of NLS, N=7; all parameters equal to 0, the Peregrine breather of order 7, $P_7$.

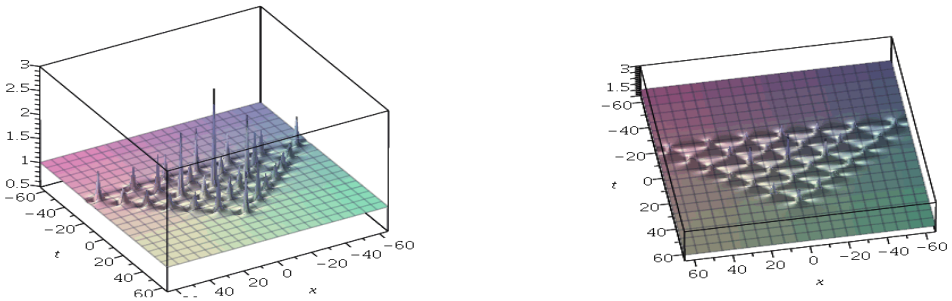


Figure 2: Solution of NLS, N=7; $\tilde{b}_1 = 10^4$; we obtain a regular triangle with 28 peaks; on the right, sight of top.

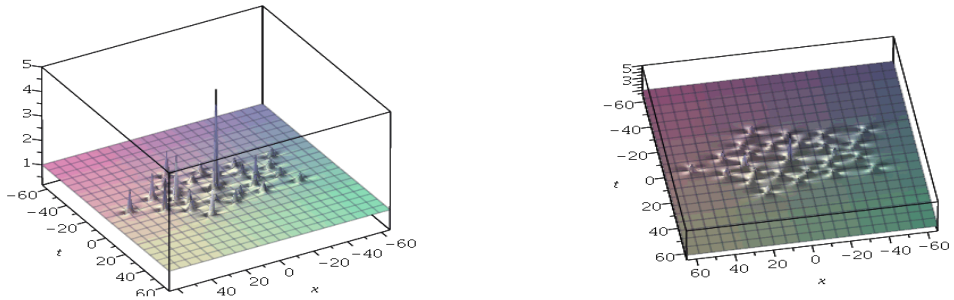


Figure 3: Solution of NLS, N=7; $\tilde{a}_2 = 10^6$, 3 rings with respectively 5, 10, 10 peaks with in the center the Peregrine of order 2, $P_2$; on the right, sight of top.

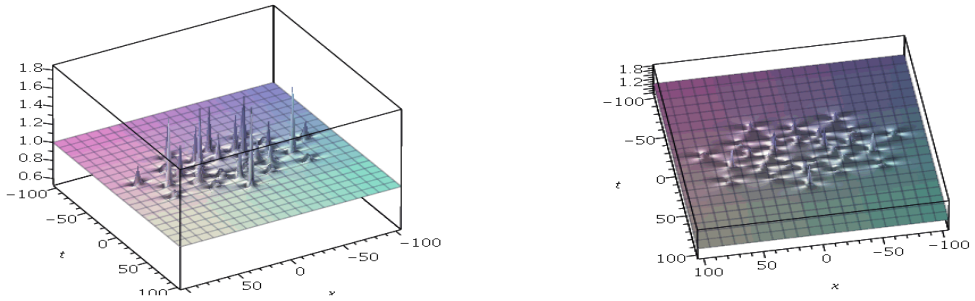


Figure 4: Solution of NLS, N=7; $\tilde{b}_3 = 10^{10}$, 4 rings with 7 peaks on each of them without central peak; on the right, sight of top.

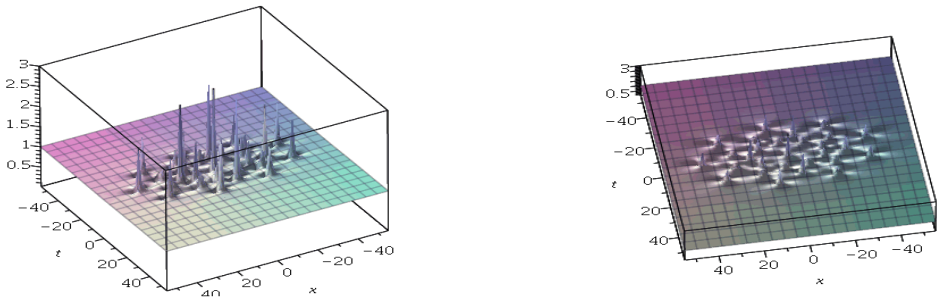


Figure 5: Solution of NLS, N=7; $\tilde{a}_4 = 10^{10}$, 3 rings with 9 peaks on each of them with in the center one peak; on the right, sight of top.

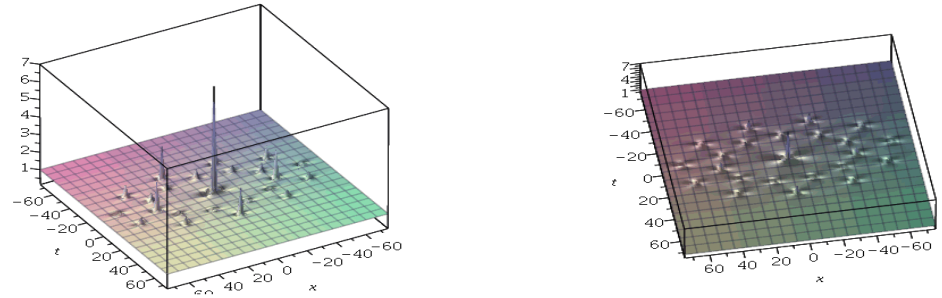


Figure 6: Solution of NLS, N=7; $\tilde{b}_5 = 10^{15}$, 2 rings of 11 peaks with in the center the Peregrine breather of order 3, $P_3$; on the right, sight of top.

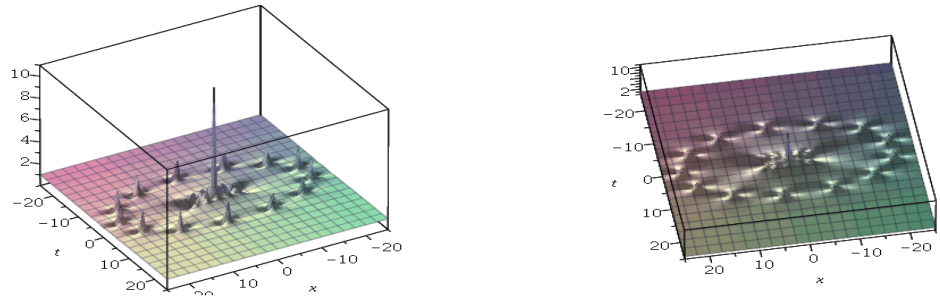



Figure 7: Solution of NLS, N=7; $\tilde{a}_6 = 10^{12}$, $\tilde{b}_6 = -10^8$, a ring with 13 peaks and in the center the Peregrine breather of order 5, $P_5$; on the right, sight of top.

# 4 Conclusion

We have constructed explicitly solutions of the NLS equation of order $N$ with $2N-2$ real parameters. The expressions in terms of polynomials in $x$ and $t$ are too monstrous to be published in this paper.
It is important to note the symmetrical role played by the parameters $\tilde{a}_j$ and $\tilde{b}_j$; the configurations obtained for one of these two parameters $\tilde{a}_j$ or $\tilde{b}_j$ for the index $j$ are the same ones. Thus for each couple $(\tilde{a}_j;\ \tilde{b}_j)$ we have only built one associated figure for only for one parameter, $\tilde{a}_j \neq 0$, or $\tilde{b}_j \neq 0$.
In the cases $a_1 \neq 0$ or $b_1 \neq 0$ we obtain triangles with a maximum of 28 peaks; for $a_2 \neq 0$ or $b_2 \neq 0$, we have 3 concentric rings with two of them with 10 peaks and an other with 5 peaks with in the center the Peregrine $P_2$ with 3 peaks. For $a_3 \neq 0$ or $b_3 \neq 0$, we obtain 4 concentric rings without central peak with 7 peaks on each of them. For $a_4 \neq 0$ or $b_4 \neq 0$, we have 3 concentric rings with 9 peaks, with a in the center one peak. For $a_5 \neq 0$ or $b_5 \neq 0$, we obtain 2 concentric rings without central peak with 11 peaks on each of them and the appearance in the center of the Peregrine breather $P_3$ with 6 peaks. For $a_6 \neq 0$ or $b_6 \neq 0$, we have only one ring with 13 peaks with inside the appearance of the Peregrine breather of order 5 with 15 peaks.
We obtained new patterns in the $(x;t)$ plane, by different choices of these parameters; we recognized rings configurations as already observed in the case of deformations depending on two parameters [7, 10]. We get news triangular shapes and multi-concentric rings.

This study at the order 7 was never still carried out; it is completely new and makes it possible to provide a better understanding of the phenomena of rogue waves.

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