Directed acyclic graphs with the unique dipath property

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Abstract

Let $\mathscr{P}$ be a family of dipaths of a DAG (Directed Acyclic Graph) G. The load of an arc is the number of dipaths containing this arc. Let $\pi(G,\mathscr{P})$ be the maximum of the load of all the arcs and let $w(G,\mathscr{P})$ be the minimum number of wavelengths (colors) needed to color the family of dipaths $\mathscr{P}$ in such a way that two dipaths with the same wavelength are arc-disjoint.

There exist DAGs such that the ratio between $w(G,\mathscr{P})$ and $\pi(G,\mathscr{P})$ cannot be bounded. An internal cycle is an oriented cycle such that all the vertices have at least one predecessor and one successor in G (said otherwise every cycle contains neither a source nor a sink of G). We prove that, for any family of dipaths $\mathscr{P}$, $w(G,\mathscr{P}) = \pi(G,\mathscr{P})$ if and only if G is without internal cycle.

We also consider a new class of DAGs, called UPP-DAGs, for which there is at most one dipath from a vertex to another. For these UPP-DAGs we show that the load is equal to the maximum size of a clique of the conflict graph. We prove that the ratio between $w(G,\mathscr{P})$ and $\pi(G,\mathscr{P})$ cannot be bounded (a result conjectured in an other article). For that we introduce "good labelings" of the conflict graph associated to G and $\mathscr{P}$, namely labelings of the edges such that for any ordered pair of vertices (x,y) there do not exist two paths from x to y with increasing labels.

Keywords-: DAG (Directed acyclic graphs); load; wavelengths; dipaths; good labelings; conflict graphs; intersection graphs; chromatic number.

1 Introduction

The problem we consider is motivated by routing, wavelength assignment and grooming in optical networks. But it can be of interest for other applications in parallel computing, where the graph will represent for example the precedence graph of a program or for scheduling complex operations on pipelined operators. A generic problem in the design of optical networks (see [11]), consists in satisfying a family of requests (or a traffic matrix) under various constraints like capacity constraints. Satisfying a request in a WDM optical network consists in assigning to it a route (dipath), but also a wavelength, which shall stay unchanged if no conversion is allowed. The constraint is that two requests, having the same wavelength, have to be routed by two arc disjoint dipaths or, equivalently, two requests whose associated dipaths share an arc, have to be assigned different wavelengths. One optimization problem associated consists in minimizing the number of wavelengths for a given family of requests. This problem is known in the literature as the RWA (Routing and Wavelength Assignment) problem and has been extensively studied (see for example the survey [2]).

The RWA problem is very difficult to analyze, therefore it is often split into two separate problems. The first one solves the routing problem by determining dipaths which are easy to compute like shortest paths or which minimize the load. The load of an arc is the number of routes (dipaths) containing it and the load of the routing is the maximum load of the arcs. Then in a second step, the routing being given, the wavelength assignment problem is solved. In that case the input of the problem is not a family of requests, but a family of dipaths $\mathscr{P}$. We will denote by $\pi(G, \mathscr{P})$ the maximum of the load of all the arcs of the digraph G for the family $\mathscr{P}$. Determining the minimum number $w(G, \mathscr{P})$ of wavelengths (colors) needed to color a family of dipaths $\mathscr{P}$ in such a way that two dipaths with the same wavelength are arc-disjoint is still NP-hard in that case. Indeed it corresponds to finding the chromatic number of the **conflict graph** (also called the intersection graph), associated to the digraph G and the family of dipaths $\mathscr{P}$, whose vertices represent the dipaths and where two vertices are joined if the corresponding dipaths are in conflict (that is share an arc).

There are examples of topologies and families of dipaths where there are at most 2 dipaths using an arc ($\pi(G, \mathscr{P}) = 2$), but where we need an arbitrarily large number of wavelengths. Such an example is given by considering k dipaths from s_i to t_i. The dipaths starts in s_i, then go alternatively right and down till they arrive at the bottom where they go right and up till they arrive at the destination t_i. Figure 1 shows the example for $k = 4$ wavelengths. Any two dipaths intersect; so the conflict graph is complete and we need k colors. However the load of an arc is at most 2. Therefore the ratio between $w(G, \mathscr{P})$ and $\pi(G, \mathscr{P})$ is unbounded in general.

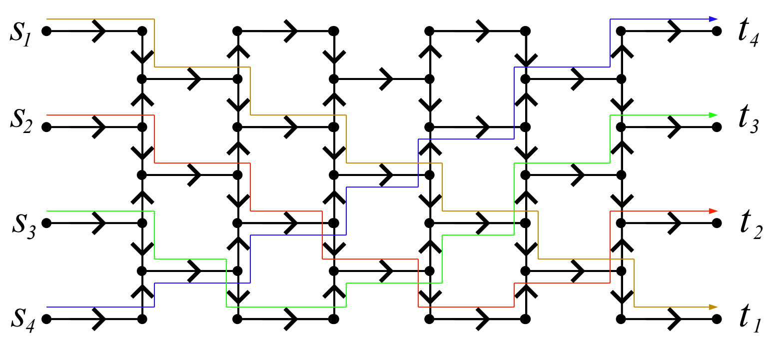

Figure 1: A pathological example

Here we consider the class of Directed Acyclic Graphs, DAGs, which plays a central role in Parallel and Distributed Computing. Part of our motivation came when we tried to extend the results obtained in [5] for paths motivated by grooming problems ([3, 9]).

The example given above in Figure 1 being a DAG there is no hope to bound the ratio between $w(G, \mathscr{P})$ and $\pi(G, \mathscr{P})$ for DAGs. In [4] we fully characterize when $w(G, \mathscr{P}) = \pi(G, \mathscr{P})$ for a DAG. In fact the necessary and sufficient condition is that G does not contain an internal cycle, i.e. a cycle, such that all the vertices have at least one predecessor and one successor in G.

Note that a rooted tree is a directed tree where there is a unique dipath from the root to any vertex. Here, we consider a new (to our best knowledge) class of DAGs, which is of interest in itself, those for which there is at most one dipath from a vertex to another. Note that, if the original digraph has the property that for any request (x, y) there is a unique dipath from x to y, then routing is very easy and it is equivalent to consider a family of requests or a family of dipaths. We call this property the UPP (Unique diPath Property) and call these digraphs UPP-DAGs. For these UPP-DAGs we show that the load is equal to the maximum size of a clique of the conflict graph. In [4] we proved that if an UPP-DAG has only one internal cycle, then for any family of dipaths $w(G, \mathscr{P}) \leq \left\lceil \frac{4}{3}\pi(G, \mathscr{P}) \right\rceil$ and we exhibit an UPP-DAG and a family of dipaths reaching the bound. The main result of this article is that for UPP-DAGs with load 2, the ratio between $w(G, \mathscr{P})$ and $\pi(G, \mathscr{P})$ cannot be bounded (a result conjectured in [4]).

The paper is organized as follows. In section 2, we give the basic definitions and show that for an UPP-DAG the load is equal to the clique number of the conflict graph $\pi(G, \mathscr{P}) = \Omega(C(G, \mathscr{P}))$. In section 3, we give a new shorter proof of the fact that for a DAG, $w(G, \mathscr{P}) = \pi(G, \mathscr{P})$ if and only G does not contain an internal cycle. We also give examples of UPP-DAGs with one internal cycle and where there is no equality. Section 4 is devoted to prove our main result: for UPP-DAGs, with load 2, the ratio between $w(G, \mathscr{P})$ and $\pi(G, \mathscr{P})$ cannot be bounded. For that we introduce the notion of "good labelings" of the conflict graph associated to G and $\mathscr{P}$, namely labelings of the edges such that for any ordered pair of vertices (x, y) there do not exist two paths from x to y with increasing labels. In section 5 we give some open problems.

2 Definitions

We model the network by a digraph G. The *outdegree* of a vertex x is the number of arcs with initial vertex x (that is the number of vertices y such that (x, y) is an arc of G). The *indegree* of a vertex x is the number of arcs with terminal vertex x (that is the number of vertices y such that (y, x) is an arc of G). A *source* is a vertex with indegree 0 and a *sink* a vertex with outdegree 0. A dipath is a sequence of vertices $x_1, x_2, \ldots, x_k$ such that (x_i, x_{i+1}) is an arc of G. If $x_k = x_1$ the dipath is called a directed cycle.

A **DAG (Directed Acyclic Graph)** is a digraph with no directed cycle. However the underlying (undirected) graph obtained by deleting the orientation can have cycles. An (oriented) cycle in a DAG consists therefore of an even sequence of dipaths $P_1, P_2, \ldots, P_{2k}$ alternating in direction (see Figure 2a). The vertices inside the dipaths have indegree and outdegree 1; those where there is a change of orientation have either indegree 2 and outdegree 0 or indegree 0 and outdegree 2.

An **internal cycle** of a DAG G is an oriented cycle, such that all its vertices have in G an indegree > 0 and an outdegree > 0; said otherwise no vertex is a source or a sink. Hence the vertices where there is a change of orientation in the cycle have a predecessor (resp. a successor) in G, if they are of indegree 0 (resp. outdegree 0) in the cycle (see Figure 2b).

We will say that a DAG has the **Unique Path Property** if between two vertices there is at most one dipath. A digraph satisfying this property will be called an **UPP-DAG**.

If G is an UPP-DAG, then any internal cycle contains at least $2k \geq 4$ vertices where there is a change of orientation. Otherwise it would consist of a dipath from x to y and a reverse dipath from y to x and so there would be two dipaths from x to y.

Finally a DAG with no cycles is an **oriented tree** (its underlying graph has no cycles and so is a tree).

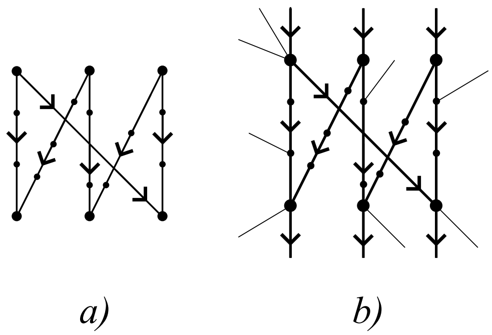

Figure 2: An oriented cycle (a) and an internal cycle (b)

Given a digraph G and a family of dipaths $\mathscr{P}$, the **load of an arc** e is the number of dipaths of the family containing e:

$$\text{load}(G, \mathscr{P}, e) = |\{P : P \in \mathscr{P}; e \in P\}|$$

The **load of** G **for** $\mathscr{P}$ will be the maximum of the loads of all the arcs of G and will be denoted by $\pi(G, \mathscr{P})$.

We will say that two dipaths are in conflict (or intersect) if they share at least one arc. We will denote by $w(G, \mathscr{P})$ the **minimum number of colors** needed to color the dipaths of $\mathscr{P}$ in such a way that two dipaths in conflict (sharing an arc) have different colors. Note that $\pi(G, \mathscr{P}) \leq w(G, \mathscr{P})$.

The **conflict graph** (also called the intersection graph) associated to the digraph G and the family of dipaths $\mathscr{P}$ has as vertices the dipaths of $\mathscr{P}$, two vertices being joined if their associated dipaths are in conflict. It will be denoted $C(G, \mathscr{P})$. Then $w(G, \mathscr{P})$ is the chromatic number χ of the conflict graph: that is $w(G, \mathscr{P}) = \chi(C(G, \mathscr{P}))$. Note that π is only upper bounded by the clique number of the conflict graph; indeed the π dipaths containing an arc e of maximum load are pairwise in conflict. The following property shows that if G is an UPP-DAG then $\pi(G, \mathscr{P})$ is exactly the clique number $\Omega(C(G, \mathscr{P}))$ of the conflict graph.

Property 1 *If G is an UPP-DAG then the dipaths in conflict have the following Helly property: if a set of dipaths are pairwise in conflict, then their intersection is a dipath. Therefore $\pi(G, \mathscr{P}) = \Omega(C(G, \mathscr{P}))$*

Proof: If two dipaths intersect, then their intersection is a dipath. Indeed suppose their intersection contains two different dipaths (x_1, y_1) and (x_2, y_2) in this order. Then between y_1 and x_2 there are two dipaths, one via P_1 and the other via P_2 (see Figure 3 a).

So suppose P_1 and P_2 intersect in only one interval (x_1, y_1), and P_3 intersects P_1 in an arc disjoint interval (u_1, v_1). W.l.o.g. we may assume that v_1 is before x_1. Let P_3 intersects P_2 in the interval (u_2, v_2).

Case 1: v_2 is before u_1 on P_3. v_2 cannot be after y_1 on P_2 otherwise there will be a directed cycle. So v_2 is before x_1 on P_2 and we have two dipaths from v_2 to x_1, one via P_2 and the other one via P_3 till u_1 and then via P_1 (see Figure 3 b).

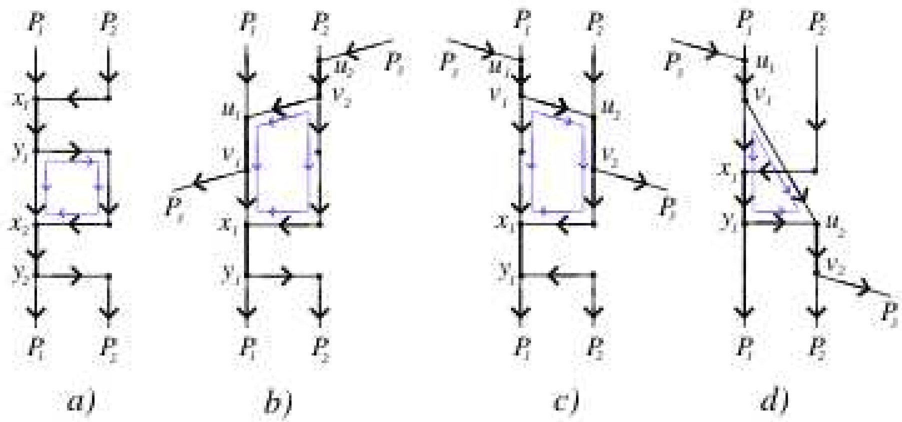

Figure 3: Helly property

Case 2: u_2 is after v_1 on P_3. If u_2 is before x_1 on P_2 we have two dipaths from v_1 to x_1 one via P_1 and the other going from v_1 to u_2 via P_3 and to x_1 via P_2. If u_2 is after y_1, we have two dipaths from v_1 to u_2 one via P_3 and the other via P_1 till y_1 and P_2 (see Figure 3 c and d). $\square$

3 Relations between $\pi(G, \mathscr{P})$ and $w(G, \mathscr{P})$

As we have seen in the introduction, there exist DAGs G and a set of dipaths $\mathscr{P}$ such that $\pi(G, \mathscr{P}) = 2$ and $w(G, \mathscr{P})$ is arbitrarily large (see Figure 1). These DAGs have many internal cycles.

In [4] we showed that the DAGs for which for any family $\mathscr{P}$ of dipaths, $w(G, \mathscr{P}) = \pi(G, \mathscr{P})$ are exactly those with no internal cycles. We give here a new simpler proof.

Theorem 2 *Let G be a DAG. Then, for any family of dipaths $\mathscr{P}$, $w(G, \mathscr{P}) = \pi(G, \mathscr{P})$ if and only if G does not contain an internal cycle.*

Proof: Let G be a DAG without internal cycles. If G is an oriented tree, this is a known result as the conflict graph is a perfect graph (see for example [10] or for a polynomial algorithm in $O(n, |\mathscr{P}|)$ [8]). If G is not an oriented tree, let G' be the digraph obtained as follows: replace each source s with $d^+(s)$ neighbors v_i ($i = 1, \dots, d^+(s)$) by $d^+(s)$ sources s_i and join s_i to v_i. If a dipath P of $\mathscr{P}$ contains the arc (s, v_i) associate in G' the dipath obtained by replacing (s, v_i) by (s_i, v_i). Do also the same transformation for all the sinks replacing the sink t with $d^-(t)$ neighbors w_j ($j = 1, \dots, d^-(t)$) by $d^-(t)$ sinks t_j and (w_j, t) by (w_j, t_j). Let $\mathscr{P}'$ be the resulting family of dipaths obtained. G' is an oriented tree; indeed, as there is no internal cycle, all the cycles in G contain either a source or a sink. So we have $w(G', \mathscr{P}') = \pi(G', \mathscr{P}')$.

By construction $\pi(G, \mathscr{P}) = \pi(G', \mathscr{P}')$. To conclude let us color the dipaths P of $\mathscr{P}$ following the coloring of $\mathscr{P}'$. If P does not contain a source or a sink it belongs to $\mathscr{P}'$ and we keep its color. If it contains (s, v_i) (resp. (w_j, t)) we give to it the color of the associated path in $\mathscr{P}'$ obtained by replacing (s, v_i) by (s_i, v_i) (resp. (w_j, t) by (w_j, t_j)). We get a valid coloring as there are no conflicts between two arcs (s, v_i) and (s, v_j) (resp. (w_i, t) and (w_j, t)). So we also have $w(G, \mathscr{P}) = w(G', \mathscr{P}')$ and therefore $w(G, \mathscr{P}) = \pi(G, \mathscr{P})$.

□

In [4], we also show that if a DAG G contains an internal cycle, there exists a set $\mathscr{P}$ of dipaths, such that $\pi(G, \mathscr{P}) = 2$ and $w(G, \mathscr{P}) = 3$. A particular case gives an UPP-DAG G with $\pi = 2$ and a set of 5 dipaths such that the conflict graph is a C_5 and therefore $w = 3$. Replacing each of these dipaths with h identical dipaths we obtain a family of $5h$ dipaths with $\pi = 2h$ and $w = \lceil \frac{5h}{2} \rceil$ giving a ratio $\frac{w}{\pi} = \frac{5}{4}$. In fact, in [4] we showed that the bound can be improved to $\frac{4}{3}$ with the following example that we give again as it will illustrate the proof of Theorem 9:

Theorem 3 *There exists an UPP-DAG G with one internal cycle and a family $\mathscr{P}$ of dipaths such that*

$$w(G, \mathscr{P}) = \left\lceil \frac{4}{3}\pi(G, \mathscr{P}) \right\rceil$$

.

Proof: The following example is due to Frédéric Havet. It consists of 8 dipaths generating the conflict graph consisting of a cycle of length 8 plus chords between the antipodal vertices (see Figure 4). Here again $\pi = 2$ and $w = 3$; but if we replace each of these dipaths with h identical dipaths we obtain a family $\mathscr{P}$ of $8h$ dipaths with $\pi = 2h$ and $w = \lceil \frac{8h}{3} \rceil$; indeed in the conflict graph an independent set has at most 3 vertices and so we need at least $\frac{8h}{3}$ colors. Therefore this family satisfies the theorem (the reader can see the relation with fractional coloring).

□

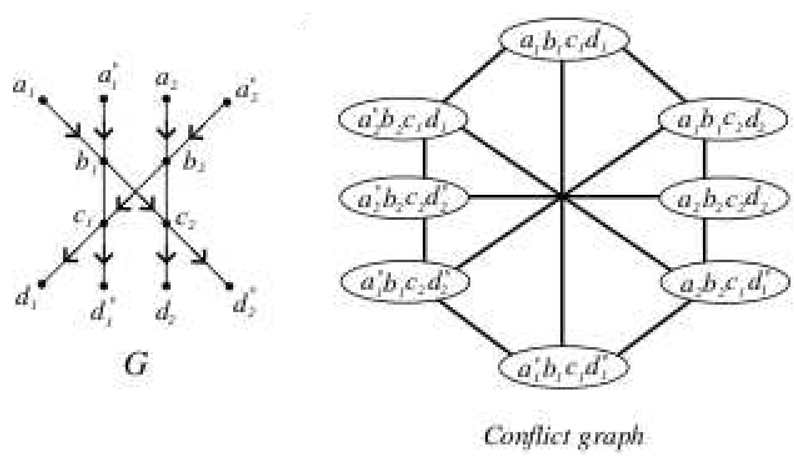

Figure 4: An other UPP-DAG with $\pi = 2$ and $w = 3$.

In [4] we asked the question whether for any UPP-DAG the ratio $\frac{w}{\pi}$ was bounded by some constant. We were only able to prove (with an involved proof) that it is the case when there is only one internal cycle obtaining a ratio $\frac{w}{\pi} = \frac{4}{3}$ which is the best possible in view of the example of Theorem 3. We refer the reader to [4] for a proof.

Theorem 4 *[4] Let G be an UPP-DAG with only one internal cycle. Then for any family of dipaths $\mathscr{P}$,*

$$w(G,\mathscr{P}) \le \left\lceil \frac{4}{3}\pi(G,\mathscr{P}) \right\rceil$$

and the bound is the best possible.

In the next section we show that the ratio is unbounded at least for UPP-DAGs with load 2. For that we will characterize their conflict graphs.

4 UPP-DAGS with load 2

We will now prove our main result: for UPP-DAGs with load 2, the ratio between $w(G,\mathscr{P})$ and $\pi(G,\mathscr{P})$ cannot be bounded. For that we introduce the notion of "good labeling" of the edges of a graph. We first show that, if G is an UPP-DAG with load 2, then for any family of dipaths $\mathscr{P}$, the conflict graph $H = C(G,\mathscr{P})$ admits a "good labeling" of its edges. Then we show a kind of converse property, namely that, if H is a graph with a good labeling, then there exists an UPP-DAG G with load 2 and a family of dipaths $\mathscr{P}$ such that $H = C(G,\mathscr{P})$. This property combined with the existence of graphs with good labelings and arbitrarily large chromatic number gives us the existence of UPP-DAGS and family of dipaths $\mathscr{P}$ with $\pi(G,\mathscr{P}) = 2$ and with an arbitrarily large $w(G,\mathscr{P})$.

We give two definitions of a **good labeling of the edges** of a graph H which can be used interchangeabily.

Definition 1: Let us label the edges of a graph H with distinct real labels. This labeling is said to be good if for any ordered pair of vertices (x, y) there do not exist 2 paths from x to y with increasing labels.

Definition 2: Let us label the edges of a graph H with non necessarily distinct labels. This labeling is said to be good if for any ordered pair of vertices (x, y) there do not exist 2 paths from x to y with non-decreasing labels. (Note that as there are m edges, we can in definition use as labels the integers from 1 to m keeping the order between labels).

Clearly a labeling satisfying Definition 1 satisfies Definition 2.

Conversely if a graph has a good labeling L satisfying Definition 2 it has also a labeling L' satisfying Definition 1. Let the distinct labels used in L be a_i, $1 \le i \le k$, with $a_1 < a_2 < \ldots < a_k$ and suppose a_i is repeated λ_i times. Let us define the labeling L' as follows: we label the λ_i edges having label a_i in L with the distinct labels $a_i, a_i + \epsilon_i, \ldots, a_i + (\lambda_i - 1)\epsilon_i$ where $\lambda_i\epsilon_i \le a_{i+1} - a_i$. All the labels of L' are distinct (see examples after for this transformation). Now consider any ordered pair of vertices (x, y). By definition of L there exists at most one path from x to y with non-decreasing labels. So any other path contains two consecutive edges e_k and e_{k+1} with labels b_k and b_{k+1}, where $b_k = a_j$ for some j and $b_{k+1} = a_i$ and such that $a_j > a_i$. In the labeling L', these two edges have also decreasing labels as $a'_j \ge a_j \ge a_{i+1} > a_i + (\lambda_i - 1)\epsilon_j \ge a'_i$. So L' is a labeling with distinct labels satisfying Definition 1.

Let us give now some examples of graphs having a good labeling or not:

Property 5 *The cycles C_4, C_5 and the conflict graph of Figure 4 with 8 vertices have a good labeling*

Proof: Let the C_4 be (a, b, c, d), then a good labeling with Definition 2 is obtained by labeling the edges $\{a, b\}$ and $\{c, d\}$ with label 1 and the edges $\{b, c\}$ and $\{a, d\}$ with label 3. The labeling L' is obtained by giving label 1 to $\{a, b\}$, 2 to $\{c, d\}$, 3 to $\{b, c\}$ and 4 to $\{a, d\}$.

Let the C_5 be (a, b, c, d, e), then a good labeling with Definition 2 is obtained by labeling the edges $\{a, b\}$ $\{c, d\}$ and $\{a, e\}$ with label 1 and the edges $\{b, c\}$ and $\{d, e\}$ with label 4. The labeling L' is obtained by giving label 1 to $\{a, b\}$, 2 to $\{c, d\}$, 3 to $\{a, e\}$, 4 to $\{b, c\}$ and 5 to $\{d, e\}$.

For the conflict graph of Figure 4 with 8 vertices, a good labeling (Definition 2) is obtained by labeling the edges of the external cycle alternately with labels 1 and 3 and the 4 diagonals with label 2. □

Property 6 *$K_{2,3}$ does not admit a good labeling.*

Proof: This proof is due to J-S Sereni. Let the vertices of $K_{2,3}$ be respectively a, b and c, d, e. Suppose it admits a good labeling L with Definition 1. W.l.o.g. we can suppose that $L(a, c) < L(a, d) < L(a, e)$.

Then $L(b, c) > L(a, c)$; otherwise, there will be two increasing paths from b to d: (b, d) and (b, c, a, d).

Then $L(a, d) > L(b, d)$; otherwise, there will be two increasing paths from a to b: (a, d, b) and (a, c, b).

Then $L(a, e) > L(b, e)$; otherwise, there will be two increasing paths from a to b: (a, e, b) and (a, c, b).

But we get a contradiction, as there are two increasing paths from b to a: (b, d, a) and (b, e, a). □

Theorem 7 *Let G be an UPP-DAG with load 2. Then for any family of dipaths $\mathscr{P}$, the conflict graph $C(G, \mathscr{P})$ has a good labeling.*

Proof: Recall (see the proof of property 1) that, if two dipaths P and Q intersect, they intersect in an interval $[x, y]$. As the load of G is 2, the arcs of this interval belong only to these 2 dipaths. Therefore, if G' is the digraph obtained from G by replacing the interval $[x, y]$ by a single arc (x, y), then G' has the same conflict graph H as G. The edge of $H = C(G, \mathscr{P})$ joining the two vertices P and Q will correspond to the intersection interval $[x, y]$ of P and Q in G, that is, to the arc (x, y) of G'. (Note that if in G' we delete the arcs with load at most 1, that is covered by at most one path of $\mathscr{P}$, then there is a one to one mapping between the remaining arcs of G' and the edges of the conflict graph H).

Now we label the arcs of G' according to the topological order; that is we label with 1 the arcs leaving a source; then we delete the arcs labeled 1 getting a digraph G'_2 and label with 2 the arcs leaving a source in G'_2 and so on. As G and therefore G' is a DAG we can label all the arcs of G'. This implies a labeling of the edges of the conflict graph H, by giving to the edge joining the two vertices P and Q the label of the arc (x, y) of G' associated to the intersection interval $[x, y]$ of P and Q.

Let us now show that this is a good labeling of H. Consider a non-decreasing path in H, from P to Q, $P = P_1, P_2, \ldots, P_k = Q$ and let (x_i, y_i) be the arc of G' associated to the intersection of P_i and P_{i+1}. As the labels are non decreasing, then (x_i, y_i) is in the topological order before (x_{i+1}, y_{i+1}) and so y_i is before x_{i+1} in P_{i+1}. Stherefore, we can associate to this non-decreasing path in H the following dipath in G': $x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}$ (in fact that implies that the labels are strictly increasing). Suppose we have two non-decreasing paths in H from a vertex P to a vertex Q, then we have in G' two dipaths $x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}$ and $x'_1, y'_1, x'_2, y'_2, \ldots, , x'_{m-1}, y'_{m-1}$ with x_1, y_1 and x'_1, y'_1 belonging to P and x_{k-1}, y_{k-1} and x'_{m-1}, y'_{m-1} belonging to Q. W.l.o.g. we can suppose x'_1 is after (or equal to) y_1 on P.

If x'_{m-1} is after y_{k-1} on Q, we have two dipaths joining y_1 and x'_{m-1} namely $y_1, x_2, y_2, , \dots, x_{k-1}, y_{k-1}, x'_{m-1}$ and $y_1, x'_1, y'_1, x'_2, y'_2, \dots, , x'_{m-2}, y'_{m-2}, x'_{m-1}$.
If y'_{m-1} is before x_{k-1} on Q, we have two dipaths joining y_1 and x_{k-1} namely $y_1, x_2, y_2, , \dots, x_{k-1}$ and $y_1, x'_1, y'_1, x'_2, y'_2, \dots, , x'_{m-1}, y'_{m-1}, x_{k-1}$.
Therefore G' and so G cannot be an UPP-DAG. $\square$

Likewise, using the property 6 we obtain the following corollary proved in [4]:

Corollary 8 *Let G be an UPP-DAG with load 2. Then its conflict graph cannot contain a $K_{2,3}$.*

Theorem 9 *Let H be a graph with a good labeling. Then there exists an UPP-DAG G with load 2 and a family of dipaths $\mathscr{P}$, such that $H = C(G, \mathscr{P})$.*

Proof: To each edge $\{P, Q\}$ in H let us associate in G two vertices x_{PQ} and y_{PQ} joined by the arc (x_{PQ}, y_{PQ}). Now for each vertex P in H, order its neighbors $Q_1, Q_2, \dots, Q_h$ according to the labels of the edges $\{P, Q_i\}$ that is $L(P, Q_1) < L(P, Q_2) < \dots < L(P, Q_h)$. Then, for $i = 1, 2, \dots, h-1$, let us identify the vertex y_{PQ_i} to $x_{PQ_{i+1}}$, and associate to the vertex P in H the dipath $P = (x_{PQ_1}, y_{PQ_1} = x_{PQ_2}, y_{PQ_2}, \dots, y_{PQ_{h-1}} = x_{PQ_h}, y_{PQ_h})$ in G. The family of dipaths $\mathscr{P}$ consists of the dipaths associated to each vertex of H. The conflict graph associated to the graph G and the family of dipaths $\mathscr{P}$ is exactly the graph H. G has load 2 as an arc x_{PQ}, y_{PQ} of G belongs exactly to the two dipaths P and Q. G is an UPP-DAG as a dipath in G corresponds to an increasing path in H and so if there were two dipaths in G joining some y_{PQ} to $x_{P'Q'}$ there will be two increasing paths in H from P to P'. $\square$

If we apply the construction of the proof to the conflict graph H of Figure 4, we get exactly the graph G and the dipaths of the example.

Remark: there is no equivalence between the two properties: G is an UPP digraph with load 2 and its conflict graph has a good labeling. Indeed there exist digraphs which are not UPP, but whose conflict graph has a good labeling: for example consider the graph consisting of the 4 dipaths a_1, b_1, c_1; b_1, c_1, d_1; c_1, d_1, e_1; and a_1, b_1, f_1, d_1, e_1. It has C_4 as conflict graph, but is not UPP as there are two dipaths from b_1 to d_1.

Theorem 10 *There exists a family of graphs with a good labeling and an arbitrarily large chromatic number.*

Proof: Consider a regular graph H of degree at most k, girth $> 2k+2$ and with a large chromatic number. The existence of such graphs has been shown in [7, 12].

The edges of H can be partitioned in at most $k+1$ matchings; indeed we know by Vizing's theorem that we can color the edges of a graph with maximum degree Δ with $\Delta + 1$ colors. Let us give to the edges of the ith matching the label i, $1 \leq i \leq k+1$. Then any non-decreasing (in fact increasing as there cannot be two consecutive edges with the same label) path in H has at most $k+1$ edges. But there cannot exist two increasing paths between any pair of nodes; otherwise there will be a cycle in H of length $\leq 2k+2$ contradicting the value of the girth. Therefore H has a good labeling. $\square$

Now using Theorems 9 and 10 we are able to answer the question asked in [4].

Theorem 11 *There exist UPP-DAG and a family of dipaths with load* $\pi(G, \mathscr{P}) = 2$ *and with an arbitrarily large* $w(G, \mathscr{P})$.

5 Conclusions

We list below some open problems worth of being investigated. Some concern the relations between the maximum of the load $\pi(G, \mathscr{P})$ of all the arcs and $w(G, \mathscr{P})$ the minimum number of wavelengths in particular for the class of UPP-DAGs. In particular we have the following conjecture (a specialization of a problem asked in [2]).

Question 1: Let G be a DAG and consider the All To All (ATA) family of dipaths (i.e for each couple (x,y) connected by a dipath we have a request). We conjecture that $w(G, ATA) = \pi(G, ATA)$.

Question 2: For which digraphs $\pi(G, \mathscr{P}) = \Omega(C(G, \mathscr{P}))$?

Question 3: Given an undirected graph when is it possible to orient its edges such that the digraph obtained is UPP?

This problem should be NP-Hard. However classes of such graphs could be exhibited.

For graphs with one or a small number of internal cycles we have the following questions.

Question 4: Is Theorem 4 true also for DAGs (not necessarily UPP) with exactly one internal cycle?

Question 5: Is there a simple proof of Theorem 4?

Question 6: What is the bound for graphs with k internal cycles?

In a preliminary version of this paper we asked for a characterization of graphs with a good labeling. In particular we asked if there exists a polynomial algorithm to decide if a graph has a good labeling. In [1], the authors exhibit infinite families of graphs for which no such labeling can be found. They also show that deciding if a graph admits a good labeling is NP-complete. Finally, they give large classes of graphs admitting a good labeling like forests, C_3-free outerplanar graphs, planar graphs of girth at least 6. In [6] they proved that if the girth is at least 5, the average degree of G is less than 3 and G is minimal without a good labeling, then G is either C_3 or $K_{2,3}$.

A last question consists in extending the results to UPP-DAGs with load > 2 perhaps by using hypergraphs.

Question 7: Characterize UPP-DAGs with load 3 or load h.

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