On the solutions of the Aubin equation and the K-energy of Einstein-Fano manifolds

Nefton Pali

► To cite this version:

Nefton Pali. On the solutions of the Aubin equation and the K-energy of Einstein-Fano manifolds. 2005. hal-00868965

HAL Id: hal-00868965

<https://hal.science/hal-00868965>

Preprint submitted on 2 Oct 2013

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

On the solutions of the Aubin equation and the K-energy of Einstein-Fano manifolds.

Nefton Pali

Abstract

We propose an improvement to the bifurcation technique considered by Bando-Mabuchi for the construction of the solutions of the Aubin equation over Einstein-Fano manifolds. We introduce also a simplification in Tian's proof of the properness of the K-energy functional over Einstein-Fano manifolds with trivial holomorphic automorphisms group. This result combined with Perelman's estimates for the Kähler-Ricci flow allows us to give a sharp version of the Hamilton-Tian conjecture on the convergence of the Kähler-Ricci flow in the case of Einstein-Fano manifolds with trivial holomorphic automorphisms group.

Contents

1	Some notations and definitions	2
1.1	The functionals of Kähler-Einstein Geometry.	3
2	Introduction	4
3	Basics of Kähler-Einstein geometry	5
3.1	II and III order estimates for the solutions of the complex Monge-Ampère equations	5
3.2	Projections to a Kähler-Einstein orbit	10
4	The solutions of the Aubin equation	13
5	A Tian C^0-estimate for the solutions of the Aubin equation.	23
5.1	Smoothing with the Kähler-Ricci Flow	25
6	Proof of the properness of the K-energy	27
6.1	On the existence of Kähler-Einstein metrics.	30
7	Convergence of the Kähler-Ricci flow	31
8	Appendix. A proof of the short time existence of the Kähler-Ricci flow equation.	34

Key words : Einstein-Fano manifolds, Properness of the K-energy, Convergence of the Kähler-Ricci flow.

AMS Classification : 53C25, 53C55, 32J15.

1 Some notations and definitions

Let X be a Fano manifold of complex dimension n , let $\text{Aut}_J^0(X)$ be the identity component of the group of holomorphic automorphisms and let $\mathcal{E}(X, \mathbb{R})$ be the space of smooth real valued functions. We will note by $\mathcal{E}^{p,p}(X, \mathbb{R})$ the space of real smooth (p, p) -forms on X and by $\mathcal{K} := \{\omega \in 2\pi c_1 \mid \omega > 0\}$ the space of Kähler metrics in the anticanonical class $2\pi c_1 \subset \mathcal{E}^{1,1}(X, \mathbb{R})$ and $\mathcal{KE} \subset \mathcal{K}$ the subset of Kähler-Einstein metrics. For any $\omega \in \mathcal{K}$ we define the space of potentials $\mathcal{P}_\omega := \{\varphi \in \mathcal{E}(X, \mathbb{R}) \mid i\partial\bar{\partial}\varphi > -\omega\}$ and we set $\omega_\varphi := \omega + i\partial\bar{\partial}\varphi$ for every $\varphi \in \mathcal{P}_\omega$. More in general we consider the space $C_{p,p}^{k,1/2}(X, \mathbb{R})$ of real (p, p) -forms with coefficients of class $C^{k,1/2}$. We define

$$(2\pi c_1)_k := \left\{ \alpha \in C_{1,1}^{k,1/2}(X, \mathbb{R}) \mid \exists \beta \in 2\pi c_1, \exists f \in C^{k+2,1/2} : \alpha - \beta = i\partial\bar{\partial}f \right\},$$

and $\mathcal{K}_k := \{\omega \in (2\pi c_1)_k \mid \omega > 0\}$. The symbol $\mathcal{P}_\omega^{k,1/2} \subset C^{k,1/2}(X, \mathbb{R})$ has an obvious meaning. Set

$$C_{\omega,0}^{k,1/2}(X, \mathbb{R}) := \left\{ f \in C^{k,1/2}(X, \mathbb{R}) \mid \int_X f \omega^n = 0 \right\},$$

$k \in \mathbb{N}$ and let $\mathcal{P}_{\omega,0}^{k,1/2} \subset C_{\omega,0}^{k,1/2}(X, \mathbb{R})$ be the corresponding set of potentials. We will use the notation $f_X := (\int_X \omega^n)^{-1} \int_X$ for the average operator. Set also

$$C_\omega^{k,1/2}(X, \mathbb{R})_1 := \left\{ f \in C^{k,1/2}(X, \mathbb{R}) \mid \int_X f \omega^n = 1 \right\}.$$

In all this paper a linear map $f : E \rightarrow F$ between Banach spaces is called an isomorphism if it is an isomorphism of Banach spaces. Let E_k be a closed affine subspace of the Banach space of differential forms with coefficients of class $C^{k,\alpha}$ with $E_{k+1} \subset E_k$ and let $E := \cap_{0 \leq k < +\infty} E_k$ equipped with the direct limit topology. Let U be an open set in a Banach space. We say that a map $f : U \rightarrow E$ is smooth if the induced maps $f : U \rightarrow E_k$ are smooth for all $k \geq 0$. More in general let F_k, F as before, $U_k \subset E_k$ open sets, with $U_{k+1} \subset U_k$ and let $U := \cap_{0 \leq k < +\infty} U_k \subset E$. We say that map $f : U \rightarrow F$ is smooth if for some integer p it extends to a smooth map $f : U_p \rightarrow F_0$ such that its resection defines a smooth map from U_{p+k} to F_k , for all $k > 0$. In a similar way we define the notion of smooth diffeomorphism.

Given a real $(1, 1)$ -form α , we define the trace of α respect to ω as

$$\text{Tr}_\omega(\alpha) = \frac{2n\alpha \wedge \omega^{n-1}}{\omega^n},$$

We remind that the scalar curvature $\text{Sc}(\omega) \in \mathcal{E}(X, \mathbb{R})$ of ω is defined by the formula $\text{Sc}(\omega) := \text{Tr}_\omega(\text{Ric}(\omega))$, where $\text{Ric}(\omega)$ is the Ricci form. This definition coincides with the usual definition of scalar curvature of the Riemannian metric $g = \omega(\cdot, J\cdot)$. We define the Laplacian of a function f by the formula $\Delta_\omega f := \text{Trace}_\omega(i\partial\bar{\partial}f)$. Our Laplacian differs by a minus sign from the usual Laplace-Beltrami operator associated to the Riemannian metric $g = \omega(\cdot, J\cdot)$. Let also $\langle \alpha, \beta \rangle_\omega := \text{Tr}_\omega(i\alpha \wedge \bar{\beta})/2$ be the induced hermitian product over the complex vector bundle $\Lambda_J^{1,0}T_X^*$.

1.1 The functionals of Kähler-Einstein Geometry.

Aubin's functionals. The Aubin [Aub2] functionals $I_\omega, J_\omega : \mathcal{P}_\omega \rightarrow [0, +\infty)$ are defined by the formulas

$$I_\omega(\varphi) := \int_X \varphi (\omega^n - \omega_\varphi^n) = \sum_{k=0}^{n-1} \int_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^k \wedge \omega_\varphi^{n-k-1}$$

$$\begin{aligned} J_\omega(\varphi) &:= \sum_{k=0}^{n-1} \frac{k+1}{n+1} \int_X i\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^k \wedge \omega_\varphi^{n-k-1} \\ &= \int_X \varphi \omega^n - \frac{1}{n+1} \sum_{k=0}^n \int_X \varphi \omega^k \wedge \omega_\varphi^{n-k}. \end{aligned}$$

Set $G_\omega := I_\omega - J_\omega$. We have the obvious inequalities,

$$0 \leq J_\omega \leq I_\omega \leq (n+1)J_\omega, \quad (1.1)$$

$$0 \leq I_\omega \leq (n+1)G_\omega \leq nI_\omega. \quad (1.2)$$

If $(\varphi_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathcal{P}_\omega$ is a $\mathcal{C}^\infty$ path then we have the well known formula

$$\frac{d}{dt} \sum_{k=0}^n \int_X \varphi_t \omega^k \wedge \omega_t^{n-k} = (n+1) \int_X \dot{\varphi}_t \omega_t^n, \quad (1.3)$$

where $\dot{\varphi}_t := \frac{\partial}{\partial t} \varphi_t$ and $\omega_t := \omega_{\varphi_t}$. See for example [Ti2] or [Pal-1]. This formula implies the equality

$$\frac{d}{dt} G_\omega(\varphi_t) = -\frac{1}{2} \int_X \dot{\varphi}_t \Delta_t \varphi_t \omega_t^n, \quad (1.4)$$

The K-energy functional of the anticanonical class $2\pi c_1$. Consider the function $h_\omega \in \mathcal{E}(X, \mathbb{R})$ defined by the conditions $\text{Ric}(\omega) = \omega + i\partial\bar{\partial}h_\omega$, with normalization $\int_X e^{h_\omega} \omega^n = 1$. The K-energy functional $\nu_\omega : \mathcal{P}_\omega \rightarrow \mathbb{R}$ of the anticanonical class $2\pi c_1$ is given by the formula

$$\nu_\omega(\varphi) := \int_X \left(\log \frac{\omega_\varphi^n}{\omega^n} - h_\omega \right) \omega_\varphi^n - G_\omega(\varphi) + \int_X h_\omega \omega^n,$$

which satisfies

$$\frac{d}{dt} \nu_\omega(\varphi_t) = -\frac{1}{2} \int_X \dot{\varphi}_t \left(\text{Sc}(\omega_t) - 2n \right) \omega_t^n, \quad (1.5)$$

for every $\mathcal{C}^\infty$ path $(\varphi_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathcal{P}_\omega$. All the previous functionals naturally extends to the space of Kähler metrics $\mathcal{K}$. Consider also $\Lambda_\omega := \text{Ker}(\Delta_\omega + 2) \subset \mathcal{E}(X, \mathbb{R})$, $\mathbb{C}\Lambda_\omega := \text{Ker}(\Delta_\omega + 2) \subset \mathcal{E}(X, \mathbb{C})$ and

$$\Lambda_{\omega, k}^\perp := \left\{ f \in C^{k, 1/2}(X, \mathbb{R}) \mid \int_X f u \omega^n = 0, \forall u \in \Lambda_\omega \right\},$$

for some nonnegative integer k . The symbol $\Lambda_\omega^\perp \subset \mathcal{E}(X, \mathbb{R})$ has an obvious meaning.

Definition 1 A $(1, 1)$ -form $\alpha \in 2\pi c_1$ is called θ -orthogonal respect to a Kähler-Einstein metric $\theta \in 2\pi c_1$ if $\int_X u \alpha \wedge \theta^{n-1} = 0$ for all $u \in \Lambda_\theta$. We will note by $\mathcal{K}_\theta^\perp$ the set of θ -orthogonal Kähler metrics in $\mathcal{K}$ respect to the Kähler-Einstein metric θ .

If we write $\alpha = \theta_\varphi$ then the θ -orthogonality condition is equivalent to the condition $\int_X \varphi u \theta^n = 0$ for all $u \in \Lambda_\theta$. At this point the symbols $(2\pi c_1)_{\theta, k}^\perp \subset (2\pi c_1)_k$ and $\mathcal{K}_{\theta, k}^\perp \subset \mathcal{K}_k$ have an obvious meaning.

2 Introduction

Inspired from the work of Donaldson on the existence of Hermite-Einstein metrics, Mabuchi introduced the K-energy functional [Mab]. In a joint work with Bando, [Ba-Ma] they proved that the existence of a Kähler-Einstein metric implies a lower bound of the K-energy functional and the uniqueness of the Kähler-Einstein metrics modulo the action of the identity component of the holomorphic automorphisms of the manifold.

This last result is proved by applying the backward continuity method in order to construct the solutions of the Aubin equation. The solution at time $t = 0$ correspond to Yau's solution of the Calabi conjecture. The solution at time $t = 1$ correspond to a Kähler-Einstein metric. Then the uniqueness of Kähler-Einstein metrics follows from the uniqueness of the solution of the Calabi equation combined with the uniqueness of the solutions in the implicit function theorem (see the proof of part (B) of the theorem 1 for the precise argument).

The main difficulty in the backward continuity method is at time $t = 1$ since the implicit function theorem does not apply directly. In order to handle this difficulty Bando-Mabuchi consider a remarkable bifurcation technique. In [Ba-Ma] the authors reduce the application of the implicit function theorem on the eigenspace corresponding to the eigenvalue 2 of the Laplacian of the Kähler-Einstein metric at time $t = 1$. However even after this reduction the implicit function theorem does not apply for all Kähler data ω orthogonal to the Kähler-Einstein metric at time $t = 1$.

In this paper we introduce a new density method which allows to overcome this difficulty (see section 4) and simplifies the argument and avoid the computation of a Hessian in [Ba-Ma]. We prove the following result which is the optimal version of the main ingredient in [Ba-Ma].

Theorem 1 (The Aubin equation). *Let X be an Einstein-Fano manifold.*
(A). *Let $\omega \in \mathcal{K}$. There exist a unique solution $(\omega_t)_{t \in [0, 1]} \subset \mathcal{K}$ of Aubin's equation $\text{Ric}(\omega_t) = t\omega_t + (1-t)\omega$. Moreover ω_1 is the unique Kähler-Einstein metric such that $\omega \in \mathcal{K}_{\omega_1}^\perp$ and the map $t \mapsto \omega_t$ is smooth.*
(B). *Let $\hat{\omega} \in \mathcal{KE}$. Then $\mathcal{KE} = \text{Aut}_J^0(X) \cdot \hat{\omega}$.*

We can assume in this statement that if $\omega \in \mathcal{K}_k$, $k > 0$, then $\omega_t \in \mathcal{K}_{\omega_1, k+2}^\perp$ for all $t \in [0, 1]$. We suppose for simplicity $k > 0$ since Calabi's C^3 -uniform estimate for the potential of the complex Monge-Ampère equation requires derivatives of order V of the potential [Cal], [Yau]. We remark also that a Kähler-Einstein metric $\hat{\omega} \in \mathcal{K}_0$ is always smooth by elliptic regularity. (See [Aub1], Th. 3.56, pag. 86.) In the proof of the theorem 1 (A) we reverse the point of view (see

prop 4.1). We fix a Kähler-Einstein metric $\hat{\omega} \in \mathcal{K}$ and we consider all the metrics $\omega \in \mathcal{K}_{\hat{\omega}}^{\perp}$.

We introduce also a simplification in the computations of the following result due to Tian [Ti2] (see also the statements in [P-S-S-W]).

Theorem 2 (The properness of the K-energy). *Let X be a Einstein-Fano manifold with $H^0(T_{X,J}) = 0$ and let $\hat{\omega} \in \mathcal{K}$ be the unique Kähler-Einstein metric. Then there exists two constants $A > 0$, $B > 0$ such that the inequality $\nu_{\hat{\omega}}(\omega) \geq AJ_{\hat{\omega}}(\omega) - B$ hold for all Kähler metrics $\omega \in \mathcal{K}$.*

In our proof we use directly the K-energy functional instead of passing by Tian's functional F as in [Ti2]. It is well known that the meaningful functional from the algebraic stability point of view is the K-energy functional. We do this by using the backward continuity method in an efficient way until $t = 0$. It turns out that this approach simplifies some of the computations in a consistent way.

Theorem 2 allows to prove the C^0 -uniform estimate for the convergence of the Kähler-Ricci flow. Namely the fact that the K-energy is nonincreasing along the Kähler-Ricci flow provides a uniform bound for the Aubin's functional J along the flow. This combined with Perelman's uniform estimate for the potential of the Kähler-Ricci flow allows us to obtain the uniform C^0 estimate along the Kähler-Ricci flow, see [Ti-Zh] and [Pal-2]. The previous results allows us to give the following sharp version of the Hamilton-Tian conjecture in the case of Einstein-Fano manifolds with $H^0(T_{X,J}) = 0$. (Compare also with [Ti-Zh].)

Theorem 3 (An Einstein case of the Hamilton-Tian conjecture). *Let X be a Einstein-Fano manifold with $H^0(T_{X,J}) = 0$ and let $\hat{\omega} \in \mathcal{K}$ be the unique Kähler-Einstein metric. Then any Kähler-Ricci flow $(\omega_t)_{t \geq 0} \subset \mathcal{K}$ converge in the smooth topology to the Kähler-Einstein metric $\hat{\omega}$.*

We remind that the convergence obtained in [Ti-Zh] is by one sequence in the Cheeger-Gromov topology. After the work of X.X. Chen [Che] there has been a recent interest (see [Pal-1], [C-L-W]) in the Chen-Tian energy functional E_1 introduced in [Ch-Ti] in order to get a new existence criteria for Kähler-Einstein metrics over a Fano manifold. By combining the estimate

$$E_{1,\hat{\omega}}(\omega) \geq 2\nu_{\hat{\omega}}(\omega) + C_{\hat{\omega}},$$

$\omega \in \mathcal{K}$, proved in [Pal-1] with the theorem 2 we deduce immediately the following corollary.

Corollary 1 (The properness of the energy functional E_1). *Let X be a Einstein-Fano manifold with $H^0(T_{X,J}) = 0$ and let $\hat{\omega} \in \mathcal{K}$ be the unique Kähler-Einstein metric. Then there exists two constants $A > 0$, $B > 0$ such that the inequality $E_{1,\hat{\omega}}(\omega) \geq AJ_{\hat{\omega}}(\omega) - B$ hold for all Kähler metrics $\omega \in \mathcal{K}$.*

3 Basics of Kähler-Einstein geometry

3.1 II and III order estimates for the solutions of the complex Monge-Ampère equations

We explain now the Aubin-Yau C^2 and Calabi C^3 uniform estimates for the solutions of the complex Monge-Ampère equations. We will refer to the notations

introduced in the section 6 of [Pal-2].

Proposition 3.1 *Let (X, ω) be a compact Kähler manifold of complex dimension $n \geq 2$, let $\mathcal{S} \subset \mathcal{P}_\omega$ be a subset of potentials for the Kähler metric ω and let $\lambda : \mathcal{S} \longrightarrow [-1, 1]$, $\varphi \mapsto \lambda_\varphi$ be a map. Suppose that the functions*

$$f_\varphi := \log \frac{\omega_\varphi^n}{\omega^n} + \lambda_\varphi \varphi$$

satisfy the uniform estimates $f_\varphi \leq K_0$ and $\Delta_\omega f_\varphi \geq -K_0$, for some constant $K_0 > 0$ independent of $\varphi \in \mathcal{S}$. Then there exists constants $C > 0, k > 0$ such that for all $\varphi \in \mathcal{S}$ we have the estimates

$$0 < 2n + \Delta_\omega \varphi \leq C e^{k\varphi - (\lambda_\varphi + k) \min_X \varphi} =: C_\varphi,$$

$|\partial\bar{\partial}\varphi|_\omega < (C_\varphi + 2\sqrt{n})/2$ and $\omega_\varphi < (C_\varphi/2)\omega$. Moreover if also $f_\varphi \geq -K_0$ then there exist a constant $k_0 > 0$ such that $k_0^{-1} C_\varphi^{1-n} e^{-\lambda_\varphi \varphi} \omega < \omega_\varphi$ for all $\varphi \in \mathcal{S}$.

(In the case $n = 1$ is obvious to find an uniform estimate for $0 < 2n + \Delta_\omega \varphi$.)

Proof. Consider the smooth function $A := \log(2n + \Delta_\omega \varphi) - k\varphi$ where the constant k will be chosen later. By deriving the formula

$$\bar{\partial}A = \frac{\bar{\partial}\Delta_\omega \varphi}{2n + \Delta_\omega \varphi} - k\bar{\partial}\varphi,$$

we obtain

$$\begin{aligned} i\partial\bar{\partial}A &= \frac{i\partial\bar{\partial}\Delta_\omega \varphi}{2n + \Delta_\omega \varphi} + i\partial\left(\frac{1}{2n + \Delta_\omega \varphi}\right) \wedge \bar{\partial}\Delta_\omega \varphi - ki\partial\bar{\partial}\varphi \\ &= \frac{i\partial\bar{\partial}\Delta_\omega \varphi}{2n + \Delta_\omega \varphi} - \frac{i\partial\Delta_\omega \varphi \wedge \bar{\partial}\Delta_\omega \varphi}{(2n + \Delta_\omega \varphi)^2} - ki\partial\bar{\partial}\varphi. \end{aligned}$$

Taking the trace with respect to the metric ω_φ we get the identity

$$\Delta_\varphi A = \frac{\Delta_\varphi \Delta_\omega \varphi}{2n + \Delta_\omega \varphi} - \frac{2|\partial\Delta_\omega \varphi|_\varphi^2}{(2n + \Delta_\omega \varphi)^2} - k\Delta_\varphi \varphi. \quad (3.1)$$

Set $C_\omega := \min\{-1/2, \min_{x \in X} \lambda_1^\omega(x)\}$ (see [Pal-2]). Using the inequality (19) in the proposition 6.1 in [Pal-2] and the inequality $\text{Tr}_\varphi \omega > 0$ we find the inequality

$$2 \text{Tr}_\omega \text{Ric}(\omega_\varphi) \geq -\Delta_\varphi \Delta_\omega \varphi + 2C_\omega(2n + \Delta_\omega \varphi) \text{Tr}_\varphi \omega + \frac{2|\partial\Delta_\omega \varphi|_\varphi^2}{2n + \Delta_\omega \varphi}$$

which combined with the equality (3.1) gives

$$\Delta_\varphi A \geq -\frac{2 \text{Tr}_\omega \text{Ric}(\omega_\varphi)}{2n + \Delta_\omega \varphi} + 2C_\omega \text{Tr}_\varphi \omega - k\Delta_\varphi \varphi. \quad (3.2)$$

Applying the $i\partial\bar{\partial}$ operator to the definition of f_φ and taking the trace respect to ω we find the equality

$$\Delta_\omega f_\varphi - \lambda_\varphi \Delta_\omega \varphi = \text{Sc}(\omega) - \text{Tr}_\omega \text{Ric}(\omega_\varphi). \quad (3.3)$$

Moreover we have the trivial identity $\Delta_\varphi \varphi = -\text{Tr}_\varphi \omega + 2n$. Using this identity with the equality (3.3) in the inequality (3.2), we find

$$\begin{aligned}\Delta_\varphi A &\geq 2 \frac{\Delta_\omega f_\varphi - \lambda_\varphi \Delta_\omega \varphi - \text{Sc}(\omega)}{2n + \Delta_\omega \varphi} + (2C_\omega + k) \text{Tr}_\varphi \omega - 2kn \\ &= 2 \frac{\Delta_\omega f_\varphi + 2\lambda_\varphi n - \text{Sc}(\omega)}{2n + \Delta_\omega \varphi} + (2C_\omega + k) \text{Tr}_\varphi \omega - 2kn - 2\lambda_\varphi. \quad (3.4)\end{aligned}$$

Let $C_1, C_2 > 0$ be two constants independents of φ such that $\Delta_\omega f_\varphi + 2\lambda_\varphi n - \text{Sc}(\omega) \geq -C_1$ and $-2kn - 2\lambda_\varphi \geq -C_2$. Using ω -orthogonal and ω_φ -diagonal coordinates in a point x we find the inequality

$$\frac{\text{Tr}_\varphi \omega}{2} = \sum_l \frac{1}{1 + 2\varphi_{l,\bar{l}}} \geq \frac{2n}{2n + \Delta_\omega \varphi}.$$

This combined with (3.4) implies

$$\Delta_\varphi A \geq \left(-\frac{C_1}{2n} + 2C_\omega + k \right) \text{Tr}_\varphi \omega - C_2.$$

We choose k such that $(-C_1/(2n) + 2C_\omega + k) = 2^{-1}$. In particular $k > 1 + 1/2 > 0$ and $\lambda_\varphi + k > 1/2 > 0$. So we have obtain the inequality

$$\Delta_\varphi A \geq \frac{\text{Tr}_\varphi \omega}{2} - C_2. \quad (3.5)$$

Consider now the trivial inequality $\sum_{l=1}^n a_1 \dots \hat{a}_l \dots a_n \leq (\sum_{l=1}^n a_l)^{n-1}$ for any positive a_l . Taking the $1/(n-1)$ -th power of this inequality with the terms $a_l := 1/(1 + 2\varphi_{l,\bar{l}})$ we find the expressions

$$\frac{\text{Tr}_\varphi \omega}{2} = \sum_l \frac{1}{1 + 2\varphi_{l,\bar{l}}} \geq \left(\frac{\sum_l (1 + 2\varphi_{l,\bar{l}})}{\prod_l (1 + 2\varphi_{l,\bar{l}})} \right)^{\frac{1}{n-1}} = K_n e^{\frac{\lambda_\varphi - f}{n-1}} (2n + \Delta_\omega \varphi)^{\frac{1}{n-1}},$$

where $K_n := 2^{\frac{-1}{n-1}} > 0$. Consider the function $u := e^A = (2n + \Delta_\omega \varphi)e^{-k\varphi}$. Then the previous inequality gives

$$\frac{\text{Tr}_\varphi \omega}{2} \geq K_n e^{\frac{(\lambda+k)\varphi - f}{n-1}} u^{\frac{1}{n-1}} \geq C_0 e^{\frac{(\lambda+k)\varphi}{n-1}} u^{\frac{1}{n-1}}$$

for some constant $C_0 > 0$ independent of φ . Then the inequality (3.5) gives

$$\Delta_\varphi A \geq -C_2 + C_0 e^{\frac{(\lambda+k)\varphi}{n-1}} u^{\frac{1}{n-1}}.$$

Let x_0 be a maximum point for A . Then x_0 is also a maximum point for u and $\Delta_\varphi A(x_0) \leq 0$. We deduce the existence of a constant $C > 0$ independent of φ such that

$$u(x_0) \leq C e^{-(\lambda_\varphi + k) \min_X \varphi}.$$

So in conclusion we have found the required estimate

$$0 < 2n + \Delta_\omega \varphi \leq e^{k\varphi} u(x_0) \leq C e^{k\varphi - (\lambda_\varphi + k) \min_X \varphi} =: C_\varphi.$$

Moreover $2|\omega_\varphi|_\omega < \text{Tr}_\omega \omega_\varphi$, since $\omega_\varphi > 0$. This implies the a priori estimate $|\partial\bar{\partial}\varphi|_\omega < (C_\varphi + 2\sqrt{n})/2$. The inequality $0 < 2 + 4\varphi_{l,\bar{l}} < 2n + \Delta_\omega \varphi \leq C_\varphi$ implies $\omega_\varphi < (C_\varphi/2)\omega$. By definition of f_φ we have

$$e^{f_\varphi - \lambda_\varphi \varphi} = \omega_\varphi^n / \omega^n = \prod_l (1 + 2\varphi_{l,\bar{l}}) < (C_\varphi/2)^{n-1} (1 + 2\varphi_{j,\bar{j}}),$$

for all j . If we have also $f_\varphi \geq -K_0$ then there exist a uniform constant $k_0 > 0$ such that $k_0^{-1} C_\varphi^{1-n} e^{-\lambda_\varphi \varphi} \omega < \omega_\varphi$. $\square$

Lemma 1 *Let (X, ω) be a compact Kähler manifold of complex dimension n and let $\mathcal{S} \subset \mathcal{P}_\omega$ be a subset of potentials for the Kähler metric ω . Suppose that there exist a constant $k > 0$ such that $k^{-1}\omega \leq \omega_\varphi \leq k\omega$ for all $\varphi \in \mathcal{S}$. Then there exist a constant $C > 0$ depending only on the constant k , such that for every potential $\varphi \in \mathcal{S}$ hold the uniform estimate*

$$\begin{aligned} \Delta_\varphi |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 \geq & - \left(6k |\text{Ric}(\omega_\varphi)|_\omega + C |R_\omega|_\omega \right) |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 \\ & - \left(4 |\nabla_\omega^{1,0} \text{Ric}(\omega_\varphi)|_\varphi + nC |\nabla_\omega^{1,0} R_\omega|_\omega \right) |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi. \end{aligned}$$

Proof. The proof follows the lines of the computation in the lemma 6 of [Pal-2] until the definition of the tensor $\text{Tr}_\varphi \nabla_\omega^{1,0} R_\omega$. By applying the Cauchy-Schwartz inequality to the expression (31) of $\Delta_\varphi |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2$ in the lemma 6 of [Pal-2] we find the following intrinsic inequality.

$$\begin{aligned} \Delta_\varphi |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 \geq & - 6k |\text{Ric}(\omega_\varphi)|_\omega |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 - 4 |\nabla_\omega^{1,0} \text{Ric}(\omega_\varphi)|_\varphi |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi \\ & - 8 |\text{Tr}_\varphi \nabla_\omega^{1,0} R_\omega|_{\varphi,\omega} |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_{\varphi,\omega} - C_1 |R_\omega|_\omega |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2. \end{aligned}$$

Then the required inequality of lemma 1 follows from the inequality $|\text{Tr}_\varphi \nabla_\omega^{1,0} R_\omega|_\omega \leq 2kn |\nabla_\omega^{1,0} R_\omega|_\omega$. $\square$

Proposition 3.2 *Let (X, ω) be a compact Kähler manifold of complex dimension n , let $\mathcal{S} \subset \mathcal{P}_\omega$ be a subset of potentials for the Kähler metric ω and let $\lambda : \mathcal{S} \rightarrow [-1, 1]$, $\varphi \mapsto \lambda_\varphi$ be a map. Assume the existence of a uniform constant $C > 0$ such that $\text{Osc}(\varphi) \leq C$ for all $\varphi \in \mathcal{S}$. Suppose also that the functions*

$$f_\varphi := \log \frac{\omega_\varphi^n}{\omega^n} + \lambda_\varphi \varphi$$

satisfy the uniform estimates $|f_\varphi| \leq K_0$, $|\partial\bar{\partial}f_\varphi|_\omega \leq K_0$ and $|\nabla_\omega^{1,0} \partial\bar{\partial}f_\varphi|_\omega \leq K_0$, for some constant $K_0 > 0$ independent of $\varphi \in \mathcal{S}$. Let $\alpha \in (0, 1)$. Then there exist positive constants $k, K, K', K_\alpha > 0$ such that for all $\varphi \in \mathcal{S}$ hold the uniform estimates $0 < 2n + \Delta_\omega \varphi \leq K$, $|\partial\bar{\partial}\varphi|_\omega < (K + 2\sqrt{n})/2$, $k^{-1}\omega < \omega_\varphi < (K/2)\omega$, $|\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\omega \leq K'$ and $\|d\varphi\|_{C^{1,\alpha}} \leq K_\alpha$.

Remark. By definition of f follows the equality $\int_X \omega^n = \int_X e^{f - \lambda_\varphi \varphi} \omega^n$. So $f_\varphi - \lambda_\varphi \varphi = 0$ or $f_\varphi - \lambda_\varphi \varphi$ change signs. In the first case $|\lambda_\varphi \varphi| \leq K_0$. In the second $|f_\varphi - \lambda_\varphi \varphi| \leq \text{Osc}(f_\varphi - \lambda_\varphi \varphi)$ and so $|\lambda_\varphi \varphi| \leq \text{Osc}(\varphi) + 3K_0 \leq C$. In particular $-\lambda_\varphi \min_X \varphi \leq C$.

Proof. Using the inequality $|\Delta_\omega f_\varphi| \leq 2\sqrt{n}|\partial\bar{\partial}f_\varphi|_\omega \leq 2\sqrt{n}K_0$, we can apply proposition 3.1 to obtain the first three uniform estimates of the proposition 3.2. Applying the $i\partial\bar{\partial}$ operator to the definition of f_φ and taking respectively the trace respect to ω and deriving again we find the equalities

$$i\partial\bar{\partial}f_\varphi - \lambda_\varphi i\partial\bar{\partial}\varphi = \text{Ric}(\omega) - \text{Ric}(\omega_\varphi), \quad (3.6)$$

$$\Delta_\omega f_\varphi - \lambda_\varphi \Delta_\omega \varphi = \text{Sc}(\omega) - \text{Tr}_\omega \text{Ric}(\omega_\varphi), \quad (3.7)$$

$$\nabla_\omega^{1,0} \partial\bar{\partial}f_\varphi - \lambda_\varphi \nabla_\omega^{1,0} \partial\bar{\partial}\varphi = \nabla_\omega^{1,0} \text{Ric}(\omega) - \nabla_\omega^{1,0} \text{Ric}(\omega_\varphi). \quad (3.8)$$

Then using the identity (3.8), we find the inequality

$$|\nabla_\omega^{1,0} \text{Ric}(\omega_\varphi)|_\varphi \leq |\lambda_\varphi| \cdot |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi + C_3,$$

for some constant $C_3 > 0$. The lemma 1 implies in conclusion the uniform estimate

$$\Delta_\varphi |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 \geq -C_4 |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 - C_5, \quad (3.9)$$

where $C_4, C_5 > 0$ and the constant C_5 is chosen sufficiently large. On the other hand

$$2 \text{Tr}_\varphi(\omega_\varphi \cdot \text{Rm}_\omega) \geq 2\lambda_1^\omega(2n + \Delta_\omega \varphi) \text{Tr}_\varphi \omega \geq -C_6,$$

with $C_6 > 0$. Then combining the identity (18) in the proposition 6.1 in [Pal-2] with the equality (3.7) we find the uniform estimate

$$\Delta_\varphi \Delta_\omega \varphi \geq (4/k) |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 - C_7, \quad (3.10)$$

with $C_7 > 0$. Define now the positive constant $C_8 := k(C_4 + 1)/4 > 0$. Then combining the estimates (3.9) and (3.10) we find the estimate

$$\Delta_\varphi (|\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 + C_8 \Delta_\omega \varphi) \geq |\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 - C_9,$$

with $C_9 > 0$. This implies that at the point where $|\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 + C_8 \Delta_\omega \varphi$ achieves its maximum we have the estimate

$$|\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\varphi^2 + C_8 \Delta_\omega \varphi \leq C_9 + C_8 \Delta_\omega \varphi \leq C_{10},$$

which implies the uniform estimate $|\nabla_\omega^{1,0} \partial\bar{\partial}\varphi|_\omega \leq C$, since the metrics ω_φ are uniformly equivalent to the metric ω for all $\varphi \in \mathcal{S}$. We infer that the family of $(1,1)$ -forms $(\partial\bar{\partial}\varphi)_{\varphi \in \mathcal{S}}$ is uniformly bounded in $C^\alpha(X)$ -topology. In particular we have the uniform estimate $|\Delta_\omega \varphi|_{C^\alpha(X)} \leq C$. By deriving the Green Formula (see [Aub1], Th. 4.13 pag. 108) we deduce the identity

$$d_x \varphi = - \int_X d_x \mathcal{G}_\omega(x, \cdot) \Delta_\omega \varphi \omega^n.$$

By the classic Newtonian potential theory (see [Gi-Tru], Lemm. 4.4 pag. 56) we deduce the uniform estimate

$$|d\varphi|_{C^{1,\alpha}(X)} \leq C |\Delta_\omega \varphi|_{C^\alpha(X)} \leq C',$$

which allows to conclude. $\square$

3.2 Projections to a Kähler-Einstein orbit

Lemma 2 (Bochner-Kodaira formula). *Let (X, ω) be a compact Kähler manifold of complex dimension n and let $u \in \mathcal{E}(X, \mathbb{R})$ be a smooth real function. Then hold the identity*

$$2 \int_X |\bar{\partial} \nabla_\omega^{1,0} u|_\omega^2 \omega^n = \int_X (\Delta_\omega u)^2 \omega^n - 2 \int_X \text{Ric}(\omega)(\nabla_\omega u, J \nabla_\omega u) \omega^n. \quad (3.11)$$

Proof. Consider the decomposition of the norm square of the Hessian in the classic Bochner Formula

$$\Delta |\nabla u|_t^2 = 2g(\nabla \Delta u, \nabla u) + 4|\nabla^{1,0} \partial u|^2 + 4|\partial \bar{\partial} u|^2 + 2\text{Ric}(\nabla u, J \nabla u).$$

(We drop the dependence on ω .) Combining this with the identities

$$4n(n-1) \frac{(i\partial \bar{\partial} u)^2 \wedge \omega^{n-2}}{\omega^n} = (\Delta u)^2 - 4|\partial \bar{\partial} u|^2, \quad (3.12)$$

$2|\bar{\partial} \nabla^{1,0} u|^2 = 4|\nabla^{1,0} \partial u|^2$ and integrating by parts we get the formula (3.11). $\square$

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of the Laplacian Δ_ω if there exists a function $u \in \mathcal{E}(X, \mathbb{C})$, not identically zero, such that $\Delta_\omega u + \lambda u = 0$. We deduce the following corollary.

Corollary 2 *Let (X, ω) be a compact Kähler manifold such that $\text{Ric}(\omega) > t\omega$ for some $t > 0$. Then the first eigenvalue λ_1 of the Laplacian Δ_ω satisfies the estimate $\lambda_1 > 2t$. Moreover if $\text{Ric}(\omega) \geq t\omega$ then $\lambda_1 \geq 2t$.*

In fact let u be an eigenfunction of λ_1 . By the Bochner formula we deduce the inequality

$$\lambda_1^2 \int_X u^2 \omega^n > 2t \int_X |\nabla_\omega u|_\omega^2 \omega^n = 2t \lambda_1 \int_X u^2 \omega^n.$$

We remind also the following well known theorem which is a consequence of the Bochner formula for differential forms.

Theorem 4 *Let (M, g) be a compact Riemannian manifold such that $\text{Ric}(g) > 0$. Then the first Betti number is zero.*

We remind now a very useful lemma [Ba-Ma].

Lemma 3 *Let (X, ω) be a compact Kähler manifold of complex dimension n and let $\psi_t \in \mathcal{P}_\omega$, $t > 0$ be a potential such that the metric $\omega_t := \omega + i\partial \bar{\partial} \psi_t$ satisfies $\text{Ric}(\omega_t) \geq t\omega_t$. Then there exist a constant $C_n > 0$ depending only on n and a constant $C_\omega > 0$ depending only on ω such that $\text{Osc}(\psi_t) \leq I_\omega(\psi_t) + C_n t^{-1} + C_\omega$.*

We need now a very elementary lemma.

Lemma 4 *Let (X, J, ω) be a Kähler manifold and $\xi \in \mathcal{O}(T_{X,J}^{1,0})(U)$ be a $(1, 0)$ holomorphic vector field over an open set $U \subset X$. Then hold the identity over U*

$$2\xi \lrcorner \text{Ric}(\omega) = -i\bar{\partial} \text{Tr}_\omega[\partial(\xi \lrcorner \omega)]. \quad (3.13)$$

Let $f \in \mathcal{E}(U, \mathbb{C})$ be a smooth function such that $\nabla_\omega^{1,0} f \in \mathcal{O}(T_{X,J}^{1,0})(U)$. Then hold the identity over U

$$2\nabla_\omega^{1,0} f \lrcorner \text{Ric}(\omega) = -i\bar{\partial}\Delta_\omega f. \quad (3.14)$$

Proof. Let first remind the local expression of the Ricci tensor

$$\text{Ric}(\omega) = i\bar{\partial}(\omega^{l,\bar{r}} \partial \omega_{r,\bar{l}}) = -i\partial_{\bar{k}}(\omega^{l,\bar{r}} \partial_j \omega_{r,\bar{l}}) dz_j \wedge d\bar{z}_k.$$

Set $\xi = \xi_k \frac{\partial}{\partial z_k}$. Then we have the equalities

$$\xi \lrcorner \omega = \frac{i}{2} \omega_{j,\bar{l}} \xi_j d\bar{z}_l, \quad \partial(\xi \lrcorner \omega) = \frac{i}{2} \partial_r(\omega_{j,\bar{l}} \xi_j) dz_r \wedge d\bar{z}_l,$$

$$\text{Tr}_\omega[\partial(\xi \lrcorner \omega)] = 2\omega^{l,\bar{r}} \partial_r(\omega_{j,\bar{l}} \xi_j) = 2\omega^{l,\bar{r}} \partial_r \omega_{j,\bar{l}} \xi_j + 2\partial_j \xi_j.$$

Using the hypothesis $\bar{\partial}\xi_k = 0$ and the fact that ω is Kähler we find the equality

$$-i\bar{\partial}\text{Tr}_\omega[\partial(\xi \lrcorner \omega)] = -2i\bar{\partial}(\omega^{l,\bar{r}} \partial_j \omega_{r,\bar{l}}) \xi_j = 2\xi \lrcorner \text{Ric}(\omega).$$

The splitting of the gradient $\nabla_\omega f = \nabla_\omega^{1,0} f + \nabla_\omega^{0,1} f$ of the function f implies the identities $\nabla_\omega^{1,0} f \lrcorner \omega = i\bar{\partial}f$ and $\nabla_\omega^{0,1} f \lrcorner \omega = -i\partial f$. Then by using the identity (3.13) we deduce the identity (3.14). $\square$

Let X be a Fano manifold admitting a Kähler-Einstein metric $\omega \in 2\pi c_1$. It is well known that the isotropy group $K_\omega \triangleleft \text{Aut}_J^0(X)$ of the Kähler-Einstein metric ω is maximal compact. Let $\kappa_\omega = \{\xi \in H^0(T_{X,J}) \mid L_\xi \omega = 0\}$ its (real) Lie algebra. We remind the following well known fact in Kähler-Einstein geometry, which follows from the Bochner-Kodaira formula [Mat].

Theorem 5 (Matsushima). *Let X be a Fano manifold admitting a Kähler-Einstein metric $\omega \in 2\pi c_1$. Then the first eigenvalue λ_1 of the Laplacian Δ_ω satisfies the estimate $\lambda_1 \geq 2$. If $H^0(T_{X,J}^{1,0}) = 0$, then $\lambda_1 > 2$. If $H^0(T_{X,J}^{1,0}) \neq 0$, then $\lambda_1 = 2$. In this case the $(1,0)$ -component of the gradient map*

$$\nabla_\omega^{1,0} : \mathbb{C}\Lambda_\omega \longrightarrow H^0(T_{X,J}^{1,0})$$

is well defined and gives a $\mathbb{C}$ -isomorphism of vector spaces. Moreover this induces the $\mathbb{R}$ -isomorphisms $\nabla_\omega : \Lambda_\omega \longrightarrow J\kappa_\omega$, $J\nabla_\omega : \Lambda_\omega \longrightarrow \kappa_\omega$ and the decomposition $H^0(T_{X,J}) = \kappa_\omega \oplus J\kappa_\omega$.

Proof. The estimate $\lambda_1 \geq 2$ follows immediately from the corollary 2. Assume first that $H^0(T_{X,J}^{1,0}) \neq 0$ and let prove that the $(1,0)$ -component of the gradient map is an isomorphism. In some sense is more natural to start with the inverse map, so consider $\xi \in H_\omega^0(T_{X,J}^{1,0})$. The fact that ω is Kähler and ξ holomorphic implies the equality $\bar{\partial}(\xi \lrcorner \omega) = 0$. Then the theorem 4 implies the existence of a function $u \in \mathcal{E}(X, \mathbb{C})$, $\int_X u \omega^n = 0$, such that $\xi \lrcorner \omega = i\bar{\partial}u$. In other terms $\xi = \nabla_\omega^{1,0} u$. Then the identity 3.14 implies the identity $\bar{\partial}(\Delta_\omega u + 2u) = 0$, which in his turn implies the equality $\Delta_\omega u + 2u = 0$.

Consider now an eigenfunction with real values $u \in \mathcal{E}(X, \mathbb{R})$, such that $\Delta_\omega u + 2u = 0$. Then the Bochner formula (3.11) implies $\nabla_\omega u \in H^0(T_{X,J})$. The fact that u is a function with real values implies $\nabla_\omega^{1,0} u \in H^0(T_{X,J}^{1,0})$. In general

if $u \in \mathcal{E}(X, \mathbb{C})$ the same conclusion holds by the fact that the Laplacian is a real operator. So in conclusion we have show the required isomorphism. Now is clear that If $H^0(T_{X,J}^{1,0}) = 0$, then $\lambda_1 > 2$. The existence of the $\mathbb{R}$ -isomorphisms is now also obvious. $\square$

Remark 1.

Let $O := \text{Aut}_J^0(X) \cdot \omega$ be the orbit of a Kähler-Einstein metric $\omega \in 2\pi c_1$ and let $\mathcal{P}_\omega^O$ the corresponding set of potentials φ , normalized by the condition $\int_X e^{-\varphi} \omega^n = 1$. Moreover $O \simeq \text{Aut}_J^0(X)/K_\omega$ has the structure of a homogeneous space. A posteriori the uniqueness result of Bando-Mabuchi [Ba-Ma] implies that O is the set of all Kähler-Einstein metrics. We have a natural identification $\eta_\omega : O \rightarrow \mathcal{P}_\omega^O$,

$$\theta \mapsto u_\theta := \log \frac{\omega^n}{\theta^n}.$$

Let $\omega_t = \omega + i\partial\bar{\partial}u_t$, $u_t \in \mathcal{P}_\omega^O$, $\omega_0 = \omega$ a Kähler-Einstein variation of ω , so we have the identity $\omega_t^n = e^{-u_t} \omega^n$. By time deriving the equivalent identity

$$\log \frac{\omega_t^n}{\omega^n} + u_t = 0,$$

at $t = 0$ we get $\Delta_\omega \dot{u}_0 / 2 + \dot{u}_0 = 0$ i.e $\dot{u}_0 \in \Lambda_\omega$. The differential at ω of the map η_ω is the isomorphism

$$d_\omega \eta_\omega : T_{O, \omega} \rightarrow T_{\mathcal{P}_\omega^O, 0} \subset \Lambda_\omega$$

given by $i\partial\bar{\partial}\dot{u}_0 \mapsto \dot{u}_0$. We remark that $T_{\mathcal{P}_\omega^O, 0} = \Lambda_\omega$. In fact consider the composition

$$\begin{aligned} \Lambda_\omega &\xrightarrow{\nabla_\omega} J\kappa_\omega \xrightarrow{\mu_\omega} T_{O, \omega} \\ v &\longmapsto \nabla_\omega v \longmapsto \frac{1}{2} L_{\nabla_\omega v} \omega. \end{aligned}$$

Let $(\sigma_t)_{t \in \mathbb{R}} \triangleleft \text{Aut}_J^0(X)$ the one parameter subgroup of automorphisms corresponding to $\nabla_\omega v$ and $\omega_t := \sigma_t^* \omega = \omega + i\partial\bar{\partial}u_t$, $u_t \in \mathcal{P}_\omega^O$, its Kähler-Einstein variation. As before $\dot{u}_0 \in \Lambda_\omega$. This combined with the identities

$$2i\partial\bar{\partial}v = L_{\nabla_\omega v} \omega = \frac{d}{dt}|_0 \omega_t = i\partial\bar{\partial}\dot{u}_0$$

implies $2v = \dot{u}_0$, which shows that $T_{\mathcal{P}_\omega^O, 0} = \Lambda_\omega$ and so $(d_\omega \eta_\omega \circ \mu_\omega)^{-1} = \nabla_\omega$. In particular μ_ω is also an isomorphism.

Remark 2.

Consider now the Banach space and $\hat{\Lambda}_{\omega, k}^\perp := \Lambda_{\omega, k}^\perp \cap C_{\omega, 0}^{k, 1/2}(X, \mathbb{R})$ for some nonnegative integer k . The isomorphism

$$\Delta_\omega : C_{\omega, 0}^{k+2, 1/2}(X, \mathbb{R}) \rightarrow C_{\omega, 0}^{k, 1/2}(X, \mathbb{R}),$$

induces the isomorphism $\Delta_\omega : \hat{\Lambda}_{\omega, k+2}^\perp \rightarrow \hat{\Lambda}_{\omega, k}^\perp$ and by construction the kernel of the operator

$$\Delta_\omega + 2 : \Lambda_{\omega, k+2}^\perp \rightarrow \Lambda_{\omega, k}^\perp,$$

is zero. Then by a classic result of elliptic Theory (see Th. 12.4, pag 688 in [A-D-N]) we deduce that the previous operator is surjective and so an isomorphism of Banach spaces by the Banach theorem.

Orthogonal projection of Kähler metrics.

We remind the following projection lemma due to Bando-Mabuchi [Ba-Ma].

Lemma 5 *Let $(X, \hat{\omega})$ be an Einstein-Fano manifold. Then for all orbits $O := \text{Aut}_J^0(X) \cdot \hat{\omega} \subset \mathcal{KE}$ and for all Kähler metrics $\omega \in \mathcal{K}$ there exist a Kähler-Einstein metric $\theta \in O$ such that $\omega \in \mathcal{K}_\theta^\perp$.*

Proof. Set $\theta = \omega_\phi \in O$ and consider the functional $\Psi_\omega := G_{\omega|O}$. Consider a Kähler-Einstein variation of θ , $\theta_t = \theta + i\partial\bar{\partial}u_t = \omega + i\partial\bar{\partial}(u_t + \phi)$, $u_t \in \mathcal{P}_\theta^O$, $u_0 = 0$. Moreover $\dot{u}_0 \in \Lambda_\theta$ as in remark 1. Using the identity (1.4) we get

$$\frac{d}{dt}|_0 G_\omega(\theta_t) = -\frac{1}{2} \int_X \phi \Delta_\theta \dot{u}_0 \theta^n = \int_X \phi \dot{u}_0 \theta^n.$$

Then the fact that the differential $d_\theta \eta_\theta : T_{O, \theta} \rightarrow \Lambda_\theta$ of the map $\eta_\theta : O \rightarrow \mathcal{P}_\theta^\mathcal{E}$ in remark 1 is an isomorphism implies that a metric $\theta \in O$ is a critical point of the functional Ψ_ω if and only if $\omega \in \mathcal{K}_\theta^\perp$. We show now that the smooth function Ψ_ω is an exhaustion, which implies that it always admits a minimum. Consider $\omega_\psi = \hat{\omega}_{\varphi+\psi} \in O$. The Einstein condition is equivalent to the equation $\omega_\psi^n = e^{h_\omega - \psi} \omega^n$, which implies

$$\int_X e^{-\psi} e^{h_\omega} \omega^n = \int_X e^{h_\omega} \omega^n,$$

and so ψ change signs. Then using lemma 3 and the inequality (1.2) we get the estimate

$$\|\psi\|_{C^0} \leq \text{Osc}(\psi) \leq I_\omega(\psi) + C_\omega \leq (n+1)\Psi_\omega(\omega_\psi) + C_\omega,$$

for some constants $C_\omega > 0$ depending only on ω . Now all the topologies induced from a Banach norm coincident on $\mathcal{P}_\omega^O$ since it is a finite dimensional submanifold. Moreover the fact that $\mathcal{P}_\omega^O$ is homeomorphic with the homogeneous space $O \simeq \text{Aut}_J^0(X)/K_{\hat{\omega}}$ implies that it is also closed in any holder space $C^{k, \alpha}$. This combined with the previous inequality implies that for all $r > 0$ the sets $\eta_{\hat{\omega}}(\{\Psi_\omega \leq r\})$ are compact and so the sets $\{\Psi_\omega \leq r\}$ are compact, which shows that Ψ_ω is an exhaustion. $\square$

4 The solutions of the Aubin equation

Let $\hat{\omega} \in 2\pi c_1$ be a Kähler-Einstein metric. For all $\varphi \in \mathcal{P}_{\hat{\omega}} \cap \Lambda_{\hat{\omega}}^\perp$ consider $\omega := \hat{\omega} + i\partial\bar{\partial}\varphi$ and the smooth family of solutions $\psi_t \in \mathcal{P}_\omega$ of the complex Monge-Ampère equation

$$(\omega + i\partial\bar{\partial}\psi_t)^n = e^{h_\omega - t\psi_t} \omega^n, \quad t \in [0, 1], \quad (4.1)$$

which is equivalent to the Aubin equation

$$\text{Ric}(\omega_t) = t\omega_t + (1-t)\omega, \quad t \in [0, 1], \quad (4.2)$$

with $\omega_t := \omega + i\partial\bar{\partial}\psi_t$. The potential $\psi_1 := -\varphi + c$ is a solution of the equation (4.1) for $t = 1$ since $\hat{\omega}$ is a Kähler-Einstein metric. It has been pointed out by Bando-Mabuchi [Ba-Ma] that a necessary condition for the existence of solutions of (4.1) for t in a neighborhood of 1, with $\omega_1 = \hat{\omega}$ is $\varphi \in \mathcal{P}_{\hat{\omega}} \cap \Lambda_{\hat{\omega}}^{\perp}$. In fact differentiating at $t = 1$ the equation (4.1) we get $\Delta_{\hat{\omega}}\dot{\psi}_1 + 2\dot{\psi}_1 + 2\psi_1 = 0$. Then

$$2 \int_X \psi_1 u \hat{\omega}^n = - \int_X (\Delta_{\hat{\omega}} + 2) \dot{\psi}_1 u \hat{\omega}^n = - \int_X \dot{\psi}_1 (\Delta_{\hat{\omega}} + 2) u \hat{\omega}^n = 0,$$

for all $u \in \Lambda_{\hat{\omega}}$. The following proposition (the main argument is due to [Ba-Ma]) shows that the condition $\varphi \in \mathcal{P}_{\hat{\omega}} \cap \Lambda_{\hat{\omega}}^{\perp}$ is also sufficient for the existence of solutions of (4.1).

Proposition 4.1 *Let X be a Fano manifold admitting a Kähler-Einstein metric $\hat{\omega} \in 2\pi c_1$ and let $\omega = \hat{\omega}_{\varphi}$ with $\varphi \in \mathcal{P}_{\hat{\omega}} \cap \Lambda_{\hat{\omega}}^{\perp}$. Then the following hold.*

(A). *There exist a unique map $[0, 1] \rightarrow \mathcal{K}$, $t \mapsto \omega_t$, solution of the Aubin equation (4.2). Moreover this map is smooth and $\omega_1 = \hat{\omega}$.*

(B). *There exist a unique map $\psi : [0, 1] \rightarrow \mathcal{P}_{\omega}$, $t \mapsto \psi_t$, solution of the complex Monge-Ampère equation (4.1). This map is continuous over the interval $[0, 1]$ and smooth over $(0, 1]$. Moreover $\omega_{\psi_1} = \hat{\omega}$.*

Proof. We start by considering the solution $\psi_1 = -\varphi - \log f_X e^{-\varphi} \hat{\omega}^n$, which gives $\omega_{\psi_1} = \hat{\omega}$. By comparing the Monge-Ampère equation (4.1) with the equation (4.1) at time $t = 1$ we find $e^{t\psi_t} \omega_t^n = e^{h_{\omega}} \omega^n = e^{\psi_1} \hat{\omega}^n$. This implies that (4.1) can be written in the equivalent ways

$$\log \frac{\hat{\omega}_{\theta_t}^n}{\hat{\omega}^n} + t\theta_t = (1-t)\psi_1, \quad (4.3)$$

$$\log \frac{\hat{\omega}_{\theta_t}^n}{\hat{\omega}^n} + \theta_t = (1-t)\psi_t, \quad (4.4)$$

$$\hat{\omega}^n = e^{\theta_t + (t-1)\psi_t} \omega_t^n, \quad (4.5)$$

where $\theta_t := \psi_t - \psi_1 \in \mathcal{P}_{\hat{\omega}}$. Remark that $\omega_t = \hat{\omega}_{\theta_t}$. The last two expressions will be useful in the section 5.

Step I: Short backward time existence. Let $P : L^2 \rightarrow \Lambda_{\hat{\omega}}$ be the L^2 projection on $\Lambda_{\hat{\omega}}$. In the following considerations we will always consider the natural embedding $\mathcal{P}_{\hat{\omega}}^{k, 1/2} \subset \Lambda_{\hat{\omega}} \oplus \Lambda_{\hat{\omega}, k}^{\perp}$. Consider now the smooth maps

$$\Phi : [0, 1] \times \mathcal{P}_{\hat{\omega}}^{k+2, 1/2} \rightarrow \Lambda_{\hat{\omega}, k}^{\perp},$$

$$\Psi : [0, 1] \times \mathcal{P}_{\hat{\omega}}^{k+2, 1/2} \rightarrow \Lambda_{\hat{\omega}},$$

with $k \geq 3$, given by the expressions

$$\Phi(t, u, \mu) := (1-P) \log \frac{\hat{\omega}_{u+\mu}^n}{\hat{\omega}^n} + t\mu - (1-t)\psi_1,$$

$$\Psi(t, u, \mu) := P \log \frac{\hat{\omega}_{u+\mu}^n}{\hat{\omega}^n} + tu = 0.$$

So if we decompose $\theta_t = u_t + \mu_t$, $u_t \in \Lambda_{\hat{\omega}}$ and $\mu_t \in \Lambda_{\hat{\omega}}^{\perp}$ we deduce that the equation (4.3) is equivalent to the system $\Phi(t, u_t, \mu_t) = 0$, $\Psi(t, u_t, \mu_t) = 0$. Remark that $\Phi(1, 0, 0) = 0$. The partial Frechet derivative

$$\begin{aligned} D_{\mu}\Phi(1, 0, 0) : \Lambda_{\hat{\omega}, k+2}^{\perp} &\longrightarrow \Lambda_{\hat{\omega}, k}^{\perp} \\ f &\longmapsto \frac{1}{2}\Delta_{\hat{\omega}}f + f, \end{aligned}$$

is an isomorphism by the remark 2 in the last section. Then the implicit function theorem implies the existence of a unique and smooth map

$$\mu : (1 - \varepsilon_1, 1] \times B_{\Lambda_{\hat{\omega}, k+2}}(0, R_1) \longrightarrow B_{\Lambda_{\hat{\omega}, k+2}^{\perp}}(0, R_2),$$

such that $\Phi(t, u, \mu(t, u)) = 0$. In particular $\mu(1, 0) = 0$. (Here $B_E(0, R)$ is the ball of center 0 and radius R in the space E .) Set

$$\hat{\Psi}(t, u) := \Psi(t, u, \mu(t, u)).$$

Then the equation (4.3) is equivalent to the equation $\hat{\Psi}(t, u_t) = 0$, $u_1 = 0$ by setting $\mu_t := \mu(t, u_t)$. In fact is obvious that this last equation implies (4.3). The other direction follows by applying the uniqueness in the statement of the implicit function theorem to the equality

$$\Phi(t, u_t, \mu_t) = \Phi(t, u_t, \mu(t, u_t)) = 0,$$

which gives $\mu_t = \mu(t, u_t)$. On the other hand for any $\theta \in \mathcal{P}_{\hat{\omega}}^O$ the identity

$$\log \frac{\hat{\omega}_{\theta}^n}{\hat{\omega}^n} + \theta = 0,$$

is equivalent to $\Phi(1, u, \mu) = 0$, $\Psi(1, u, \mu) = 0$, where $\theta = u + \mu$, $u \in \Lambda_{\hat{\omega}}$ and $\mu \in \Lambda_{\hat{\omega}}^{\perp}$. For $r > 0$ sufficiently small

$$\begin{aligned} (1 - P)(\mathcal{P}_{\hat{\omega}}^O \cap B_{C^{k+2}}(0, r)) &\subset B_{\Lambda_{\hat{\omega}, k+2}^{\perp}}(0, R_2), \\ P(\mathcal{P}_{\hat{\omega}}^O \cap B_{C^{k+2}}(0, r)) &\supset B_{\Lambda_{\hat{\omega}, k+2}}(0, r_1), \end{aligned}$$

$r_1 \in (0, R_1)$. The last inclusion follows from the fact that $T_{\mathcal{P}_{\hat{\omega}}^O, 0} = \Lambda_{\hat{\omega}}$ as pointed out in remark 1 of the previous section. So the identity

$$\Phi(1, u, \mu(1, u)) = 0, \quad u \in B_{\Lambda_{\hat{\omega}, k+2}}(0, r_1),$$

combined with $\Phi(1, u, \mu) = 0$ implies $\mu = \mu(1, u)$. Then from the identity

$$\Psi(1, u, \mu) = 0,$$

we get $\hat{\Psi}(1, u) = 0$ for all $u \in B_{\Lambda_{\hat{\omega}, k+2}}(0, r_1)$. We deduce that the function $\hat{\Psi}(t, u)/(t - 1)$ extends to a smooth function

$$\tilde{\Psi} : (1 - \varepsilon_1, 1] \times B_{\Lambda_{\hat{\omega}, k+2}}(0, r_1) \longrightarrow \Lambda_{\hat{\omega}}.$$

By considering the first order Taylor expansion of $\hat{\Psi}$ in the variable t we get

$$\tilde{\Psi}(1, u) = \frac{\partial \hat{\Psi}}{\partial t}(1, u).$$

In particular for $u = 0$ we have

$$\tilde{\Psi}(1, 0) = \frac{\partial \hat{\Psi}}{\partial t}(1, 0) = \frac{d}{dt}|_1 P \log \frac{\hat{\omega}_{\mu(t,0)}^n}{\hat{\omega}^n} = \frac{1}{2} P \Delta_{\hat{\omega}} \frac{d}{dt}|_1 \mu(t, 0) = 0, \quad (4.6)$$

since $\frac{d}{dt}|_1 \mu(t, 0) \in \Lambda_{\hat{\omega}}^\perp$ and $\Delta_{\hat{\omega}}(\Lambda_{\hat{\omega}}^\perp) = \Lambda_{\hat{\omega}}^\perp$. We remark that the orthogonality condition $\varphi \in \Lambda_{\hat{\omega}}^\perp$ on the potential plays a crucial role in the equality (4.6). In conclusion the equation (4.3) is equivalent to the equation $\tilde{\Psi}(t, u_t) = 0, u_1 = 0$. We will apply now the implicit function theorem to the function $\tilde{\Psi}$. Consider now for all $\varphi \in \Lambda_{\hat{\omega}}^\perp$ the quadratic form $\mathcal{Q}_\varphi^{\hat{\omega}} : \Lambda_{\hat{\omega}} \times \Lambda_{\hat{\omega}} \rightarrow \mathbb{R}$ given by the formula

$$\mathcal{Q}_\varphi^{\hat{\omega}}(u, v) := \int_X \left(1 - \frac{1}{4} \Delta_{\hat{\omega}} \varphi\right) uv \hat{\omega}^n,$$

for all $u, v \in \Lambda_{\hat{\omega}}$. The following lemma due to [Ba-Ma] gives an expression of the partial derivative $D_u \tilde{\Psi}(1, 0) : \Lambda_{\hat{\omega}} \rightarrow \Lambda_{\hat{\omega}}$.

Lemma 6 *For all $f_1, f_2 \in \Lambda_{\hat{\omega}}$ hold the equality*

$$n! \left\langle D_u \tilde{\Psi}(1, 0) f_1, f_2 \right\rangle_{L^2(X, \hat{\omega})} = \mathcal{Q}_\varphi^{\hat{\omega}}(f_1, f_2),$$

Set $U_{\hat{\omega}} := \{\varphi \in \Lambda_{\hat{\omega}}^\perp \mid \mathcal{Q}_\varphi^{\hat{\omega}} \text{ is non degenerate}\}$. Clearly $U_{\hat{\omega}}$ is non empty since $\mathcal{Q}_0^{\hat{\omega}}$ is the induced $L^2(X, \hat{\omega})$ product on $\Lambda_{\hat{\omega}}$. Any choice of the initial potential $\varphi \in \mathcal{P}_{\hat{\omega}} \cap U_{\hat{\omega}}$ implies the invertibility of the partial derivative $D_u \tilde{\Psi}(1, 0)$. We deduce that so far we have prove the existence of a unique smooth family of solutions $\psi_t \in \mathcal{P}_{\omega}$ of (4.1) for all $t \in (1 - \varepsilon, 1]$, $\varepsilon \in (0, \varepsilon_1]$, with $\omega_{\psi_1} = \hat{\omega}$ and $\varphi \in \mathcal{P}_{\hat{\omega}} \cap U_{\hat{\omega}}$.

Proof of the lemma 6. Let $(u_s)_s \subset \Lambda_{\hat{\omega}}$ be a variation of 0 i.e $u_0 = 0$, with $\dot{u}_0 = f$. Then

$$\begin{aligned} \frac{d}{ds}|_0 \tilde{\Psi}(1, u_s) &= \frac{d}{ds}|_0 \frac{\partial \hat{\Psi}}{\partial t}(1, u_s) \\ &= \frac{d}{ds}|_0 \left[P \frac{n i \partial \bar{\partial} \left(\frac{d}{dt}|_1 \mu(t, u_s) \right) \wedge \hat{\omega}_{u_s + \mu(1, u_s)}^{n-1}}{\hat{\omega}_{u_s + \mu(1, u_s)}^n} + u_s \right] \\ &= P \left[\frac{n i \partial \bar{\partial} \left(\frac{\partial^2}{\partial s \partial t} \Big|_{(s,t)=(0,1)} \mu(t, u_s) \right) \wedge \hat{\omega}^{n-1}}{\hat{\omega}^n} \right. \\ &\quad + n(n-1) \frac{i \partial \bar{\partial} \left(\frac{d}{dt}|_1 \mu(t, 0) \right) \wedge i \partial \bar{\partial} \left(f + \frac{d}{ds}|_0 \mu(1, u_s) \right) \wedge \hat{\omega}^{n-2}}{\hat{\omega}^n} \\ &\quad \left. - \frac{1}{4} \Delta_{\hat{\omega}} \left(\frac{d}{dt}|_1 \mu(t, 0) \right) \Delta_{\hat{\omega}} \left(f + \frac{d}{ds}|_0 \mu(1, u_s) \right) \right] + f \\ &= f - P \left\langle i \partial \bar{\partial} \left(\frac{d}{dt}|_1 \mu(t, 0) \right), i \partial \bar{\partial} f \right\rangle_{\hat{\omega}}. \end{aligned}$$

In the last equality we have use the facts that

$$\frac{\partial^2}{\partial s \partial t}|_{(s,t)=(0,1)} \mu(t, u_s) \in \Lambda_{\hat{\omega}}^{\perp},$$

$\Delta_{\hat{\omega}}(\Lambda_{\hat{\omega}}^{\perp}) = \Lambda_{\hat{\omega}}^{\perp}$ and $\frac{d}{ds}|_0 \mu(1, u_s) = 0$. This last one need an explanation. Set

$$h := \frac{d}{ds}|_0 \mu(1, u_s) \in \Lambda_{\hat{\omega}, k+2}^{\perp}.$$

Deriving at $s = 0$ the identity $\Phi(1, u_s, \mu(1, u_s)) = 0$ we get $\Delta_{\hat{\omega}} h/2 + h = 0$ which implies $h = 0$, so $D_u \mu(1, 0) = 0$.

So far we have obtain the formula

$$\begin{aligned} & \left\langle D_u \tilde{\Psi}(1, 0) f_1, f_2 \right\rangle_{L^2(X, \hat{\omega})} \\ &= \int_X \left[f_1 f_2 - f_2 \left\langle i \partial \bar{\partial} \left(\frac{d}{dt}|_1 \mu(t, 0) \right), i \partial \bar{\partial} f_1 \right\rangle_{\hat{\omega}} \right] \frac{\hat{\omega}^n}{n!}. \end{aligned} \quad (4.7)$$

We need other formulas in order to conclude the computation. By time differentiating at $t = 1$ the identity $\Phi(t, 0, \mu(t, 0)) = 0$ we get

$$(\Delta_{\hat{\omega}} + 2) \left(\frac{d}{dt}|_1 \mu(t, 0) \right) = -2\psi_1. \quad (4.8)$$

Moreover for all $u, v \in \Lambda_{\hat{\omega}}$ and $f \in \mathcal{E}(X, \mathbb{R})$ we have

$$\Delta_{\hat{\omega}} \langle \partial f, \partial u \rangle_{\hat{\omega}} = 2 \langle \partial \bar{\partial} f, \partial \bar{\partial} u \rangle_{\hat{\omega}} + \langle \partial \Delta_{\hat{\omega}} f, \partial u \rangle_{\hat{\omega}}, \quad (4.9)$$

$$(\Delta_{\hat{\omega}} + 2) \langle \partial u, \partial v \rangle_{\hat{\omega}} = 2 \langle \partial \bar{\partial} u, \partial \bar{\partial} v \rangle_{\hat{\omega}} = (\Delta_{\hat{\omega}} + 2) \langle \partial v, \partial u \rangle_{\hat{\omega}}. \quad (4.10)$$

The equality (4.10) is an immediate consequence of the equality (4.9). Let prove the equality (4.9). Let $(z_1, \dots, z_n)$ holomorphic $\hat{\omega}$ -geodesic coordinates centered in an arbitrary point x . Then at the point x we have

$$\begin{aligned} \Delta_{\hat{\omega}} \langle \partial f, \partial u \rangle_{\hat{\omega}} &= 4 \partial_{r\bar{r}}^2 (2 \hat{\omega}^{k\bar{l}} \partial_l f \partial_{\bar{k}} u) \\ &= -8 \partial_{r\bar{r}}^2 \hat{\omega}_{k\bar{l}} \partial_l f \partial_{\bar{k}} u + 2 \partial_k \Delta_{\hat{\omega}} f \partial_{\bar{k}} u + 2 \partial_k f \partial_{\bar{k}} \Delta_{\hat{\omega}} u \\ &\quad + 2 \langle \partial \bar{\partial} f, \partial \bar{\partial} u \rangle_{\hat{\omega}} + 2 \left\langle \nabla_{\hat{\omega}}^{1,0} \partial f, \nabla_{\hat{\omega}}^{1,0} \partial u \right\rangle_{\hat{\omega}}. \end{aligned}$$

Then using the identities $-\partial_{r\bar{r}}^2 \hat{\omega}_{k\bar{l}} = R_{k\bar{l}} = \delta_{k\bar{l}}$, $\Delta_{\hat{\omega}} u + 2u = 0$ and $\nabla_{\hat{\omega}}^{1,0} \partial u = 0$, which is equivalent to say that $\nabla_{\hat{\omega}} u$ is a holomorphic vector field, we deduce the equality (4.9). We need to prove also the identity

$$-2 \int_X u \langle \partial \bar{\partial} f, \partial \bar{\partial} u \rangle_{\hat{\omega}} \hat{\omega}^n = \int_X \left(uv - \frac{1}{2} \langle \partial u, \partial v \rangle_{\hat{\omega}} - \frac{1}{2} \langle \partial v, \partial u \rangle_{\hat{\omega}} \right) (\Delta_{\hat{\omega}} + 2) f \hat{\omega}^n.$$

for all u, v, f as before. In fact set $h := (\Delta_{\hat{\omega}} + 2)f$. Then

$$\begin{aligned}
& \int_X \left(uv - \frac{1}{2} \langle \partial u, \partial v \rangle_{\hat{\omega}} - \frac{1}{2} \langle \partial v, \partial u \rangle_{\hat{\omega}} \right) h \hat{\omega}^n \\
&= \int_X (uv - \langle \partial u, \partial v \rangle_{\hat{\omega}}) h \hat{\omega}^n \quad (\text{by (4.10)}) \\
&= -n \int_X (u i \partial \bar{\partial} v + i \partial u \wedge \bar{\partial} v) h \wedge \hat{\omega}^{n-1} \quad (\text{since } v = -\Delta_{\hat{\omega}} v / 2) \\
&= -n \int_X h i \partial (u \bar{\partial} v) \wedge \hat{\omega}^{n-1} = n \int_X u i \partial h \wedge \bar{\partial} v \wedge \hat{\omega}^{n-1} = \int_X u \langle \partial h, \partial v \rangle_{\hat{\omega}} \hat{\omega}^n \\
&= \int_X u (\langle \partial \Delta_{\hat{\omega}} f, \partial v \rangle_{\hat{\omega}} + 2 \langle \partial f, \partial v \rangle_{\hat{\omega}}) \hat{\omega}^n \\
&= \int_X u (\Delta_{\hat{\omega}} \langle \partial f, \partial v \rangle_{\hat{\omega}} - 2 \langle \partial \bar{\partial} f, \partial \bar{\partial} v \rangle_{\hat{\omega}} + 2 \langle \partial f, \partial v \rangle_{\hat{\omega}}) \hat{\omega}^n \quad (\text{by (4.9)}) \\
&= \int_X [(\Delta_{\hat{\omega}} + 2)u \langle \partial f, \partial v \rangle_{\hat{\omega}} - 2u \langle \partial \bar{\partial} f, \partial \bar{\partial} u \rangle_{\hat{\omega}}] \hat{\omega}^n \\
&= -2 \int_X u \langle \partial \bar{\partial} f, \partial \bar{\partial} u \rangle_{\hat{\omega}} \hat{\omega}^n. \quad (\text{since } (\Delta_{\hat{\omega}} + 2)u = 0)
\end{aligned}$$

Combining the identity (4.7) with the identity just proved we get

$$\begin{aligned}
& n! 2 \left\langle D_u \tilde{\Psi}(1, 0) f_1, f_2 \right\rangle_{L^2(X, \hat{\omega})} = 2 \int_X f_1 f_2 \hat{\omega}^n \\
&+ \int_X \left(f_1 f_2 - \frac{1}{2} \langle \partial f_1, \partial f_2 \rangle_{\hat{\omega}} - \frac{1}{2} \langle \partial f_2, \partial f_1 \rangle_{\hat{\omega}} \right) (\Delta_{\hat{\omega}} + 2) \left(\frac{d}{dt} \Big|_1 \mu(t, 0) \right) \hat{\omega}^n \\
&= 2 \int_X \left[f_1 f_2 - \left(f_1 f_2 - \frac{1}{2} \langle \partial f_1, \partial f_2 \rangle_{\hat{\omega}} - \frac{1}{2} \langle \partial f_2, \partial f_1 \rangle_{\hat{\omega}} \right) \psi_1 \right] \hat{\omega}^n \quad (\text{by (4.8)}) \\
&= 2 \int_X \left[f_1 f_2 + \frac{1}{4} \Delta_{\hat{\omega}} (f_1 f_2) \psi_1 \right] \hat{\omega}^n = 2 \int_X \left(1 - \frac{1}{4} \Delta_{\hat{\omega}} \varphi \right) f_1 f_2 \hat{\omega}^n.
\end{aligned}$$

□

Step II: All time existence. We continue assuming $\varphi \in \mathcal{P}_{\hat{\omega}} \cap U_{\hat{\omega}}$. Let

$$\tau := \inf \{ t \in (0, 1) \mid \exists \psi \in C^\infty((t, 1], \mathcal{P}_\omega), \psi_s \text{ sol. of (4.1)} \forall s \in (t, 1], \omega_{\psi_1} = \hat{\omega} \}.$$

We prove first that $\tau = 0$ by a contradiction argument. So assume $\tau \in (0, 1)$. By deriving the Monge-Ampère equation (4.1) we get

$$\Delta_t \dot{\psi}_t + 2t \dot{\psi}_t + 2\psi_t = 0. \quad (4.11)$$

By plugging this in the equality (1.4) we get

$$\frac{d}{dt}G_\omega(\psi_t) = -\frac{1}{2}\int_X \Delta_t \dot{\psi}_t \psi_t \omega_t^n = \frac{1}{4}\int_X \Delta_t \dot{\psi}_t (\Delta_t \dot{\psi}_t + 2t\dot{\psi}_t) \omega_t^n.$$

Let $(e_k)_{k=0}^{+\infty} \subset \mathcal{E}(X, \mathbb{R})$ be an $L^2(X, \omega_t^n)$ -orthonormal base of eigenfunctions of Δ_t . i.e $\Delta_t e_k + \lambda_k e_k = 0$, with $e_0 = 1$, $\lambda_0 = 0$ and $0 < \lambda_1 \leq \lambda_2$. The equation (4.2) implies $\text{Ric}(\omega_t) > t\omega_t$ for all $t \in (\tau, 1)$. By the Bochner formula we get $\lambda_1 > 2t > 0$. The equalities in sense L^2

$$\dot{\psi}_t = \sum_{k=0}^{+\infty} c_k e_k, \quad \Delta_t \dot{\psi}_t = -\sum_{k=0}^{+\infty} c_k \lambda_k e_k, \quad \Delta_t \dot{\psi}_t + 2t\dot{\psi}_t = -\sum_{k=0}^{+\infty} c_k (\lambda_k - 2t) e_k$$

imply

$$\int_X \Delta_t \dot{\psi}_t (\Delta_t \dot{\psi}_t + 2t\dot{\psi}_t) \omega_t^n = \sum_{k=1}^{+\infty} c_k^2 (\lambda_k - 2t) \lambda_k \geq 0 \quad (4.12)$$

for all $t \in (\tau, 1)$. We deduce that the function $G(t) := G_\omega(\omega_t) \equiv G_\omega(\psi_t) \geq 0$ is nondecreasing in t . So we get

$$0 \leq G_\omega(\psi_t) \leq G_\omega(\psi_1) = G_\omega(-\varphi) \leq \text{Osc}(\varphi),$$

which implies

$$0 \leq I_\omega(\psi_t) \leq (n+1) \text{Osc}(\varphi). \quad (4.13)$$

Combining lemma 3 with the inequality (4.13) we deduce the estimate $\text{Osc}(\psi_t) \leq t^{-1}C$. The equation (4.1) can be written as $e^{t\psi_t} \omega_t^n = e^{h_\omega} \omega^n$. This implies

$$\int_X e^{t\psi_t} \omega_t^n = \int_X \omega_t^n,$$

and so

$$\|\psi_t\|_{C^0} \leq \text{Osc}(\psi_t) \leq t^{-1}C.$$

Then Yau's C^2 -uniform estimate and Calabi's C^3 -uniform estimate [Yau], [Cal], implies that for any sequence $(t_k)_k \subset (\tau, 1)$, $t_k \rightarrow \tau$ there exist a subsequence $(t_l)_l \subset (t_k)_k$ such that the sequence $(\psi_{t_l})_l$ converge in the $C^{2,\alpha}$ -topology to a potential $\psi_\tau \in \mathcal{P}_\omega$. The fact that ψ_τ is a potential derive from the inequality

$$\omega_{\psi_\tau} = \lim_{l \rightarrow +\infty} \omega_{t_l} > 0,$$

which hold since $\omega_t^n / \omega^n > c > 0$, for some constant $c > 0$ independent of t . Moreover ψ_τ is a solution of the equation (4.1) for $t = \tau$, so by elliptic regularity ψ_τ is smooth. Consider now the map

$$F : [0, 1] \times \mathcal{P}_\omega^{k+2, 1/2} \longrightarrow C^{k, 1/2}(X, \mathbb{R})$$

$$(t, \psi) \longmapsto \log \frac{\omega_\psi^n}{\omega^n} + t\psi - h_\omega.$$

We have $F(t, \psi_t) = 0$ for all $t \in [\tau, 1]$. Consider also the partial Frechet derivative

$$D_\psi F(t, \psi) : C^{k+2, 1/2}(X, \mathbb{R}) \longrightarrow C^{k, 1/2}(X, \mathbb{R})$$

$$f \longmapsto \frac{1}{2} \Delta_\psi f + t f.$$

The fact that $\text{Ric}(\omega_\tau) > \tau \omega_\tau$ implies $\lambda_1(\Delta_\tau) > 2\tau$, which means that the map $D_\psi F(\tau, \psi_\tau)$ is an isomorphism. So by the implicit function theorem there exist an open ball $B_r(\psi_\tau) \subset \mathcal{P}_\omega^{k+2, 1/2}$ and $\delta \in (\tau, 1)$ such that for all $t \in [\tau, \delta]$ there exist a unique potential $\hat{\psi}_t \in B_r(\psi_\tau)$ such that $F(t, \hat{\psi}_t) = 0$ and the corresponding function

$$\hat{\psi} : [\tau, \delta] \rightarrow \mathcal{P}_\omega^{k+2, 1/2},$$

is smooth. We remark now that the limit ψ_τ is independent of the sequence. In fact if $(s_k) \subset (\tau, 1)$ is an other sequence such that $s_k \rightarrow \tau$ and ψ_{s_k} converge in the $C^{2, \alpha}$ -topology to a potential $\psi'_\tau \in \mathcal{P}_\omega$ we can choose the radius $r > 0$ such that $B_r(\psi_\tau) \cap B_r(\psi'_\tau) = \emptyset$ and we can pick $s_k, t_l \in (\tau, \delta)$, $s_k < t_l$ such that $\psi_{s_k} \in B_r(\psi'_\tau)$ and $\psi_{t_l} = \hat{\psi}_{t_l} \in B_r(\psi_\tau)$. But by continuity of the map ψ there exist

$$t \in \psi^{-1}(B_r(\psi_\tau)) \subset (s_k + \varepsilon, t_l],$$

such that $\psi_t \neq \hat{\psi}_t$, which contradicts the uniqueness in the statement of the implicit function theorem. We deduce the existence of the limit $\psi_\tau = \lim_{t \rightarrow \tau} \psi_t$ in the $C^{2, \alpha}$ -topology. So by the implicit function theorem we can find a smooth extension $\psi : (\tau - \varepsilon, 1] \rightarrow \mathcal{P}_\omega$, $t \mapsto \psi_t$, $\omega_{\psi_1} = \hat{\omega}$ solution of the equation (4.1). But this contradicts the definition of τ , so $\tau = 0$.

Step III: Regularity at $t = 0$. The inequality $\|t\psi_t\|_{C^0} \leq C$ for all $t \in (0, 1]$ implies that the factor $e^{\hbar_\omega - t\psi_t}$ on the right hand side of the Monge-Ampère equation (4.1) is bounded. Then Yau's C^0 -estimate for the solution of Calabi's conjecture [Yau], implies $\text{Osc}(\psi_t) \leq C$ for all $t \in (0, 1]$ and so we deduce

$$\|\psi_t\|_{C^0} \leq \text{Osc}(\psi_t) \leq C.$$

Then Yau's C^2 -uniform estimate and Calabi's C^3 -uniform estimate implies that for any sequence $(t_k)_k \subset (0, 1)$, $t_k \rightarrow 0$ there exist a subsequence $(t_l)_l \subset (t_k)_k$ such that the sequence ω_{t_l} is convergent in the C^α -topology. So set

$$\omega_0 := \lim_{l \rightarrow +\infty} \omega_{t_l} \in 2\pi c_1.$$

We have $\omega_0 > 0$ since $\omega_t^n / \omega^n > c > 0$, for some constant $c > 0$ independent of t and ω_0 is the unique solution of Calabi's equation $\text{Ric}(\omega_0) = \omega$. By elliptic regularity ω_0 is a smooth Kähler metric. The uniqueness implies the existence of the limit $\omega_0 := \lim_{t \rightarrow 0} \omega_t$ in the C^α -topology.

So we have obtained a Hölder continuous map $[0, 1] \rightarrow \mathcal{K}$, $t \mapsto \omega_t$, which is smooth over the interval $(0, 1]$ and solution of the equation (4.2) with $\omega_1 = \hat{\omega}$. We remark now that this map is smooth over all $[0, 1]$. Set $\tilde{\omega} := \omega_0 = \omega_{\psi_0}$.

With the notations of section 1 consider the map

$$\begin{aligned} \Gamma : [0, 1] \times \mathcal{P}_{\tilde{\omega}, 0}^{k+2, 1/2} &\longrightarrow C_{\tilde{\omega}}^{k, 1/2}(X, \mathbb{R})_1 \\ (t, \rho) &\longmapsto \frac{\tilde{\omega}_\rho^n}{\tilde{\omega}^n} e^{t(\rho + \psi_0)} \left(\int_X e^{t(\rho + \psi_0)} \tilde{\omega}_\rho^n \right)^{-1}. \end{aligned}$$

Then the equation (4.2) is equivalent to the equation $\Gamma(t, \rho_t) = 1$, $t \in [0, 1]$. In fact $\omega_t = \tilde{\omega}_{\rho_t}$. The partial Frechet derivative of the map Γ at the point $(t, \rho) = (0, 0)$ is

$$D_\rho \Gamma(0, 0) : C_{\tilde{\omega}, 0}^{k+2, 1/2}(X, \mathbb{R}) \rightarrow C_{\tilde{\omega}, 0}^{k, 1/2}(X, \mathbb{R})$$

$$f \longmapsto \frac{1}{2} \Delta_{\tilde{\omega}} f,$$

which is an isomorphism. Then the map $t \mapsto \omega_t$ is also smooth in a neighborhood of 0 by the implicit function theorem. Set now

$$\tilde{\mathcal{P}}_{\omega, 0} := \left\{ \varphi \in \mathcal{P}_\omega \mid \int_X \varphi e^{h_\omega} \omega^n = 0 \right\},$$

and consider the potentials $\varphi_t \in \tilde{\mathcal{P}}_{\omega, 0}$, $t \in [0, 1]$ defined by

$$\varphi_t := \tilde{\varphi}_t - \int_X \tilde{\varphi}_t e^{h_\omega} \omega^n, \quad \text{with} \quad \tilde{\varphi}_t(x) := - \int_X (\text{Tr}_\omega \omega_t) \mathcal{G}_\omega(x, \cdot) \omega^n,$$

where $\mathcal{G}_\omega$ is the Green function. Thus we have a smooth map $[0, 1] \rightarrow \tilde{\mathcal{P}}_{\omega, 0}$, $t \mapsto \varphi_t$, $\omega_{\varphi_t} = \omega_t$, solution of the equation

$$(\omega + i\partial\bar{\partial}\varphi_t)^n = e^{h_\omega - t\varphi_t + c_t} \omega^n, \quad t \in [0, 1],$$

with

$$c_t := - \log \int_X e^{h_\omega - t\varphi_t} \omega^n.$$

For all $t \in (0, 1]$ we can pass from the solutions of this equation to the solutions of (4.1) by the formulas $\psi_t = \varphi_t - c_t/t$ and

$$\varphi_t = \psi_t - \int_X \psi_t e^{h_\omega} \omega^n.$$

Consider now the limit

$$\begin{aligned} \lim_{t \rightarrow 0} \left[\int_X e^{h_\omega - t\varphi_t} \omega^n \right]^{\frac{1}{t}} &= \exp \left(\int_X \frac{d}{dt} \Big|_0 (e^{-t\varphi_t}) e^{h_\omega} \omega^n \right) \\ &= \exp \left(- \int_X \varphi_0 e^{h_\omega} \omega^n \right) = 1. \end{aligned}$$

Then the formula $\psi_t = \varphi_t - c_t/t$ implies $\lim_{t \rightarrow 0} \psi_t = \varphi_0$. So we set $\psi_0 = \varphi_0$ in (4.1). $\square$

Step IV: The case of potentials φ with degenerate form $\mathcal{Q}_\varphi^{\hat{\omega}}$. We start by observing that $\overline{U}_{\hat{\omega}} = \Lambda_{\hat{\omega}}^\perp$ in the smooth topology. In fact consider the linear map $\mathcal{A}^{\hat{\omega}} : \Lambda_{\hat{\omega}}^\perp \rightarrow S_{\mathbb{R}}^2(\Lambda_{\hat{\omega}}^*)$, $\varphi \mapsto \mathcal{A}_\varphi^{\hat{\omega}}$ given by

$$\mathcal{A}_\varphi^{\hat{\omega}}(u, v) := -\frac{1}{4} \int_X \Delta_{\hat{\omega}} \varphi uv \hat{\omega}^n.$$

Let $A_\varphi^{\hat{\omega}}$ be the endomorphism on $\Lambda_{\hat{\omega}}$ induced by the $L^2(X, \hat{\omega})$ product. For all $v \in \Lambda_{\hat{\omega}}^\perp$ the map

$$\mathbb{R}v \ni \varphi \mapsto \det(I + A_\varphi^{\hat{\omega}})$$

is polynomial, which implies $\bar{U}_{\hat{\omega}} = \Lambda_{\hat{\omega}}^{\perp}$. So for a potential $\varphi \in \mathcal{P}_{\hat{\omega}} \cap \partial U_{\hat{\omega}}$ consider a sequence $(\varphi_k) \subset \mathcal{P}_{\hat{\omega}} \cap U_{\hat{\omega}}$ such that $\varphi_k \rightarrow \varphi$ in the smooth topology. Set also $\omega_k := \hat{\omega}_{\varphi_k}$, $\omega := \hat{\omega}_{\varphi}$. Consider also for all k the solutions $\psi_{k,t} \in \mathcal{P}_{\omega_k}$ of the complex Monge-Ampère equations

$$(\omega_k + i\partial\bar{\partial}\psi_{k,t})^n = e^{h_{\omega_k} - t\psi_{k,t}} \omega_k^n, \quad t \in [0, 1], \quad (4.14)$$

which we rewrite in the equivalent ways

$$\text{Ric}(\omega_{k,t}) = t\omega_{k,t} + (1-t)\omega_k, \quad t \in [0, 1], \quad (4.15)$$

$$(1-t)\psi_{k,1} = \log \frac{\hat{\omega}_{\theta_{k,t}}^n}{\hat{\omega}^n} + t\theta_{k,t}, \quad t \in [0, 1], \quad (4.16)$$

where

$$\omega_{k,t} := \omega_k + i\partial\bar{\partial}\psi_{k,t} = \hat{\omega}_{\theta_{k,t}}^n,$$

with $\theta_{k,t} := \psi_{k,t} - \psi_{k,1} \in \mathcal{P}_{\hat{\omega}}$ as in the beginning of the section. Set also $\psi_1 := \lim_{k \rightarrow +\infty} \psi_{k,1}$. The monotonicity of the functional G_{ω_k} along the solutions of (4.14) implies

$$G_{\omega_k}(\psi_{k,t}) \leq G_{\omega_k}(\psi_{k,1}) = G_{\omega_k}(-\varphi_k) \leq G_{\omega_k}(-\varphi) + 1,$$

for all $t \in [0, 1]$ and $k \geq c$. Moreover $G_{\omega_k}(-\varphi) \leq C$ for some uniform constant $C > 0$. We infer by lemma 3 the estimate

$$\|t\psi_{k,t}\|_{C^0} \leq \text{Osc}(t\psi_{k,t}) \leq C.$$

As before, by applying Yau's C^0 -estimate for the solution of the Calabi's conjecture to the equation (4.14), we deduce the uniform estimate

$$\|\psi_{k,t}\|_{C^0} \leq \text{Osc}(\psi_{k,t}) \leq C.$$

Here we use also the fact that all the metrics ω_k are uniformly equivalents. In particular $\|\theta_{k,t}\|_{C^0} \leq C$ for all $t \in [0, 1]$ and $k \geq c$. Then by applying Yau's C^2 and Calabi's C^3 -uniform estimates to the equation (4.16) we deduce the uniform estimate $\|\theta_{k,t}\|_{C_{\hat{\omega}}^{2,\alpha}} \leq C$. For all $t \in [0, 1]$ consider the set

$$\Theta_t := \left\{ \theta_t \mid \exists (\theta_{k_l,t})_l : \theta_t \stackrel{C^{2,\alpha}}{=} \lim_{l \rightarrow +\infty} \theta_{k_l,t} \right\}.$$

By the properties of the Monge-Ampère equation (4.16) we deduce $\Theta_t \subset \mathcal{P}_{\hat{\omega}}$ and every $\theta_t \in \Theta_t$ is solution of the equation (4.3) at time t . Consider also the corresponding set of metrics

$$\Omega_t := \left\{ \omega_t \in 2\pi c_1 \mid \exists (\omega_{k_l,t})_l : \omega_t \stackrel{C^\alpha}{=} \lim_{l \rightarrow +\infty} \omega_{k_l,t} \right\}.$$

Every $\omega_t \in \Omega_t$ is a solution of the equation (4.2). In fact for $t = 0$ at the limit the equations (4.16) becomes (4.3). Then the uniqueness of the solutions of Calabi's equation $\text{Ric}(\omega_0) = \omega$ implies $\omega_0 = \lim_{k \rightarrow +\infty} \omega_{k,0}$ in the

C^α -topology. For $t \neq 0$ we remark, by means of (4.16), (4.3) and the uniform estimate $\|\theta_{k,t}\|_{C_{\hat{\omega}}^{2,\alpha}} \leq C$, that

$$\omega_t = \hat{\omega}_{\theta_t}^n \stackrel{C^\alpha}{=} \lim_{l \rightarrow +\infty} \omega_{k_l,t}, \quad \text{if and only if} \quad \theta_t \stackrel{C^{2,\alpha}}{=} \lim_{l \rightarrow +\infty} \theta_{k_l,t}.$$

Thus the equation (4.3) implies that ω_t is a solution of the equation (4.2). The fact that the uniform estimate

$$\|\theta_t\|_{C_{\hat{\omega}}^{2,\alpha}} \leq C,$$

hold for all $\theta_t \in \Theta_t$, $t \in [0, 1]$ combined with the uniqueness of the solutions of Calabi's equation $\text{Ric}(\omega_0) = \omega$ implies that for every sequence $(\omega_{t_l})_l$, $\omega_{t_l} \in \Omega_{t_l}$ with $t_l \rightarrow 0$ we have $\omega_0 = \lim_{l \rightarrow +\infty} \omega_{t_l}$. Then using the implicit function theorem in the ways explained above we obtain the required existence of the solutions of the equation (4.2) with data $\varphi \in \mathcal{P}_{\hat{\omega}} \cap \partial U_{\hat{\omega}}$.

In conclusion we have prove the existence of the solutions for arbitrary data $\varphi \in \mathcal{P}_{\hat{\omega}} \cap \Lambda_{\hat{\omega}}^\perp$. The uniqueness of the solutions follows immediately by combining the uniqueness of the solution of the Calabi equation with the implicit function theorem. This concludes the proof of the proposition 4.1. $\square$

Proof of theorem 1.

Proof of (A). Combining the projection lemma 5 with the uniqueness of the solutions of Aubin's equation 4.2, given by proposition 4.1 we infer that for an arbitrary $\omega \in \mathcal{K}$ there exist a unique Kähler-Einstein metric $\hat{\omega} \in \mathcal{K}$ such that $\omega \in \mathcal{K}_{\hat{\omega}}^\perp$. Then the theorem 1 (A) follows immediately from the proposition 4.1.

Proof of (B). Let $O \subset \mathcal{KE}$ be an arbitrary orbit of a Kähler-Einstein metric. Let $\omega \in \mathcal{K}_{\hat{\omega}}^\perp$. The projection lemma 5 implies the existence of $\theta \in O$ such that $\omega \in \mathcal{K}_\theta^\perp$, which implies $\hat{\omega} = \theta \in O$ by the proof of the statement (A). Thus $O = \text{Aut}_J^0(X) \cdot \hat{\omega}$ and so $\mathcal{KE} = \text{Aut}_J^0(X) \cdot \hat{\omega}$. $\square$

5 A Tian C^0 -estimate for the solutions of the Aubin equation.

Set $\alpha := 1 - 1/4n > 1/2$ and consider the function f_ω over the interval $[0, 1]$ defined by the formula

$$f_\omega(t) := (1-t)^{1-\alpha} (1 + 2(1-t)\|\psi_t\|_{C^0})^\alpha.$$

The following proposition is a very slight modification of a computation in [Ti2].

Proposition 5.1 *There exist a constant $D \in (0, 1)$ depending only on $\hat{\omega}$ such that the estimate*

$$\text{Osc}(\psi_t - \psi_1) \leq 2\|\psi_t - \psi_1\|_{C^0} \leq 18(1-t)\|\psi_t\|_{C^0} + 1 \quad (5.1)$$

hold for all $t \in [t_0, 1]$, where $t_0 \in [0, 1)$ satisfies $f_\omega(t_0) = \max_{[t_0, 1]} f_\omega = D$.

Proof. The first inequality is obvious. Now consider the identity

$$h_{\omega_t} = -\log \frac{\hat{\omega}_{\theta_t}^n}{\hat{\omega}^n} - \theta_t - c_t.$$

Then the equation (4.4) implies $h_{\omega_t} = (t-1)\psi_t - c_t$, and so the integral normalization of h_{ω_t} implies the equality $c_t = \log \int_X e^{(t-1)\psi_t} \omega_t^n$, which gives $|c_t| \leq (1-t)\|\psi_t\|_{C^0}$. In conclusion we get the inequality

$$\|h_{\omega_t}\|_{C^0} \leq 2(1-t)\|\psi_t\|_{C^0}. \quad (5.2)$$

Consider now the Kähler-Ricci flow equation

$$\frac{d}{ds} U_s = \log \frac{\Omega_{U_s}^n}{\Omega^n} + U_s - h_{\Omega},$$

with initial data $U_0 = 0$ and $\Omega = \omega_t \in \mathcal{K}$. Set $u_t := U_1$ and

$$\tilde{\omega}_t := \omega_t + i\partial\bar{\partial}u_t = \hat{\omega} + i\partial\bar{\partial}(\theta_t + u_t) \in \mathcal{K}.$$

The fact that the metric $\omega_1 = \hat{\omega}$ is Kähler-Einstein implies $u_1 = 0$. The identity

$$h_{\tilde{\omega}_t} = -\log \frac{[\hat{\omega} + i\partial\bar{\partial}(\theta_t + u_t)]^n}{\hat{\omega}^n} - \theta_t - u_t - k_t,$$

combined with the equation (4.5) implies the equalities

$$1 = \int_X e^{h_{\tilde{\omega}_t}} \tilde{\omega}_t^n = \int_X e^{-\theta_t - u_t - k_t} \hat{\omega}^n = \int_X e^{-k_t - u_t + (t-1)\psi_t} \omega_t^n,$$

which gives the estimate

$$|k_t| \leq \|u_t\|_{C^0} + (1-t)\|\psi_t\|_{C^0}. \quad (5.3)$$

Set $\eta_t := \theta_t + u_t + k_t \in \mathcal{P}_{\hat{\omega}}$. We have $\hat{\omega}_{\eta_t} = \tilde{\omega}_t$. The inequality (5.3) implies $\eta_1 = 0$. We have the obvious identity

$$-h_{\tilde{\omega}_t} = \log \frac{\hat{\omega}_{\eta_t}^n}{\hat{\omega}^n} + \eta_t. \quad (5.4)$$

We will prove in the sequel the following lemma.

Lemma 7 *The inequality $\|u_t\|_{C^0} \leq 2\|h_{\omega_t}\|_{C^0}$ hold for all $t \in [0, 1]$. Moreover let $\tau \in [0, 1)$ such that $\hat{\omega}/2 \leq \tilde{\omega}_\tau \leq 2\hat{\omega}$. (such τ always exist since $\tilde{\omega}_1 = \hat{\omega}$.) Then there exist a constant $B = B(\hat{\omega}) > 1$ depending only on $\hat{\omega}$ such that*

$$\|h_{\tilde{\omega}_\tau}\|_{C_{\hat{\omega}}^{0,1/2}} \leq B(1-\tau)^{1-\alpha} (1 + \|h_{\omega_\tau}\|_{C^0})^\alpha. \quad (5.5)$$

Then using the inequality (5.2) we get $\|u_t\|_{C^0} \leq 4(1-t)\|\psi_t\|_{C^0}$, and so the inequality (5.3) implies

$$|k_t| \leq 5(1-t)\|\psi_t\|_{C^0}.$$

In conclusion we get the estimate

$$\text{Osc}(\theta_t)/2 \leq \|\theta_t\|_{C^0} \leq \|u_t\|_{C^0} + |k_t| + \|\eta_t\|_{C^0} \leq 9(1-t)\|\psi_t\|_{C^0} + \|\eta_t\|_{C^0}.$$

In order to prove the estimate $\|\eta_t\|_{C^0} < 1/2$ we consider the map

$$\begin{aligned}\Phi : \mathcal{P}_{\hat{\omega}}^{2,1/2} &\longrightarrow C^{0,1/2}(X, \mathbb{R}) \\ \eta &\longmapsto \log \frac{\hat{\omega}_\eta^n}{\hat{\omega}^n} + \eta.\end{aligned}$$

Its Frechet derivative at $\eta = 0$ is the map

$$\begin{aligned}D\Phi(0) : C^{2,1/2}(X, \mathbb{R}) &\longrightarrow C^{0,1/2}(X, \mathbb{R}) \\ f &\longmapsto \frac{1}{2}\Delta_{\hat{\omega}}f + f,\end{aligned}$$

which is an isomorphism of Banach spaces. By the Inverse function theorem we deduce the existence of $\varepsilon = \varepsilon(\hat{\omega}) \in (0, 2)$ and $\delta = \delta(\hat{\omega}) \in (0, 1/2)$ such that for all $h \in C^{0,1/2}(X, \mathbb{R})$ with $\|h\|_{C_{\hat{\omega}}^{0,1/2}} < \varepsilon$ there exist a unique potential $\eta \in \mathcal{P}_{\hat{\omega}}^{2,1/2} \cap \overline{B}_\delta(0)$ such that $\Phi(\eta) = h$. Moreover $\|\eta\|_{C_{\hat{\omega}}^{2,1/2}} < \delta$.

Claim 1 *Set $D := \varepsilon/(2B)$ and let $t_0 \in [0, 1)$ the time defined in the statement of the proposition 5.1. Then $\|\eta_t\|_{C_{\hat{\omega}}^{2,1/2}} < \delta$ for all $t \in [t_0, 1]$.*

Proof. We assume the contrary. Since $\eta_1 = 0$ there exist a $\tau \in [t_0, 1)$ such that

$$\|\eta_\tau\|_{C_{\hat{\omega}}^{2,1/2}} = \delta < 1/2 \quad \text{and} \quad \|\eta_t\|_{C_{\hat{\omega}}^{2,1/2}} < \delta \quad \text{for all } t \in (\tau, 1].$$

In particular $-\hat{\omega}/2 \leq i\partial\bar{\partial}\eta_\tau \leq \hat{\omega}/2$, i.e $\hat{\omega}/2 \leq \hat{\omega}_{\eta_\tau} = \tilde{\omega}_\tau \leq 2\hat{\omega}$. Then by applying the estimate (5.5) of lemma 7 and the estimate (5.2) we find

$$\|h_{\tilde{\omega}_\tau}\|_{C_{\hat{\omega}}^{0,1/2}} \leq B(1-\tau)^{1-\alpha} (1 + 2(1-\tau)\|\psi_\tau\|_{C^0})^\alpha.$$

So by definition of t_0 we get $\|h_{\tilde{\omega}_\tau}\|_{C_{\hat{\omega}}^{0,1/2}} < \varepsilon$. By applying the estimates of the Inverse function theorem to the identity (5.4) we deduce the contradiction $\|\eta_\tau\|_{C_{\hat{\omega}}^{2,1/2}} < \delta$. $\square$

This concludes the proof of the proposition 5.1 modulo the proof of lemma 7 which we give in the next subsection. $\square$

5.1 Smoothing with the Kähler-Ricci Flow

Consider the Kähler-Ricci flow equation

$$\frac{d}{ds}U_s = \log \frac{\Omega_{U_s}^n}{\Omega^n} + U_s - h_\Omega,$$

with initial data $U_0 = 0$. Set $\Omega_s := \Omega + i\partial\bar{\partial}U_s$, $v_s = \dot{U}_s$ and $h_s = h_{\Omega_s}$. Then $h_s = -v_s + c_s$ for some constant c_s with $c_0 = 0$. In the following context the objects Δ_s , ∇_s , $|\cdot|_s$ are defined respect to the evolving metric Ω_s .

Lemma 8 *The inequalities*

$$|v_s| \leq e^s \|h_0\|_{C^0}, \quad (5.6)$$

$$|v_s|^2 + s|\nabla_s h_s|_s^2 \leq e^{2s} \|h_0\|_{C^0}^2, \quad (5.7)$$

$$e^{-s} \Delta_s h_s \geq \min_X \Delta_0 h_0, \quad (5.8)$$

holds for all $s \geq 0$.

Proof. Differentiating the Kähler-Ricci flow equation we get the evolving equation $2\dot{v}_s = \Delta_s v_s + 2v_s$. By applying the maximum principle we get (5.6). Using the identity $\Delta v^2 = 2|\nabla v|^2 + 2v\Delta v$ we get

$$2\frac{\partial}{\partial s} v_s^2 = \Delta_s v_s^2 - 2|\nabla_s v_s|_s^2 + 4v_s^2. \quad (5.9)$$

Moreover we have the evolution formulas

$$\begin{aligned} 2\frac{\partial}{\partial s} |\nabla_s v_s|_s^2 &= \Delta_s |\nabla_s v_s|_s^2 - 4|\nabla_s^{1,0} \partial v_s|_s^2 - 4|\partial \bar{\partial} v_s|_s^2 + 2|\nabla_s v_s|_s^2 \\ &\leq \Delta_s |\nabla_s v_s|_s^2 + 4|\nabla_s v_s|_s^2, \end{aligned} \quad (5.10)$$

$$2\frac{\partial}{\partial s} \Delta_s v_s = \Delta_s^2 v_s + 2\Delta_s v_s - 4|\partial \bar{\partial} v_s|_s^2 \leq \Delta_s^2 v_s + 2\Delta_s v_s. \quad (5.11)$$

By applying the maximum principle to (5.11) we get $\Delta_s v_s \leq e^s \max_X \Delta_0 v_0$ which implies (5.8). Moreover combining (5.9) with (5.10) we get

$$2\frac{\partial}{\partial s} (v_s^2 + s|\nabla_s v_s|_s^2) \leq \Delta_s (v_s^2 + s|\nabla_s v_s|_s^2) + 4(v_s^2 + s|\nabla_s v_s|_s^2),$$

which implies (5.7) by the maximum principle. $\square$

Lemma 9 *Assume that the initial metric $\Omega_0 \in 2\pi c_1$ satisfies $\text{Ric}(\Omega_0) > \tau \Omega_0$ for some $\tau \in [0, 1)$ and $\hat{\omega}/2 \leq \Omega_1 \leq 2\hat{\omega}$. Set $p = 2n$ and $H := h_1 - \int_X h_1 \Omega_1^n$. then there exist a constant $C > 0$ depending only on $\hat{\omega}$ such that*

$$\|H\|_{C^0} \leq C(1 - \tau)^{1/p} \|h_0\|_{C^0}^{1-1/p}. \quad (5.12)$$

Remember that in the setting of lemma 7 we have $\Omega_0 = \omega_\tau$, $\Omega_1 = \tilde{\omega}_\tau$, $h_0 = h_{\omega_\tau}$ and $h_1 = h_{\tilde{\omega}_\tau}$.

Proof. The inequalities (5.6) and (5.7) of lemma 8 imply $\|H\|_{C^0} \leq 2e\|h_0\|_{C^0}$ and $|\nabla_1 H|_1 \leq e\|h_0\|_{C^0}$. Then using the Sobolev inequality (see [Aub1] lemma 2.22 pag 45) and the Poincarre inequality we deduce

$$\begin{aligned} \|H\|_{C^0}^{p+1} &\leq C_S \left(\int_X |H|^{p+1} \Omega_1^n + \int_X |\nabla_1 H|_1^{p+1} \Omega_1^n \right) \\ &\leq C_S \left(\|H\|_{C^0}^{p-1} \int_X |H|^2 \Omega_1^n + (e\|h_0\|_{C^0})^{p-1} \int_X |\nabla_1 H|_1^2 \Omega_1^n \right) \\ &\leq C_1 \|h_0\|_{C^0}^{p-1} \int_X |\nabla_1 H|_1^2 \Omega_1^n. \end{aligned} \quad (5.13)$$

The Sobolev and Poincarre constants depends only on $\hat{\omega}$ since the metric Ω_1 is uniformly equivalent to $\hat{\omega}$. The assumption $\text{Ric}(\Omega_0) > \tau\Omega_0$ implies $\Delta_0 h_0 > 2n(\tau-1)$. Then the inequality (5.8) implies $-\Delta_1 H < 2ne(1-\tau)$. So we deduce

$$\begin{aligned} \int_X |\nabla_1 H|_1^2 \Omega_1^n &= - \int_X H \Delta_1 H \Omega_1^n = \int_X \left(H - \min_X H \right) (-\Delta_1 H) \Omega_1^n \\ &\leq 2ne(1-\tau) \int_X \left(H - \min_X H \right) \Omega_1^n \leq C(1-\tau) \|H\|_{C^0}. \end{aligned}$$

Combining this with the inequality (5.13) we deduce $\|H\|_{C^0}^p \leq C(1-\tau) \|h_0\|_{C^0}^{p-1}$, which is the inequality (5.12). $\square$

Proof of the lemma 7. The proof of this lemma is a very slight modification of a Tian's computation [Ti2] pointed out recently in [P-S-S-W]. By integrating from 0 to 1 the inequality $-e^s \|h_0\|_{C^0} \leq \dot{U}_s \leq e^s \|h_0\|_{C^0}$, (which is (5.6)) we get $|U_1| \leq 2\|h_0\|_{C^0}$, which is the inequality $|u_t| \leq 2\|h_{\omega_t}\|_{C^0}$ of lemma 7. Let $x, y \in X$ and let $d_1(x, y)$ be the distance between x and y with respect to the metric Ω_1 . Set $\delta := 1/2n$. If $d_1(x, y) \geq (1-\tau)^\delta (1 + \|h_0\|_{C^0})^{-\delta}$, then the inequality (5.12) implies

$$\begin{aligned} \frac{|h_1(x) - h_1(y)|}{d_1(x, y)^{1/2}} &\leq 2\|H\|_{C^0} (1-\tau)^{-\delta/2} (1 + \|h_0\|_{C^0})^{\delta/2} \\ &\leq C(1-\tau)^{1/p-\delta/2} (1 + \|h_0\|_{C^0})^{1-1/p+\delta/2} \\ &= C(1-\tau)^{1-\alpha} (1 + \|h_0\|_{C^0})^\alpha. \end{aligned}$$

If $d_1(x, y) < (1-\tau)^\delta (1 + \|h_0\|_{C^0})^{-\delta}$ instead, then we use the estimate (5.7) of lemma 8, $|\nabla_1 H|_1 \leq e\|h_0\|_{C^0}$ to get

$$\begin{aligned} \frac{|h_1(x) - h_1(y)|}{d_1(x, y)^{1/2}} &\leq d_1(x, y)^{1/2} \max_X |\nabla_1 H|_1 \\ &\leq C(1-\tau)^{\delta/2} (1 + \|h_0\|_{C^0})^{-\delta/2+1} \\ &= C(1-\tau)^{1-\alpha} (1 + \|h_0\|_{C^0})^\alpha. \end{aligned}$$

So in both cases we get the same estimate. Now the integral normalization $\int_X e^{h_1} \Omega_1^n = \int_X \Omega_1^n$ implies that h_1 change signs, and thus $\|h_1\|_{C^0} \leq \text{Osc}(h_1) = \text{Osc}(H) \leq 2\|H\|_{C^0}$. Then using (5.12) we get

$$\|h_1\|_{C^0} \leq C(1-\tau)^{1/p} \|h_0\|_{C^0}^{1-1/p} \leq C(1-\tau)^{1-\alpha} (1 + \|h_0\|_{C^0})^\alpha,$$

which concludes the proof of the lemma 7. $\square$

6 Proof of the properness of the K-energy

We will prove now the inequality $\nu_{\hat{\omega}}(\varphi) \geq AJ_{\hat{\omega}}(\varphi) - B$ for all potentials $\varphi \in \mathcal{P}_{\hat{\omega}} \cap \Lambda_{\hat{\omega}}^\perp$. We start by observing that the equation (4.2) can also be written in the form

$$\text{Ric}(\omega_t) = \omega_t - (1-t)i\partial\bar{\partial}\psi_t.$$

By taking the trace we get $\text{Sc}(\omega_t) = 2n - (1-t)\Delta_t\psi_t$. Plugging this in (1.5) and using (1.4) we get the well known formula

$$\frac{d}{dt}\nu_\omega(\psi_t) = -(1-t)\frac{d}{dt}G_\omega(\psi_t). \quad (6.1)$$

Set $\nu(t) := \nu_\omega(\omega_t) \equiv \nu_\omega(\psi_t)$. By integrating the equality (6.1) from t to 1 we get the identity

$$\nu(1) - \nu(t) = (1-t)G(t) - \int_t^1 G. \quad (6.2)$$

For $t = 0$ we get

$$\nu(1) = \nu(0) + G(0) - \int_0^1 G.$$

The equation (4.1) for $t = 0$ implies $\nu(0) + G(0) = f_X h_\omega \omega^n$. Then the integral normalization $f_X e^{h_\omega} \omega^n = 1$, implies $\nu(0) + G(0) \leq 0$. So we get

$$\nu(1) \leq - \int_0^1 G = - \int_0^t G - \int_t^1 G \leq - \int_t^1 G.$$

Then the monotonicity of the function G implies $\nu(1) \leq (t-1)G(t)$ for all $t \in [0, 1]$. Using the fact that $\nu(1) = \nu_\omega(\hat{\omega}) = -\nu_{\hat{\omega}}(\omega) = -\nu_{\hat{\omega}}(\varphi)$ we get the inequality $\nu_{\hat{\omega}}(\varphi) \geq (1-t)G(t)$ for all $t \in [0, 1]$. We remind the following well known fact. (See [Ti2] for example.)

Claim 2 *The inequality $|G_\omega(\varphi) - G_\omega(\psi)| \leq n \text{Osc}(\varphi - \psi)$, holds for any Kähler metric ω and any potentials $\varphi, \psi \in \mathcal{P}_\omega$.*

So using this claim and the inequality (1.2) we get

$$\nu_{\hat{\omega}}(\varphi) \geq (1-t)[G(1) - n \text{Osc}(\psi_t - \psi_1)] \geq (1-t) \left[\frac{I_\omega(\psi_1)}{n+1} - n \text{Osc}(\psi_t - \psi_1) \right].$$

It is easy to see that $I_\omega(\psi_1) = I_{\hat{\omega}}(\varphi)$. In conclusion we get the estimate

$$\nu_{\hat{\omega}}(\varphi) \geq (1-t) \left[\frac{J_{\hat{\omega}}(\varphi)}{n+1} - n \text{Osc}(\psi_t - \psi_1) \right], \quad (6.3)$$

for all $t \in [0, 1]$. We prove now an other useful estimate. Using the cocycle identity for the K-energy and the identity (6.2) we have

$$\nu_{\hat{\omega}}(\psi_t - \psi_1) = \nu(t) - \nu(1) = \int_t^1 G - (1-t)G(t) \leq (1-t)[G(1) - G(t)].$$

In the last inequality we have use the monotonicity of the function G . Combining this with the claim 2 we get the inequality

$$\nu_{\hat{\omega}}(\psi_t - \psi_1) \leq n(1-t) \text{Osc}(\psi_t - \psi_1). \quad (6.4)$$

The fact that $\omega_t = \hat{\omega} + i\partial\bar{\partial}(\psi_t - \psi_1)$ allows us to apply the lemma 3 with metric $\hat{\omega}$ and potential $\psi_t - \psi_1$. So we get for all $t \in [1/2, 1]$ the estimate

$$\text{Osc}(\psi_t - \psi_1) \leq (n+1) J_{\hat{\omega}}(\psi_t - \psi_1) + C, \quad (6.5)$$

where $C > 0$ is a constant depending only on $\hat{\omega}$. Then the inequalities (6.3) and (6.5) will allows to conclude provided we prove the following proposition.

Proposition 6.1 *There exists constants $C, c > 0$ (depending only on $\hat{\omega}$) and $t' \geq 1/2$, $1 - t' \geq c$ such that $J_{\hat{\omega}}(\psi_{t'} - \psi_1) \leq C$.*

Before proving this proposition we prove the following weaker version of theorem 2.

Proposition 6.2 *There exists constants $C_1, C_2 > 0$ depending only on $\hat{\omega}$ such that the estimate*

$$\nu_{\hat{\omega}}(\varphi) \geq \frac{C_1 J_{\hat{\omega}}(\varphi)}{(2 \text{Osc}(\varphi) + 1)^\alpha} - C_2 \quad (6.6)$$

hold for all potentials $\varphi \in \mathcal{P}_{\hat{\omega}}$.

Proof. By definition of t_0 in the proposition 5.1 we get $2^\alpha(1 - t_0)\|\psi_{t_0}\|_{C^0}^\alpha \leq D$. The fact that $1/\alpha < 2$ implies $(1 - t_0)^2\|\psi_{t_0}\|_{C^0} \leq D^{1/\alpha}$. Hence from the inequality (5.1) we get

$$(1 - t_0) \text{Osc}(\psi_{t_0} - \psi_1) \leq 18(1 - t_0)^2\|\psi_{t_0}\|_{C^0} + 1 \leq C'_2,$$

which combined with the inequality (6.3) gives

$$\nu_{\hat{\omega}}(\varphi) \geq \frac{1 - t_0}{n + 1} J_{\hat{\omega}}(\varphi) - C_2. \quad (6.7)$$

Set

$$\delta := \min\{1/18, 2^{\frac{-\alpha}{1-\alpha}} D^{\frac{1}{1-\alpha}}\} > 0.$$

We can assume $1 - t_0 \leq \delta$ otherwise we are done. This implies the inequality $(1 - t_0)\|\psi_{t_0}\|_{C^0} \geq 1/2$. In fact if not the definition of t_0 implies

$$2^{\frac{-\alpha}{1-\alpha}} D^{\frac{1}{1-\alpha}} < 1 - t_0.$$

Then the definition of t_0 implies the inequality $D \leq 2^{2\alpha}(1 - t_0)\|\psi_{t_0}\|_{C^0}^\alpha$, giving $1 - t_0 \geq C/\|\psi_{t_0}\|_{C^0}^\alpha$. So we deduce by (6.7) the estimate

$$\nu_{\hat{\omega}}(\varphi) \geq C_1 \frac{J_{\hat{\omega}}(\varphi)}{\|\psi_{t_0}\|_{C^0}^\alpha} - C_2. \quad (6.8)$$

Using the second inequality in the estimate (5.1) and the assumption $1 - t_0 \leq \delta \leq 1/18$ we get $\|\psi_{t_0}\|_{C^0} \leq 2\|\psi_1\|_{C^0} + 1$. But $\|\psi_1\|_{C^0} \leq \text{Osc}(\psi_1) = \text{Osc}(\varphi)$ since ψ_1 change signs. So we get the estimate $\|\psi_{t_0}\|_{C^0} \leq 2 \text{Osc}(\varphi) + 1$, which combined with (6.8) allows to deduce (6.6). $\square$

Proof of the Proposition (6.1). Set $J(t) := J_{\hat{\omega}}(\psi_t - \psi_1)$. Then applying the estimate (6.6) to the potential $\psi_t - \psi_1 \in \mathcal{P}_{\hat{\omega}}$ and using (6.5) we deduce for all $t \in [1/2, 1]$ the inequality

$$\frac{CJ}{(J + 1)^\alpha} - C_2 \leq \nu_{\hat{\omega}}(\psi_t - \psi_1),$$

with $C > 0$ depending only on $\hat{\omega}$. Combining this with the estimates (6.4) and (6.5) we get for all $t \in [1/2, 1]$ the estimate

$$\frac{CJ}{(J + 1)^\alpha} - C_2 \leq n(n + 1)(1 - t)J + C_3(1 - t),$$

which can also be written as

$$\frac{J}{(J+1)^\alpha} [C - n(n+1)(1-t)(J+1)^\alpha] \leq C_3(1-t) + C_2. \quad (6.9)$$

We can assume the existence of $t' \in [1/2, 1]$ such that

$$n(n+1)(1-t')(J(t') + 1)^\alpha = C/2. \quad (6.10)$$

In fact if not

$$n(n+1)(1-t)(J(t) + 1)^\alpha < C/2,$$

for all $t \in [1/2, 1]$. In particular for $t = 1/2$ this gives $J(1/2) \leq C_4$ and we will be done. So assuming (6.10) we deduce $J(t') \leq C_5$ by (6.9). Combining this with (6.10) we get $1-t' = C_6/(J(t') + 1)^\alpha \geq c > 0$. $\square$

6.1 On the existence of Kähler-Einstein metrics.

By using directly the K-energy functional we can give a rather direct proof of the following result due to Tian [Ti2].

Theorem 6 *Let X be a Fano manifold, G be a compact maximal subgroup of the identity component of the group of automorphisms of X and $\omega \in 2\pi c_1$ a G -invariant Kähler metric. Suppose that the K-energy ν_ω is proper in the space of G -invariant Kähler metrics. Then X admits a G -invariant Kähler-Einstein metric.*

Proof. The solutions ψ_t $t \in [0, \tau]$ of the complex Monge-Ampère equation (4.1) are G -invariant since ω is G -invariant. Moreover along the solutions of (4.1) the K-energy has the expression

$$\nu_\omega(\psi_t) = -t \int_X \psi_t \omega_t^n - G_\omega(\psi_t) + \int_X h_\omega \omega^n. \quad (6.11)$$

Plugging this in the identity (6.1) we get the identity

$$\frac{d}{dt} \left(t \int_X \psi_t \omega_t^n + t G_\omega(\psi_t) \right) = G_\omega(\psi_t) \geq 0,$$

which implies that the function inside the brackets is nondecreasing and so

$$t \int_X \psi_t \omega_t^n + t G_\omega(\psi_t) \geq 0. \quad (6.12)$$

Then using the expression (6.11) we get $\nu_\omega(\psi_t) \leq (t-1)G_\omega(\psi_t) \leq 0$. Then the properness of the K-energy implies $I_\omega(\psi_t) \leq C$ and so $\text{Osc}(\psi_t) \leq C/t$ by lemma 3. $\square$

Tian's α invariant. Let $\omega \in 2\pi c_1$, $\alpha > 0$ and set

$$C_\omega(\alpha) := \sup_{\varphi \in \mathcal{P}_\omega} \int_X e^{-\alpha(\varphi - \max \varphi)} \omega^n.$$

We remind the following well known fact [Hör].

Proposition 6.3 *It always exist $\alpha > 0$ such that $C_\omega(\alpha) < +\infty$.*

This allows to define the invariant $\alpha(X) := \sup\{\alpha > 0 \mid C_\omega(\alpha) < +\infty\}$. At this point we can give a simple proof of the following result due to Tian [Ti1].

Theorem 7 *Let X be a Fano manifold of complex dimension n such that $\alpha(X) > \frac{n}{n+1}$. Then X admits a Kähler-Einstein metric.*

Proof. Let ψ_t $t \in [0, \tau)$ be the solutions of the complex Monge-Ampère equation (4.1). The inequality (6.12) implies

$$-(n+1) \int_X \psi_t \omega_t^n \leq (n+1) G_\omega(\psi_t) \leq n \int_X \psi_t (\omega^n - \omega_t^n),$$

which gives

$$-\int_X \psi_t \omega_t^n \leq n \max_X \psi_t. \quad (6.13)$$

Take $\alpha \in (\frac{n}{n+1}, \alpha(X))$. Then by using the Monge-Ampère equation (4.1) we get

$$C \geq \int_X e^{-\alpha t(\psi_t - \max_X \psi_t)} \omega^n = \int_X e^{(1-\alpha)t\psi_t + \alpha t \max_X \psi_t - h_\omega} \omega_t^n.$$

So by the concavity of the log function

$$t \int_X [(1-\alpha)\psi_t + \alpha \max_X \psi_t] \omega_t^n \leq C.$$

We deduce

$$\max_X \psi_t \leq -\frac{1-\alpha}{\alpha} \int_X \psi_t \omega_t^n + C/t.$$

Combining this with the inequality (6.13) we obtain

$$\max_X \psi_t \leq \mu \max_X \psi_t + C/t,$$

where $\mu := \frac{1-\alpha}{\alpha} n < 1$, since $\alpha > \frac{n}{n+1}$. Set $\gamma := 1 - \mu > 0$. Then $\max_X \psi_t \leq C/(\gamma t)$. Combining this with (6.13) we deduce

$$-\int_X \psi_t \omega_t^n \leq C/t.$$

So in conclusion $I_\omega(\psi_t) \leq C/t$. This implies $\text{Osc}(\psi_t) \leq C/t^2$ by lemma 3. $\square$

7 Convergence of the Kähler-Ricci flow

Proof of the theorem 3. Consider the the Kähler-Ricci flow equation in terms of potentials

$$\dot{\varphi}_t = \log \frac{\omega_t^n}{\hat{\omega}^n} + \varphi_t + c_t, \quad (7.1)$$

with $\omega_t = \hat{\omega} + i\partial\bar{\partial}\varphi_t$ and normalization $\int_X e^{-\varphi_t} \omega_t^n = 1$. We remind the following fundamental result due to Perelman (see [Se-Ti] for a detailed proof).

Theorem 8 *Over a Fano manifold X of complex dimension n , the Kähler-Ricci flow $\frac{d}{dt}\omega_t = \omega_t - \text{Ric}_t = i\partial\bar{\partial}\dot{\varphi}_t$ satisfies the uniform estimates $|\dot{\varphi}_t|$, $|\nabla_t \dot{\varphi}_t|_t$, $|\Delta_t \dot{\varphi}_t|$, $\text{Diam}_t(X)$, $\text{Sc}_t \leq C$, where the Ricci potential $\dot{\varphi}_t$ is normalized by the condition $\int_X e^{-\dot{\varphi}_t} \omega_t^n = 1$.*

So combining Perelman's uniform estimate $|\dot{\varphi}_t| \leq C$ with the uniform bound $0 \leq J_{\hat{\omega}}(\varphi_t) \leq C$ deriving from the properness of the K-energy (which is decreasing along the flow) and with the C^0 -uniform estimate of the theorem 4 in [Pal-2] (see also [Ti-Zh]), we infer the C^0 -uniform estimate $|\varphi_t + c_t| \leq C$. Then the C^2 and C^3 -uniform estimates

$$|\partial\bar{\partial}\varphi_t|_{C^0(X)}, |\nabla_{\hat{\omega}}^{1,0}\partial\bar{\partial}\varphi_t|_{C^0(X)} \leq C,$$

for the Kähler-Ricci flow [Cao], [Pal-2], imply that the $(1,1)$ -forms $(\partial\bar{\partial}\varphi_t)_{t \geq 0}$ are uniformly bounded in $C^\alpha(X)$ -topology. Thus the operator Δ_t is uniformly elliptic with coefficients uniformly bounded in C^α -norm at least. Set

$$\square_t := \Delta_t - 2\frac{\partial}{\partial t}.$$

Deriving the Kähler-Ricci flow equation (7.1) respect to a holomorphic vector field $\xi \in \mathcal{O}(T_X)(U)$ over an open set U , (see [Pal-2] eq. 11) we obtain

$$\square_t(\xi \cdot \varphi_t) + 2\xi \cdot \varphi_t = (\text{Tr}_{\hat{\omega}} - \text{Tr}_t)(L_\xi \hat{\omega}). \quad (7.2)$$

The right hand side of this equation is also uniformly bounded in C^α -norm at least. By the regularity theory for parabolic equations [L-S-U] we deduce that the functions $(\xi \cdot \varphi_t)_{t \geq 0}$ are uniformly bounded in $C^{2,\alpha}$ -norm. So we obtain the uniform estimate $\|\varphi_t + c_t\|_{C_{\hat{\omega}}^{3,\alpha}} \leq C$ for all $t \geq 0$. This implies that the coefficients of the Laplacian Δ_t and the right hand side of the equation (7.2) are uniformly bounded in $C^{1,\alpha}$ -norm at least. By iteration we get the uniform estimates $\|\varphi_t + c_t\|_{C_{\hat{\omega}}^{k,\alpha}} \leq C_{k,\alpha}$ for all $t \geq 0$. We remind now that along the Kähler-Ricci flow hold the identity

$$\frac{d}{dt}\nu_{\hat{\omega}}(\varphi_t) = - \int_X |\partial\dot{\varphi}_t|_t^2 \omega_t^n =: -f(t),$$

which implies the convergence of the integral

$$\int_0^{+\infty} dt \int_X |\partial\dot{\varphi}_t|_t^2 \omega_t^n = \nu_{\hat{\omega}}(\varphi_0) - \lim_{T \rightarrow +\infty} \nu_{\hat{\omega}}(\varphi_T) \leq \nu_{\hat{\omega}}(\varphi_0) + C < +\infty$$

by the lower bound of the K-energy. Combining the identity (3.12) with the Stokes formula and Perelman's theorem 8 we deduce

$$4 \int_X |\partial\bar{\partial}\dot{\varphi}_t|_t^2 \omega_t^n = \int_X (\Delta_t \dot{\varphi}_t)^2 \omega_t^n \leq C, \quad (7.3)$$

for all times $t \geq 0$. By time deriving the Kähler-Ricci flow equation (7.1) we deduce

$$2\ddot{\varphi}_t = \Delta_t \dot{\varphi}_t + 2\dot{\varphi}_t + 2\dot{c}_t. \quad (7.4)$$

By deriving the function f , using the identities (7.4), (3.12) and the Stokes formula we get

$$\begin{aligned}
\dot{f}(t) &= 2n \int_X i \partial \ddot{\varphi}_t \wedge \bar{\partial} \dot{\varphi}_t \wedge \omega_t^{n-1} \\
&+ n(n-1) \int_X i \partial \dot{\varphi}_t \wedge \bar{\partial} \dot{\varphi}_t \wedge i \partial \bar{\partial} \dot{\varphi}_t \wedge \omega_t^{n-2} \\
&= 2 \int_X |\partial \dot{\varphi}_t|_t^2 \omega_t^n + n \int_X i \partial \Delta_t \dot{\varphi}_t \wedge \bar{\partial} \dot{\varphi}_t \wedge \omega_t^{n-1} \\
&- n(n-1) \int_X \dot{\varphi}_t (i \partial \bar{\partial} \dot{\varphi}_t)^2 \wedge \omega_t^{n-2} \\
&= 2 \int_X |\partial \dot{\varphi}_t|_t^2 \omega_t^n - \frac{1}{2} \int_X (\Delta_t \dot{\varphi}_t)^2 \omega_t^n \\
&+ \int_X \dot{\varphi}_t |\partial \bar{\partial} \dot{\varphi}_t|_t^2 \omega_t^n - \frac{1}{4} \int_X \dot{\varphi}_t (\Delta_t \dot{\varphi}_t)^2 \omega_t^n.
\end{aligned}$$

Then Perelman's theorem 8 combined with the estimate (7.3) implies the uniform estimate $|\dot{f}(t)| \leq C$ for all times $t \geq 0$. So the function $f : [0, +\infty) \rightarrow [0, K]$ satisfies $\int_0^{+\infty} f(t) dt < +\infty$ and $|\dot{f}(t)| \leq C, t \geq 0$. But this implies the existence of the limit $\lim_{t \rightarrow +\infty} f(t) = 0$. In fact by contradiction assume there exist an $\varepsilon > 0$ and a sequence of times $(t_j) \subset [0, +\infty), t_j \rightarrow +\infty$ such that $f(t_j) > \varepsilon$. Define

$$r_j := \sup\{r > 0 \mid f|_{(t_j-r, t_j+r)} > \varepsilon/2\}.$$

Then the equality $f(t_j \pm r_j) = \varepsilon/2$ hold for at least one of the signs "+, -" and the integral condition $\int_0^{+\infty} f(t) dt < +\infty$ implies $r_j \rightarrow 0$. Moreover for at least one of the signs "+, -"

$$0 < \frac{\varepsilon}{2r_j} < \frac{f(t_j) - f(t_j \pm r_j)}{r_j} = |\dot{f}(\tau_j)|,$$

with $\tau_j \in (t_j - r_j, t_j + r_j)$, which contradicts the uniform bound $|\dot{f}(t)| \leq C$.

By the other hand the uniform estimate $\|\varphi_t + c_t\|_{C_{\hat{\omega}}^{h,\alpha}} \leq C_{h,\alpha}, h \geq 3$ implies that for every sequence $t_j \rightarrow +\infty$ there exist a subsequence $(t_k)_k$ such that the sequence $(\varphi_{t_k} + c_{t_k})_k$ converges to a potential $\varphi_\infty \in \mathcal{P}_{\hat{\omega}}^{h,\alpha}$ in the $C^{h,\alpha}$ -topology. Moreover we deduce the existence of the limits

$$\psi := \lim_{k \rightarrow +\infty} \dot{\varphi}_{t_k} = \log \frac{\hat{\omega}_{\varphi_\infty}^n}{\hat{\omega}^n} + \varphi_\infty$$

and $\partial \psi = \lim_{k \rightarrow +\infty} \partial \dot{\varphi}_{t_k}$ in the $C^{h-3,\alpha}$ -topology at least. Then the existence of the limit $\lim_{t \rightarrow +\infty} f(t) = 0$ implies

$$0 = \lim_{k \rightarrow +\infty} \int_X |\partial \dot{\varphi}_{t_k}|_{t_k}^2 \omega_{t_k}^n = \int_X |\partial \psi|_{\varphi_\infty}^2 \omega_{\varphi_\infty}^n,$$

which means $\psi = 0$, by the integral normalization of $\dot{\varphi}_t$. So we have a solution φ_∞ of class $C^{h,\alpha}$ of the Einstein equation

$$\log \frac{\hat{\omega}_{\varphi_\infty}^n}{\hat{\omega}^n} + \varphi_\infty = 0.$$

By elliptic regularity the solution φ_∞ is smooth. Moreover $\hat{\omega}_{\varphi_\infty} = \hat{\omega}$ by the uniqueness of the Kähler-Einstein metric in our case. This implies the convergence of the limit $\lim_{t \rightarrow +\infty} \omega_t = \hat{\omega}$ in the smooth topology. $\square$

8 Appendix. A proof of the short time existence of the Kähler-Ricci flow equation.

We will use some elements of abstract parabolic theory in a similar spirit as in [Ch-He].

Short time existence and uniqueness of fully nonlinear parabolic equations.

We remind here some basic facts about the short time existence and uniqueness of abstract fully nonlinear parabolic equations. See for example [Ang], [Lun]. Let $T > 0$, $I := [0, T]$, E_0 be a Banach space and $E_1 \subset E_0$ be a dense subvector space equipped with a Banach norm such that the embedding $E_1 \hookrightarrow E_0$ is continuous. We define also the Banach space $E_0(I) := C^0(I, E_0)$ and

$$E_1(I) := C^0(I, E_1) \cap C^1(I, E_0),$$

equipped with the Banach norm

$$\|u\|_{E_1(I)} := \max_{t \in I} (\|u(t)\|_{E_1} + \|\dot{u}(t)\|_{E_0})$$

An operator $A \in \mathcal{L}(E_1, E_0)$ can be considered as an unbounded operator on E_0 with domain E_1 . It can happen that $A \in \mathcal{L}(E_1, E_0)$ seen as an unbounded operator on E_0 , is closed and generates a strongly continuous analytic semigroup denoted by $(e^{tA})_{t \geq 0} \subset \mathcal{L}(E_0, E_0)$. We will note by $\mathcal{A}(E_1, E_0)$ the subset of $\mathcal{L}(E_1, E_0)$ for which this property holds. It is well known that $\mathcal{A}(E_1, E_0)$ is open in $\mathcal{L}(E_1, E_0)$ and $(e^{tA})_{t \geq 0} \subset \mathcal{L}(E_0, E_1)$. Let C^{ra} the real analytic class. It is easy to see that if $A \in \mathcal{A}(E_1, E_0)$, the function $t \mapsto e^{tA}x$ belongs to the space

$$C^0([0, +\infty), E_0) \cap C^{ra}((0, +\infty), E_0) \cap C^0((0, +\infty), E_1),$$

if $x \in E_0$ and to

$$C^{ra}((0, +\infty), E_0) \cap C^0([0, +\infty), E_1),$$

if $x \in E_1$. For any $A \in \mathcal{A}(E_1, E_0)$ we define the bounded operator

$$\hat{A} : E_1(I) \longrightarrow E_0(I) \oplus E_1,$$

by $\hat{A}(u) := (\dot{u} - Au, u(0))$ and the set $\mathcal{M}(E_1, E_0) \subset \mathcal{A}(E_1, E_0)$ of operators A such that $\hat{A}$ is an isomorphism. In other words, $\mathcal{M}(E_1, E_0)$ contains those infinitesimal generators A of strongly continuous analytic semigroups for which the problem

$$\begin{cases} \dot{u}(t) - Au(t) = f(t), & t \in [0, 1] \\ u(0) = x, \end{cases}$$

has a unique solution $u \in E_1(I)$ for all $f \in E_0(I)$ and $x \in E_1$. In this case the solution is given by the "variation of constants formula"

$$u(t) = e^{tA}x + \int_0^t e^{(t-s)A}f(s) ds =: e^{tA}x + (K_A f)(t).$$

An operator $A \in \mathcal{A}(E_1, E_0)$ belongs to $\mathcal{M}(E_1, E_0)$ if and only if $K_A E_0(I) \subset E_1(I)$. It is also well known that $\mathcal{M}(E_1, E_0)$ does not depend on the interval $I \subset [0, +\infty)$. With this notations we have the following short time existence and uniqueness theorem for abstract fully nonlinear parabolic equations [Ang].

Theorem 9 *Let $U \subset E_1$ be an open set and $\varphi : U \rightarrow E_0$ be a map of class C^1 such that the $D\varphi(x) \in \mathcal{M}(E_1, E_0)$ for all $x \in U$. Then for all $x \in U$ there exist $T > 0$ such that the problem*

$$\begin{cases} \dot{u}(t) = \varphi(u(t)), & t \in I := [0, T] \\ u(0) = x, \end{cases}$$

has a unique solution $u \in E_1(I)$.

Continuous interpolation spaces.

The interpolation space theory is very usefull in order to prove that many interesting operators belongs to $\mathcal{M}(E_1, E_0)$. We remind here some very basic concepts. Let $\theta \in (0, 1)$ and set $\dot{I} := (0, T]$. We define the Banach spaces

$$\begin{aligned} E_0^\theta(I) &:= \left\{ u \in C^0(\dot{I}, E_0) \mid \lim_{t \rightarrow 0^+} t^\theta \|u(t)\|_{E_0} = 0 \right\}, \\ E_1^\theta(I) &:= \left\{ u \in C^0(\dot{I}, E_1) \cap C^1(\dot{I}, E_0) \mid \lim_{t \rightarrow 0^+} t^\theta (\|u(t)\|_{E_1} + \|\dot{u}(t)\|_{E_0}) = 0 \right\}, \end{aligned}$$

equipped respectively with the norms

$$\begin{aligned} \|u\|_{E_0^\theta(I)} &:= \sup_{t \in \dot{I}} t^\theta \|u(t)\|_{E_0}, \\ \|u\|_{E_1^\theta(I)} &:= \sup_{t \in \dot{I}} t^\theta (\|u(t)\|_{E_1} + \|\dot{u}(t)\|_{E_0}). \end{aligned}$$

The fact that for all $u \in E_1^\theta(I)$ there exist a constant $C > 0$ such that $\|\dot{u}(t)\|_{E_0} \leq C/t^{-\theta}$ on $\dot{I}$ implies the existence of $u(0) := u(T) - \int_0^T \dot{u}(t) dt \in E_0$. We define the interpolation space of the couple $E := (E_1, E_0)$ as the Banach space

$$E_\theta := \{x \in E_0 \mid \exists u \in E_1^\theta(I) : x = u(0)\},$$

equipped with the norm

$$\|x\|_{E_\theta} := \inf \left\{ \|u\|_{E_1^\theta(I)} \mid u \in E_1^\theta(I) : x = u(0) \right\}.$$

Moreover $\overline{E_1}^{E_\theta} = E_\theta$. It is well known that if $A \in \mathcal{A}(E_1, E_0)$ with $\text{type}(A) < 0$ then

$$E_\theta = \left\{ x \in E_0 \mid \sup_{t > 0} t^\theta \|Ae^{tA}x\|_{E_0} < +\infty, \lim_{t \rightarrow 0^+} t^\theta \|Ae^{tA}x\|_{E_0} = 0 \right\},$$

and an equivalent norm is $\|x\|'_{E_\theta} := \sup_{t>0} t^\theta \|Ae^{tA}x\|_{E_0}$. We remind now a fundamental construction due to Da Prato and Grisvard. Let $F_1 \subset F_0$ as before and $A \in \mathcal{A}(F_1, F_0)$. We define the Banach space

$$F_2 := \{x \in F_1 \mid Ax \in F_1\} = \text{Dom}(A^2),$$

equipped with the graph norm $\|x\|_{F_2} := \|x\|_{F_1} + \|Ax\|_{F_1}$. In this way we get the continuous and dense embeddings $F_2 \hookrightarrow F_1 \hookrightarrow F_0$. For $\theta \in (0, 1)$ we define $E_0 := (F_1, F_0)_\theta \subset F_0$ and $E_1 := (F_2, F_1)_\theta \subset F_1$. With this notations we have the following fundamental theorem [Da-Gr].

Theorem 10 (Da Prato-Grisvard). $A \in \mathcal{M}(E_1, E_0)$ for every $A \in \mathcal{A}(F_1, F_0)$.

Short time existence of the Kähler-Ricci flow.

We apply now the previous considerations to our setting. We remind that the space of smooth functions is not dense in the Hölder space $C^{k,\alpha}$. One defines the little Hölder space $c^{k,\alpha}$ to be the closure of the space of smooth functions in the Hölder space $C^{k,\alpha}$. For $k \leq l$ and $0 < \alpha \leq \beta < 1$ we have the continuous and dense embedding of Banach spaces $c^{l,\beta} \hookrightarrow c^{k,\alpha}$. There is a Banach space isomorphism (see [Tri])

$$(c^{l,\beta}, c^{k,\alpha})_\theta \simeq c^{\theta l + (1-\theta)k + \theta\beta + (1-\theta)\alpha},$$

for all $\theta \in (0, 1)$ such that $\theta l + (1-\theta)k + \theta\beta + (1-\theta)\alpha$ is not an integer. For all $\omega \in \mathcal{K}$ the operator

$$A := \frac{1}{2}\Delta_\omega + 1 : c^{k+2,\alpha} \rightarrow c^{k,\alpha},$$

is strictly elliptic. Then a result in [Lun] implies $A \in \mathcal{A}(c^{k+2,\alpha}, c^{k,\alpha})$. We show now that $A \in \mathcal{M}(c^{k+2,\alpha}, c^{k,\alpha})$. In fact consider $0 < \gamma < \alpha < 1$ and define $\theta := (\alpha - \gamma)/2$, $F_0 := c^{k,\gamma}$, $F_1 := c^{k+2,\gamma}$. In this case hold the Banach space equality $F_2 = c^{k+4,\gamma}$ since the map $\Delta_\omega : c^{k+2,\alpha} \rightarrow c^{k,\alpha}$ is an isomorphism. Moreover we have the Banach space isomorphisms

$$E_0 := (c^{k+2,\gamma}, c^{k,\gamma})_\theta \simeq c^{k,\alpha} \quad \text{and} \quad E_1 := (c^{k+4,\gamma}, c^{k+2,\gamma})_\theta \simeq c^{k+2,\alpha}.$$

Then the Da Prato-Grisvard theorem implies the required conclusion on A . The Frechet derivative at a point φ of the smooth map

$$\begin{aligned} \Phi : \mathcal{P}_\omega^{k+2,\alpha} \cap c^{k+2,\alpha} &\longrightarrow c^{k,\alpha} \\ \varphi &\longmapsto \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega, \end{aligned}$$

is given by the map

$$\begin{aligned} D\Phi(\varphi) : c^{k+2,\alpha} &\longrightarrow c^{k,\alpha} \\ f &\longmapsto \frac{1}{2}\Delta_{\omega'} f + f, \end{aligned}$$

where $\omega' := \omega_\varphi \in \mathcal{K}$ and $D\Phi(\varphi) \in \mathcal{M}(c^{k+2,\alpha}, c^{k,\alpha})$, by the previous considerations. Then theorem 9 implies the existence of a unique solution

$$\varphi \in C^0([0, \varepsilon], \mathcal{P}_\omega^{k+2,\alpha} \cap c^{k+2,\alpha}) \cap C^1([0, \varepsilon], c^{k,\alpha}),$$

of the Kähler-Ricci flow equation $\dot{\varphi}_t = \log \frac{\omega_{\varphi_t}^n}{\omega^n} + \varphi_t - h_\omega$ with initial data $\varphi_0 \in \mathcal{P}_\omega$. The fact that the map Φ is smooth implies

$$\varphi \in C^1([0, \varepsilon), c^{k, \alpha}) \cap C^\infty((0, \varepsilon), c^{k, \alpha}), \quad (8.1)$$

by classic parabolic regularity (see [Lun] prop 8.4.5 pag 312). Moreover the fact that the initial data φ_0 is smooth implies that the relation (8.1) hold for $k+2$ and $\varepsilon' \in (0, \varepsilon]$. By time deriving the Kähler-Ricci flow equation we get the identity

$$\ddot{\varphi}_t = \frac{1}{2} \Delta_t \dot{\varphi}_t + \dot{\varphi}_t. \quad (8.2)$$

Then the regularity of the Green formula for the Laplacian Δ_t implies $\varepsilon' = \varepsilon$ and so

$$\varphi \in C^1([0, \varepsilon), C^\infty(X)) \cap C^\infty((0, \varepsilon), C^\infty(X)).$$

But using again the equation (8.2) we deduce

$$\varphi \in C^2([0, \varepsilon), C^\infty(X)) \cap C^\infty((0, \varepsilon), C^\infty(X)).$$

In general by time deriving k -times the Kähler-Ricci flow equation we get the identity

$$\varphi_t^{(k+1)} = \frac{1}{2} \Delta_t \varphi_t^{(k)} + \varphi_t^{(k)} + f_k(\partial \bar{\partial} \varphi_t, \dots, \partial \bar{\partial} \varphi_t^{(k-1)}),$$

where f_k is a smooth function over an open set of $(\Lambda^{1,1} T_X^*)^k$ and $\varphi_t^{(k)} := \partial_t^k \varphi_t$. This implies the smoothness of the unique solution $\varphi : [0, \varepsilon) \rightarrow \mathcal{P}_\omega$ of the Kähler-Ricci flow over the interval $[0, \varepsilon)$. $\square$

The long time existence of the Kähler-Ricci flow follows by combining the short time existence just proved with the arguments in [Cao] or [Pal-2]. We infer the existence of a unique and smooth solution $\varphi : [0, +\infty) \rightarrow \mathcal{P}_\omega$ of the Kähler-Ricci flow equation with $\omega \in \mathcal{K}$ and with initial data $\varphi_0 \in \mathcal{P}_\omega$.

References

- [Ang] ANGENENT, S. Nonlinear Analysis Semiflow, Proc. Roy. Soc. Edin. 115A, 91-107, 1990.
- [Aub1] AUBIN, T. Nonlinear Analysis on Manifolds. Monge-Ampère Equations, Springer-Verlag, Berlin, New-York, 1982.
- [Aub2] AUBIN, T. Réduction du cas positif de l'équation de Monge-Ampère sur les variétés Kähleriennes compactes à la démonstration d'une inégalité, J. Funct. Anal., 57, 1984, 143-153.
- [A-D-N] AGMON, S., DOUGLIS, A., NIRENBERG, L. Estimates Near the Boundary for solutions of Elliptic Partial Differential Equations satisfying General Boundary Conditions. I. Comm. Pur. Appl. Math. 12, 1959, 623-727.
- [Ba-Ma] BANDO, S., MABUCHI, T. Uniqueness of Einstein Kähler metrics modulo connected group actions, Adv. Stud. in Pure Math. 10, 1987, 11-40.

- [Cal] CALABI, E. Improper affine hyperspheres and generalisation of a theorem of K. Jörgens, *Mich. Math. J.* 5., 105-126, 1958.
- [Cao] CAO, H.D. Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds. *Invent. Math.*, 81, 1985, 359-372.
- [Che] CHEN, X.X. On the lower bound of the energy functional E_1 , (I)- a stability theorem on the Kähler-Ricci flow, preprint, arXiv/math.DG/050219.
- [Ch-Ti] CHEN, X.X., TIAN, G. Ricci flow on Kähler-Einstein surfaces, *Invent. Math.* 147, 2002, 487-544.
- [Ch-He] CHEN, X.X., HE, W. On the Calabi flow, arXiv/DG/0603523, 2006.
- [C-L-W] CHEN, X.X., LI, H., WANG, B. On the lower bound of the energy functional E_1 , (II), preprint, arXiv/math.DG/0609694.
- [Da-Gr] DA PRATO, G., GRISVARD, P. Équations d'évolution abstraites non-linéaires de type parabolique, *Ann. Mat. Pura Appl.* (4) 120, 329-396, 1979.
- [L-S-U] LADYZENSKAJA, O.A., SOLONNIKOV, V.A., URAL'CEVA, N.N. Linear and quasilinear equations of parabolic type. Providence, Amer.Math. Soc. 1968.
- [Lun] LUNARDI, A. analytic semigroups and optimal regularity in parabolic problems, *Progress in nonlinear differential equations and their applications*, Vol. 16, 1995.
- [Mab] MABUCHI, T. K-energy maps integrating Futaki invariants, *Tohoku Math. Journ.* 38, 1986, 575-593.
- [Mat] MATSUSHIMA, Y. Sur la structure de groupe d'homéomorphismes analytiques d'une certaine variété kählérienne, *Nagoya Math. J.*, 11, 145-150, 1957.
- [P-S-S-W] PHONG, D.H., SONG, J., STURM, J., WEINKOVE, B. The Moser-Trudinger inequality on Kähler-Einstein manifolds, arXiv/DG/0604076, 2006.
- [Se-Ti] SESUM, N., TIAN, G. Bounding scalar curvature and diameter along the Kähler-Ricci flow (after Perelman), preprint.
- [Ti1] TIAN, G., On Kähler-Einstein metrics on certain Kähler manifolds with $c_1(M) > 0$, *Invent. Math.* 89 (1987), no. 2, 225-246.
- [Ti2] TIAN, G. Kähler-Einstein metrics with positive scalar curvature, *Invent. Math.* 137, 1997, 1-37.
- [Ti-Zh] TIAN, G., ZHU, X. Convergence of the Kähler-Ricci Flow, To appear in *Journal of Amer. Math. Soc.*
- [Tri] TRIEBEL, H. Interpolation theory, function spaces, differential operators, (rev.ed.) Johann Ambrosius Barth Verlag, Heidelberg, 1995.

- [Yau] YAU, S.-T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm.Pure Appl. Math. 31, 1978, 339-411.
- [Gi-Tru] GILBARG, D., TRUDINGER, N. (1977) Elliptic partial differential equations of second order. Berlin Heidelberg New York: Springer
- [Hör] HÖRMANDER, L., *An introduction to Complex Analysis in several variables*, (1st edition 1966) 3rd edition, North-Holland Math. Libr., vol. 7, Amsterdam, London(1990).
- [Kol] KOŁODZIEJ, S., *The complex Monge-Ampère equation*, Acta Math. 180 (1998), no. 1, 69-117.
- [Pal-1] PALI,N. (2005) A consequence of a lower bound of the K-energy. International Mathematics Research Notices, (2005), no. 50, 3081 - 3090
- [Pal-2] PALI,N. (2006) Characterization of Einstein-Fano manifolds via the Kähler-Ricci flow. To appear in Indiana Univ. Math. J. (2008).

Nefton Pali
 Université Paris Sud, Département de Mathématiques
 Bâtiment 425 F91405 Orsay, France
 E-mail: *nefton.pali@math.u-psud.fr*