Sym-Bobenko formula for minimal surfaces in Heisenberg space

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Abstract

We give an immersion formula, the Sym-Bobenko formula*, for minimal surfaces in the* 3*-dimensional Heisenberg space. Such a formula can be used to give a generalized Weierstrass type representation and construct explicit examples of minimal surfaces.*

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1 Introduction

A Sym-Bobenko formula is the expression of an immersion in terms of a one-parameter family of moving frames, called the *extended frame*. This idea was first used by A. Sym [9] in the case of surfaces with negative constant (Gauss) curvature in euclidean space. A. I. Bobenko applied the method to numerous cases [1] [2] [3], including constant mean curvature (*CMC* for short) surfaces in space forms — euclidean 3-space, 3-sphere and hyperbolic 3-space — and T. Taniguchi applied it to CMC spacelike surfaces in Minkowski 3-space [10]. In the mean time, the work of J. Dorfmeister, F. Pedit and H. Wu [8] and D. Brander, W. Rossman and N. Schmitt [4] show that Sym-Bobenko formulæ can be seen as generalized Weierstrass type representations for CMC surfaces, extended frames coming from holomorphic data.

In Heisenberg 3-space, the classical method does not apply, since the isometry group is of dimension only 4 — contrary to the ones of space forms that are 6-dimensional — and does not act transitively on orthonormal frames; there are “not enough” isometries to define a moving frame. We show that nevertheless, for minimal immersions a Sym-Bobenko formula can be established using an *ad-hoc* matrix-valued map.

In [7], J. F. Dorfmeister, J. Inoguchi and S. Kobyashi link this formula with pairs of meromorphic and anti-meromorphic 1-forms, which they call *pairs of normalized potentials*, in a way to get a generalized Weierstrass type representation for minimal surfaces.

*This work is part of the author’s Ph. D. thesis [5].

2 Surfaces in Heisenberg space

We see the 3-dimensional Heisenberg space Nil_3 as $\mathbb{R}^3$, with generic coordinates (x_1, x_2, x_3), endowed with the following riemannian metric:

$$\langle \cdot, \cdot \rangle = dx_1^2 + dx_2^2 + \left(\frac{1}{2}(x_2 dx_1 - x_1 dx_2) + dx_3 \right)^2 .$$

We call *canonical frame* the orthonormal frame (E_1, E_2, E_3) defined by:

$$E_1 = \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad E_2 = \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} \quad \text{and} \quad E_3 = \frac{\partial}{\partial x_3},$$

and the Levi-Civita connection ∇ writes:

$$\begin{array}{lll} \nabla_{E_1} E_1 = 0 & \nabla_{E_2} E_1 = -\frac{1}{2} E_3 & \nabla_{E_3} E_1 = -\frac{1}{2} E_2 \\ \nabla_{E_1} E_2 = \frac{1}{2} E_3 & \nabla_{E_2} E_2 = 0 & \nabla_{E_3} E_2 = \frac{1}{2} E_1 \\ \nabla_{E_1} E_3 = -\frac{1}{2} E_2 & \nabla_{E_2} E_3 = \frac{1}{2} E_1 & \nabla_{E_3} E_3 = 0. \end{array}$$

Note that the vector field E_3 is a Killing field and that the projection $\pi : (x_1, x_2, x_3) \in \mathrm{Nil}_3 \mapsto (x_1, x_2) \in \mathbb{R}^2$ on the first two coordinates is a Riemannian submersion. From now on, we identify $\mathbb{R}^2$ with $\mathbb{C}$.

We may also write Nil_3 as a subset of $\mathcal{M}_2(\mathbb{C})$. Consider the matrices:

$$\sigma_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The identification is the following:

$$\begin{aligned} (x_1, x_2, x_3) \in \mathrm{Nil}_3 &\longleftrightarrow x_1\sigma_1 + x_2\sigma_2 + x_3\sigma_3 \\ &= \begin{pmatrix} x_3 & x_1 + ix_2 \\ x_1 - ix_2 & x_3 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}). \quad (1) \end{aligned}$$

Note that this identification is purely formal and does not involve any manifold related structure.

Let Σ be a simply connected Riemann surface and z be a conformal parameter on Σ. A conformal immersion is denoted $f : \Sigma \to \mathrm{Nil}_3$ with unit normal N and conformal factor $\rho : \Sigma \to (0, +\infty)$ meaning:

$$\langle f_z, f_z \rangle = \langle f_{\bar z}, f_{\bar z} \rangle = 0, \quad \langle f_z, f_{\bar z} \rangle = \frac{\rho}{2},$$
$$\langle f_z, N \rangle = \langle f_{\bar z}, N \rangle = 0 \quad \text{and} \quad \langle N, N \rangle = 1.$$

Consider also $\varphi = \langle N, E_3 \rangle : \Sigma \to (-1, 1)$ denote the angle function of N, $A = \langle f_z, E_3 \rangle : \Sigma \to \mathbb{C}$ the vertical part of f_z and $p dz^2 = \langle \nabla_{f_z} f_z, N \rangle dz^2$ the Hopf differential of f.

The Abresch-Rosenberg differential expresses $Qdz^2 = (ip + A^2)dz^2$, and a necessary and sufficient condition for f to be minimal is $\nabla_{f_z} f_{\bar{z}} = 0$.

We also decompose f into $f = (F, h)$ with $F = \pi \circ f : \Sigma \to \mathbb{C}$ the horizontal projection of f and $h : \Sigma \to \mathbb{R}$ its height function. We can express A in terms of F and h:

$$A = h_z - \frac{i}{4}\left(F\overline{F}_z - \overline{F}F_z\right). \tag{2}$$

In the matrix model (1) of Nil_3, the map F is given by the non-diagonal coefficients — precisely the $(1,2)$-coefficient — and h by the diagonal ones.

The intuitive idea behind Sym-Bobenko formulæ in space forms is that, up to ambient isometries, the unit normal — or Gauss map — would locally determine the immersion up to ambient isometries. In Nil_3 such a map is defined as follows; see [6] for details. Since Nil_3 is a Lie group, the map $f^{-1}N$ takes values in the unit sphere $\mathbb{S}^2$ of the Lie algebra. Moreover, for a local study, we can suppose $\varphi > 0$ so that the values of $f^{-1}N$ are actually in the northern hemisphere of $\mathbb{S}^2$. If s denotes the stereographic projection centered at the South Pole, we call *Gauss map* of an immersion f the map $g = s \circ (f^{-1}N)$ with values in the unit disk. Actually, endowing the unit disk with the Poincaré metric, we see the Gauss map g as a map with values into the hyperbolic disk $\mathbb{H}^2$.

We use the following criterion to show that a conformal immersion in Heisenberg space is minimal:

Proposition 2.1 (Daniel [6])**.** *A conformal immersion $f = (F, h) : \Sigma \to \mathrm{Nil}_3$ is minimal if and only if:*

$$F_{z\bar{z}} = \frac{i}{2}\left(\overline{A}F_z + AF_{\bar{z}}\right) \quad \textit{and} \quad A_{\bar{z}} + \overline{A}_z = 0.$$

Furthermore, when f is minimal its Gauss map $g : \Sigma \to \mathbb{H}^2$ is harmonic.

3 The Sym-Bobenko formula

Consider the family $(\Psi_t)_{t\in\mathbb{R}}$ of matrix fields over Σ which are solutions of the system:

$$\begin{cases} \Psi_t^{-1} d\Psi_t = \dfrac{1}{4}\begin{pmatrix} (\log\rho_0)_z & i\sqrt{\rho_0} \\ -\dfrac{4iQ_0}{\sqrt{\rho_0}}e^{2it} & -(\log\rho_0)_z \end{pmatrix} dz \\ \qquad\qquad\qquad + \dfrac{1}{4}\begin{pmatrix} -(\log\rho_0)_{\bar{z}} & \dfrac{4i\overline{Q_0}}{\sqrt{\rho_0}}e^{-2it} \\ -i\sqrt{\rho_0} & (\log\rho_0)_{\bar{z}} \end{pmatrix} d\bar{z} \\ \Psi_t(z=0) = \sigma_3 \end{cases},$$

where $\rho_0 : \Sigma \to (0, +\infty)$ and $Q_0 : \Sigma \to \mathbb{C}$ are smooth. Such a family (Ψ_t) exists if and only if:

$$(\log \rho_0)_{z\bar{z}} = \frac{\rho_0}{8} - \frac{2|Q_0|^2}{\rho_0} \quad \text{and} \quad (Q_0)_{\bar{z}} = 0.$$

Theorem 3.1 (Sym-Bobenko formula)**.** *Using the matrix model* (1)*, define the map $f_t : \Sigma \to Nil_3$ for any $t \in \mathbb{R}$ as:*

$$f_t = -\frac{1}{2}\left(\sigma_0 \frac{\partial \widehat{f}_t}{\partial t}\right)^d + \left(\widehat{f}_t\right)^{nd} \quad \textit{with} \quad \widehat{f}_t = -2\frac{\partial \Psi_t}{\partial t}\Psi_t^{-1} + 2\Psi_t \sigma_0 \Psi_t^{-1}, \quad (3)$$

where the superscripts $\cdot^d$ and $\cdot^{nd}$ denote respectively the diagonal an non-diagonal terms. Then f_t is a conformal minimal immersion in Heisenberg space and the family (f_t) is the so-called associated family*.*

Proof. Fix $t \in \mathbb{R}$. From Equation (3), we get:

$$(f_t)^{nd} = (\widehat{f}_t)^{nd}, \quad ((f_t)_z)^d = \frac{1}{2}\left(\sigma_0\left[(\widehat{f}_t)_z, \widehat{f}_t\right]\right)^d + 2\left(\sigma_0(\widehat{f}_t)_z\right)^d$$
$$\text{and} \quad (\widehat{f}_t)_{z\bar{z}} = \frac{i}{4}\left[(\widehat{f}_t)_z, (\widehat{f}_t)_{\bar{z}}\right], \quad (4)$$

where $[\cdot,\cdot]$ denotes the commutator. From the first equation in (4), we have that matrices f_t and $\widehat{f}_t$ write:

$$f_t = \begin{pmatrix} h & F \\ \overline{F} & h \end{pmatrix} \quad \text{and} \quad \widehat{f}_t = \begin{pmatrix} i\widehat{h} & F \\ \overline{F} & -i\widehat{h} \end{pmatrix},$$

with $F : \Sigma \to \mathbb{C}$ and $h, \widehat{h} : \Sigma \to \mathbb{R}$ smooth. We show that F and h verify the conditions of Proposition 2.1. Using Equation (2) and the second identity in (4), we deduce $A = i\widehat{h}_z$ and since $\widehat{h}$ is real-valued, we obtain $A_{\bar{z}} + \overline{A}_z = 0$. Finally, the $(1,2)$-coefficient of the third equation in (4) verify:

$$F_{z\bar{z}} = \frac{1}{2}\left(\widehat{h}_{\bar{z}}F_z - \widehat{h}_z F_{\bar{z}}\right) = \frac{i}{2}\left(\overline{A}F_z + AF_{\bar{z}}\right),$$

which concludes the proof. □

References

[1] A. I. Bobenko, *All constant mean curvature tori in $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$ in terms of theta functions*, Math. Ann. **290** (1991), no. 2, 209–245.

[2] A. I. Bobenko, *Constant mean curvature surfaces and integrable equations*, Russian Math. Surveys **46** (1991), no. 4, 1–45.

[3] A. I. Bobenko, *Surfaces in terms of* 2 *by* 2 *matrices. Old and new integrable cases*, Harmonic maps and integrable systems, Aspects Math., E23, Friedr. Vieweg, Braunschweig, 1994, pp. 83–127.

[4] D. Brander, W. Rossman, and N. Schmitt, *Holomorphic representation of constant mean curvature surfaces in Minkowski space: consequences of non-compactness in loop group methods*, Adv. Math. **223** (2010), 949–986.

[5] S. Cartier, *Surfaces des espaces homogènes de dimension* 3, Ph.D. thesis, Université Paris-Est, 2011, tel-00672332.

[6] B. Daniel, *The Gauss map of minimal surfaces in the Heisenberg group*, Int. Math. Res. Not. IMRN (2011), no. 3, 674–695.

[7] J. F. Dorfmeister, J. Inoguchi, and S. Kobayashi, *A loop group method for minimal surfaces in the three-dimensional Heisenberg group*, preprint arXiv:1210.7300, 2012.

[8] J. F. Dorfmeister, F. Pedit, and H. Wu, *Weierstrass type representation of harmonic maps into symmetric spaces*, Comm. Anal. Geom. **6** (1998), no. 4, 633–668.

[9] A. Sym, *Soliton surfaces and their applications (soliton geometry from spectral problems)*, Geometric aspects of the Einstein equations and integrable systems (Scheveningen, 1984), Lecture Notes in Phys., vol. 239, Springer, Berlin, 1985, pp. 154–231.

[10] T. Taniguchi, *The Sym-Bobenko formula and constant mean curvature surfaces in Minkowski* 3*-space*, Tokyo J. Math. **20** (1997), no. 2, 463–473.

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