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On the homogeneity at infinity of the stationary probability for an affine random walk.

by Y. Guivarc'h, E. Le Page

Abstract
We consider an affine random walk on $\mathbb{R}$. We assume the existence of a stationary probability ν on $\mathbb{R}$ and we describe the shape at infinity of ν, if ν has unbounded support. We discuss the connections of the result with geometrical or probabilistic problems.

I - Introduction

Let G be the affine group of the line. For $g \in G$, $x \in \mathbb{R}$, we write $gx = a(g)x + b(g)$ with $a(g) \in \mathbb{R}^*, b(g) \in \mathbb{R}$. Let μ be a probability on G. We denote by P the Markov operator on $\mathbb{R}$ defined by $P\varphi(x) = \int \varphi(gx)\mu(dg)$ where φ is a bounded Borel function. Our hypothesis H_μ is stated below and we observe that $H_\mu(1)$ and $H_\mu(2)$ imply that P has a unique stationary probability ν (see [16]) ; if $H_\mu(4)$ is also valid, then $supp\nu$ is unbounded. Here we are interested in the "shape at infinity" of ν ; we will show that for some $\alpha > 0$, the quantities $|t|^\alpha \nu[t, \infty)$ and $|t|^\alpha \nu(-\infty, t]$ have limits at infinity, we discuss their positivity and we illustrate the possible uses of this result by two corollaries in two different contexts. This "homogeneity at infinity" of ν plays an essential role in extreme value theory (see [19]), for random variables associated with the Markov chain X_n^x with kernel P on $\mathbb{R}$. Also, for random walk in a random medium on $\mathbb{Z}$ (see [21]) the slow diffusion property is closely related to this homogeneity (see [6], [17]). Furthermore the construction of ν given here provides a natural construction of a large class of heavy tailed measures which generates "anomalous" random walks on the additive group $\mathbb{R}$. This class of measures appears now to be of great interest from the physical point of view (see [2]). In the geometrical context of excursions of geodesic flows on manifolds of negative curvature the "logarithm law" is well known (see [22], [18]), and we will discuss analogous properties for the Markov chain X_n^x.

We assume that μ satisfies the following set of conditions H_μ.

$H_\mu(1)$: $\int (|\ell n|a(g)|| + |(\ell n|b(g)||))\mu(dg) < \infty$.

$H_\mu(2)$: For some $\alpha > 0$ $\int |a(g)|^\alpha \mu(dg) = 1$.

$H_\mu(3)$: $\int |a(g)|^\alpha \ell n|a(g)|\mu(dg) < \infty$, $\int |b(g)|^\alpha \mu(dg) < \infty$.

$H_\mu(4)$: The elements of $supp\mu$ have no common fixed point in $\mathbb{R}$.

$H_\mu(5)$: The set $\{\ell n|a(g)|\ ;\ g \in supp\mu\}$ generates a dense subgroup of $\mathbb{R}$.

Then we have the

Theorem 1

Assume that μ satisfies H_μ. Then

1) There exists $c_+ \geq 0$, $c_- \geq 0$ such that $\lim\limits_{t\to\infty} |t|^\alpha \nu(t, \infty) = c_+$, $\lim\limits_{t\to-\infty} |t|^\alpha \nu(-\infty, t) = c_-$. Moreover $c = c_+ + c_- > 0$.

2) If $\mu\{g \in G\ ;\ a(g) < 0\} > 0$, then $c_+ = c_- > 0$.

3) If $\mu\{g \in G \ ; \ a(g) > 0\} = 1$, then $c_+ > 0$ (resp $c_- > 0$) if and only if the action of $supp\mu$ on $\mathbb{R}$ has no invariant half-line of the form $]-\infty, k]$, (resp $[k, \infty[$) .
4) If $\mu\{g \in G \ ; \ a(g) < 0\} > 0$, then $supp\nu = \mathbb{R}$. Otherwise the set $supp\nu$ is a half-line if and only if $supp\mu$ preserves a half-line of the same form.

We denote by $\mathbb{P}$ the product probability $\mu^{\otimes\mathbb{N}}$ on $\Omega = G^{\mathbb{N}}$, where $\mathbb{N}$ is the set of positive integers. For $\omega \in \Omega$, we write $g_k = (a(g_k), b(g_k)) = (a_k, b_k)$. Then X_n^x satisfies the stochastic recursion :

$$X_n^x = a_n X_{n-1}^x + b_n, \ X_0^x = x.$$

The Markov chain X_n^x on $\mathbb{R}$ will be called "affine random walk". It is well known that, for the existence and uniqueness of the stationary measure ν, it is sufficient to assume $H_\mu(1)$ and $\int \ell n|a(g)|\mu(dg) < 0$; then X_n^x converges in law to $R = \sum_1^\infty a_1 \dots a_{k-1} b_k$ and the law of R is ν. Also if $\int(|a(g)|^\beta + |b(g)|^\beta)\mu(dg) < \infty$ for some $\beta > 0$, then $\int |x|^\beta d\nu(x) < \infty$. We observe that R can be interpreted as the sum of a "random geometric series", hence its interest for collective risk theory ([19]).
The validity of 1) and 2) was proved in [10], [16] ; in particular implicit expression for c_+, c_- were given in [10] and the relation $c_+ + c_- > 0$ was obtained. Here we restrict our study to 3) and 4), a result which is new under hypothesis H_μ. A different proof was sketched in [12], where a survey of the multidimensional situation was also given. We observe that the main difficulty of the proof occurs when $supp\mu$ do not preserve a half line and $a(g) > 0$ $\mu - a.e$; in this case we have $c_+ > 0, c_- > 0$. If $supp\mu$ has compact support a short complex analytic proof of this fact, depending of a Lemma of E. Landau, well known in analytic number theory, is given in [11] (see also [5]).
We recall that Fréchet's law with parameter γ is the probability Φ_α^γ on $\mathbb{R}_+$ given by $\Phi_\alpha^\gamma(0, t) = e^{-\gamma t^{-\alpha}}$ where $\gamma > 0$, $\alpha > 0$. This family of laws is one of the three families of max-infinitely divisible laws of extreme value theory ([9], [19]). The following is shown in [14].

Corollary 2

For $x \in \mathbb{R}$ we denote

$$M_n^x = sup\{|X_k^x| \ ; \ 1 \leq k \leq n\}, \ {}_+M_n^x = sup\{X_n^x \ ; \ 1 \leq k \leq n\}.$$

Then the sequence $n^{-\alpha} M_n^x (resp \ n^{-\alpha} \ {}_+M_n^x)$ converges in $\mathbb{P}$-law to $\Phi_\alpha^{c\theta}$ with $0 < \theta < 1$ (resp $\Phi_\alpha^{c_+\theta_+}$ with $0 < \theta_+ < 1$, if $c_+ > 0$).

Closely related properties have been intensively studied in the context of extreme value theory (see [19]). The positive number θ is the so-called extremal index of the stochastic process X_n^x ; its inverse θ^{-1} gives a measure of the clustering of the exceptionally large values of the process. If the random variables X_n^x were i.i.d. with law ν, one would have $\theta = 1$ (see [9]). If $a(g) > 0$, $b(g) > 0$ for $g \in supp\mu$, the above corollary is proved in [15]. It is also known (see [13]) that, under hypothesis H_μ, the normalized Birkhoff sum of X_n^x

converges in law to a stable law of index α if $\alpha < 2$. As mentioned in ([3], remark 4.8), this convergence is a consequence of extreme value properties of X_n^x, at least for $\alpha < 1$. The analysis of random walk in a random medium on $\mathbb{Z}$ developed in [6] is closely related to such properties for the sojourn time of the particle at a site in $\mathbb{Z}$, instead of its hitting time as in [17], where Birkhoff sums as above played a dominant role.
The following logarithm law is an easy consequence of Corollary 2.

Corollary 3

For any $x \in \mathbb{R}$, we have the following $\mathbb{P}-a.e$ convergences :

$$\limsup_{n\to\infty} \frac{\ell n|X_n^x|}{\ell n(n)} = \frac{1}{\alpha}, \ \limsup_{n\to\infty} \frac{\ell n^+(X_n^x)}{\ell n(n)} = \frac{1}{\alpha} \text{ if } c_+ > 0.$$

The so-called "logarithm law" for excursions of geodesic flow around the cusps on hyperbolic manifolds was proved in [22] and extended to more general situations in [18]. It was observed in [20] that in case of the modular surface, it is a simple consequence of Fréchet's law for geodesic flow which follows from already known extreme value properties of the continuous fraction expansion of a number x uniformly distributed on $[0,1]$ (see [8]).

II - Calculation of invariant measures on $\mathbb{R}$ in a special case

The Lie algebra of G is generated by the vector fields $X = a\frac{\partial}{\partial a}$, $Y = \frac{\partial}{\partial b}$. We consider the convolution semi-group of probability measures on G with infinitesimal generator $D = X^2 + Y^2 - (\beta+1)X$. This operator is elliptic and we denote by $p^t (t \geq 0)$ the associated semi-group of probability measures.
We have $\int ln\ a(g) p^t(dg) = -t(\beta+1)$ in particular $\int \ell n\ a(g)p(dg)$ is negative if $\beta > -1$, hence p^t has a stationary probability ν on $\mathbb{R}$ in this case. We consider more generally, for any β, the action of p^t and X, Y, D on positive measures of the form $\nu = f(x)dx$ on the line. We denote by X^*, Y^*, D^* the operators adjoint to X, Y, D. Then the extremal solutions of the equation $D^* f = 0$ $(f \geq 0)$ are described by the

Proposition 1

With the above notations, the equation $D^* f = 0$ has the following normalized extremal solutions :

$\beta \geq -1 :\ f(x) = (1+x^2)^{-(1+\beta/2)}$,

$\beta < -1 :\ f_+(x) = (1+x^2)^{-(1+\beta/2)} \int_{-\infty}^x (1+t^2)^{\beta/2} dt$,

and $f_-(x) = (1+x^2)^{-(1+\beta/2)} \int_x^\infty (1+t^2)^{\beta/2} dt$.

If $\beta > -1$, then $\int f(x)dx < \infty$. If $\beta \leq -1$ then $\int f(x)dx = \int f_+(x)dx = \int_- f(x)dx = \infty$.

Proof

We calculate the action of X, Y on the measure $\nu = f dx$ as follows.
Since dx is translation-invariant and the action of the one parameter group $x \to x + b$ is by translation we get $Y^* f = -f'$.
Since $X\varphi(x) = x\varphi'(x)$, we get also $X^* f(x) = -(xf(x))'$. It follows $D^* f(x) = (x(xf)')' +$

$f'' + (\beta+1)(xf)'$, so that the equation $D^*f = 0$ implies :

$x(xf)' + f'(x) + (\beta+1)(xf) = k,$

for a certain constant k, i.e :

$(1+x^2)f' + (\beta+2)(xf) = k.$

With $u(x) = (1+x^2)^{-(1+\beta/2)}$ we have $(1+x^2)u'(x) + (\beta+2)xu(x) = 0$, hence the above differential equation has the solutions : $f = u(d+kv)$ with $v(x) = \int_0^x (1+t^2)^{\beta/2} dt$ and d is a constant.

For $\beta \geq -1$, we have $\lim_{x\to\infty} v(x) = \infty$, hence the condition $f \geq 0$ implies $k = 0$. In this case the equation $D^*f = 0$ has only positive extremal solutions of the form $f(x) = d(1+x^2)^{-(1+\beta/2)}$. For $\beta = 0$, D is the hyperbolic Laplacian and we recover the Cauchy law on $\mathbb{R}$ with density $\frac{1}{\pi}\ \frac{1}{1+x^2}$. For $\beta > -1$, we get a probability law with density proportional to $(1+x^2)^{-(1+\beta/2)}$.

We verify that for $\beta < -1$, the equation $D^*f = 0$ has two basic extremal solutions :

$f_+(x) = (1+x^2)^{-(1+\beta/2)} \int_{-\infty}^x (1+t^2)^{\beta/2} dt,$

$f_-(x) = (1+x^2)^{-(1+\beta/2)} \int_x^{\infty} (1+t^2)^{\beta/2} dt.$

The measure ν corresponding to f_+ has infinite mass and satisfies :

$\lim_{t\to-\infty} |t|^{2+\beta} \nu(-\infty, t) = c- > 0$

$At + \infty$ $f_+(x)$ is asymptotic to $c_+ x^{-1}$ with $c_+ > 0$. Analogous properties are valid fo f_-.
Also, at ∞, $f(x)$ is asymptotic to $c|x|^{-1}(c > 0)$ $\square$

Remark

The case $\beta > -1$ corresponds to the situation of the theorem with $\alpha = \beta + 1$.
The case $\beta = -1$ corresponds to the (critical) situation of [1], [4]. Then the unique basic extremal solution behaves at infinity like multiplicative Lebesgue measure on $\mathbb{R}^*$.
The situation $\beta < -1$, with two extremal solutions, corresponds to a so-called phase transition in $P.D.E$ theory, for example in the context of non linear Schroëdinger equations.

III - Proof of theorem 1

The proofs of 1) and 2) in [10] are based on the first renewal equation in Lemma 1 below. A delicate point in [10] for the use of the renewal theorem (see [7]) is solved by replacing ${}^{\alpha}f(t) = e^{\alpha t} f(t)$ by a related directly Riemann-integrable function. Here we give only the proofs of 3) and 4). We will now assume $\mu\{g\ ;\ a(g) > 0\} = 1$ and we will only study the non vanishing of c_+. To do that we need some preliminary notations and results.
Let T be the stopping time on Ω defined by :

$T = \{n \geq 1\ ;\ g_1 g_2 \cdots g_n \in G_+\},\ \ T = \infty$ if $\{n \geq 1\ ;\ g_1 g_2 \cdots g_n \in G_+\} = \phi,$

where $G_+ = \{b(g) > 0\}$.
We denote by $\bar{\mu}$ the probability on the additive group $\mathbb{R}$ given by $\bar{\mu}(A) = \mu\{\ell n a(g) \in A\}$
Moreover we denote by μ_T the positive measure on $\mathbb{R}$ defined by :

$\mu_T(A) = \mathbb{P}\{T < +\infty\ ;\ \ell n(a_1 a_2 \cdots a_T) \in A\},$

where A is a Borel subset of $\mathbb{R}$. We have $\mu_T(\mathbb{R}) = \mathbb{P}(T < +\infty) \leq 1$, and we denote by μ_T^n the n^{th} convolution power of μ_T on the additive group $\mathbb{R}$. Define f by

$$f(t) = \mathbb{P}\{R > e^t\} = \nu(]e^t, +\infty[) \quad t \in \mathbb{R},$$

and write $R_n = \sum_{k=1}^{n} a_1 a_2 \cdots a_{k-1} b_k$, $S_n = \sum_{k=1}^{n} \ell n(a_k)$.

Then we have the :

Lemma 1

1) For every real t, we have $f(t) = \bar{\mu} * f(t) + f_1(t) = \mu_T * f(t) + h_1(t)$ where :

$$f_1(t) = \mathbb{P}\{R - b_1 > e^t\} - \mathbb{P}\{\mathbb{R} > e^t\},\ h_1(t) = \mathbb{E}\{1_{[T<+\infty]} \nu(]e^{-S_T}(e^t - R_T),\ e^{t-S_T}])\}$$

2) For every real t, we have $f(t) = \sum_{n=0}^{+\infty} \mu_T^n * h_1(t) = \sum_{n=0}^{\infty} \mu^n * f_1(t)$.

If p is a bounded measure on $\mathbb{R}$ and φ is a positive Borel function, we write

$$p * \varphi(t) = \int \varphi(t - x) p(dx), \quad t \in \mathbb{R}.$$

We denote by ${}^{\alpha}\mu$, the probability measure on G defined by : ${}^{\alpha}\mu(dg) = a^{\alpha}(g)\mu(dg)$.

We define the probability ${}^{\alpha}\mathbb{P}$ on $G^{\mathbb{N}}$ by ${}^{\alpha}\mathbb{P} = {}^{\alpha}\mu^{\otimes \mathbb{N}}$ and we write ${}^{\alpha}\mathbb{E}$ for the corresponding expectation.

The measure ${}^{\alpha}\mu_T$ on $\mathbb{R}$ is defined by ${}^{\alpha}\mu_T(A) = {}^{\alpha}\mathbb{E}(1_A(\ell n(a_1 \cdots a_T))$,

and we write ${}^{\alpha}h_1(t) = e^{\alpha t} h_1(t) \qquad t \in \mathbb{R}$.

Then from lemma 1 we get :

Lemma 2

For every real t we have ${}^{\alpha}f(t) = \sum_{n=0}^{+\infty} {}^{\alpha}\mu_T * {}^{\alpha}h_1(t)$

Now we are going to study some properties of T and $\ell n(a_1 a_2 \cdots a_T)$ under ${}^{\alpha}\mathbb{P}$. For that purpose we consider the new random variables $g_i'(i \geq 1)$ defined by $g_i' = (a_i^{-1}, b_i a_i^{-1})$. Under ${}^{\alpha}\mathbb{P}$, there random variables are i.i.d with law ${}^{\alpha}\mu'$. We have :

$$g_n' g_{n-1}' \cdots g_1' = ((a_1 a_2 \cdots a_n)^{-1}, R_n(a_1 \cdots a_n)^{-1}),$$

hence for $T' = Inf\{n\ ;\ g_n' g_{n-1}' \cdots g_1' \in G_+\}$ we have $T' = T$. It follows that T can be interpreted as the entrance time in $\mathbb{R}_+ =]0, \infty[$ of the affine random walk on $\mathbb{R}$ defined by ${}^{\alpha}\mu'$, starting from 0. We denote by ${}^{\alpha}Q$ the Markov kernel of this affine random walk, and for $p \in \mathbb{R}$ we write $p_n = g_n' g_{n-1}' \cdots g_1' p$.

Lemma 3

1) There exists a unique probability measure ${}^{\alpha}\nu'$ on $\mathbb{R}$ such that ${}^{\alpha}Q({}^{\alpha}\nu') = {}^{\alpha}\nu'$. The probability ${}^{\alpha}\nu'$ has no atoms.

2) If ${}^{\alpha}\nu'(]0, +\infty] > 0$ then $0 < {}^{\alpha}\mathbb{E}(T') < \infty$.

Now we complete the proof of Theorem 1 using the above Lemmas.

For assertion 3, there are two cases.

First case ${}^{\alpha}\nu'(]0, +\infty]) > 0$.

Then by Lemma 3 and the observation before Lemma 3, ${}^{\alpha}\mathbb{E}(T) = {}^{\alpha}\mathbb{E}(T') < \infty, {}^{\alpha}\mu_T(\mathbb{R}) = 1$. By Wald's lemma (see [7]), since $T' < \infty$ ${}^{\alpha}\mathbb{P} - a.e$:

${}^{\alpha}\mathbb{E}\{\ell n(a_1 a_2 \cdots a_T) = {}^{\alpha}\mathbb{E}(\ell n(a_1))^{\alpha}\mathbb{E}(T)$

where ${}^{\alpha}\mathbb{E}(\ell n(a_1)) = \mathbb{E}(a_1^{\alpha}\ell n(a_1))$ is finite and positive, hence ${}^{\alpha}\mathbb{E}(S_T)$ is finite and positive. Assume $c_+ = 0$, hence $\lim_{t\to\infty} {}^{\alpha}f(t) = 0$. Then, if we denote by ${}^{\alpha}h_{1,L}$ $(L > 0)$ the function $t \to {}^{\alpha}h_1(t)1_{[-L,L]}(t)$, we have using Lemma 2 and Proposition A below : for every $L > 0$,

$$0 = \lim_{t\to+\infty} \frac{1}{t}\int_0^t \sum_{n=0}^{+\infty} {}^{\alpha}\mu_T^n * {}^{\alpha}h_{1,L}(s)ds = \frac{1}{{}^{\alpha}\mathbb{E}(\ell n(a_1))^{\alpha}\mathbb{E}(T)} \int_{-L}^{L} {}^{\alpha}h_1(s)ds,$$

hence $0 = \int_{\mathbb{R}} {}^{\alpha}h_1(s)ds$. Since ${}^{\alpha}h_1$ and h_1 are non negative we get $h_1 = 0$ $a.e$, hence Lemma 1 implies $f(t) = 0$, $dt - a.e.$

We conclude that for almost every real s :

$f(s) = \mathbb{P}(R > e^s) = 0,$

and so $\mathbb{P}(R \leq 0) = 1$, hence $supp\nu \subset]-\infty, 0]$. It follows that $supp\mu$ preserves an interval $(-\infty, v_0)$ with $v_0 \leq 0$.

<u>Second case</u> ${}^{\alpha}\nu'(]0, +\infty[) = 0$.

Denote by $v_0 \leq 0$ the upper bound of the support of the probability ${}^{\alpha}\nu'$. Then by the stationarity property of ${}^{\alpha}\nu'$ we can write that for every $n \geq 1$:

${}^{\alpha}\mathbb{P}\{g'_n g'_{n-1} \cdots g'_1 v_0 \leq v_0\} = 1, \quad 1 = \mathbb{E}(a_1^{\alpha} \cdots a_n^{\alpha} 1_{\{v_0 + R_n \leq a_1 \cdots a_n v_0\}}),$

which implies that for every integer $n \geq 1$, $\mathbb{P}(R_n \leq -v_0) = 1$ since $\mathbb{E}(a_1^{\alpha}) = 1$. Since R_n converges $\mathbb{P} - a.e$ to R we have $\mathbb{P}(R \leq -v_0) = 1$ hence $c_+ = 0$.

In conclusion we see that $c_+ = 0$ if and only if the upper bound of $supp\nu$ is finite i.e if $supp\mu$ preserves an interval $]-\infty, -v_0]$.

In order to show assertion 4 we will distinguish the 2 cases $c_+ > 0$, $c_- = 0$, $c_+ > 0$, and $c_- > 0$. We observe that $supp\nu$ is invariant under $supp\mu$ and condition $\int \ell n(a(g))d\mu(g) < 0$ implies that for some $g \in (supp\mu)^2$ we have $0 < a(g) < 1$. Also the complement of $supp\nu$ is invariant under $(supp\mu)^{-1}$. We denote by T_μ the closed subsemigroup of G generated by $supp\mu$, and by $\Delta \subset \mathbb{R}$ the closure of the set of attractive fixed points of the elements of T_μ. We observe that $T_\mu\Delta \subset \Delta$. Since for any $x \in \Delta$ the law of $g_n \cdots g_1 x$ is supported by Δ and converges to ν, we obtain that $\Delta \supset supp\nu$. Since the attractive fixed points of T_μ belong to $supp\nu$, we conclude that $\Delta = supp\nu$. Then, for any open interval $I = [a, b] \subset \mathbb{R}$, $n < 0$, $g^n(I)$ is an interval of length $a^n(g)(b-a)$ which converges to $+\infty, -\infty$ or $\mathbb{R}$, depending of the relative positions of I and the fixed point x_0 of g. If $c_+ > 0$ and $c_- = 0$, then from above $supp\mu$ preserves the interval $[\tau, \infty[$ with $\tau = Inf(supp\nu)$. Since $\Delta = supp\nu$ we can choose $g \in (supp\mu)^2$ such that its fixed point $x_0 \in supp\nu$ is arbitrary close to τ, and in particular $\tau \leq x_0 < a$. If $I \subset]\tau, \infty[$ satisfies $\nu(I) = 0$ then $\nu(g^n(I)) = 0$ for $n < 0$; since the length of the interval $g^n(I)$ is $a^n(g)(b-a)$ and $\lim_{n\to-\infty} a^n(g) = \infty$ this contradicts $c_+ > 0$.

If $c_+ > 0$, $c_- > 0$ the same argument is valid for any interval I with $\nu(I) = 0$. $\square$

We now give the proofs of the above lemmas.

Proof of Lemma 1

1) Denote $R^n = \sum_{k=n}^{+\infty} a_{n+1} \cdots a_k b_{k+1}$. Under $\mathbb{P}$ the law of R_n is ν and moreover R^n is independant of the random variables $g_i (1 \leq i \leq n)$.

The formula $R = R_n + a_1 \cdots a_n R^n$ gives $R - b_1 = a_1 R^1$, hence :

$\mathbb{P}\{R - b_1 > e^t\} = \mathbb{P}\{R^1 > e^t a_1^{-1}\} = \mu * f(t), \quad f(t) = \mu * f(t) + f_1(t)$

We have also from above

$\{R > e^t\} = \{R_T + a_1 a_2 \cdots a_\tau R^T > e^t,\ T < \infty\}$

$= \{R^T > e^{t - \ell n(a_1 a_2 \cdots a_T)}\ ;\ T < \infty\}\ U \{e^{t - \ell n(a_1 a_2 \cdots a_T)} < R^T \leq e^{t - \ell n(a_1 a_2 \cdots a_T)}\ ;\ T < \infty\}$

Using the fact that T is a stopping time we have

$f(t) = \mathbb{P}\{R > e^t\} = \mathbb{P}\{R > e^t,\ T < \infty\} = \mu_T * f(t) + h_1(t)$

where $h_1(t) = \mathbb{E}(1_{\{T<\infty\}} \nu] e^{t - \ell n(a_1 \cdots a_T)} - R_T, e^{t - \ell n(a_1 \cdots a_T)}]$,

It follows :

$f = \sum_{k=0}^{n} \bar{\mu}^k * f_1 + \bar{\mu}^{n+1} * f$

where $\bar{\mu}^{n+1} * f(t) = \mathbb{P}\{R > e^t (a_1 \cdots a_{n+1})^{-1}\}$. The condition $\mathbb{E}(\ell n(a_1)) < 0$ implies the $\mathbb{P} - a.e$ convergence of $(a_1 \cdots a_{n+1})^{-1}$ to ∞, hence $\lim_{n\to\infty} \bar{\mu}^{n+1} * f(t) = 0$. The first part of the formula follows.

2) From above we deduce that for every integer n and $t \in \mathbb{R}$.

$f(t) = \sum_{j=0}^{n} \mu_T^j * h_1(t) + \mu_T^{n+1} * f(t)$.

We now prove that $\quad \lim_{n\to+\infty} \mu_T^{n+1} * f_1(t) = 0$.

There are two cases

Case 1) $\mathbb{P}(T < \infty) < 1$

We have

$0 \leq \mu_T^{n+1} * f(t) \leq (\mathbb{P}(T < \infty))^n$

hence $\lim_{n\to\infty} \mu_T^{n+1} * f(t) = 0$.

Case 2) $\mathbb{P}(T < \infty) = 1$

Define the shift θ on Ω by $\theta(\omega) = (g_{i+1}(\omega), i \geq 1)$ where $\omega = (g_i(\omega), i \geq 1)$ and consider the sequence $(T_n(\omega))_{n\geq 1}$ of random times defined $\mathbb{P} - a.e$ by $T_{n+1} = T_0 \theta^{T_n}$, $\ T_1 = T$. Under $\mathbb{P}$ the sequence of random variables $[(T_1, S_{T_1}), \cdots, (T_{n+1} - T_n\ ,\ S_{T_{n+1}} - S_{T_n})]$, is i.i.d and the law of S_{T_n} is μ_T^n. Because $\mathbb{E}(\ell n(a_1)) < 0$, we have $\mathbb{P} - a.e$ $\lim_{n\to\infty} S_n = -\infty$ and moreover $\lim_{n\to\infty} T_n = \infty$ hence $\mathbb{P} - a.e$, $\lim_{n\to\infty} S_{T_n} = -\infty$. We have that

$\mu_T^{n+1} * f(t) = \mathbb{E}(f(t - S_{T_{n+1}}))$,

and $\lim_{t\to\infty} f(t) = 0$. So, using Lebesgue's theorem, we can conclude that $\lim_{n\to\infty} \mu_T^{n+1} * f(t) = 0$
□

Proof of lemma 2

Lemma 2 us a direct consequence of the formula ${}^{\alpha}\mu_T^n * {}^{\alpha}h(t) = e^{\alpha t} \mu_T^n * h_1(t)$, Lemma 1 part 2, and the fact that h_1 is non negative. □

Proof of lemma 3

The definition of ${}^{\alpha}\mu'$ and the condition $H_\mu(3)$ imply $\int |\ell n(a(g))|\ {}^{\alpha}\mu'(dg) < \infty$. The strict convexity of the function $\ell n \int a^s(g)\mu(dg)(s > 0)$ gives $\int \ell n(a(g))\ {}^{\alpha}\mu'(dg) < 0$.
It follows $\int |\ell n|b(g)|\ {}^{\alpha}\mu'(dg) < \infty$.
As observed above, the existence and uniqueness of ${}^{\alpha}\nu'$ follows.
If x_0 is a fixed point of $supp^{\alpha}\mu'$ then for any $(a,b) \in supp\mu$;

$a^{-1}x_0 + ba^{-1} = x_0$, i.e $x_0(a-1) = b$.

This implies that $-x_0$ is a fixed point of $supp\mu$, which contradicts $H_\mu(4)$. Hence, as it well known (see [5]), ${}^{\alpha}\nu'$ has no atom.
In order to show ${}^{\alpha}\mathbb{E}(T') < \infty$ we consider the space ${}^{a}\Omega^{\#} = \mathbb{R} \times G^{\mathbb{Z}}$ and the extended bilateral shift defined by ${}^{a}\theta(p,\omega) = (p_1, \theta\omega)$ where $p_1 = g_1'(p)$ and θ is the bilateral shift on $G^{\mathbb{Z}}$. We endow ${}^{a}\Omega^{\#}$ with the Markov measure $\kappa^{\#}$ associated with the ${}^{\alpha}Q$-invariant probability ${}^{\alpha}\nu'$. Clearly $\kappa^{\#}$ is ${}^{a}\theta$-invariant and ergodic. Also we consider the fibered bilateral Markov chain (p_n, V_n) on $\mathbb{R} \times \mathbb{R}^*$ where $V_n = p^{-1}p_n(a_1 a_2 \cdots a_n) = p^{-1}(p + R_n)$. Let τ be the first "ladder epoch" of (p_n, V_n) (see [7]), i.e $\tau = Inf\{n \geq 1\ ;\ V_n > 1\}$, hence $p^{-1}p_\tau > 0$ and $\tau = T$ if $p > 0$. We observe that the conditions in $H_\mu(3)$ implies $\int |p|^{\varepsilon}\ {}^{\alpha}\nu'(dp) < \infty$ for some $\varepsilon > 0$, hence $\limsup_{|n|\to\infty} \frac{1}{|n|} \ell n|p_n| \leq 0$. Since ${}^{\alpha}\mathbb{E}(\ell n(a_1)) > 0$ the ergodic theorem gives $\lim_{n\to\infty} |V_n| = \infty$, $\lim_{n\to-\infty} |V_n| = 0$, in particular τ is finite $\kappa^{\#} - a.e.$ Since ${}^{\alpha}\nu'(\mathbb{R}_+) > 0$ we can consider the Markov kernel ${}^{\alpha}Q_+$ induced by ${}^{\alpha}Q$ on $\mathbb{R}_+$; the normalized restriction ${}^{\alpha}\nu'_+$ of ${}^{\alpha}\nu'$ to $\mathbb{R}_+$ is ${}^{\alpha}Q_+$-invariant and ergodic. We denote by ${}^{a}\Omega_+^{\#}$ the subset of ${}^{a}\Omega^{\#}$ defined by the conditions $p_n > 0$ infinitely often for $n = n_k > 0$ and $n = n_{-k} < 0$. Since ${}^{\alpha}\nu'(\mathbb{R}_+) > 0$, ${}^{a}\Omega_+^{\#}$ has positive $\kappa^{\#}$-measure and we denote by $\kappa_+^{\#}$ the normalized restriction of $\kappa^{\#}$ to ${}^{a}\Omega_+^{\#}$; then $\kappa_+^{\#}$ is invariant and ergodic under the corresponding induced shift ${}^{a}\theta_+$. From above we know that $\lim_{k\to\infty} V_{n_{-k}} = 0$, hence the time $\tau_+(\omega) = n_{-j}$ $(j \geq 0)$, of the last strict maximum of $V_{n_{-k}}$ is finite $\kappa_+^{\#} - a.e.$ We define ${}^{a}\Omega_0^{\#} = \{\sup_{k>0} V_{n_{-k}} < 1\} = \{\tau_+ = 0\}$. Then we have ${}^{a}\kappa_+^{\#}\ ({}^{a}\Omega_0^{\#}) > 0$ since, by ${}^{a}\theta_+$-invariance of $\kappa_+^{\#}$:

$$1 = \kappa_+^{\#}\{\tau_+ > -\infty\} = \sum_{n\geq 0} \kappa_+^{\#}\{\tau_+ = -n\} \leq \sum_{n\geq 0} \kappa_+^{\#}\{V_{-n} > \sup_{n_{-k}<-n} V_{n_{-k}}\} =$$
$$\sum_{n\geq 0} \kappa_+^{\#}\{0 > \sup_{k>0} V_{n_{-k}}\} \leq \infty \kappa_+^{\#}({}^{a}\Omega_0^{\#}).$$

On the other hand, the definition of τ shows that for $\omega \in {}^{a}\Omega_0^{\#}$, $\tau(\omega)$ is the first return time of ${}^{a}\theta^k(\omega)$ to ${}^{a}\Omega_0^{\#}$, so that ${}^{a}\theta^{\tau}$ is the transformation of ${}^{a}\Omega_0^{\#}$ induced by ${}^{a}\theta$ on ${}^{a}\Omega_0^{\#}$. Then Kac's theorem (see [23]) implies that ${}^{a}\theta^{\tau}$ is ergodic with respect to the normalized restriction $\kappa_0^{\#}$ of $\kappa_+^{\#}$ to ${}^{a}\Omega_0^{\#}$ and ${}^{\alpha}\mathbb{E}(\tau) = \int \tau(\omega)\kappa_0^{\#}(d\omega) < \infty$. Also we denote by ${}^{\alpha}\nu_+^{\tau}$ the push forward of $\kappa_0^{\#}$ to $\mathbb{R}_+$ under the map $\omega \to p_0(\omega)$. Since the stopped kernel ${}^{\alpha}Q_+^{\tau}$ and the map τ commute with p_0, the measure κ_+^{τ} is ${}^{\alpha}Q_+^{\tau}$-invariant, ergodic and absolutely continuous with respect to ${}^{\alpha}\nu_+$ with ${}^{\alpha}\mathbb{E}_0(\tau) = {}^{\alpha}\mathbb{E}(T) < \infty$. □

Remark

A different proof of ${}^{\alpha}\mathbb{E}(T) < \infty$ uses the interpretation of $T = T'$ as hitting time of the open set $\mathbb{R}_+$ by the Markov chain with kernel ${}^{\alpha}Q$ starting from 0.
Since $\int a^{\delta}(g)\ {}^{\alpha}\mu'(dg) < \infty$, $\int |b^{\delta}(g)|\ {}^{\alpha}\mu'(dg) < \infty$ for $0 < \delta < \alpha$, the operator defined by ${}^{\alpha}Q$ on a space of Hölder functions on $\mathbb{R}$ (as in [13]) has a spectral gap. This implies ${}^{\alpha}\mathbb{E}(T') < \infty$. The proof given above extends to the multidimensional case.

IV - Appendix : a weak renewal theorem

Proposition A Let $(Z_n)_{n\geq 1}$ a sequence of independant, identically distributed real random variables on $\mathbb{R}$ with law η. Assume that $\int |z|\eta(dz) < +\infty$ and that $\gamma = \int z\eta(dz) > 0$.
Let ψ a bounded non negative Borel function which is supported on $[-a, a]$.
Then the potential $U\psi = \sum_{n=0}^{+\infty} \eta^n * \psi$ is a bounded function and we have

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t U\psi(s)ds = \frac{1}{\gamma}\int_{\mathbb{R}} \psi(t)dt$$

Proof

If $\Sigma_n = \sum_{i=1}^{n} Z_i$, we have : $U\psi(s) = \sum_{n=0}^{+\infty} E[\psi(s - \Sigma_n)]$.

Because $\gamma = \int z\eta(dz) > 0$ the random walk on $\mathbb{R}$ with law η is transient and using the maximum principle we have that $\sup_{s\in\mathbb{R}} U\psi(s) < +\infty$.

For $\varepsilon > 0$, $t > 0$ denote

$$n_1(t) = [\tfrac{1}{\gamma}\varepsilon t] = n_1, \qquad n_2(t) = [\tfrac{1}{\gamma}(1-\varepsilon)t] = n_2,$$

$$U_n\psi = \sum_{0}^{n-1} \eta^k * \psi, \qquad U^n\psi = \sum_{n+1}^{\infty} \eta^k * \psi, \qquad U_n^m\psi = \sum_{n}^{m} \eta^k * \psi.$$

Then we have

$$I(t) = \tfrac{1}{t}\int_0^t U\psi(s)ds = \sum_{1}^{3} I_k(t) - I_4(t)$$

where

$$I_1(t) = \tfrac{1}{t}\int_{\mathbb{R}} U_{n_1}^{n_2}\psi(s)ds, \quad I_2(t) = \tfrac{1}{t}\int_{\mathbb{R}} U_{n_1}\psi(s)ds,$$

$$I_3(t) = \tfrac{1}{t}\int_{\mathbb{R}} U^{n_2}\psi(s)ds, \quad I_4(t) = \tfrac{1}{t}\int_{\mathbb{R}[0,t]} U_{n_1}^{n_2}\psi(s)ds.$$

We have

$$I_1(t) = \tfrac{n_2-n_1+1}{t}(\int_{\mathbb{R}} \psi(s)ds) \text{ hence } \lim_{t\to+\infty} I_1(t) = \frac{(12\varepsilon)}{\gamma}\int_{\mathbb{R}} \psi(s)ds,$$

$$0 \leq I_4(t) \leq (\tfrac{(n_2-n_1+1}{t}\sup_{s\in\mathbb{R}}|\psi(s)|)\sup_{n_1\leq n\leq n_2}[\mathbb{P}(\Sigma_n \leq a) + \mathbb{P}(t - \Sigma_n \leq a)].$$

By the law of large numbers we know that $\mathbb{P} - a.e$, $\lim_{n\to+\infty}\frac{\Sigma_n}{n} = \gamma > 0$, hence :

$$\lim_{t\to+\infty}\sup_{n_1\leq n\leq n_2}(\mathbb{P}\{\Sigma_n \leq a\} + P\{t - \Sigma_n \leq a\}) = 0.$$

Hence $\lim_{t\to+\infty} I_4(t) = 0$ and : $0 \leq \frac{I_2(t)}{t} \leq \frac{\varepsilon}{\gamma} \times \int \psi(s)ds$.

Consider now $I_3(t)$ and denote for $n \in \mathbb{N}$, $s > 0 : \rho_n^s = Inf\{k \geq n \,;\, |V_n - s| \leq a\}$.
We use the interpretation of $U^n\psi$ as the expected number of visits to ψ after time n : $U^n\psi(x) \leq (U\psi)\mathbb{P}\{\rho_n^s < \infty\}$ with $n\frac{[(1+\varepsilon)t]}{\gamma} = n_2$, hence :

$I_3(t) \leq |U\psi|\mathbb{P}\{\Sigma_k \leq t + a$ for some $k \geq \frac{(1+\varepsilon)t}{\gamma}\}$.

Since $\frac{\Sigma_n}{n}$ converges to γ, $\mathbb{P} - a.e$, we get $\lim\limits_{t\to\infty} I_3(t) = 0$.
Since ε is arbitrary we get finally : $\lim\limits_{t\to\infty} I(t) = \frac{1}{\gamma}\int \psi(s)ds$. □

References

[1] Babillot M., Bougerol P., Elie L. : The random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. Ann. Probab. 25,1 (478-493) (1997).

[2] Bardou F., Bouchaud J.P., Aspect A., Cohen-Tannoudji C. : "Levy Statistics and Laser Cooling". Cambridge Univ. Press (2002).

[3] Basrak B., Segers J. : Regularly varying time series. Stochastic Pocess Appl. 119, 1055-1080 (2009).

[4] Brofferio S., Buraczewski D., Damek E. : On the invariant measure of the random difference equation $X_n = A_n X_{n-1} + B_n$ in the critical case. Ann. I.H.P. 48, 377-395 (2012).

[5] Buraczewski D. and Damek E.,Guivarc'h Y., Hulanicki A., Urban R. : Tail-homogeneity of stationary measures for some multidimensional stochastic recursions. Probab. Theory Relat. Fields 145 n°3-4 385-420 (2009).

[6] Dolgopyat D., Goldsheid I. : Quenched limit theorems for nearest neighbour random walks in 1D random environment. Comm. Math. Phys. 315, 1, 247-277 (2012).

[7] Feller W. : "An Introduction to Probability Theory and its Applications". vol. II, 2nd edn. Wiley New-York (1971).

[8] Galambos J. : The distribution of the largest coefficient in the continued fraction expansion. Quart. J. Math. 23, 147-151 (1972).

[9] Gnedenko B. : Sur la distribution limite d'une série aléatoire. Annals of Math. 44 n°3, 423-453 (1943).

[10] Goldie C.M. : Implicit renewal theory and tails of solutions of random equations. Ann. Appl. Probab. 1(1) 126-166 (1991).

[11] Guivarc'h Y. : Heavy tail properties of stationary solutions of multidimensional stochastic recursions. In "Dynamics and Stochastics". IMS Lecture Notes Monograph Series, vol. 48, pp 85-99. Institute of Mathematical Statistics, Beachwood (2006).

[12] Guivarc'h Y., Le Page E. : Homogeneity at infinity of stationary solutions of Multivariate Affine Stochastic Recursions. In "Random Matrices and Iterated Random functions". Alsmeyer G. - Löwe M. Eds. Springer Proceedings in Mathematics and Statistics 53, pp 119-135 (2013).

[13] Guivarc'h Y., Le Page E. : On spectral properties of a family of transfer operators and convergence to stable laws for affine random walks, Ergod. Th. Dynam. Sys. 28, 423-446 (2008).

[14] Guivarc'h Y., Le Page E. : Asymptotique des valeurs extrêmes pour les marches aléatoires affines. C.R.A.S (2013).

[15] Haan L., Resnick S.I., Rootzen H., Vries C.G. : Extremal behaviour of solutions to a stochastic difference equation with applications to ARCH processes. Stochastic Process. Appl. 32, 213-224 (1989).

[16] Kesten H. : Random difference equations and renewal theory for products of random matrices. Acta Mathematica 131, 207-248 (1973).

[17] Kesten H., Kozlov M., Spitzer F. : A limit law for random walk in a random environment, Compositio Math. 30 145-168 (1975).

[18] Kleinbock D.Y., Margulis G.A. : Logarithm laws for flows on homogeneous spaces, Inv. Math. 138, 451-494 (1999).

[19] Leadbetter M.R., Lindgren G., Rootzen H. : "Extremes and related properties of random sequences and processes", Springer, Berlin (1983).

[20] Pollicott M. : Limiting distributions for geodesic excursions on the modular surface, Contemp. Math. 469, 117-185 (2008).

[21] Solomon F. : Random walks in a random environment. Ann. Proba. 3, 1-31 (1975).

[22] Sullivan D. : Disjoint spheres, approximation by imaginary quadratic numbers and the logarithm law for geodesics, Acta Math. 149, 215-237 (1982).

[23] Wolfowitz J. : Remarks on the notion of recurrence. Bull A.M.S. 55, 394-395 (1949).